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## GENERALIZED DEDEKIND COMPLETION OF A LATTICE ORDERED GROUP

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The notions of the generalized Dedekind completion  $D_1(G)$  and of the archimedean kernel  $A(G)$  of a lattice ordered group  $G$  were introduced in [10]. In this paper some further properties of  $D_1(G)$  are established.

In § 2 it is shown that to each direct decomposition of  $G$  there corresponds a direct decomposition of  $D_1(G)$ . Namely, if  $G$  is a direct product of its  $l$ -subgroups  $G_i$  ( $i \in I$ ), then  $D_1(G)$  is a direct product of its  $l$ -subgroups  $D_1(G_i)$  ( $i \in I$ ). This generalizes a result from [6] concerning archimedean lattice ordered groups. An analogous assertion is valid for direct sums of lattice ordered groups. If  $G$  is epiarchimedean and conditionally orthogonally complete, then  $D_1(G)$  is epiarchimedean. If  $G$  is strongly projectable, then so is  $D_1(G)$ . If  $G$  is projectable and  $A(G)$  is strongly projectable, then  $D_1(G)$  is projectable. If  $G$  is projectable, then  $D_1(G)$  need not be projectable. If  $G$  is conditionally orthogonally complete, then so is  $D_1(G)$ .

Pairwise splitting lattice ordered groups have been studied by MARTINEZ [12]. Generalized Dedekind completions of pairwise splitting lattice ordered groups are dealt with in § 3. It is proved that if  $G$  is a pairwise splitting abelian lattice ordered group such that the archimedean kernel  $A(G)$  of  $G$  is conditionally orthogonally complete, then  $D_1(G)$  is pairwise splitting; the assumption of the conditional orthogonal completeness of  $A(G)$  cannot be omitted.

In § 4 the relations between higher degrees of distributivity of a lattice ordered group  $G$  and those of  $D_1(G)$  are investigated. Let  $\beta$  be a cardinal. For a lattice ordered group  $H$  we write  $d(H) = \alpha$  if  $H$  is  $\gamma$ -distributive for each  $\gamma < \alpha$  and if  $H$  fails to be  $\alpha$ -distributive. Let  $d(G) = \alpha$ . If either  $A(G)$  is completely distributive or  $A(G)$  is projectable, then  $d(D_1(G)) = \alpha$ . If  $A(G)$  is not completely distributive and  $d(G_1) = \beta$ , where  $G_1$  is the Dedekind completion of  $A(G)$ , then  $d(D_1(G)) = \min \{\alpha, \beta\}$ .

A lattice ordered group  $G$  is called  $g$ -complete if  $D_1(G) = G$ . In § 5 it is shown that each lattice ordered group possesses a largest  $g$  complete convex  $l$ -subgroup. This implies that the class of all  $g$ -complete lattice ordered groups is a radical class [11].

## 1. PRELIMINARIES

The standard terminology for lattices and lattice ordered groups will be used (cf. BIRKHOFF [1], CONRAD [2] and FUCHS [5]). The group operation is written additively, the commutativity of this operation is not assumed.

Let us recall some notions and some results from [10]. Let  $G$  be a lattice ordered group. An element  $0 < a \in G$  is called *archimedean* in  $G$  if for each  $0 < x \in G$  there exists a positive integer  $n$  such that  $nx \text{ non } \leq a$ . We denote by  $A(G)$  the  $l$ -subgroup of  $G$  generated by the set of all archimedean elements of  $G$ . Then  $A(G)$  is a closed  $l$ -ideal of  $G$  and  $A(G)$  is archimedean (i.e., each element  $0 < a \in A(G)$  is archimedean in  $A(G)$ ). If  $H$  is a convex  $l$ -subgroup of  $G$  and if  $H$  is archimedean, then  $H \subseteq A(G)$ . We shall often write  $A$  instead of  $A(G)$ , when no ambiguity can occur.

For any archimedean lattice ordered group  $K$  we denote by  $D(K)$  the Dedekind closure of  $K$  (cf. e.g. [1], Chap. XIII, § 13).

For each lattice ordered group  $G$  there exists a lattice ordered group  $D_1(G)$  fulfilling the following conditions:

- (i)  $G$  is an  $l$ -subgroup of  $D_1(G)$ ;
- (ii)  $D(A(G))$  is an  $l$ -ideal of  $D_1(G)$ ;
- (iii) if  $x \in G$  and if  $X$  is a nonempty subset of  $x + A(G)$  such that  $X$  is upper bounded in  $x + A(G)$ , then there is  $x_0 \in D_1(G)$  with  $\sup X = x_0$ ;
- (iv) for each  $x_0 \in D_1(G)$  there exist  $x \in G$  and  $X \subseteq x + A(G)$  such that  $X$  is upper bounded in  $x + A(G)$  and  $x_0 = \sup X$  holds in  $D_1(G)$ .

The lattice ordered group  $D_1(G)$  is determined uniquely up to isomorphisms. More precisely, if  $D'$  is a lattice ordered group fulfilling the conditions (i)–(iv) (with  $D'$  instead of  $D_1(G)$ ), then there exists an isomorphism  $\varphi$  of  $D_1(G)$  onto  $D'$  such that  $\varphi(x) = x$  for each  $x \in G$  and each  $x \in D(A(G))$ .

If  $X$  is a subset of  $G$  and if  $\sup X = x_0$  exists in  $G$ , then  $x_0$  is the least upper bound of  $X$  in  $D_1(G)$  (and dually).  $D_1(G)$  coincides with  $D(G)$  if and only if  $G$  is archimedean. The lattice ordered group  $D_1(G)$  is said to be the generalized Dedekind completion of  $G$ . We have  $A(D_1(G)) = D(A(G))$ . The  $l$ -ideal  $D(A(G))$  is closed in  $D_1(G)$ . If  $G$  is abelian, then  $D_1(G)$  is abelian as well.

For each  $x_0 \in D_1(G)$  there is  $x \in G$  and  $a \in D(A(G))$  such that  $x_0 = x + a$ . If  $0 \leq x_0 \in D_1(G)$ , then there are elements  $0 \leq x_1 \in G$ ,  $0 \leq a_1 \in D(A(G))$  with  $x_0 = x_1 + a_1$ . In fact, if  $D(A(G)) = \{0\}$ ,  $x_0 = x + a$ ,  $x \in G$ ,  $a \in D(A(G))$ , then  $a = 0$  and  $x \geq 0$ . Let  $D(A(G)) \neq \{0\}$ ; then  $D(A(G))$  has no least element. Hence there is  $x' \in x + D(A(G))$  with  $x' \in G$ ,  $x' \leq x_0$  (cf. the condition (iv) above). Put  $x_1 = x' \vee 0$ ,  $a_1 = -x_1 + x_0$ . Then  $x_1 \in x + D(A(G))$ ,  $0 \leq x_1 \leq x_0$ ,  $0 \leq a_1 \in D(A(G))$ ,  $x_0 = x_1 + a_1$ .

Let  $X \subseteq G$ . The set

$$X^\delta = \{g \in G : |g| \wedge |x| = 0 \text{ for all } x \in X\}$$

is called a *polar* of  $G$ . The set  $X^{\delta\delta}$  is said to be a *polar generated by  $X$* ; if  $\text{card } X = 1$ , then  $X^{\delta\delta}$  is called a *principal polar*.

## 2. DIRECT DECOMPOSITIONS

Let us recall some notions concerning direct products and direct sums of lattice ordered groups (cf. e.g. [6]).

Let  $I$  be a nonempty set and for each  $i \in I$  let  $G_i$  be a lattice ordered group. We denote by  $G_1 = \prod_{i \in I} G_i$  the direct product of the lattice ordered groups  $G_i$ . Thus  $G_1$  is the set of all mappings  $f: I \rightarrow \bigcup G_i$  such that  $f(i) \in G_i$  for each  $i \in I$ , the lattice operations and the group operations being defined coordinatewise. For  $i \in I$  we denote  $G^i = \{f \in G_1 : f(j) = 0 \text{ for all } j \in I, j \neq i\}$ .

Let  $G$  be a lattice ordered group and let  $\varphi$  be an isomorphism of  $G$  onto  $G_1$ . For each  $i \in I$  we put  $G_i^0 = \varphi^{-1}(G^i)$ . Each  $G_i^0$  is said to be a *direct factor* of  $G$ . We write also  $G = \prod_{i \in I}^0 G_i^0$ . The  $I$ -subgroup of  $G$  generated by the set  $\bigcup_{i \in I} G_i^0$  will be denoted by  $\sum_{i \in I}^0 G_i^0$  and called *the direct sum of  $G_i^0$  ( $i \in I$ )*. If  $I$  is finite,  $I = \{1, \dots, n\}$ , then  $\prod_{i \in I}^0 G_i^0 = \sum_{i \in I}^0 G_i^0$  and we denote it also by  $G_1^0 \oplus \dots \oplus G_n^0$ .

Each direct factor of  $G$  is a closed  $I$ -ideal in  $G$ . A convex  $I$ -subgroup  $H$  of  $G$  is a direct factor of  $G$  if and only if it fulfils the following conditions:

(a) For each  $0 < g \in G$  the set  $S = \{0 \leq h \in H : h \leq g\}$  possesses a greatest element.

