Generalized derivations and multilinear polynomials in prime rings

Basudeb Dhara, Shuliang Huang* and Atanu Pattanayak

ABSTRACT: Let R be a prime ring with Utumi quotient ring U and extended centroid C, g a nonzero generalized derivation of R, I a nonzero right ideal of R, $f(r_1, \dots, r_k)$ a multilinear polynomial over C and $n \geq 2$ be a fixed integer. If $g(f(r_1, \dots, r_k)^n) = g(f(r_1, \dots, r_k))^n$ for all $r_1, \dots, r_k \in I$, then one of the following holds:

(1) $IC = eRC$ for some idempotent $e \in soc(RC)$ and $f(x_1, \dots, x_k)$ is central-valued on $eRCe$;

(2) there exist $a, b \in U$ such that $g(x) = ax + xb$ for all $x \in R$ and $(a - \alpha)I = (0)$, $(b - \beta)I = (0)$ for some $\alpha, \beta \in C$ with $(\alpha + \beta)^{n-1} = 1$;

(3) there exists $a \in U$ such that $g(x) = ax$ for all $x \in R$ with $aI = (0)$.

Mathematics Subject Classification: 16W25, 16R50, 16N60.

Key words and phrases: Prime ring, derivation, generalized derivation, extended centroid. [1]

1 Introduction

Let R be an associative prime ring with center $Z(R)$. Throughout this paper, U will denote the Utumi quotient ring of R and $C = Z(U)$, the center of U, which is called

[1]*The second author is the corresponding author. Address correspondence to Shuliang Huang, Department of Mathematics, Chuzhou University, Chuzhou 239012, P. R. China.

extended centroid of R. For $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The concept of derivation is extended to generalized derivation. The generalized derivation means an additive mapping $g : R \to R$ such that $g(xy) = g(x)y + xd(y)$ for all $x, , y \in R$, where d is a derivation of R. For some fixed $a, b \in R$, the maps $g(x) = ax + xb$ for all $x \in R$, is an example of generalized derivation. This kind of generalized derivations are called generalized inner derivations.

Let S be a nonempty set of R and $F : R \to R$ be an additive mapping. Then we say that F acts as homomorphism or anti-homomorphism on S if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping F acts as a Jordan homomorphism on S if $F(x^2) = F(x)^2$ holds for all $x \in S$. Obviously, any additive mapping acting as homomorphism or anti-homomorphism is a surjective Jordan homomorphism, but the converse is not true in general. In [11, Theorem 3.1], Herstein proved that in a 2-torsion free prime ring, any Jordan homomorphism is either a homomorphism or an anti-homomorphism.

In [2], Bell and Kappe proved that if a derivation d of a prime ring R acts as a homomorphism or anti-homomorphism on a nonzero right ideal of R, then $d = 0$ on R. Recently, Ali, Rehman and Ali in [1] proved a similar result to Lie ideal case. They proved that if R is a 2-torsion free prime ring, L a nonzero Lie ideal of R such that $u^2 \in L$ for all $u \in L$ and d acts as a homomorphism or anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z(R)$. In [22], Wang and You eliminated the assumption $u^2 \in L$ for all $u \in L$ and obtain the same conclusion of [1].

On the other hand, the authors developed above results, replacing the derivation d with a generalized derivation g of R. In [15], Rehman proved that the 2-torsion free prime ring R must be commutative, if there is a generalized derivation g admitting a nonzero associated derivation, that acts as homomorphism or anti-homomorphism on a nonzero ideal of R. Gusic in [10] showed that the result of Rehman is not in complete form. He proved the following: let R be a prime ring, I a nonzero ideal of R and d, g any two functions on R (not necessary to be additive and d not necessary to be a derivation) such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$. If g acts as a homomorphism or an anti-homomorphism on I, then $d = 0$ and either $g = 0$ or $g(x) = x$ for all $x \in R$; in addition, when g acts as an anti-homomorphism on I, then R must be commutative.

In the same line of investigation, recently in [7] De Filippis studied the situation when generalized derivation g acts as a Jordan homomorphism on a noncentral Lie ideal L of R and on the set $[I, I]$, where I is a nonzero right ideal of a prime ring R. More precisely, De Filippis proved the following two theorems:

Theorem A: *Let R be a prime ring, L a non-central Lie ideal of R and g a non-zero generalized derivation of R. If g acts as a Jordan homomorphism on L, then either $g(x) = x$ for all $x \in R$, or char$(R) = 2$, R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$, L is commutative and $u^2 \in Z(R)$ for any $u \in L$.*

