

Generalized derivations centralizing on Jordan ideals of rings with involution

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Abstract: A classical result of Posner states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. In this paper we extend Posner's result to generalized derivations centralizing on Jordan ideals of rings with involution and discuss the related results. Moreover, we provide examples to show that the assumed restriction cannot be relaxed.

Key words: Rings with involution, generalized derivations, Jordan ideals

1. Introduction

Throughout R will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and we will make use of the following basic commutator identities without any specific mention: $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$. R is 2-torsion free if $2x = 0$ yields $x = 0$. We recall that R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$.

A ring with involution $(R, *)$ is $*$ -prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$ -prime but the converse is in general not true. For example, if R^o denotes the opposite ring of a prime ring R , then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$ -prime but not prime. This example shows that every prime ring can be injected in a $*$ -prime ring and from this point of view $*$ -prime rings constitute a more general class of prime rings.

An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$ for all $u \in J$ and $r \in R$. A Jordan ideal J that satisfies $J^* = J$ is called a $*$ -Jordan ideal. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$. An additive mapping $F : R \rightarrow R$ is said to be a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds for all pairs $x, y \in R$. A mapping f of R into itself is called centralizing if $[f(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[f(x), x] = 0$ holds for all $x \in R$, the mapping f is said to be commuting. The history of commuting and centralizing mappings goes back to 1955 when Divinsky [5] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later, Posner [12] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). Several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations that are centralizing or commuting on an appropriate subset of the ring (see [2–4, 6–8] for a partial bibliography). Recently, Oukhtite et al. generalized Posner's second theorem to rings with involution in the

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case of characteristic not 2 : Let R be a 2-torsion free $*$ -prime ring and U a square closed $*$ -Lie ideal. If R admits a nonzero derivation d centralizing on U , then $U \subseteq Z(R)$ [10, Theorem 1].

In the present paper we shall attempt to generalize Posner's second theorem to *generalized derivations* centralizing on Jordan ideals in rings with involution.

Throughout, $(R, *)$ will be a 2-torsion free ring with involution and $Sa_*(R) := \{r \in R / r^* = \pm r\}$ the set of symmetric and skew symmetric elements.

2. Jordan ideals and generalized derivations

We shall use without explicit mention the fact that if J is a nonzero Jordan ideal of a ring R , then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$ [14, Lemma 1]. Moreover, from [1] we have $4jRj \subset J$, $4j^2R \subset J$, and $4Rj^2 \subset J$ for all $j \in J$.

In order to prove our main theorem, we shall need the following lemmas.

Lemma 1 [11, Lemma 2] *Let R be a 2-torsion free $*$ -prime ring and J a nonzero $*$ -Jordan ideal of R . If $aJb = a^*Jb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2 [9, Lemma 3] *Let R be a 2-torsion free $*$ -prime ring and J a nonzero $*$ -Jordan ideal of R . If $J \subseteq Z(R)$, then R is commutative.*

We first fix the following facts, which will be used in the sequel.

Fact 1. If $aJ = 0$ or $Ja = 0$, then $a = 0$. Indeed, if $aJ = 0$ (resp. $Ja = 0$), then $aJa = 0 = aJa^*$ (resp. $aJa = 0 = a^*Ja$) and Lemma 1 yields $a = 0$.

Fact 2. Every $*$ -prime ring is semiprime. Indeed, if $aRa = 0$ then $aRaRa^* = 0$ so that $a = 0$ or $aRa^* = 0$. But $aRa^* = 0$ together with $aRa = 0$ force $a = 0$.

Fact 3. If $x \in Z(R)$ is such that $x^2 = 0$, then $x = 0$. Indeed, as $xRx = 0$ then Fact 2 forces $x = 0$.

Lemma 3 *Let R be a 2-torsion free $*$ -prime ring and J a nonzero $*$ -Jordan ideal of R . If d is a derivation such that $d(x^2) = 0$ for all $x \in J$, then $d = 0$.*

Proof From $d(x \circ y) = 0$ it follows that

$$d(x) \circ y + d(y) \circ x = 0 \text{ for all } x, y \in J. \tag{1}$$

Replacing y by $x \circ y$ in (1) and using (1) we find that

$$d(x) \circ (xy) + d(x) \circ (yx) = 0 \text{ for all } x, y \in J. \tag{2}$$

Since $d(x)x = -xd(x)$, then (2) becomes

$$[[d(x), y], x] = 0 \text{ for all } x, y \in J. \tag{3}$$

Substituting $2y[r, uv]$ for y in (3) with $u, v \in J$, as $2[r, uv] \in J$, then equation (3) assures us that

$$[y, x][d(x), [r, uv]] + [d(x), y][[r, uv], x] = 0 \text{ for all } u, v, x, y \in J \text{ and } r \in R. \tag{4}$$