If  $H$  is a direct factor in  $G$  and  $0 \leq g \in G$ , then the greatest element of the set  $S$  will be denoted by  $g(H)$  and it is said to be *the component of  $g$  in  $H$* . For any  $g_1 \in G$  we put  $g_1(H) = g_1^+(H) - g_1^-(H)$ . Let  $H$  be a direct factor of  $G$ ; then  $H^\delta$  is also a direct factor of  $G$  and the mapping  $\psi(g_1) = (g_1(H), g_1(H^\delta))$  is an isomorphism of  $G$  onto  $H \times H^\delta$ . Let  $G = \prod_{i \in I}^0 G_i^0$  and let  $\psi$  be a mapping of  $G$  into  $\prod_{i \in I} G_i^0$  such that  $\psi(g_1)(i) = g_1^i(G_i^0)$  for each  $g_1 \in G$  and each  $i \in I$ . Then  $\psi$  is an isomorphism of  $G$  onto  $\prod_{i \in I} G_i^0$ .

The following two assertions are easy to verify.

**2.1. Lemma.** *Let  $G$  be a lattice ordered group and let  $\{G_j\}_{j \in J}$  be a system of direct factors of  $G$  such that*

- (i)  $G_j \cap G_k = \{0\}$  whenever  $j$  and  $k$  are distinct elements of  $J$ ;
- (ii)  $g = \bigvee_{j \in J} g(G_j)$  for each  $0 \leq g \in G$ ;
- (iii) if  $0 \leq h_j \in G_j$  for each  $j \in J$ , then  $\bigvee_{j \in J} h_j$  exists in  $G$ .

*Then  $G = \prod_{j \in J}^0 G_j$ . Conversely, if  $G = \prod_{j \in J}^0 G_j$ , then (i), (ii) and (iii) are valid.*

**2.2. Lemma.** *Let  $G$  be a lattice ordered group and let  $\{G_j\}_{j \in J}$  be a system of direct factors of  $G$  such that the conditions (i), (ii) from Lemma 2.1 are valid and*

- (iv) *for each  $g \in G$ , the set  $\{j \in J : g(G_j) \neq 0\}$  is finite.*

*Then  $G = \sum_{j \in J}^0 G_j$ . Conversely, if  $G = \sum_{j \in J}^0 G_j$ , then (i), (ii) and (iv) are valid.*

The condition (a) yields

**2.3. Lemma.** *Let  $G$  be a lattice ordered group, let  $H$  be a direct factor of  $G$  and let  $K$  be a convex  $I$ -subgroup of  $G$ . Then  $H \cap K$  is a direct factor of  $K$ .*

**2.4. Lemma.** Let  $G = \prod_{i \in I}^0 G_i$  and let  $K$  be a closed convex  $l$ -subgroup of  $G$ . Then  $K = \prod_{i \in I}^0 (K \cap G_i)$ .

This follows from Lemma 2.1 and Lemma 2.3.

Analogously, from Lemma 2.2 and Lemma 2.3 we obtain

**2.5. Lemma.** Let  $G = \sum_{i \in I}^0 G_i$  and let  $K$  be a convex  $l$ -subgroup of  $G$ . Then  $K = \sum_{i \in I}^0 (K \cap G_i)$ .

Let  $G$  be a lattice ordered group and  $\emptyset \neq X \subseteq D_1(G)$ . We denote by  $c_1(X)$  the convex  $l$ -subgroup of  $D_1(G)$  generated by the set  $X$ . If  $X$  is an  $l$ -subgroup of  $D_1(G)$ , then  $c_1(X)$  is the set of all  $y \in D_1(G)$  with the property that there are elements  $x_1, x_2 \in X$  with  $x_1 \leq y \leq x_2$ . If  $G$  is archimedean and  $\emptyset \neq X \subseteq D(G)$ , then we denote by  $c_0(X)$  the convex  $l$ -subgroup of  $D(G)$  generated by  $X$ . If we do not suppose that  $G$  is archimedean and if  $\emptyset \neq X \subseteq A(G)$ , then  $c_0(X) = c_1(X)$  (here the symbol  $c_0$  is taken with respect to  $D(A(G))$ ).

**2.6. Proposition.** Let  $G$  be an archimedean lattice ordered group and let  $H$  be a direct factor of  $G$ . Then  $c_0(H)$  is a direct factor of  $D(G)$ . The lattice ordered group  $c_0(H)$  is the Dedekind closure of  $H$ . For each  $g \in G$ ,  $g(H) = g(c_0(H))$ .

*Proof.* Let  $0 \leq d \in D(G)$ . There exists  $g \in G$  with  $d \leq g$ . Put  $g_1 = g(H)$  and  $g_1 \wedge d = d_1$ . Then  $d_1 \in c_0(H)$  and  $d_1 \leq d$ . Let  $0 \leq x \in c_0(H)$ ,  $x \leq d$ . Hence  $x \leq g$  and there is  $g_2 \in H$  with  $x \leq g_2$ . Thus  $x \leq g \wedge g_2$ . Since  $H$  is convex in  $G$ , we obtain  $g \wedge g_2 \in H$  and hence,  $H$  being a direct factor of  $G$ ,  $g \wedge g_2 \leq g_1$ . Therefore  $x \leq g_1 \wedge d = d_1$ . This shows that  $c_0(H)$  is a direct factor of  $D(G)$ .

From the construction of the Dedekind closure it follows immediately that for each convex  $l$ -subgroup  $H_1$  of  $G$ ,  $c_0(H_1)$  is the Dedekind closure of  $H_1$ .

Let  $0 \leq g \in G$ . Put  $g_1 = g(H)$ . Then  $g_1 \in c_0(H)$  and  $g_1 \leq g$ . Assume that there exists  $h \in c_0(H)$  with  $g_1 < h \leq g$ . There is  $g_0 \in H$  with  $h \leq g_0$ . Hence  $g_1 < h \leq g_0 \wedge g \leq g$  and  $g_0 \wedge g \in H$ . Since  $H$  is a direct factor of  $G$ , we have a contradiction. Thus  $g_1 = g(c_0(H))$ . Since each element  $g_2 \in G$  can be written as  $g_2 = g_3 - g_4$  with  $g_3, g_4 \in G^+$ , we get  $g_2(H) = g_2(c_0(H))$ .

**2.7. Proposition.** Let  $G$  be an archimedean lattice ordered group,  $G = \prod_{i \in I}^0 G_i$ . Then  $D(G) = \prod_{i \in I}^0 c_0(G_i)$ .

*Proof.* According to 2.6, each  $c_0(G_i)$  is a direct factor of  $D(G)$ . We have to verify that the conditions (i) (ii) and (iii) from 2.1 are fulfilled with  $G_i, G, J$  replaced by  $c_0(G_i), D(G), I$ .

Let  $j, k \in I, j \neq k$ . From  $G_j \cap G_k = \{0\}$  we obtain  $c_0(G_j) \cap c_0(G_k) = \{0\}$ , hence (i) is valid. Let  $0 \leq g \in D(G)$ . For  $j \in I$  denote  $g_j = g(c_0(G_j))$ . There exists  $h \in G$  with  $g \leq h$ . Put  $h_j = h(c_0(G_j))$ . Hence  $h_j \geq g_j$  for each  $j \in I$ . By 2.6,  $h_j = h(G_j)$  for each  $j \in I$ . Thus by 2.1,  $h = \bigvee h_j$ . Since  $g \geq g \wedge h_j \geq g_j$  and since  $g \wedge h_j \in c_0(G_j)$ , we have  $g \wedge h_j = g_j$ .

Therefore

$$g = g \wedge h = g \wedge (\bigvee h_j) = \bigvee (g \wedge h_j) = \bigvee g_j.$$

Hence (ii) holds.

Let  $0 \leq g_j \in c_0(G_j)$  for each  $j \in I$ . Then for each  $j \in I$  there is  $h_j \in G_j$  with  $g_j \leq h_j$ . According to 2.1 there exists  $\bigvee h_j = h$  in  $G$ . Hence the set  $\{g_j\}_{j \in I}$  is upper bounded in  $D(G)$  and so  $\bigvee g_j$  exists in  $D(G)$ . Therefore (iii) is fulfilled.

Remark 1. According to 2.6 we can also write  $D(G) = \prod_{i \in I}^0 D(G_i)$ .

Remark 2. In [6] it was shown that if  $G_i$  ( $i \in I$ ) are integrally closed directed groups, then  $D(\prod G_i)$  is isomorphic with  $\prod (D(G_i))$ .

The proof of the following proposition is similar to that of Prop. 2.7.

**2.8. Proposition.** *Let  $G$  be an archimedean lattice ordered group and let  $G = \sum_{i \in I}^0 G_i$ . Then  $D(G) = \sum_{i \in I}^0 c_0(G_i)$ .*

**2.9. Lemma.** *Let  $G$  be a lattice ordered group and let  $H$  be a direct factor of  $G$ . Then  $c_1(H)$  is a direct factor of  $D_1(G)$ .*

Proof. Let  $0 \leq x_0 \in D_1(G)$ . There are elements  $x' \in G^+$ ,  $a \in D(A(G))^+$  with  $x_0 = x' + a$ . Denote  $x_1 = x'(H)$ ,  $x_2 = x'(H^\delta)$ . Then  $x' = x_1 + x_2$ ,  $x_1 \wedge x_2 = 0$ .