Theorem B: *Let R be a prime ring, I a non-zero right ideal of R and g a non-zero generalized derivation of R. If g acts as a Jordan homomorphism on the set $[I, I]$, then one of the following holds: (i) char $(R) = 2$ and I satisfies the identity $s_4(x_1, \ldots, x_4)x_5$; (ii) $[I, I]I = 0$; (iii) there exists $a \in R$ such that $g(x) = ax$ for all $x \in R$ and $aI = 0$; (iv) $g(x) = x$ for all $x \in I$; (v) there exists $q \in R$ such that $g(x) = xq$ and $qx = x$ for all $x \in I$.*

It is natural to generalize above results considering the generalized derivation g acts as Jordan homomorphism on the set $\{f(x_1, \ldots, x_k) | x_1, \ldots, x_k \in I\}$, where I is a nonzero right ideal of R and $f(x_1, \ldots, x_k)$ is a multilinear polynomial on R over C. In the present paper, our aim is to study this situation in more generalized form by considering n-power values.

Let R be a prime ring and U be the Utumi quotient ring of R and $C = Z(U)$, the center of U. Note that U is also a prime ring with C a field. Let $f(x_1, \ldots, x_k)$ be a multilinear polynomial over C. We can write it as

$$f(x_1, \ldots, x_k) = x_1 x_2 \ldots x_k + \sum_{I \neq \sigma \in S_k} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(k)},$$

where S_k is the permutation group over k elements and any $\alpha_\sigma \in C$. We denote by $f^d(x_1, \ldots, x_k)$ the polynomial obtained from $f(x_1, \ldots, x_k)$ by replacing each coefficient α_σ with $d(\alpha_\sigma.1)$. In this way we have

$$d(f(x_1, \ldots, x_k)) = f^d(x_1, \ldots, x_k) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_k).$$

Now we include some facts which will be used to prove our theorems.

Fact 1. It is well known that any derivation of R can be uniquely extended to a derivation of U (see [18, Lemma 2].

Fact 2. Let ρ be a nonzero right ideal of R. Then ρ, ρC, ρU satisfy the same generalized polynomial identities with coefficients in U (see [5]).

Fact 3. Let ρ be a nonzero right ideal of R. Then ρ, ρR and ρU satisfy the same differential identities with coefficients in U (see [18, Theorem 2].

Fact 4. Let ρ be a nonzero right ideal of R. If ρ satisfies a nontrivial polynomial identity, then RC is a primitive ring with $soc(RC) \neq 0$ and $\rho C = eRC$ for some idempotent $e = e^2 \in soc(RC)$ (see [17, Proposition])

Fact 5. Let R be a dense ring of linear transformations of a vector space V over a division ring D and $a \in R$. If for any $v \in V$, av and v are linearly D-dependent, then there exists a $\beta \in D$ such that $av = v\beta$ for all $v \in V$.

Proof. For any $v \in V$, $av = v\alpha_v$ for some $\alpha_v \in D$. Now we prove that α_v is independent of the choice of $v \in V$. Let u be a fixed vector of V. Then $au = u\alpha$. Let v be any vector of V. Then $av = v\alpha_v$, where $\alpha_v \in D$. If u and v are linearly D-dependent, then $u = v\beta$, for $\beta \in D$. In this case, we see that $u\alpha = au = av\beta = (v\alpha_v)\beta = (v\beta)\alpha_v = u\alpha_v$, implying $\alpha = \alpha_v$.

Now if u and v are linearly D-independent, then we have $(u+v)\alpha_{u+v} = a(u+v) = au + av = u\alpha + v\alpha_v$, which implies $u(\alpha_{u+v} - \alpha) + v(\alpha_{u+v} - \alpha_v) = 0$. Since u and v are linearly D-independent, we have $\alpha_{u+v} - \alpha = 0 = \alpha_{u+v} - \alpha_v$ and so $\alpha = \alpha_v$. Thus $av = v\alpha$ for all $v \in V$, where $\alpha \in D$ independent of the choice of $v \in V$.

Fact 6. Let I be a nonzero right ideal of R and $a \in U$. If for every $x \in I$, ax and x are linearly C-dependent, then there exists $\alpha \in C$ such that $(a - \alpha)I = (0)$.

The proof of Fact 6 is similar to that of Fact 5, so we omit it here.

Remark 1. Now we mention a result of Lee in [16] which will be used to prove our main theorem. In [16], Lee extended the definition of generalized derivation as follows: generalized derivation means an additive mapping $g : \rho \to U$ such that $g(xy) = g(x)y + x\delta(y)$ for all $x, y \in \rho$, where ρ is a dense right ideal of R and δ ia a derivation from ρ into U. The author proved that every generalized derivation of R can be uniquely extended to generalized derivation of U and has the form $g(x) = ax + \delta(x)$ for all $x \in U$, where $a \in U$ and δ is a derivation of U [16, Theorem

3]. For more details about generalized derivations we refer to [3], [12], [16] and [19].

