Bull. Korean Math. Soc. **48** (2011), No. 6, pp. 1253–1259
http://dx.doi.org/10.4134/BKMS.2011.48.6.1253

GENERALIZED DERIVATIONS ON SEMIPRIME RINGS

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Abstract. Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a derivation d such that $(F([x,y]))^n = [x,y]$ for all $x,y \in I$. Then either R is commutative or $n=1$, $d=0$ and F is the identity map on R. Moreover in case R is a semiprime ring and $(F([x,y]))^n = [x,y]$ for all $x,y \in R$, then either R is commutative or $n=1$, $d(R) \subseteq Z(R)$, R contains a non-zero central ideal and $F(x) - x \in Z(R)$ for all $x \in R$.

1. Introduction

Let R be a prime ring with center $Z(R)$ and extended centroid C, U the Utumi quotients ring (for more details on these objects we refer the reader to [3]). We denote by $[a,b] = ab - ba$ the simple commutator of the elements $a,b \in R$ and by $a \circ b = ab + ba$ the simple anti-commutator of a,b. Recall that a ring R is prime if for any $a,b \in R$, $aRb = (0)$ implies either $a=0$ or $b=0$, and it is semiprime if for any $a \in R$, $aRa = (0)$ implies $a=0$. Let $f : R \to R$ be an additive mapping on R. It is a derivation if $f(xy) = f(x)y + xf(y)$ holds for all $x,y \in R$. It is a left multiplier if $f(xy) = f(x)y$ for all $x,y \in R$.

An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x,y \in R$, and d is called the associated derivation of F. Hence, the concept of generalized derivations covers both the concepts of derivations and left multipliers. Basic examples of generalized derivations are mappings of type $x \to ax + xb$ for some $a,b \in R$. These maps are called inner generalized derivations. More informations on generalized derivations can be found in [8]. We would like to point out that in [11] Lee proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U. In particular Lee proves the following result:

Received July 16, 2010; Revised April 20, 2011.
2010 *Mathematics Subject Classification.* 16N60, 16W25.
Key words and phrases. prime and semiprime rings, differential identities, generalized derivations.
The second author was supported by the Natural Science Research Foundation of Anhui Provincial Education Department(No. KJ2010B144).

Fact 1 ([11, Theorem 4])**.** Let R be a semiprime ring. Then every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U.

In [5, Theorem 2], Daif and Bell showed that if R is a semiprime ring, I is a nonzero ideal of R and $d : R \to R$ is a derivation such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$.

Later in [14], Quadri et al. discussed the commutativity of prime rings with generalized derivations. More precisely, they proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative.

In [2, Theorem 4.1], Ashraf and Rehman obtained a similar result in case R is a prime ring, replacing the simple commutator by the simple anti-commutator. They proved that if I is a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative.

More recently in [1, Theorem 1], Argac and Inceboz generalized the above result as follows: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer; if R admits a derivation d with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.

Motivated by these results, we study prime and semiprime rings admitting a generalized derivation F satisfying a condition $(F([x, y]))^n = [x, y]$.

2. The results

Firstly we consider the case when R is a prime ring and begin with the following:

Remark 1. If I is a non-zero ideal of the prime ring R, then:

(1) I, R and U satisfy the same generalized polynomial identities with coefficients in U (Theorem 2 in [4]);

(2) I, R and U satisfy the same differential identities (Theorem 2 in [12]).

Theorem 1. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a derivation d such that $(F([x, y]))^n = [x, y]$ for all $x, y \in I$, then either R is commutative or $n = 1$, $d = 0$ and F is the identity map on R.*

Proof. Assume first that $n = 1$. In view of Theorem 2.1 in [14], we have either R is commutative or $d = 0$. Consider now this last case and assume that R is not commutative. Thus $F(xy) = F(x)y$ for all $x, y \in R$. Let $x, y, z \in I$, then $xz \in I$. By the hypothesis it follows $F([xz, y]) = [xz, y]$ and expanding this we have $(F(x) - x)[z, y] = 0$. Replace now z by $zr \in I$ for any $r \in R$. Thus one has $0 = (F(x) - x)[zr, y] = (F(x) - x)z[r, y]$, which means $(F(x) - x)I[R, I] = (0)$ for all $x \in R$. Thus, by the primeness of R and since R is assumed not commutative, it follows that $F(x) = x$ for all $x \in I$. Hence, for any $s \in R$ we have $sx \in I$ and $sx = F(sx) = F(s)x$, i.e., $(s - F(s))I = 0$ which implies $F(s) = s$ for all $s \in R$ and F is the identity map on R.