According to [10], Thm. 2.18,  $A$  is a closed convex  $l$ -subgroup of  $G$  and hence by Lemma 2.4 we have

$$A = (A \cap H) \oplus (A \cap H^\delta),$$

thus according to Prop. 2.7

$$(1) \quad D(A) = c_0(A \cap H) \oplus c_0(A \cap H^\delta),$$

where  $c_0(A \cap H)$  is the convex  $l$ -subgroup of  $D(A)$  generated by the set  $A \cap H$ , and analogously for  $c_0(A \cap H^\delta)$ . Clearly

$$(2) \quad c_0(A \cap H) \subseteq c_1(H), \quad c_0(A \cap H^\delta) \subseteq c_1(H^\delta).$$

From (1) it follows that  $a = a_1 + a_2$  with  $0 \leq a_1 \in c_0(A \cap H)$ ,  $0 \leq a_2 \in c_0(A \cap H^\delta)$ , thus  $a_1 \wedge a_2 = 0$ . By (2),  $a_1 \in c_1(H)$ ,  $a_2 \in c_1(H^\delta)$ , hence  $x_1 \wedge a_2 = 0$ ,  $x_2 \wedge a_1 = 0$ . Thus  $x_2 + a_1 = a_1 + x_2$  and so  $x_0 = x_1 + a_1 + x_2 + a_2$ . Because  $(x_1 + a_1) \wedge (x_2 + a_2) = 0$ , we get

$$(3) \quad x_0 = (x_1 + a_1) \vee (x_2 + a_2).$$

Clearly  $x_1 + a_1 \in c_1(H)$ ,  $x_2 + a_2 \in c_1(H^\delta)$ . Let  $0 \leq x'' \in c_1(H)$ ,  $x'' \leq x_0$ . Then  $x'' \wedge (x_2 + a_2) = 0$ , thus from (3) we obtain

$$x'' = x'' \wedge x_0 = x'' \wedge (x_1 + a_1). \quad *$$

Hence  $x_1 + a_1$  is the greatest element of the set  $\{0 \leq h_1 \in c_1(H) : h_1 \leq x_0\}$ . Therefore in view of (a),  $c_1(H)$  is a direct factor of  $D_1(G)$ .

**2.10. Proposition.** *Let  $G$  be a lattice ordered group,  $G = \prod_{i \in I}^0 G_i$ . Then  $D_1(G) = \prod_{i \in I}^0 c_1(G_i)$ .*

*Proof.* According to Lemma 2.9, each  $c_1(G_i)$  is a direct factor of  $D_1(G)$ . We have to verify that the conditions (i), (ii) and (iii) from Lemma 2.1 are satisfied for the system  $\{c_1(G_i)\}_{i \in I}$  in  $D_1(G)$ . If  $j, k \in I$  are distinct, then  $G_j \cap G_k = \{0\}$  and hence  $c_1(G_j) \cap c_1(G_k) = \{0\}$ . Thus (i) holds. Let  $0 \leq g \in D_1(G)$ . Suppose that  $g = \bigvee_{j \in J} g(c_1(G_j))$  does not hold. Then there is  $x_1 \in D_1(G)$  with  $0 \leq x_1 < g$  such that  $g(c_1(G_j)) \leq x_1$  is valid for each  $j \in J$ . Put  $x_0 = -x_1 + g$ . There is  $0 < x \in G$  with  $x \leq x_0$ . Since  $G = \prod_{i \in I}^0 G_i$ , there is  $i \in I$  such that  $x(G_i) > 0$ . Hence  $x(c_1(G_i)) > 0$  and thus

$$g(c_1(G_i)) < g(c_1(G_i)) + x(c_1(G_i)) \leq x_1 + x \leq g.$$

Since  $g(c_1(G_i)) + x(c_1(G_i)) \in c_1(G_i)$ , in view of (a) we must have  $g(c_1(G_i)) + x(c_1(G_i)) \leq g(c_1(G_i))$ , which is a contradiction. Therefore (ii) is valid.

Let  $0 \leq g_i \in c_1(G_i)$  for each  $i \in I$ . There are elements  $0 \leq x_i \in G$ ,  $0 \leq a_i \in D(A)$  with  $g_i = x_i + a_i$ . Further, there are elements  $y_i \in G_i$  with  $g_i \leq y_i$ . Hence  $x_i, a_i \in G_i$  for each  $i \in I$ . From  $G = \prod_{i \in I}^0 G_i$  and from (iii) it follows that there exists  $x = \bigvee_{i \in I} x_i$  in  $G$ . Let  $i \in I$  be fixed. According to Lemma 2.4 and Prop. 2.7 we have

$$(4) \quad D(A) = \prod_{j \in I}^0 c_0(A \cap G_j),$$

where the symbol  $c_0$  has the same meaning as in the proof of Lemma 2.9. Hence

$$a_i = \bigvee_{j \in I} a_i(c_0(A \cap G_j)).$$

Since  $a_i \in c_1(G_i)$  we have  $a_i(c_0(A \cap G_j)) \in c_1(G_i)$ . From this and from the relations  $a_i(c_0(A \cap G_j)) \in c_1(G_j)$ ,  $c_1(G_i) \cap c_1(G_j) = \{0\}$  we obtain  $a_i(c_0(A \cap G_j)) = 0$  for each  $j \in I$ ,  $j \neq i$ . This implies that  $a_i = a_i(c_0(A \cap G_i)) \in c_0(A \cap G_i)$ . Thus according to (4) and Lemma 2.1 there exists  $a \in D(A)$  with  $a = \bigvee_{i \in I} a_i$ .

We have

$$x + a = (\bigvee_{i \in I} x_i) + a = \bigvee_{i \in I} (x_i + a) = \bigvee_{i \in I} \bigvee_{j \in I} (x_i + a_j).$$

If  $i, j \in I$ ,  $i \neq j$ , then  $x_i \wedge a_j = 0$ , thus

$$x_i + a_j = x_i \vee a_j \leq (x_i + a_i) \vee (x_j + a_j).$$

Therefore

$$x + a = \bigvee_{i \in I} (x_i + a_i) = \bigvee_{i \in I} g_i.$$

Hence (iii) is valid and the proof is complete.

**2.11. Proposition.** *Let  $G$  be a lattice ordered group,  $G = \sum_{i \in I}^0 G_i$ . Then  $D_1(G) = \sum_{i \in I}^0 c_1(G_i)$ .*

Proof. According to Lemma 2.9, each  $c_1(G_i)$  is a direct factor of  $D_1(G)$ . Analogously as in the proof of Prop. 2.10 we can verify that the conditions (i) and (ii) hold (we use Prop. 2.8 instead of Prop. 2.7). Let  $x_0 \in D_1(G)$ . There are elements  $x \in G$ ,  $a \in D(A)$  with  $x_0 = x + a$ . Both the sets  $\{i \in I : x(G_i) \neq 0\}$  and  $\{i \in I : a(c_0(A \cap G_i)) \neq 0\}$  are finite. For each  $i \in I$  we have  $x(G_i) = x(c_1(G_i))$ ,  $a(c_1(G_i)) = a(c_0(A \cap G_i))$ . Hence  $x_0(c_1(G_i)) = x(G_i) + a(c_0(A \cap G_i))$  and thus the set  $\{i \in I : x_0(c_1(G_i)) \neq 0\}$  is finite as well.

A lattice ordered group  $G$  is said to be epiarchimedean, if each homomorphic image of  $G$  is archimedean. Epiarchimedean lattice ordered groups were investigated by CONRAD [4].

Let  $x \in G$ . The least convex  $l$ -subgroup of  $G$  containing the element  $x$  will be denoted by  $c(x)$ ; it is said to be a principal convex  $l$ -subgroup of  $G$ . Similarly, if  $x_0 \in D_1(G)$ , then we put  $c_1(x_0) = c_1(\{x_0\})$ . The following result has been proved in [3]:

**2.12. Theorem.** *A lattice ordered group  $G$  is epiarchimedean if and only if each principal convex  $l$ -subgroup of  $G$  is a direct factor of  $G$ .*

For  $Y \subseteq D_1(G)$  we denote

$$Y^\beta = \{g \in D_1(G) : |g| \wedge |y| = 0 \text{ for each } y \in Y\}.$$

A set  $S \neq \emptyset$  of strictly positive elements of  $G$  will be said to be *disjoint* if  $s_1 \wedge s_2 = 0$  for each pair of distinct elements  $s_1, s_2$  of  $S$ . The lattice ordered group  $G$  is called (*conditionally*) *orthogonally complete* if each (upper bounded) disjoint subset of  $G$  possesses the least upper bound in  $G$ .