2 Main Results

First we study the case when g is inner generalized derivation of R, that is, for some $a, b \in U$, $g(x) = ax + xb$ for all $x \in R$.

Lemma 2.1. *Let $R = M_m(F)$, $m \geq 2$, be the set of all $m \times m$ matrices over a field F and $f(x_1, \ldots, x_k)$ be a noncentral multilinear polynomial over F. If for some $a, b \in R$, $af(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b = (af(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b)^n$ for all $x_1, \ldots, x_k \in R$, then $a, b \in F.I_m$ with $(a + b)^n - (a + b) = 0$.*

Proof. Let $a = (a_{ij})_{m \times m}$, $b = (b_{ij})_{m \times m}$. Since $f(x_1, \ldots, x_k)$ is not central valued on R, by [20, Lemma 2, Proof of Lemma 3] there exists a sequence of matrices $r = (r_1, \ldots, r_k)$ in R such that $f(r_1, \ldots, r_k) = \gamma e_{ij}$ with $0 \neq \gamma \in F$ and $i \neq j$. Since the set $f(R) = \{f(x_1, \ldots, x_k), x_i \in R\}$ is invariant under the action of all inner automorphisms of R, for all $i \neq j$ there exists a sequence of matrices $r = (r_1, \ldots, r_k)$ such that $f(r) = \gamma e_{ij}$. Thus

$$af(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b = (af(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b)^n$$

gives $0 = (a\gamma e_{ij} + \gamma e_{ij} b)^n$ i.e., $0 = (ae_{ij} + e_{ij}b)^n$. Left multiplying by e_{ij} yields $a_{ji}^n = 0$ and right multiplying by e_{ij} yields $b_{ji}^n = 0$. Thus, we have $a_{ji} = 0$ and $b_{ji} = 0$ for any $i \neq j$, that is, a and b are diagonal matrices.

Now for any F-automorphism θ of R, we have

$$a^\theta f(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b^\theta = (a^\theta f(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b^\theta)^n$$

for all $x_1, \ldots, x_k \in R$. Then by above argument a^θ and b^θ must be diagonal. Write, $a = \sum_{i=0}^{m} a_{ii}e_{ii}$ and $b = \sum_{i=0}^{m} b_{ii}e_{ii}$; then for $s \neq t$, we have

$$(1 + e_{ts})a(1 - e_{ts}) = \sum_{i=0}^{m} a_{ii}e_{ii} + (a_{ss} - a_{tt})e_{ts}$$

diagonal and

$$(1 + e_{ts})b(1 - e_{ts}) = \sum_{i=0}^{m} b_{ii}e_{ii} + (b_{ss} - b_{tt})e_{ts}$$

diagonal, implying $a_{ss} = a_{tt}$, $b_{ss} = b_{tt}$ and so $a, b \in F.I_m$.

Then our assumption

$$af(x_1,\dots,x_k)^n + f(x_1,\dots,x_k)^n b = (af(x_1,\dots,x_k) + f(x_1,\dots,x_k)b)^n$$

for all $x_1,\dots,x_k \in R$, reduces to $((a+b)^n - (a+b))f(x_1,\dots,x_k)^n = 0$. This implies either $(a+b)^n - (a+b) = 0$ or $f(x_1,\dots,x_k)^n = 0$ for all $x_1,\dots,x_k \in R$. But by [20, Corollary 5], $f(x_1,\dots,x_k)^n = 0$ for all $x_1,\dots,x_k \in R$, implies that $f(x_1,\dots,x_k) = 0$ for all $x_1,\dots,x_k \in R$, a contradiction.

Hence, the lemma is proved.

Proposition 2.2. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, and $f(r_1,\dots,r_k)$ be a multilinear polynomial over C which is not central valued on R. If for some $a,b \in U$, $af(r)^n + f(r)^n b = (af(r) + f(r)b)^n$ for all $r = (r_1,\dots,r_k) \in R^k$, where $n \geq 2$ is a fixed integer, then $a,b \in C$ with $(a+b)^n - (a+b) = 0$.*

Proof. Since R and U satisfy same generalized polynomial identity (see [5]), U satisfies

$$\begin{aligned} &h(x_1,\dots,x_k)\\ &= af(x_1,\dots,x_k)^n + f(x_1,\dots,x_k)^n b - (af(x_1,\dots,x_k) + f(x_1,\dots,x_k)b)^n = 0. \end{aligned}$$