Taking $y = x$ in (4) we arrive at

$$[d(x), x][[r, uv], x] = 0 \text{ for all } u, v, x \in J \text{ and } r \in R. \tag{5}$$

Replacing r by xr in (5) we obtain

$$[d(x), x]x[[r, uv], x] + [d(x), x][x, uv][r, x] = 0 \text{ for all } u, v, x \in J \text{ and } r \in R. \tag{6}$$

Since $[d(x), x]x = x[d(x), x]$ by (3), in light of (5), equation (6) yields

$$[d(x), x][x, uv][r, x] = 0 \text{ for all } u, v, x \in J \text{ and } r \in R. \tag{7}$$

Writing $rd(x)$ instead of r in (7), we get

$$[d(x), x][x, uv]r[d(x), x] = 0$$

and thereby

$$[d(x), x][x, uv]R[d(x), x][x, uv] = 0 \text{ for all } u, v, x \in J \text{ and } r \in R. \tag{8}$$

In view of Fact 2, equation (8) yields $[d(x), x][x, uv] = 0$ and thus

$$[d(x), x]u[x, v] + [d(x), x][x, u]v = 0 \text{ for all } u, v \in J. \tag{9}$$

Replacing v by $2v[r, s]$ in (9) we find that $[d(x), x]uv[x, [r, s]] = 0$ and thus

$$[d(x), x]uJ[x, [r, s]] = 0 \text{ for all } x, u \in J \text{ and } r, s \in R. \tag{10}$$

If $x \in J \cap Sa_*(R)$, then $[d(x), x]uJ[x, [r, s]]^* = 0$ and Lemma 1 shows that $[x, [r, s]] = 0$ or $[d(x), x]J = 0$, in which case, because of Fact 1, we find that $[d(x), x] = 0$.

Suppose that

$$[x, [r, s]] = 0 \text{ for all } r, s \in R. \tag{11}$$

Substituting rx for r in (11) and using (11) we get

$$[x, r][x, s] = 0 \text{ for all } r, s \in R. \tag{12}$$

Replacing s by sr in (12), we get $[x, r]s[x, r] = 0$ for all $r, s \in R$ and thus

$$[x, r]R[x, r] = 0 \text{ for all } r \in R. \tag{13}$$

According to Fact 2, equation (13) forces $x \in Z(R)$ and therefore $[d(x), x] = 0$. Hence, in all the cases we have

$$[d(x), x] = 0 \text{ for all } x \in J \cap Sa_*(R). \tag{14}$$

Let $x \in J$, and since $x - x^*, x + x^* \in J \cap Sa_*(R)$, then (14) yields

$$[d(x + x^*), x + x^*] = 0 \text{ and } [d(x - x^*), x - x^*] = 0$$

in such a way that

$$[d(x^*), x^*] = -[d(x), x] \text{ for all } x \in J. \tag{15}$$

Replacing x by x^* in (10), in light of (15), $[d(x), x]uJ[x^*, [r, s]] = 0$ so that

$$[d(x), x]uJ[x, [r, s]]^* = 0 \text{ for all } x, u \in J \text{ and } r, s \in R. \tag{16}$$

Using (10) together with (16) we conclude that $[d(x), x] = 0$ or $[x, [r, s]] = 0$, which, as above, leads to $[d(x), x] = 0$. Consequently,

$$[d(x), x] = 0 \text{ for all } x \in J. \tag{17}$$

Hence, d is centralizing on J , which, in view of [9, Theorem 1], assures us that either $d = 0$ or $J \subseteq Z(R)$. Suppose that $J \subseteq Z(R)$; then $2rj = j \circ r \in J$ for all $r \in R, j \in J$.