**2.13. Theorem.** *Let  $G$  be an epiarchimedean lattice ordered group. Suppose that  $G$  is conditionally orthogonally complete. Then  $D(G)$  is epiarchimedean as well.*

Proof. The lattice ordered group  $G$  is archimedean and hence  $D(G)$  exists. Moreover,  $D(G) = D_1(G)$ . Let  $0 < g_0 \in D(G)$ . There exists  $g_1 \in G$  with  $g_0 \leq g_1$ . By using the Axiom of Choice we infer that there exists a disjoint subset  $S$  of  $G$  such that (i)  $s \leq g_0$  for each  $s \in S$ , and (ii) if  $0 < h_3 \in G$ ,  $h_3 \wedge s = 0$  for each  $s \in S$ , then  $h_3 \wedge g_0 = 0$ . The least upper bound of  $S$  in  $G$  will be denoted by  $g_2$ . Clearly  $g_2 \leq g_0$ . From the construction of  $g_2$  it follows that

$$(5) \quad \{g_2\}^\beta = \{g_0\}^\beta.$$

Since  $G$  is epiarchimedean,  $c(g_2)$  is a direct factor of  $G$ . Thus according to Prop. 2.6,  $c_1(c(g_2))$  is a direct factor of  $D(G)$ . Clearly  $c_1(c(g_2)) = c_1(g_2)$ . Further, we have

$$(6) \quad c_1(g_2)^\beta = \{g_2\}^\beta.$$

From (5) and (6) we obtain

$$c_1(g_2) = \{g_2\}^{\beta\beta} = \{g_0\}^{\beta\beta}.$$



Since  $g_0 \in \{g_0\}^{\beta\beta}$ , we have  $c_1(g_0) \subseteq c_1(g_2)$ . On the other hand,  $g_2 \leq g_0$  yields  $c_1(g_2) \subseteq c_1(g_0)$ . Hence  $c_1(g_0) = c_1(g_2)$ . Therefore  $c_1(g_0)$  is a direct factor of  $D(G)$ . Thus according to Thm. 2.12,  $D(G)$  is epiarchimedean.

*Remark.* It can be shown by examples that if  $G$  is epiarchimedean, then  $D(G)$  need not be epiarchimedean. (Cf. Example 6.4 below.)

The following remark will be useful in the sequel: if  $X$  is a lattice ordered group and if  $Y_1, Y_2$  are  $l$ -subgroups of  $X$  with  $Y_1^+ \subseteq Y_2^+$ , then  $Y_1 \subseteq Y_2$ .

**2.14. Lemma.** *Let  $G$  be a lattice ordered group,  $G = \prod_{i \in I}^0 G_i$ . Then  $c_1(G_i) \cap G = G_i$  for each  $i \in I$ .*

*Proof.* Let  $i \in I$ . We have to verify that  $c_1(G_i) \cap G \subseteq G_i$ . Let  $0 < x \in c_1(G_i) \cap G$ . Then  $x = \bigvee_{j \in I} x(G_j)$ ,  $x(G_j) \geq 0$ , hence  $x(G_j) \in c_1(G_i)$  for each  $j \in I$ . If  $j \neq i$ , then  $x(G_j) \in G_j \subseteq c_1(G_j)$ ; according to Prop. 2.11 we have  $c_1(G_i) \cap c_1(G_j) = \{0\}$ , thus  $x(G_j) = 0$ . Therefore  $x = x(G_i) \in G_i$ .

**2.15. Lemma.** *Let  $G$  be a lattice ordered group,  $G = \prod_{i \in I}^0 G_i$ . Then  $D(A(G_i)) = c_1(G_i) \cap D(A)$ .*

*Proof.* Clearly  $A(G_i) = A \cap G_i$ . According to 2.4 we have  $A = \prod_{i \in I}^0 (A \cap G_i)$ , hence  $A = \prod_{i \in I}^0 A(G_i)$ . In view of Prop. 2.7 we obtain  $D(A) = \prod_{i \in I}^0 D(A(G_i))$ . Thus  $D(A(G_i)) \subseteq D(A)$ . Let  $0 \leq x \in D(A(G_i))$ . There exists  $y \in A(G_i)$  with  $x \leq y$ . Then  $y \in G_i$ , hence  $x \in c_1(G_i)$  and therefore

$$D(A(G_i)) \subseteq c_1(G_i) \cap D(A).$$

Let  $0 < x \in c_1(G_i) \cap D(A)$ . There exists a subset  $\{a_k\} \subseteq A$  and an element  $a \in A$  such that  $0 \leq a_k \leq a$  holds for each  $a_k$ , and  $\bigvee a_k = x$  is valid in  $D(A)$ . From the convexity of  $c_1(G_i)$  we obtain  $a_k \in c_1(G_i)$  and hence, in view of Lemma 2.14,  $a_k \in G_i$  for each  $a_k$ . Moreover,  $a_k = a_k(G_i) \leq a(G_i)$ , hence  $\{a_k\}$  is an upper bounded subset of  $A(G_i)$ . Thus  $\{a_k\}$  is an upper bounded subset of  $D(A(G_i))$ . Since  $D(A(G_i))$  is a direct factor of  $D(A)$ , it is a closed  $l$ -subgroup of  $D(A)$  and hence  $x \in D(A(G_i))$ . Therefore

$$c_1(G_i) \cap D(A) \subseteq D(A(G_i)).$$

**2.16. Lemma.** *Let  $G$  be a lattice ordered group,  $G = \prod_{i \in I}^0 G_i$ . Then  $c_1(G_i) = D_1(G_i)$  for each  $i \in I$ .*

*Proof.* Let  $i \in I$  be fixed. We have to verify that the conditions (i)–(iv) from the definition of  $D_1(G)$  (cf. § 1) are fulfilled with  $G$  and  $D_1(G)$  replaced by  $G_i$  and  $c_1(G_i)$ , respectively. The validity of (i) is obvious. From Lemma 2.15 it follows that (ii) holds.

Let  $0 < y_0 \in c_1(G_i)$ . There are elements  $0 \leq y \in G, 0 \leq a \in D(A)$  with  $y_0 = y + a$ . By the convexity of  $c_1(G_i)$ , both  $y$  and  $a$  belong to  $c_1(G_i)$ . According to Lemma 2.14 we have  $y \in G_i$ . Further, from Lemma 2.15 we obtain  $a \in D(A(G_i))$ .

Now let  $x_0 \in c_1(G_i)$ . There are elements  $y_0, z_0 \in (c_1(G_i))^+$  with  $x_0 = y_0 - z_0$ . Let  $y, a$  be as above. Analogously, there are elements  $z \in G_i$  and  $a_1 \in D(A(G_i))$  with  $z_0 = z + a_1$ . Further, there is  $a_2 \in A(G_i)$  such that  $a_1 \leq a_2$ . Put  $a_3 = a_2 - a_1$ . Then we have  $a_3 \in D(A(G_i))$ ,  $a_3 \geq 0$ ,  $z_0 = z_1 - a_3$ ,  $z_1 = z + a_2 \in G_i$ . Hence

$$x_0 = y + a + a_3 - z_1 = y - z_1 + a_4$$

with  $0 \leq a_4 \in D(A(G_i))$ . Hence there exists an upper bounded subset  $\{a_k\}$  of  $A(G_i)$  with  $a_4 = \bigvee a_k$  (holding in  $D(A(G_i))$ , and hence also in  $D_1(G)$ ). Thus  $\{y - z_2 + a_k\}$  is an upper bounded subset of  $y - z_1 + A(G_i)$  and  $x_0 = \bigvee (y - z_1 + a_k)$ . Therefore (iv) is valid.

Let  $x \in G_i$  and let  $\{x_k\}$  be an upper bounded subset of  $x + A(G_i)$ . Hence  $\{x_k\}$  is an upper bounded subset in  $x + A$ . Thus the least upper bound  $x_0$  of  $\{x_k\}$  in  $D_1(G)$  exists. Since  $c_1(G_i)$  is convex in  $D_1(G)$ , the element  $x_0$  must belong to  $c_1(G_i)$ . Hence the condition (iii) holds.

From Prop. 2.10 and Lemma 2.16 we obtain

$$\mathbf{2.17. Theorem.} \text{ Let } G \text{ be a lattice ordered group, } G = \prod_{i \in I}^0 G_i. \text{ Then } D_1(G) = \prod_{i \in I}^0 D_1(G_i).$$

Analogously we can verify the following assertion:

$$\mathbf{2.18. Proposition.} \text{ Let } G \text{ be a lattice ordered group, } G = \sum_{i \in I}^0 G_i. \text{ Then } D_1(G) = \sum_{i \in I}^0 D_1(G_i).$$

A lattice ordered group  $G$  is said to be *projectable* (strongly projectable) if each principal polar (each polar) of  $G$  is a direct factor of  $G$ .

**2.19. Theorem.** *Let  $G$  be a strongly projectable lattice ordered group. Then  $D_1(G)$  is strongly projectable.*

*Proof.* Let  $X_0 \subseteq D_1(G)$ . We have to verify that  $X_0^\beta$  is a direct factor of  $D_1(G)$ . Without loss of generality we can assume that  $X_0 \subseteq (D_1(G))^+$ . Put  $X = \{x \in G : 0 \leq x \leq x_0 \text{ for some } x_0 \in X_0\}$ . In [10] (Proof of 3.4) it has been shown that  $X_0^\beta = X^\beta$  is the set of all  $y \in D_1(G)$  with the property that there is a subset  $\{y_i\} \subseteq (X^\delta)^+$  with  $|y| = \bigvee y_i$ .

Since  $G$  is strongly projectable, we have

$$G = X^{\delta\delta} \oplus X^\delta$$

and hence, in view of Prop. 2.10,

$$D_1(G) = c_1(X^{\delta\delta}) \oplus c_1(X^\delta).$$

It suffices to verify that  $X_0^\beta = c_1(X^\delta)$ .

Because  $X^\delta \subseteq X^\beta = X_0^\beta$ , we have  $c_1(X^\delta) \subseteq c_1(X_0^\beta) = X_0^\beta$ , hence  $c_1(X^\delta) \subseteq X_0^\beta$ .