Suppose that $h(x_1,\dots,x_k)$ is a trivial GPI for U. Let $T = U *_C C\{x_1,\dots,x_k\}$, the free product of U and $C\{x_1,\dots,x_k\}$, the free C-algebra in noncommuting indeterminates $x_1,\dots,x_k$. Then,

$$af(x_1,\dots,x_k)^n + f(x_1,\dots,x_k)^n b - (af(x_1,\dots,x_k) + f(x_1,\dots,x_k)b)^n$$

is zero element in T. If $a \notin C$, then a and 1 are linearly independent over C. Then expanding the above identity, it will imply

$$af(x_1,\dots,x_k)^n - af(x_1,\dots,x_k)(af(x_1,\dots,x_k) + f(x_1,\dots,x_k)b)^{n-1} = 0$$

that is,

$$af(x_1,\dots,x_k)\{f(x_1,\dots,x_k)^{n-1} - (af(x_1,\dots,x_k) + f(x_1,\dots,x_k)b)^{n-1}\} = 0$$

in T. Again, since a and 1 are linearly independent over C, this implies that

$$af(x_1,\dots,x_k)\{af(x_1,\dots,x_k)(af(x_1,\dots,x_k) + f(x_1,\dots,x_k)b)^{n-2}\} = 0$$

and so $(af(x_1,\dots,x_k))^n = 0$, implying $a = 0$, a contradiction. Hence, $a \in C$. Then our generalized polynomial identity (GPI) reduces to $f(x_1,\dots,x_k)^n(a+b) - (f(x_1,\dots,x_k)(a+b))^n = 0$ in T. If $a+b \notin C$, then $a+b$ and 1 are linearly independent over C. Then by same argument as above, $(f(x_1,\dots,x_k)(a+b))^n = 0$, which is a nontrivial generalized polynomial identity for R, a contradiction. Thus, $a+b \in C$ and hence $b \in C$. Then our GPI becomes $\{(a+b)-(a+b)^n\}f(x_1,\dots,x_k)^n = 0$, which is trivial GPI for R, implying $(a+b)-(a+b)^n = 0$.

Next suppose that $h(x_1,\dots,x_k)$ is a nontrivial GPI for R and so for U. In case C is infinite, we have $h(x_1,\dots,x_k) = 0$ for all $x_1,\dots,x_k \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [8, Theorems 2.5 and 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1,\dots,x_k) = 0$ for all $x_1,\dots,x_k \in R$. By Martindale's theorem [21], R is then a primitive ring with nonzero socle $soc(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [13, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1,\dots,r_k)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Lemma 2.1, we obtain our required conclusion.

Now, if V is infinite dimensional over C, then as in lemma 2 in [23], the set $f(R)$ is dense on R and so from

$$af(r_1,\dots,r_k)^n + f(r_1,\dots,r_k)^n b - (af(r_1,\dots,r_k) + f(r_1,\dots,r_k)b)^n = 0$$

for all $r_1,\dots,r_k \in R$, we have

$$ar^n + r^n b - (ar+rb)^n = 0$$

for all $r \in R$. Let v and bv be linearly C-independent for some $v \in V$. Then by density there exists $r \in R$ such that $rv = 0$, $rbv = v$. Therefore, we have $0 = \{ar^n + r^n b - (ar+rb)^n\}v = -v$ for $n \geq 2$, contradiction. Hence, v and bv are linearly C-dependent for all $v \in V$. By Fact 5, we can write $bv = v\alpha$ for all $v \in V$ and $\alpha \in C$ fixed.

Now let $r \in R$, $v \in V$. Since $bv = v\alpha$,

$$[b,r]v = (br)v - (rb)v = b(rv) - r(bv) = (rv)\alpha - r(v\alpha) = 0.$$

Thus $[b,r]v = 0$ for all $v \in V$ i.e., $[b,r]V = 0$. Since $[b,r]$ acts faithfully as a linear transformation on the vector space V, $[b,r] = 0$ for all $r \in R$. Therefore, $b \in C$. Then we obtain

$$(a+b)r^n - ((a+b)r)^n = 0$$

for all $r \in R$. Let v and $(a+b)v$ be linearly C-independent for some $v \in V$. By density, we may choose $r \in R$ such that $rv = v$, $r(a+b)v = 0$. Then we get $0 = \{(a+b)r^n - ((a+b)r)^n\}v = (a+b)v$ for $n \geq 2$, a contradiction. Hence, v and $(a+b)v$ are linearly C-dependent for all $v \in V$, which implies as before that $a+b \in C$ and so $a \in C$. Therefore, $\{(a+b)^n - (a+b)\}r^n = 0$ for all $r \in R$. Since V is infinite dimensional over C, $(a+b)^n - (a+b) = 0$.