Since $d(xy + yx) = 0$ by hypothesis, then by replacing y by $2jr$ we obtain $xjd(r) = 0$. Hence

$$xJd(r) = x^*Jd(r) = 0$$

and Lemma 1 forces $d = 0$. □

Lemma 4 *Let F be an additive mapping that is centralizing on a $*$ -Jordan ideal J . If R is 2-torsion free $*$ -prime, then $F(J \cap Z(R)) \subset Z(R)$.*

Proof Linearizing $[F(x), x] \in Z(R)$, we obtain

$$[F(x), y] + [F(y), x] \in Z(R) \text{ for all } x, y \in J. \tag{18}$$

Now if $x \in J \cap Z(R)$, then (18) yields

$$[F(x), y] \in Z(R) \text{ for all } y \in J. \tag{19}$$

Replacing y by $2y[t, s]$ in (19), where $t, s \in R$, we get

$$y[F(x), [t, s]] + [F(x), y][t, s] \in Z(R) \text{ for all } x, y \in J, r, s \in R. \tag{20}$$

Writing st instead of s in (20), where $t \in R$, we find that

$$y[t, s][F(x), t] + y[F(x), [t, s]]t + [F(x), y][t, s]t \in Z(R). \tag{21}$$

Substituting $F(x)$ for t in (21) we obtain

$$y[F(x), [F(x), s]]F(x) + [F(x), y][F(x), s]F(x) \in Z(R) \text{ for all } y \in J, s \in R. \tag{22}$$

Taking $s = y$ in (22), because of $[F(x), y] \in Z(R)$, we arrive at

$$[F(x), y]^2F(x) \in Z(R) \text{ for all } y \in J. \tag{23}$$

Using the fact that $[[F(x), y]^2F(x), y] = 0$, by (23), we find that

$$[F(x), y]^3 = 0 \text{ for all } y \in J. \tag{24}$$

From $[F(x), y] \in Z(R)$, equation (24) yields $[F(x), y]^2R[F(x), y]([F(x), y])^* = 0$. Since $[F(x), y]([F(x), y])^*$ is invariant under $*$ and R is $*$ -prime, then we get $[F(x), y]^2 = 0$ or $[F(x), y]([F(x), y])^* = 0$.

If $[F(x), y]([F(x), y])^* = 0$, then $[F(x), y]^2 R([F(x), y])^* = 0 = [F(x), y]^2 R[F(x), y]$ and either $[F(x), y] = 0$ or $[F(x), y]^2 = 0$. Hence, in all cases we have $[F(x), y]^2 = 0$ and by virtue of Fact 3 it follows that

$$[F(x), y] = 0 \quad \text{for all } y \in J. \tag{25}$$

Replacing y by $2y[r, s]$ in (25), where $r, s \in R$, and using (25) we get

$$y[F(x), [r, s]] = 0 \quad \text{for all } y \in J, r, s \in R.$$

Accordingly

$$J[F(x), [r, s]] = 0 \quad \text{for all } r, s \in R. \tag{26}$$

In view of Fact 1, equation (26) forces

$$[F(x), [r, s]] = 0 \quad \text{for all } r, s \in R. \tag{27}$$

Substituting zt for r in (27), where $z \in J$, in view of (25) we arrive at

$$[z, s][F(x), t] = 0 \quad \text{for all } z \in J, s, t \in R. \tag{28}$$

Replacing t by wt in (28), with $w \in R$ we obtain

$$[z, s]w[F(x), t] = 0 \quad \text{for all } z \in J, s, t, w \in R,$$

and thereby we conclude that

$$[z, s]R[F(x), t] = 0 \quad \text{for all } z \in J, s, t \in R. \tag{29}$$

Since J is invariant under $*$, then (29) assures us that

$$([z, s])^* R[F(x), t] = 0 \quad \text{for all } z \in J, s, t \in R. \tag{30}$$

As R is $*$ -prime, then (29) together with (30) yields $[z, s] = 0$ for all $s \in R, z \in J$, in which case $J \subset Z(R)$, or $[F(x), t] = 0$ for all $t \in R$ so that $F(x) \in Z(R)$.

Now assume that $J \subset Z(R)$; from $j \circ r = 2jr \in J$ for all $r \in R, j \in J$, then replacing y by $2jr$ in (25) we find that $j[F(x), r] = 0$ for all $j \in J, r \in R$. Hence

$$J[F(x), r] = 0 \quad \text{for all } r \in R. \tag{31}$$

Once again using Fact 1, equation (31) leads to $[F(x), r] = 0$ for all $r \in R$ and thus $F(x) \in Z(R)$. In conclusion, $F(J \cap Z(R)) \subset Z(R)$. □

Now we are ready to state the main result of this paper.