Let  $0 \leq z \in X_0^\beta$ . There is a subset  $\{z_i\} \subseteq (X^\delta)^+$  such that  $z = \bigvee z_i$  holds in  $D_1(G)$ .

We have  $\{z_i\} \subseteq c_1(X^\delta)$  and since  $c_1(X^\delta)$  is a direct factor of  $D_1(G)$ , it is a closed  $l$ -subgroup of  $D_1(G)$ . Thus  $z \in c_1(X^\delta)$  and hence  $X_0^\beta \subseteq c_1(X^\delta)$ .

**2.20. Theorem.** *Let  $G$  be a projectable lattice ordered group. Suppose that  $A(G)$  is strongly projectable. Then  $D_1(G)$  is projectable.*

*Proof.* Let  $g_0 \in D_1(G)$ . We have to verify that  $\{g_0\}^{\beta\beta}$  is a direct factor of  $D_1(G)$ . Since  $\{g_0\}^{\beta\beta} = \{|g_0|\}^{\beta\beta}$ , we may assume that  $g_0 \geq 0$ . There are elements  $0 \leq g \in G$ ,  $0 \leq a \in D(A)$  with  $g_0 = g + a$ . The  $l$ -subgroup  $\{g_0\}^{\beta\beta}$  is a direct factor of  $D_1(G)$  if and only if  $\{g_0\}^\beta$  is a direct factor of  $D_1(G)$ .

Since  $G$  is projectable, we have

$$G = \{g\}^{\delta\delta} \oplus \{g\}^\delta.$$

This implies by 2.10

$$D_1(G) = c_1(\{g\}^{\delta\delta}) \oplus c_1(\{g\}^\delta).$$

Denote  $c_1(\{g\}^{\delta\delta}) = F_1$ ,  $c_1(\{g\}^\delta) = F_2$ . Then  $g$  is a weak unit in  $F_1$ . Put  $a_i = a(F_i)$  ( $i = 1, 2$ ),  $g_1 = g + a_1$ . Clearly  $g_1 \in F_1$ ,  $a_2 \in D(A)$ .

There is  $a_3 \in A$  with  $a_2 \leq a_3$ . Put  $X = [0, a_2] \cap A$  (the interval  $[0, a_2]$  being taken with respect to  $D(A)$ ). Because  $A$  is strongly projectable, we obtain

$$A = X^{\delta\delta} \oplus X^\delta.$$

Denote  $a_3(X^{\delta\delta}) = a_4$ . Since  $a_4 \in A \subseteq G$ ,  $\{a_4\}^{\delta\delta}$  is a direct factor of  $G$  and thus  $F_3 = c_1(\{a_4\}^{\delta\delta})$  is a direct factor of  $D_1(G)$ . It is not difficult to verify that  $a_2$  is a weak unit in  $F_3$ . From this we infer that  $F_3 \subseteq F_2$ . Hence there is a direct factor  $F_4$  of  $D_1(G)$  such that  $F_2 = F_3 \oplus F_4$ ; thus

$$D_1(G) = F_1 \oplus F_3 \oplus F_4.$$

Let  $0 \leq h \in \{g_0\}^\beta$ . Hence  $g_0 \wedge h = 0$  and thus  $g \wedge h = 0$ ,  $a_2 \wedge h = 0$ . Since  $g$  and  $a_2$  are weak units in  $F_1$  and  $F_3$ , respectively, we have  $h(F_1) = 0 = h(F_3)$ . Thus  $h = h(F_4) \in F_4$ . Therefore  $\{g_0\}^\beta \subseteq F_4$ .

Conversely, let  $0 \leq h \in F_4$ . Then  $t \wedge h = 0$  for each  $0 \leq t \in F_1 \oplus F_3$ . By putting  $t = g_1 + a_2 = g_0$  we obtain  $g_0 \wedge h = 0$  and hence  $h \in \{g_0\}^\beta$ . Thus  $F_4 \subseteq \{g_0\}^\beta$ . Therefore  $\{g_0\}^\beta = F_4$  is a direct factor of  $D_1(G)$ .

If both  $G$  and  $A(G)$  are projectable lattice ordered groups, then  $D_1(G)$  need not be projectable (cf. Example 6.2 below).

The following result has been obtained by ROTKOVIČ [13].

(\*) *Let  $G$  be a conditionally orthogonally complete archimedean lattice ordered group. Then  $G$  is projectable.*

**2.21. Theorem.** *Let  $G$  be a conditionally orthogonally complete lattice ordered group. Then  $D_1(G)$  is conditionally orthogonally complete.*

*Proof.* Let  $Z = \{z_i\}_{i \in I}$  be a bounded disjoint subset of  $D_1(G)$ . Let  $z_1 \in D_1(G)$  be an upper bound of  $Z$ . There exists  $z \in G$  with  $z_1 \leq z$ . For each  $i \in I$  there are elements  $x_i \in G$ ,  $a_i \in D(A)$  such that  $0 \leq x_i$ ,  $0 \leq a_i$ ,  $z_i = x_i + a_i$ . If  $i, j \in I$ ,  $i \neq j$ , then  $x_i \wedge x_j = 0 = a_i \wedge a_j$ . Hence there exists  $x = \bigvee x_i$  in  $G$ .

Let  $i \in I$  be fixed. If  $a_i = 0$ , we put  $X_i = \{0\}$ . If  $a_i > 0$ , then we choose a maximal disjoint subset  $X_i$  of the set  $[0, a_i] \cap A$ . The set  $X_i$  is upper bounded in  $G$ , hence there exists  $c_i = \sup X_i$  in  $G$ . Since  $A$  is a closed  $l$ -subgroup of  $G$ , we have  $c_i \in A$ . If  $i, j$  are distinct elements of  $I$ , then  $c_i \wedge c_j = 0$ . Because  $A$  is a convex  $l$ -subgroup of  $G$ , it is conditionally orthogonally complete and hence, according to (\*), it is projectable. Let  $D_i$  be the principal polar in  $A$  generated by the element  $c_i$ ; thus  $D_i$  is a direct factor of  $A$ . We denote by  $E_i$  the convex  $l$ -subgroup of  $D(A)$  generated by the set  $D_i$ . By 2.10,  $E_i$  is a direct factor of  $D(A)$ .

For each  $i \in I$  there is  $b_i \in A$  with  $a_i \leq b_i \leq z$ . Denote  $d_i = b_i(D_i)$ . Then  $0 \leq d_i \leq z$  for each  $i \in I$  and  $d_i \wedge d_j = 0$  whenever  $i, j$  are distinct elements of  $I$ . Hence there is  $d = \bigvee d_i$  in  $G$ ; since  $A$  is closed in  $G$ , we have  $d \in A$ .

If  $a_i(E_i) < a_i$ , then there is  $0 < a \in A$  with  $a \leq a_i - a_i(E_i)$ ; but then  $0 < a' = x \wedge a$  for some  $x \in X_i$  and hence  $a' \in D_i \subseteq E_i$ , thus  $a_i(E_i) < a_i(E_i) + a' \leq a_i$  and  $a_i(E_i) + a' \in E_i$ , which is a contradiction. Hence  $a_i = a_i(E_i) \in E_i$ . We have  $t(E_i) = t(D_i)$  for each  $t \in A$ . Thus

$$a_i = a_i(E_i) \leq b_i(E_i) = b_i(D_i) = d_i \leq d.$$

Hence it follows that  $a = \bigvee a_i$  exists in  $D(A)$ . Put  $z_0 = x + a$ . Clearly  $z_i \leq z_0$  for each  $i \in I$ . In the same way as in the proof of 2.10 we can now verify that  $z_0 = \bigvee z_i$ . Hence  $D_1(G)$  is conditionally orthogonally complete.

**2.22. Theorem.** *Let  $G$  be an orthogonally complete lattice ordered group. Then  $D_1(G)$  is orthogonally complete.*

The proof is analogous to that of 2.22.

### 3. PAIRWISE SPLITTING LATTICE ORDERED GROUPS

Let  $G$  be a lattice ordered group,  $0 \leq x, y \in G$ . We write  $x \ll y$  if  $nx \leq y$  for each positive integer  $n$ . We say that  $x$  splits by  $y$  if there are elements  $x_1, x_2 \in G$  such that  $x = x_1 + x_2$ ,  $x_1 \wedge x_2 = 0$ ,  $x_1 \in c(y)$  and  $x_2 \wedge y \ll x_2$ .

Let us consider the following condition for  $G$ :

(p) For each pair  $0 \leq x, y \in G$ , the element  $x$  splits by  $y$ .

A lattice ordered group  $G$  fulfilling (p) is said to be *pairwise splitting*; lattice ordered groups with this property were investigated by MARTINEZ [12]. It is easy to verify that an archimedean lattice ordered group is pairwise splitting if and only if it is epiarchimedean. Let  $\mathcal{P}$  be the class of all pairwise splitting lattice ordered groups. If  $G$  is pairwise splitting, then each convex  $l$ -subgroup of  $G$  is pairwise splitting.