Proposition 2.3. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R and $f(r_1,\ldots,r_k)$ a multilinear polynomial over C. If for some $a,b \in U$, $af(r)^n + f(r)^n b = (af(r)+f(r)b)^n$ for all $r = (r_1,\ldots,r_k) \in I^k$, then one of the following holds:*

(1) $IC = eRC$ for some idempotent $e \in soc(RC)$ and $f(x_1,\ldots,x_k)$ is central-valued on $eRCe$;

(2) there exist $\alpha,\beta \in C$ such that $(a-\alpha)I = (0)$ and $(b-\beta)I = (0)$ with $(\alpha+\beta)^{n-1} = 1$;

(3) $b \in C$ and $(a+b)I = (0)$.

Proof. Let $u \in I$. Then R satisfies the GPI

$$\begin{aligned} &af(ux_1,\ldots,ux_k)^n + f(ux_1,\ldots,ux_k)^n b \\ &= (af(ux_1,\ldots,ux_k) + f(ux_1,\ldots,ux_k)b)^n. \end{aligned} \tag{1}$$

Now we consider following two cases:

Case-I: R does not satisfy any nontrivial GPI

Then (1) is a trivial GPI for R, that is,

$$af(ux_1,\ldots,ux_k)^n + f(ux_1,\ldots,ux_k)^n b - (af(ux_1,\ldots,ux_k) + f(ux_1,\ldots,ux_k)b)^n \tag{2}$$

is zero element in $R *_C C\{x_1,\ldots,x_k\}$. Suppose first that there exists $u \in I$ such that $\{bu,u\}$ are linearly C-independent. Then $b \notin C$, and hence above GPI implies that

$$f(ux_1,\ldots,ux_k)^n b - (af(ux_1,\ldots,ux_k) + f(ux_1,\ldots,ux_k)b)^{n-1} f(ux_1,\ldots,ux_k)b = 0.$$

Now since $\{bu, u\}$ are linearly C-independent, we see expanding the above expression that $(f(ux_1, \ldots, ux_k)b)^n$ appears nontrivially, a contradiction. Hence bu and u are linearly C-dependent for all $u \in I$. Then by Fact 6, there exists $\beta \in C$ such that $(b - \beta)I = (0)$. Next suppose that there exists $u \in I$ such that $\{au, u\}$ are linearly C-independent. Then from above (2), we obtain that

$$af(ux_1, \ldots, ux_k)^n - af(ux_1, \ldots, ux_k)\{af(ux_1, \ldots, ux_k) + f(ux_1, \ldots, ux_k)b\}^{n-1} = 0. \quad (3)$$

Expanding the above expression we find that the term $\{af(ux_1, \ldots, ux_k)\}^n$ appears nontrivially, a contradiction. Hence we conclude that au and u are linearly C-dependent for all $u \in I$. By Fact 6, there exists $\alpha \in C$ such that $(a - \alpha)I = (0)$.

Then (1) reduces to

$$f(ux_1, \ldots, ux_k)^n(\alpha + b) = (f(ux_1, \ldots, ux_k)(\alpha + b))^n. \quad (4)$$

Using $(b - \beta)I = (0)$, it follows that

$$f(ux_1, \ldots, ux_k)^n(\alpha + b) = f(ux_1, \ldots, ux_k)^n(\alpha + \beta)^{n-1}(\alpha + b) \quad (5)$$

that is

$$f(ux_1, \ldots, ux_k)^n\{1 - (\alpha + \beta)^{n-1}\}(\alpha + b) = 0. \quad (6)$$

Since this is trivial GPI for R, either $1 - (\alpha + \beta)^{n-1} = 0$ or $b = -\alpha \in C$. These two cases gives conclusion (2) and (3) respectively.

Case-II: R satisfy a nontrivial GPI

Now assume first that $[f(I), I]I = 0$, that is $[f(x_1, \ldots, x_k), x_{k+1}]x_{k+2} = 0$ for all $x_1, x_2, \ldots, x_{k+2} \in I$. Then by Fact 4, $IC = eRC$ for some idempotent $e \in soc(RC)$. Since $[f(I), I]I = 0$, we have $[f(IR), IR]IR = 0$ and hence $[f(IU), IU]IU = 0$ by [5, Theorem 2]. In particular, $[f(IC), IC]IC = 0$, or equivalently, $[f(eRC), eRC]eRC = 0$. Then $[f(eRCe), eRCe] = 0$, that is, $f(x_1, \ldots, x_k)$ is central-valued on $eRCe$ and hence conclusion (1) is obtained.