Theorem 1 *Let R be a 2-torsion free $*$ -prime ring and F a generalized derivation associated with a nonzero derivation d . If F is centralizing on a nonzero $*$ -Jordan ideal J , then R is commutative.*

Proof Let F be a generalized derivation associated with a derivation $d \neq 0$. Suppose first that $J \cap Z(R) = 0$; in light of

$$4[F(x), x]^2 = (4xF(x)x) \circ F(x) - 4x(F(x))^2x - F(x) \circ (4x^2F(x)) + 4x^2(F(x))^2$$

it follows that $4[F(x), x]^2 \in J$ and thus $4[F(x), x]^2 = 0$. Using 2-torsion freeness, then we get $[F(x), x]^2 = 0$ for all $x \in J$, which, according to Fact 3, assures us that

$$[F(x), x] = 0 \text{ for all } x \in J. \tag{32}$$

Linearizing (32) we obtain

$$[F(x), y] + [F(y), x] = 0 \text{ for all } x, y \in J. \tag{33}$$

Substituting $4zx^2$ for y in (33), where $z \in J$, and using (33) we find that

$$z[d(x^2), x] + [z, x]d(x^2) = 0 \text{ for all } z, x \in J. \tag{34}$$

Replacing z by $2[r, s]z$ in (34), where $r, s \in R$, we arrive at $[[r, s], x]zd(x^2) = 0$ and therefore

$$[[r, s], x]Jd(x^2) = 0 \text{ for all } x \in J, r, s \in R. \tag{35}$$

For $x \in J \cap Sa_*(R)$, equation (35) together with Lemma 1 assures us that either $d(x^2) = 0$ or $[[r, s], x] = 0$ for all $r, s \in R$.

Assume that

$$[[r, s], x] = 0 \text{ for all } r, s \in R. \tag{36}$$

Writing xt instead of s in (36) we get $[r, x][t, x] = 0$ so that

$$[r, x]R[t, x] = 0 \text{ for all } r, t \in R. \tag{37}$$

Since $x \in J \cap Sa_*(R)$, then (37) yields

$$[r, x]^*R[t, x] = 0 \text{ for all } r, t \in R. \tag{38}$$

Using the $*$ -primeness of R , from equations (37) and (38), it follows that $x \in Z(R)$. As $J \cap Z(R) = 0$, then $x = 0$, in which case $d(x^2) = 0$. In conclusion,

$$d(x^2) = 0 \text{ for all } x \in J \cap Sa_*(R). \tag{39}$$

Let $x \in J$; the fact that $x - x^*, x + x^* \in J \cap Sa_*(R)$, implies that $d((x - x^*)^2) = 0 = d((x + x^*)^2)$, which, because of $\text{char } R \neq 2$, forces $d(x^2) = -d((x^*)^2)$.

Replacing x by x^* in (35), then we get $[[r, s], x^*]Jd(x^2) = 0$ so that

$$[[r, s], x]^*Jd(x^2) = 0 \text{ for all } x \in J, r, s \in R. \tag{40}$$

Using (35) together with (40), Lemma 1 assures us that for all $x \in J$, either $d(x^2) = 0$ or $[[r, s], x] = 0$ for all $r, s \in R$.

If $d(x^2) = 0$ for all $x \in J$, then Lemma 3 yields $d = 0$, a contradiction. If $[[r, s], x] = 0$ for all $r, s \in R$, then it is obvious that $x \in J \cap Z(R)$ and thus $J = 0$, a contradiction.

Accordingly, we are forced to $J \cap Z(R) \neq 0$. Let us consider $0 \neq u \in J \cap Z(R)$; linearizing $[F(x), x] \in Z(R)$ we get

$$[F(x), y] + [F(y), x] \in Z(R) \text{ for all } x, y \in J. \tag{41}$$

Replacing y by $2u^2r$ in (41) we get

$$2u^2[F(x), r] + [F(2u^2)r, x] + 2u^2[d(r), x] \in Z(R) \text{ for all } x \in J, r \in R. \tag{42}$$

Since $2u^2 \in J \cap Z(R)$, applying Lemma 4, (42) then becomes

$$2u^2[F(x), r] + F(2u^2)[r, x] + 2u^2[d(r), x] \in Z(R) \text{ for all } x \in J, r \in R. \tag{43}$$