**3.1. Lemma.** *Let  $G$  be a pairwise splitting abelian lattice ordered group. Suppose that  $A = A(G)$  is conditionally orthogonally complete. Let  $0 \leq x \in G$ ,  $0 \leq y_0 \in D_1(G)$ . Then  $x$  splits by  $y_0$  in  $D_1(G)$ .*

*Proof.* There are elements  $y \in G$ ,  $b \in D(A)$  such that  $0 \leq y$ ,  $0 \leq b$ ,  $y_0 = y + b$ . If  $b = 0$ , then  $x$  splits by  $y_0$ . Suppose that  $b > 0$ . There exists  $b_1 \in A$  with  $b \leq b_1$ . From the Axiom of Choice it follows that there exists a disjoint subset  $\{b_i\}$  of  $A$  such that

- (i)  $b_i \leq b$  for each  $b_i$ ,
- (ii) if  $0 < a_1 \in A$ ,  $a_1 \leq b$ , then  $a_1 \wedge b_i > 0$  for some  $b_i$ .

The set  $\{b_i\}$  is upper bounded in  $A$ , hence there exists  $\bigvee b_i = b_2$  in  $A$  and by (i),  $b_2 \leq b$ .

Since  $A$  is a convex  $l$ -subgroup of  $G$  and because  $\mathcal{P}$  is a torsion class,  $A$  must be pairwise splitting and hence  $A$  is epiarchimedean. Thus by 2.13,  $D(A)$  is epiarchimedean. Hence  $c(b_2)$  is a direct factor of  $A$  and  $c_1(c(b_2)) = c_1(b_2)$  is a direct factor of  $D(A)$ . From (i) and (ii) it follows that  $c_1(b)^\beta = c_1(b_2)^\beta$ . Hence we obtain (because  $D(A)$  is epiarchimedean)

$$c_1(b_2) = c_1(b).$$

Put  $b_3 = b_1(c_1(b_2))$ . From the construction of the convex  $l$ -subgroup  $c_1(b_2)$  it follows that  $b_3 = \bigvee_{m \geq 0} (mb_2 \wedge b_1)$  and that there exists a positive integer  $n$  with  $b_3 \leq nb_2 \wedge b_1$ . Thus  $b_3 = nb_2 \wedge b_1$ , hence  $b_3 \in G$ . We have  $b \leq b_1$  and thus

$$b = b(c_1(b)) = b(c_1(b_2)) \leq b_1(c_1(b_2)) = b_3.$$

This implies that  $c_1(b) = c_1(b_3)$ . From this and from the commutativity of  $G$  we get  $c_1(y_0) = c_1(y + b) = c_1(y + b_3)$ .

Since  $y + b_3 \in G$ , the element  $x$  splits by  $y + b_3$ . Thus there are elements  $x_1, x_2 \in G$  such that  $x = x_1 + x_2$ ,  $x_1 \wedge x_2 = 0$ ,  $x_1 \in c(y + b_3)$ ,  $(y + b_3) \wedge x_2 \ll x_2$ . Therefore  $x_1 \in c_1(y_0)$ ,  $y_0 \wedge x_2 \ll x_2$ . Hence  $x$  splits by  $y_0$  in  $D_1(G)$ .

**3.1.1. Corollary.** *Let  $G$  be as in 3.1. Let  $0 \leq a \in D(A)$ ,  $0 \leq y_0 \in D_1(G)$ . Then  $a$  splits by  $y_0$ .*

*Proof.* There is  $b \in A$  with  $a \leq b$ . According to Lemma 3.1,  $b$  splits by  $y_0$ . Hence there are elements  $b_1, b_2 \in D_1(G)$  such that  $b = b_1 + b_2$ ,  $b_1 \wedge b_2 = 0$ ,  $b_1 \in c(y_0)$  and  $b_2 \wedge y_0 \ll b_2$ . Since  $D(A)$  is archimedean,  $b_2 \wedge y_0 = 0$ . Put  $a_1 = b_1 \wedge a$ ,  $a_2 = b_2 \wedge a$ . Then  $a = a_1 + a_2$ ,  $a_1 \wedge a_2 = 0$ ,  $a_1 \in c(y_0)$ ,  $a_2 \wedge y_0 = 0$ . Hence  $a$  splits by  $y_0$ .

**3.2. Lemma.** *Let  $G$  be a pairwise splitting abelian lattice ordered group. Suppose that  $A(G)$  is conditionally orthogonally complete. Let  $0 \leq x \in D_1(G)$ ,  $0 \leq z \in D_1(G)$ ,  $0 \leq a \in D(A)$ ,  $z \ll x + a$ . Then  $z \ll x$ .*

*Proof.* According to Corollary 3.1.1, the element  $a$  splits by  $z$ . Thus there are elements  $a_1, a_2 \in D_1(G)$  such that  $a = a_1 + a_2$ ,  $a_1 \wedge a_2 = 0$ ,  $a_1 \in c_1(z)$ ,  $a_2 \wedge z \ll$

$\ll a_2$ . Then  $a_1, a_2 \in D(A)$  and since  $D(A)$  is archimedean, we have  $a_2 \wedge z = 0$ . Hence  $a_2 \wedge nz = 0$  for each positive integer  $n$ .

Since  $nz \leq x + a_1 + a_2$ , there are elements  $0 \leq z_1, z_2 \in D_1(G)$  with  $nz = z_1 + z_2$ ,  $0 \leq z_1 \leq x + a_1$ ,  $0 \leq z_2 \leq a_2$ . If  $z_2 > 0$ , then  $a_2 \wedge nz \geq z_2 > 0$ , a contradiction. Thus  $z_2 = 0$ ,  $nz \leq x + a_1$  for each positive integer  $n$ . If  $a_1 = 0$ , then the assertion of the lemma is valid; suppose that  $a_1 > 0$ .

There exists a maximal disjoint subset  $\{a_i\} \subset A$  with  $a_i \leq a_1$ . The set  $\{a_i\}$  is upper bounded in  $A$ , hence there exists  $\bigvee a_i = a_3$  in  $A$  and  $a_3 \leq a_1$ . From the construction of  $a_3$  it follows that  $c_1(a_3)^\beta = c_1(a_1)^\beta$ ; from this and from the fact that  $D(A)$  is epiarchimedean we obtain  $c_1(a_3) = c_1(a_1)$ . Hence there is a positive integer  $n_1$  with  $n_1 a_3 \geq a_1$ .

Since  $a_1 \in c_1(z)$ , there is a positive integer  $m$  with  $a_1 \leq mz$ . Then for each positive integer  $n$  we have

$$(n + m)z \leq x + a_1,$$

$$nz \leq x + a_1 - mz \leq x.$$

**3.3. Theorem.** *Let  $G$  be a pairwise splitting abelian lattice ordered group. Suppose that  $A(G)$  is conditionally orthogonally complete. Then  $D_1(G)$  is pairwise splitting.*

*Proof.* Let  $0 \leq x_0, y_0 \in D_1(G)$ . There are elements  $0 \leq x \in G$ ,  $0 \leq a_1 \in D(A)$  with  $x_0 = x + a_1$ . Further, there is  $a \in A$  such that  $a_1 \leq a$ . Then  $x + a \in G$  and hence, according to Lemma 3.1,  $x + a$  splits by  $y_0$ . Hence there are elements  $x_1, x_2 \in G$  with  $x + a = x_1 + x_2$ ,  $x_1 \wedge x_2 = 0$ ,  $x_1 \in c_1(y_0)$ ,  $x_2 \wedge y_0 \ll x_2$ . Denote  $x'_1 = x_1 \wedge x_0$ ,  $x'_2 = x_2 \wedge x_0$ . We have  $x'_1 \wedge x'_2 = 0$  and

$$x_0 = x_0 \wedge (x_1 + x_2) = x_0 \wedge (x_1 \vee x_2) = (x_0 \wedge x_1) \vee (x_0 \wedge x_2) =$$

$$= x'_1 \vee x'_2 = x'_1 + x'_2, \quad x'_1 \in c_1(y_0), \quad x'_2 \wedge y_0 \leq x_2 \wedge y_0 \ll x_2.$$

Since  $x + a \in x_0 + D(A)$ , we have  $x_2 = (x + a) \wedge x_2 \in x_0 \wedge x_2 + D(A)$ . Hence there is  $0 \leq a_2 \in D(A)$  such that  $x_2 = x'_2 + a_2$ . We have  $x'_2 \wedge y_0 \ll x'_2 + a_2$ , thus according to Lemma 3.2,  $x'_2 \wedge y_0 \ll x'_2$ . Therefore  $x_0$  splits by  $y_0$ .

**Problem.** Does the assertion of Thm. 3.3 remain valid without the assumption of commutativity of  $G$ ?

#### 4. THE $\alpha$ -DISTRIBUTIVITY

Let  $\alpha$  be a cardinal and let  $L$  be a lattice. Consider the following condition for  $L$ :

( $\alpha$ ) If  $\{x_{t,s}\}_{t \in T, s \in S}$  is a subset of  $L$  such that both  $\bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s}$  and  $\bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t, \varphi(t)}$  exist in  $L$  and if  $\text{card } T \leq \alpha$ ,  $\text{card } S \leq \alpha$ , then

$$\bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t, \varphi(t)}.$$

If  $L$  fulfils the condition  $(\alpha)$  and the condition dual to  $(\alpha)$ , then it is said to be  $\alpha$ -distributive.  $L$  is called *completely distributive* if it is  $\alpha$ -distributive for each cardinal  $\alpha$ .

Let  $\beta$  be a cardinal. If  $L$  is  $\beta_1$ -distributive for each cardinal  $\beta_1 < \beta$  and if  $L$  fails to be  $\beta$ -distributive, then we write  $d(L) = \beta$ .