So, we assume that $[f(I), I]I \neq 0$, that is, $[f(x_1, \ldots, x_k), x_{k+1}]x_{k+2}$ is not an identity for I. In this case R is a prime GPI-ring and so is U (see [5]). Since U is centrally closed over C, it follows from [21] that U is a primitive ring with $H = Soc(U) \neq 0$. Then $[f(IH), IH]IH \neq 0$. For otherwise, $[f(IU), IU]IU = 0$ by [5], a contradiction. Choose $u_1, \ldots, u_{k+2} \in IH$ such that $[f(u_1, \ldots, u_k), u_{k+1}]u_{k+2} \neq$

0. Let $u \in IH$. Since H is a regular ring, there exists $e^2 = e \in H$ such that $eH = uH + u_1H + \cdots + u_{k+2}H$. Then $e \in IH$ and $u = eu$, $u_i = eu_i$ for $i = 1, \ldots, k+2$. Thus, we have $0 \neq [f(eH), eH]eH = [f(eHe), eHe]H$ i.e., $f(r_1, \ldots, r_k)$ is not central-valued in eHe.

By our assumption and by [5], we may also assume that

$$af(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b = (af(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b)^n$$

is an identity for IU. In particular,

$$af(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b = (af(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b)^n$$

is an identity for IH and so for eH. It follows that, for all $r_1, \ldots, r_k \in H$,

$$af(er_1, \ldots, er_k)^n + f(er_1, \ldots, er_k)^n b = (af(er_1, \ldots, er_k) + f(er_1, \ldots, er_k)b)^n. \quad (7)$$

We may write

$$f(x_1, \ldots, x_k) = \sum_i t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)x_i,$$

where t_i is a suitable multilinear polynomial in $k-1$ variables and x_i never appears in any monomials of t_i. Since $f(eHe) \neq 0$, there exists some t_i which does not vanish in eHe. Without loss of generality, we assume that $t_k(eHe) \neq 0$. Let $r \in H$. Then replacing r_k with $r(1-e)$ in (7), we have

$$0 = (at_k(er_1, \ldots, er_{k-1})er(1-e) + t_k(er_1, \ldots, er_{k-1})er(1-e)b)^n. \quad (8)$$

Left multiplying by $(1-e)$, we obtain $(1-e)(at_k(er_1, \ldots, er_{k-1})er(1-e))^n = 0$, that is, $\{(1-e)at_k(er_1, \ldots, er_{k-1})er\}^{n+1} = 0$ for all $r \in H$. By [9], $(1-e)at_k(er_1e, \ldots, er_{k-1}e) = 0$ for all $r_1, \ldots, r_{k-1} \in H$. Since eHe is a simple Artinian ring and $t_k(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [6, Lemma 2], $(1-e)ae = 0$. Now again right multiplying by e in (8), we obtain $(t_k(er_1, \ldots, er_{k-1})er(1-e)b)^n e = 0$ that is, $\{(1-e)bt_k(er_1, \ldots, er_{k-1})er)^{n+1} = 0$ for all $r \in H$, implying $(1-e)bt_k(er_1e, \ldots, er_{k-1}e) = 0$ for all $r_1, \ldots, r_{k-1} \in H$. By above argument we conclude that $(1-e)be = 0$.

In particular, from (7), we can write that H satisfies

$$e\{af(er_1e, \ldots, er_ke)^n + f(er_1e, \ldots, er_ke)^n b$$
$$-(af(er_1e, \ldots, er_ke) + f(er_1e, \ldots, er_ke)b)^n\}e = 0 \quad (9)$$

and so using the facts $(1-e)ae = 0$ and $(1-e)be = 0$, we have, prime ring eHe satisfies

$$eaef(r_1,\dots,r_k)^n + f(r_1,\dots,r_k)^n ebe - (eaef(r_1,\dots,r_k) + f(r_1,\dots,r_k)ebe)^n = 0. \quad (10)$$

By Proposition 2.2, since $f(r_1,\dots,r_k)$ is not central-valued in eHe, we conclude $eae, ebe \in Ce$ with $(eae+ebe)^n - (eae+ebe) = 0$. Therefore, $ae = eae \in Ce$ and $be = ebe \in Ce$. Thus $au = aeu = eaeu \in Cu$ and hence au, u are linearly C-dependent for each $u \in I$. So $(a-\alpha)I = (0)$ for some $\alpha \in C$. Similarly, $(b-\beta)I = (0)$ for some $\beta \in C$.