Taking $r = x$ in (43) we obtain $u^2[d(x), x] \in Z(R)$ for all $x \in J$ so that

$$u^2[[d(x), x], r] = 0 \text{ for all } x \in J, r \in R. \tag{44}$$

Substituting sr for r in (44), with $s \in R$, we get $u^2s[[d(x), x], r] = 0$ and therefore

$$u^2R[[d(x), x], r] = 0 \text{ for all } x \in J, r \in R. \tag{45}$$

Since $u^* \in J \cap Z(R)$, a similar reasoning leads to

$$(u^*)^2R[[d(x), x], r] = 0 \text{ for all } x \in J, r \in R. \tag{46}$$

From equations (45) and (46), it follows, according to the $*$ -primeness of R , that $[[d(x), x], r] = 0$, because $u^2 \neq 0$ by Fact 3. Consequently, $[d(x), x] \in Z(R)$ for all $x \in J$. Applying [9, Theorem 1], we obtain $J \subseteq Z(R)$ and Lemma 2 assures the commutativity of R . □

The following example demonstrates that Theorem 1 cannot be extended to semiprime rings.

Example 1 Let $(R_1, *)$ be a noncommutative semiprime ring, with involution, which admits a generalized derivation F associated with a nonzero derivation d and let $R = R_1 \times R_1$. Consider $J = \{0\} \times R_1$ and define a generalized derivation \mathcal{F} on R by $\mathcal{F}(x, y) = (F(x), 0)$ associated with a derivation D defined by $D(x, y) = (d(x), 0)$. Obviously, J is a nonzero τ -Jordan ideal of R , where τ is the involution defined on R by $\tau(x, y) = (x^*, y^*)$. Furthermore, $[\mathcal{F}(u), u] \in Z(R)$ for all $u \in J$, but R is noncommutative.

In Theorem 1, we cannot exclude the condition “ J a $*$ -Jordan ideal” as below.

Example 2 Let R be a noncommutative prime ring that admits a generalized derivation F associated with a nonzero derivation d and let $\mathcal{R} = R \times R^0$. If we set $J = \{0\} \times R^0$, then J is a nonzero Jordan ideal of the $*_{ex}$ -prime ring \mathcal{R} . Furthermore, if we define $\mathcal{F}(x, y) = (F(x), 0)$, then \mathcal{F} is a generalized derivation of \mathcal{R} , associated with the nonzero derivation D defined by $D(x, y) = (d(x), 0)$, which satisfies $[\mathcal{F}(u), u] \in Z(\mathcal{F})$ for all $u \in J$; however, \mathcal{R} is noncommutative.

Corollary 1 *Let R be a 2-torsion free $*$ -prime ring and F be a generalized derivation associated with a nonzero derivation. If F is centralizing, then R is commutative.*

As an application of Theorem 1, the following theorem gives a version of Posner’s Second Theorem for generalized derivations on Jordan ideals.

Theorem 2 *Let R be a 2-torsion free prime ring and F a generalized derivation associated with a nonzero derivation d . If F is centralizing on a nonzero Jordan ideal J , then R is commutative.*

Proof Assume that F is a generalized derivation associated with a nonzero derivation d . Let \mathcal{F} be the additive mapping defined on $\mathcal{R} = R \times R^0$ by $\mathcal{F}(x, y) = (F(x), y)$. Clearly, \mathcal{F} is a generalized derivation associated with the nonzero derivation \mathcal{D} defined on \mathcal{R} by $\mathcal{D}(x, y) = (d(x), 0)$. Moreover, if we set $\mathcal{J} = J \times J$, the \mathcal{J} is a $*_{\text{ex}}$ -Jordan ideal of \mathcal{R} . As F is centralizing on J , it is easy to check that \mathcal{F} is centralizing on \mathcal{J} . Since \mathcal{R} is a $*_{\text{ex}}$ -prime ring, in view of Theorem 1 we deduce that \mathcal{R} is commutative and it follows that R is commutative. \square

The following theorem extends [13, Theorem 3.1] to Jordan ideals.

Theorem 3 *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal. If R admits a generalized derivation F such that $[F(u), u] = 0$ for all $u \in J$, then F is a left multiplier or R is commutative.*

Proof Assume that F is a generalized derivation associated with a derivation d . If $d = 0$, then F is a left multiplier. If $0 \neq d$, as F is commuting and a fortiori centralizing, then Theorem 2 assures the commutativity of R . \square

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