Let  $G$  be a lattice ordered group. It is easy to verify that  $G$  is  $\alpha$ -distributive if it fulfils  $(\alpha)$ .

The following assertion is easy to verify.

**4.1. Lemma.** *Let  $G$  be a lattice ordered group and let  $\alpha$  be an infinite cardinal. Suppose that  $G$  fails to be  $\alpha$ -distributive. Then there is  $0 < v \in G$  such that for each  $0 < v_1 \in G$  with  $v_1 \leq v$ , the interval  $[0, v_1]$  of  $G$  fails to be  $\alpha$ -distributive.*

We need the following result:

(A) (Cf. [7].) Let  $\alpha$  be an infinite cardinal and let  $G$  be an archimedean lattice ordered group. Suppose that  $\text{card } [0, v] \leq \alpha$  for each strictly positive element  $v$  of  $G$ . Assume that  $G$  is  $\alpha$ -distributive. Then  $D(G)$  is  $\alpha$ -distributive.

**4.2. Proposition.** *Let  $G$  be a lattice ordered group. Suppose that  $G$  is completely distributive. Then  $D_1(G)$  is completely distributive.*

*Proof.* This follows from Prop. 1.7 of the paper [15] and from the fact that each element of  $D_1(G)$  is the supremum of a certain family of elements of  $G$  (cf. the condition (iv) in § 1).

**4.3. Theorem.** *Let  $\alpha$  be an infinite cardinal and let  $G$  be a lattice ordered group. Suppose that  $\text{card } [0, v] \leq \alpha$  for each strictly positive element of  $A(G)$ . Assume that  $G$  is  $\alpha$ -distributive. Then  $D_1(G)$  is  $\alpha$ -distributive.*

*Proof.* Since  $G$  is  $\alpha$ -distributive,  $A(G) = A$  must be  $\alpha$ -distributive as well. Thus (A) implies that  $D(A)$  is  $\alpha$ -distributive.

Assume that  $D_1(G)$  is not  $\alpha$ -distributive. By Lemma 4.1 there is  $0 < v \in D_1(G)$  such that the interval  $[0, v_1]$  of  $D_1(G)$  fails to be  $\alpha$ -distributive for each  $0 < v_1 \in D_1(G)$  with  $v_1 \leq v$ .

We distinguish two cases. First suppose that there exists  $0 < a \in A$  with  $a \leq v$ . Then the interval  $[0, a]$  of  $D_1(G)$  is a sublattice of  $D(A)$  and hence it is  $\alpha$ -distributive, which is a contradiction. Now suppose that no  $0 < a \in A$  with  $a \leq v$  exists. Each element  $0 < v_1 \in D_1(G)$  with  $v_1 \leq v$  can be written as  $v_1 = x + a_1$ ,  $0 \leq x \in G$ ,  $0 \leq a_1 \in D(A)$ . Then we have  $a_1 \leq v_1 \leq v$ , hence  $a_1 = 0$  and thus  $v_1 = x \in G$ . Hence the interval  $[0, v]$  of  $D_1(G)$  is a sublattice of  $G$  and so it is  $\alpha$ -distributive, which is a contradiction.

**4.4. Theorem.** *Let  $G$  be a lattice ordered group that is not completely distributive,  $d(G) = \alpha$ . If  $A(G)$  is completely distributive, then  $d(D_1(G)) = \alpha$ . If  $A(G)$  is not completely distributive,  $d(D(A(G))) = \beta$ , then  $d(D_1(G)) = \min(\alpha, \beta)$ .*

Proof. From [10], Prop. 2.20 it follows that  $D_1(G)$  is not  $\alpha$ -distributive, hence  $d(D_1(G)) \leq \alpha$ . If  $A(G)$  is not completely distributive, then  $D(A(G))$  cannot be completely distributive; if  $d(D(A(G))) = \beta$ , then according to [10], Prop. 2.16,  $d(D_1(G)) \leq \beta$ . Now it suffices to verify that if  $\gamma$  is a cardinal and both  $G$  and  $D(A(G))$  are  $\gamma$ -distributive, then  $D_1(G)$  is  $\gamma$ -distributive as well. To prove it we can use the same method as in the proof of 4.2.

**4.5. Theorem.** *Let  $G$  be a lattice ordered group that is not completely distributive,  $d(G) = \alpha$ . Suppose that  $A(G)$  is projectable. Then  $d(D_1(G)) = \alpha$ .*

Proof. If  $A(G)$  is completely distributive, then the assertion is valid according to 4.4. Suppose that  $d(A(G)) = \beta$ . Hence  $\beta \geq \alpha$ . Since  $A(G)$  is projectable, from [8] we obtain  $d(D(A(G))) = \beta$ . Hence  $d(D_1(G)) = \alpha$  by 4.4.

### 5. $g$ -COMPLETE LATTICE ORDERED GROUPS

An archimedean lattice ordered group  $G$  is complete if and only if  $D(G) = G$ . A lattice ordered group  $H$  will be called  $g$ -complete (generalized complete) if  $D_1(H) = H$ . It was remarked in [11] that  $D_1(H) = H$  if and only if  $A(H)$  is complete.

The following assertion has been proved in [9]:

(A) Let  $G$  be a lattice ordered group. Then there exists a convex  $l$ -subgroup  $C(G)$  of  $G$  such that

- (a)  $C(G)$  is complete;
- (b) if  $H$  is a convex  $l$ -subgroup of  $G$  and if  $H$  is complete, then  $H \subseteq C(G)$ .

A class  $\mathcal{K}$  of lattice ordered groups is said to be a *radical class* [11] if it fulfils the following conditions:

- (i)  $\mathcal{K}$  is closed with respect to isomorphisms.
- (ii) If  $H_1$  is a convex  $l$ -subgroup of a lattice ordered group  $H$  and if  $H \in \mathcal{K}$ , then  $H_1 \in \mathcal{K}$ .
- (iii) If  $H_i$  is a system of convex  $l$ -subgroups of a lattice ordered group  $H$  and if each  $H_i$  belongs to  $\mathcal{K}$ , then  $\bigvee H_i \in \mathcal{K}$ .

In this paragraph it will be shown that the class of all  $g$ -complete lattice ordered groups is a radical class.

**5.1. Theorem.** *Let  $G$  be a lattice ordered group. There exists a convex  $l$ -subgroup  $B_0(G)$  of  $G$  such that*

- (a)  $B_0(G)$  is  $g$ -complete,
- (b) if  $B_1$  is a convex  $l$ -subgroup of  $G$  and if  $B_1$  is  $g$ -complete, then  $B_1 \subseteq B_0(G)$ .

Proof. Let  $\{B_i\}$  be the set of all convex  $l$ -subgroups of  $G$  fulfilling

$$B_i \cap A(G) \subseteq C(G).$$



Put  $B_0(G) = \bigvee B_i$ . Then  $B_0(G) \cap A(G) \subseteq C(G)$ . Hence  $A(B_0(G)) = A(G) \cap B_0(G) \subseteq C(G)$ . Since  $A(B_0(G))$  is complete,  $B_0(G)$  is  $g$ -complete.

Let  $B_1$  be a convex  $l$ -subgroup of  $G$  and suppose that  $B_1$  is  $g$ -complete. Then  $A(B_1) = A(G) \cap B_1$  is complete, hence  $A(G) \cap B_1 \subseteq C(G)$  and thus  $B_1 \in \{B_i\}$ . Therefore  $B_1 \subseteq B_0(G)$ .

**Remark.** It is easy to verify that  $B_0(G)$  is a characteristic  $l$ -subgroup of  $G$ . It can be shown by examples that  $B_0(G)$  need not be a closed  $l$ -subgroup of  $G$  (cf. Example 6.5 below).

**5.2. Theorem.** *The class  $K_g$  of all  $g$ -complete lattice ordered groups is a radical class.*

**Proof.**  $K_g$  obviously fulfils (i). Let  $H \in K_g$  and let  $H_1$  be a convex  $l$ -subgroup of  $H$ . Then  $A(H)$  is complete and since  $A(H_1) = H_1 \cap A(H)$ ,  $A(H_1)$  is complete as well. Thus (ii) holds. Let  $G$  be a lattice ordered group and let  $\{G_i\}$  be a system of convex  $l$ -subgroups of  $G$  such that each  $G_i$  belongs to  $K_g$ . Let  $B_0(G)$  be as in 5.1. Then each  $G_i$  is a subset of  $B_0(G)$ , hence  $\bigvee G_i \subseteq B_0(G)$ ; in view of (ii) we have  $\bigvee G_i \in K_g$  and hence (iii) is valid.

**5.3. Proposition.** *Let  $G$  be a lattice ordered group,  $G = \prod_{i \in I}^0 G_i$ . Then  $G$  is  $g$ -complete if and only if all  $G_i$  are  $g$ -complete.*

**Proof.** Assume that all  $G_i$  are  $g$ -complete. Then according to Prop. 2.17,  $G$  is  $g$ -complete. Conversely, suppose that  $G$  is  $g$ -complete. Hence  $A(G)$  is complete. We have  $A(G) = \prod_{i \in I}^0 (A(G) \cap G_i)$  and  $A(G) \cap G_i = A(G_i)$  for each  $i \in I$ . Since each direct factor of a complete lattice ordered group is complete, all  $G_i$ 's are  $g$ -complete.

An analogous proposition is valid for direct sums.