Thus our hypothesis $af(x_1,\dots,x_k)^n + f(x_1,\dots,x_k)^n b = (af(x_1,\dots,x_k) + f(x_1,\dots,x_k)b)^n$ for all $x_1,\dots,x_k \in I$, implies that $f(x_1,\dots,x_k)^n\{(\alpha+\beta)^{n-1}-1\}(\alpha+b) = 0$ for all $x_1,\dots,x_k \in I$. By Lemma 2 in [4], either $f(I)I = 0$ or $\{(\alpha+\beta)^{n-1}-1\}(\alpha+b) = 0$. If $f(I)I = 0$, then by Fact 4, conclusion (1) is obtained. If $\{(\alpha+\beta)^{n-1}-1\}(\alpha+b) = 0$, then either $(\alpha+\beta)^{n-1} = 1$ or $b = -\alpha \in C$. Both cases imply conclusions (2) and (3) respectively.

We are now ready to prove our main theorem.

Theorem 2.4. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, g a nonzero generalized derivation of R, I a nonzero right ideal of R, $f(r_1,\dots,r_k)$ a multilinear polynomial over C and $n \geq 2$ be a fixed integer. If $g(f(r_1,\dots,r_k)^n) = g(f(r_1,\dots,r_k))^n$ for all $r_1,\dots,r_k \in I$, then one of the following holds:*

(1) $IC = eRC$ for some idempotent $e \in soc(RC)$ and $f(x_1,\dots,x_k)$ is central-valued on $eRCe$;

(2) there exist $a, b \in U$ such that $g(x) = ax + xb$ for all $x \in R$ and $(a-\alpha)I = (0)$, $(b-\beta)I = (0)$ for some $\alpha, \beta \in C$ with $(\alpha+\beta)^{n-1} = 1$;

(3) there exists $a \in U$ such that $g(x) = ax$ for all $x \in R$ with $aI = (0)$.

Proof. If g is inner generalized derivation of R, then result follows by Proposition 2.3. Assume that g is not U-inner. Then by Remark 1, we may assume that for all $x \in U$, $g(x) = ax + d(x)$, where $a \in U$ and d is a derivation of U. By our assumption, I satisfies $g(f(x_1,\dots,x_k)^n) = g(f(x_1,\dots,x_k))^n$. Since I and IU satisfy the same generalized polynomial identities (see [5]) as well as the same differential identities (see [18]), we may assume for $u_1,\dots,u_k \in I$ that U satisfies

$$af(u_1x_1,\dots,u_kx_k)^n + d(f(u_1x_1,\dots,u_kx_k)^n)$$

$$= \{af(u_1x_1,\dots,u_kx_k) + d(f(u_1x_1,\dots,u_kx_k))\}^n \tag{11}$$

that is,

$$\begin{aligned}&af(u_1x_1,\dots,u_kx_k)^n\\ &+\sum_{i=0}^{n-1} f(u_1x_1,\dots,u_kx_k)^i d(f(u_1x_1,\dots,u_kx_k))f(u_1x_1,\dots,u_kx_k)^{n-i-1}\\ &= \{af(u_1x_1,\dots,u_kx_k) + d(f(u_1x_1,\dots,u_kx_k))\}^n.\end{aligned} \tag{12}$$

Since g is not inner, d can not be inner derivation of U. Then we have from (12) that

$$\begin{aligned}&af(u_1x_1,\dots,u_kx_k)^n\\ &+\sum_{i=0}^{n-1} f(u_1x_1,\dots,u_kx_k)^i\Big\{f^d(u_1x_1,\dots,u_kx_k)\\ &+\sum_j f(u_1x_1,\dots,d(u_j)x_j+u_jd(x_j),\dots,u_kx_k)\Big\}f(u_1x_1,\dots,u_kx_k)^{n-i-1}\\ &=\Big\{af(u_1x_1,\dots,u_kx_k)+f^d(u_1x_1,\dots,u_kx_k)\\ &+\sum_j f(u_1x_1,\dots,d(u_j)x_j+u_jd(x_j),\dots,u_kx_k)\Big\}^n.\end{aligned} \tag{13}$$

By Kharchenko's theorem [14], we have that U satisfies

$$\begin{aligned}&af(u_1x_1,\dots,u_kx_k)^n\\ &+\sum_{i=0}^{n-1} f(u_1x_1,\dots,u_kx_k)^i\Big\{f^d(u_1x_1,\dots,u_kx_k)\\ &+\sum_j f(u_1x_1,\dots,d(u_j)x_j+u_jy_j,\dots,u_kx_k)\Big\}f(u_1x_1,\dots,u_kx_k)^{n-i-1}\\ &=\Big\{af(u_1x_1,\dots,u_kx_k)+f^d(u_1x_1,\dots,u_kx_k)\\ &+\sum_j f(u_1x_1,\dots,d(u_j)x_j+u_jy_j,\dots,u_kx_k)\Big\}^n.\end{aligned} \tag{14}$$

In particular, putting $x_1 = 0$, we have that U satisfies

$$0 = \{f(u_1y_1,\dots,u_kx_k)\}^n. \tag{15}$$

Since I and IU satisfy the same polynomial identities, we have that I satisfies $f(x_1,\dots,x_k)^n = 0$. By [6, Main Theorem], $f(I)I = 0$ and hence conclusion (2) is obtained by using Fact 4. Hence the theorem is proved.

It is well known that if R is a prime ring and L is a non-central Lie ideal of R, then there exists a nonzero two-sided ideal I of R such that $0 \neq [I,R] \subseteq L$, unless char $(R) = 2$ and R satisfies the standard identity s_4. Thus from above theorem following corollary is straightforward.

Corollary 2.5. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, g a nonzero generalized derivation of R, L a noncentral Lie ideal of R and $n \geq 2$ be a fixed integer. If $g(u^n) = g(u)^n$ for all $u \in L$, then one of the following holds:*

(1) char $(R) = 2$ and R satisfies s_4, standard identity of four variables.

(2) there exists $\lambda \in C$ such that $g(x) = \lambda x$ for all $x \in R$ with $\lambda^{n-1} = 1$.

Now we prove our next corollary, which states that the restriction on char $(R) = 2$ and R satisfies s_4 in the Theorem B can be omitted.

Corollary 2.6. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, g a nonzero generalized derivation of R, I a nonzero right ideal of R and $f(r_1, \ldots, r_k)$ be a multilinear polynomial over C. If $g(f(r_1, \ldots, r_k)^2) = g(f(r_1, \ldots, r_k))^2$ for all $r_1, \ldots, r_k \in I$, then one of the following holds:*

(1) $IC = eRC$ for some idempotent $e \in soc(RC)$ and $f(x_1, \ldots, x_k)$ is central-valued on $eRCe$;

(2) there exists $a \in U$ such that $g(x) = xa$ for all $x \in I$ and $(a-1)I = (0)$;

(3) there exists $a \in U$ such that $g(x) = ax$ for all $x \in R$ with $aI = (0)$.

Proof. By theorem 2.4, we have only to consider the case when $g(x) = ax + xb$ for all $x \in R$ and $(a-\alpha)I = (0)$, $(b-\beta)I = (0)$ for some $\alpha, \beta \in C$ with $\alpha + \beta = 1$. Then $g(x) = ax + xb = \alpha x + xb = x(\alpha + b)$ for all $x \in I$ with $(0) = (b-\beta)I = (b+\alpha-1)I$. Hence we obtain our conclusion (2).

Corollary 2.7. *Let R be a prime ring with extended centroid C, g a nonzero generalized derivation of R and $f(r_1, \ldots, r_k)$ a noncentral multilinear polynomial over C. If $g(f(r_1, \ldots, r_k)^2) = g(f(r_1, \ldots, r_k))^2$ for all $r_1, \ldots, r_k \in R$, then $g(x) = x$ for all $x \in R$.*

Corollary 2.8. *Let R be a prime ring with extended centroid C, d a derivation of R and $f(r_1, \ldots, r_k)$ a noncentral multilinear polynomial over C. If $d(f(r_1, \ldots, r_k)^2) = d(f(r_1, \ldots, r_k))^2$ for all $r_1, \ldots, r_k \in R$, then $d = 0$.*

Example 1. Let Z be the set of all integers. Consider a ring $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | x, y \in Z \right\}$

and a multilinear polynomial $f(X,Y) = XY$ which is not central-valued on R. We define maps $g, d : R \to R$, by $g\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 2y \\ 0 & 0 \end{pmatrix}$ and $d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then g is a generalized derivation associated to the derivation d satisfying $g(f(X,Y)^2) = g(f(X,Y))^2$ for all $X, Y \in R$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0$, R is not prime ring. Since g is not an identity mapping in R, the primeness hypothesis in Corollary 2.7 is essential.

Acknowledgement. The authors would like to thank the referees for their careful reading the paper and providing very helpful comments and suggestions to improve the paper.

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Basudeb Dhara
Department of Mathematics
Belda College, Belda,
Paschim Medinipur
721424(W.B.), INDIA
e-mail: basu_dhara@yahoo.com

Shuliang Huang
Department of Mathematics
Chuzhou University
Chuzhou 239012, (CHINA)

e-mail: shulianghuang@sina.com

Atanu Pattanayak
Department of Mathematics
Belda College, Belda,
Paschim Medinipur
721424(W.B.), INDIA
e-mail: atanu.pattanayak@yahoo.com