Let  $H$  be an abelian lattice ordered group. Consider the following condition for  $H$ :

(I) it is possible to define a multiplication of elements of  $H$  by reals so that  $H$  turns out to be a vector lattice.

We denote by  $\mathcal{V}_1$  the class of all archimedean lattice ordered groups fulfilling the condition (I). Further, let  $\mathcal{V}_2$  be the class of all  $G \in \mathcal{V}_1$  that are complete. Lattice ordered groups belonging to  $\mathcal{V}_2$  are called *complete vector lattices* [1] or *K-spaces* [14]. Let us denote by  $\mathcal{V}_3$  the class of all  $G \in K_g$  fulfilling (I).

**5.4. Proposition.** *Both  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are radical classes.*

**Proof.**  $\mathcal{V}_1$  obviously fulfils the conditions (i) and (ii). Let  $G$  be a lattice ordered group and let  $\{H_i\}$  be a system of convex  $l$ -subgroups of  $G$  such that each  $H_i$  belongs to  $\mathcal{V}_1$ . Put  $H = \bigvee H_i$ . Each  $H_i$  is a convex  $l$ -subgroup of  $A(G)$ , hence  $H$  is a convex  $l$ -subgroup of  $A(G)$  as well. From Thm. 1.3, [4] it follows that each archimedean lattice ordered group possesses a largest convex  $l$ -subgroup fulfilling the condition (I). We denote by  $H_0$  the largest convex  $l$ -subgroup of  $A(G)$  fulfilling (I). Since all  $H_i$

are convex  $l$ -subgroups of  $H_0$ , we obtain that  $H$  is a convex  $l$ -subgroup of  $H_0$ . Thus  $H$  belongs to  $\mathcal{V}_1$ . Therefore  $\mathcal{V}_1$  is a radical class. Let  $\mathcal{C}$  be the class of all complete lattice ordered groups.  $\mathcal{C}$  is a radical class [12] and  $\mathcal{V}_2 = \mathcal{V}_1 \cap \mathcal{C}$ . The intersection of two radical classes being again a radical class,  $\mathcal{V}_2$  is a radical class as well.

**5.5. Corollary.** *Let  $G$  be a lattice ordered group. Then  $G$  possesses a largest convex  $l$ -subgroup  $V_i(G)$  belonging to  $\mathcal{V}_i$  ( $i = 1, 2$ ).*

**Problem.** *Is  $\mathcal{V}_3$  a radical class?*

## 6. EXAMPLES

**6.1.** If a lattice ordered group  $G$  is complete, then each polar of  $G$  is a direct factor of  $G$ . A polar of a  $g$ -complete lattice ordered group  $H$  need not be a direct factor of  $H$ .

Let  $H$  be the set of all triples  $(x, y, z)$  of reals, the operation  $+$  in  $H$  being defined componentwise. For  $(x_1, y_1, z_1), (x_2, y_2, z_2) \in H$  we put  $(x_1, y_1, z_1) \leq (x_2, y_2, z_2)$ , if either  $x_1 < x_2$ , or  $x_1 = x_2$  and  $y_1 \leq y_2, z_1 \leq z_2$ . Then  $H$  is a  $g$ -complete lattice ordered group. The set  $P$  consisting of all  $(x, y, z) \in H$  with  $x = z = 0$  is a polar of  $H$  and  $P$  fails to be a direct factor of  $H$ .

**6.2.** If a lattice ordered group  $G$  is projectable and if  $A(G)$  is projectable, then  $D_1(G)$  need not be projectable.

Let  $I = [0, 1]$  be the interval of reals and let  $F$  be the set of all real functions  $f$  defined on  $I$  with the following property: there is a finite set  $M(f) \subseteq I$  such that  $f(x_1) = f(x_2)$  whenever  $x_1, x_2 \in I \setminus M(f)$ . The partial order and the operation  $+$  on the set  $F$  are defined in the natural way. Let  $G$  be the set of all pairs  $(f, g)$  with  $f, g \in F$ . For  $(f_i, g_i) \in G$  ( $i = 1, 2$ ) we put  $(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2)$  and we set  $(f_1, g_1) \leq (f_2, g_2)$  if for each  $x \in I$  we have either  $f_1(x) < f_2(x)$ , or  $f_1(x) = f_2(x)$  and  $g_1(x) \leq g_2(x)$ . Then  $G$  is a projectable lattice ordered group.  $A(G)$  consists of all elements  $(0, g)$  with  $g \in F$ ;  $A(G)$  is projectable as well. Let  $I_1$  be an infinite subset of  $I, I_1 \neq I$ . For each  $t \in I_1$  let  $f_t \in F$  such that  $f_t(t) = 1$  and  $f_t(x) = 0$  for  $x \in I, x \neq t$ . The least upper bound  $h$  of the set  $\{(0, f_t) : t \in I_1\}$  in  $D_1(G)$  exists. Let  $f \in F, f(x) = 1$  for each  $x \in I$ . Let  $P$  be the principal polar of  $D_1(G)$  generated by the element  $h$ . Then the set

$$\{h_1 \in P : 0 \leq h_1 \leq (f, 0)\}$$

has no greatest element. Hence  $P$  fails to be a direct factor of  $G$ .

**6.3.** If a lattice ordered group  $G$  is pairwise splitting, then  $D_1(G)$  need not be pairwise splitting.

Let  $F$  be as in 6.2. Then  $F$  is pairwise splitting lattice ordered group. Put  $t_n = 1/n$  ( $n = 1, 2, \dots$ ). For each positive integer  $n$  let  $f_n \in F$  with  $f_n(t_n) = 1/n, f_n(x) = 0$

for each  $x \neq t_n$ . Since  $G$  is archimedean,  $D_1(F) = D(F)$ . The least upper bound  $h$  of the set  $\{f_n\}$  ( $n = 1, 2, \dots$ ) in  $D_1(F)$  exists. Let  $f \in F$ ,  $f(x) = 1$  for each  $x \in I$ . The element  $f$  does not split by  $h$  in  $D_1(F)$ . Hence  $D_1(F)$  is not pairwise splitting.

**6.4.** There exists an epiarchimedean lattice ordered group  $G$  such that  $D(G)$  fails to be epiarchimedean.

Let  $F, h$  be as in 6.3. The lattice ordered group  $F$  is epiarchimedean and the principal convex  $l$ -subgroup of  $D_1(F)$  generated by the element  $h$  fails to be a direct factor of  $D_1(G)$ . Hence  $D_1(F)$  is not epiarchimedean.

**6.5.** The largest  $g$ -complete  $l$ -ideal  $B_0(G)$  of a lattice ordered group  $G$  need not be a closed  $l$ -subgroup of  $G$ .

Let  $F$  be as in 6.2. Let  $F_1$  be the set of all  $f \in F$  such that (i)  $f(x)$  is an integer for each  $x \in I$ , and (ii) if  $x \in I \setminus M(f)$ , then  $f(x)$  is even.  $F_1$  is an archimedean lattice ordered group. Hence  $B_0(F_1) = C(F_1)$ . Thus  $B_0(F_1)$  consists of all  $f \in F_1$  such that  $f(x) = 0$  for each  $x \in I \setminus M(f)$ . Let  $f \in F_1$  with  $f(x) = 2$  for each  $x \in I$ . There is a subset  $X \subseteq B_0(F_1)$  such that  $\sup X = f$  holds in  $F_1$ ; hence  $B_0(F_1)$  fails to be closed in  $F_1$ .

#### References

- [1] *G. Birkhoff*: Lattice theory, third edition, Providence 1967.
- [2] *P. Conrad*: Lattice ordered groups, Tulane University 1970.
- [3] *P. Conrad*: Epiarchimedean lattice ordered groups. Czech. Math. J. 24 (1974), 192–218.
- [4] *P. Conrad*: Changing the scalar multiplication on a vector lattice. J. Austral. Math. Soc. 20 (1975), 332–347.
- [5] *Л. Фукс*: Частично упорядоченные алгебраические системы, Москва, 1965.
- [6] *J. Jakubík*: Die Dedekindschen Schnitte im direkten Produkt von halbgeordneten Gruppen. Matem. fyz. časop. 16 (1966), 329–336.
- [7] *J. Jakubík*: Distributivity in lattice ordered groups. Czech. Math. J. 22 (1972), 108–125.
- [8] *J. Jakubík*: On  $\sigma$ -complete lattice ordered groups. Czech. Math. J. 23 (1973), 164–174.
- [9] *J. Jakubík*: Conditionally  $\alpha$ -complete sublattices of a distributive lattice. Algebra univ. 2 (1972), 255–261.
- [10] *J. Jakubík*: Archimede kernel of a lattice ordered group. Czech. Math. J. 28 (1978), 140–154.
- [11] *J. Jakubík*: Radical classes and radical mappings of lattice ordered groups. Symposia Math. 31 (1977), 451–477.
- [12] *J. Martinez*: Pairwise splitting lattice ordered groups. Czech. Math. J. 27 (1977), 545–551.
- [13] *Г. Я. Роткович*: О дизъюнктно полных архимедовых полуупорядоченных группах. Czech. Math. J. 27 (1977), 523–527.
- [14] *Б. З. Вулих*: Введение в теорию полуупорядоченных пространств, Москва 1961.
- [15] *E. C. Weinberg*: Completely distributive lattice ordered groups. Pacif. J. Math. 12 (1962), 1131–1148.

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