Assume now that $n \geq 2$. By Fact 1 we have that for all $x \in R$, $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. By the given hypothesis we have now $[x,y] = (a[x,y] + d([x,y]))^n = (a[x,y] + [d(x),y] + [x,d(y)])^n$ for all $x, y \in I$. This means that I satisfies the generalized differential identity

$$(a[x_1,x_2] + [d(x_1),x_2] + [x_1,d(x_2)])^n - [x_1,x_2]. \tag{1}$$

Since I and U satisfy the same differential identities (Remark 1) we also have that U satisfies (1). We divide the proof into two cases:

Firstly we assume that d is the inner derivation induced by some element $q \in U$, that is $d(x) = [q,x]$ for all $x, y \in U$. In this case we will prove that $q \in C$.

Notice that U satisfies the generalized polynomial identity

$$(a[x_1,x_2] + [[q,x_1],x_2] + [x_1,[q,x_2]])^n - [x_1,x_2].$$

In case the center C of U is infinite, we have that $(a[x_1,x_2] + [[q,x_1],x_2] + [x_1,[q,x_2]])^n - [x_1,x_2]$ is a generalized polynomial identity for $U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Remark that, in light of Remark 1, $[q,x]$ is a generalized polynomial identity for U if and only if it is a generalized identity also for R; analogously U is commutative if and only if R is commutative. Therefore, in order to prove that either $q \in C$ or R is commutative, we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite. Moreover, since both U and $U \otimes_C \overline{C}$ are prime and centrally closed (Theorem 2.5 and Theorem 3.5 in [6]), we may assume that R is centrally closed over C (i.e., $RC = C$) which is either finite or algebraically closed and $(a[x,y]+[[q,x],y]+[x,[q,y]])^n = [x,y]$ for all $x, y \in R$. By Theorem 3 in [13], R is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D.

Assume that $\dim V_D \geq 3$. Our first aim is to show that v and qv are linearly D-dependent for all $v \in V$. Suppose there exists $v \in V$ such that v and qv are D-independent. Since $\dim V_D \geq 3$, then there exists $w \in V$ such that v, qv, w are also D-independent. By the density of R, there exist $x, y \in R$ such that: $xv = 0, xqv = w, xw = v; yv = 0, yqv = 0, yw = v$. These imply that $v = (a[x,y] + [[q,x],y] + [x,[q,y]])^n v = [x,y]v = xyv - yxv = 0$, which is a contradiction. So we conclude that v and qv are linearly D-dependent for all $v \in V$.

Our next goal is to show that there exists $b \in D$ such that $qv = vb$ for all $v \in V$. In fact, choose $v, w \in V$ linearly D-independent. Since $\dim V_D \geq 3$, then there exists $u \in V$ such that u, v, w are linearly D-independent, and so $b_u, b_v, b_w \in D$ such that $qu = ub_u$, $qv = vb_v$, $qw = wb_w$, that is $q(u+v+w) = ub_u + vb_v + wb_w$. Moreover $q(u+v+w) = (u+v+w)b_{u+v+w}$ for a suitable $b_{u+v+w} \in D$. Then $0 = u(b_{u+v+w} - b_u) + v(b_{u+v+w} - b_v) + w(b_{u+v+w} - b_w)$ and because u, v, w are linearly D-independent, $b_u = b_v = b_w = b_{u+v+w}$, that is b does not depend on the choice of v. Hence now we have $qv = vb$ for all $v \in V$.

Now for $r \in R$, $v \in V$, we have $(rq)v = r(qv) = r(vb) = (rv)b = q(rv)$, that is $[q, R]V = 0$. Since V is a left faithful irreducible R-module, hence $[q, R] = 0$, i.e., $q \in C$ and so $d = 0$.

Therefore $(a[x, y])^n = [x, y]$ for all $x, y \in R$. Suppose there exists $v \in V$ such that v and av are D-independent. Since $\dim V_D \geq 3$, there exists $w \in V$ such that v, av, w are also D-independent. By the density of R, there exist $x, y \in R$ such that $xv = 0, yv = w, xw = v, xav = 0, yav = 0$. Hence it follows the contradiction $v = [x, y]v = (a[x, y])^n v = 0$. Therefore, using the same above argument, we have that $a \in C$. This means that $a^n[x, y]^n = [x, y]$ for all $x, y \in R$. Again fix $v_1, v_2, v_3 \in V$ linearly D-independent vectors. As above there exist $x, y \in R$ such that $xv_1 = 0, yv_1 = v_2, xv_2 = v_3, xv_3 = 0, yv_3 = 0$. Finally we have the contradiction $v_3 = [x, y]v_1 = a^n[x, y]^n v_1 = 0$.

Suppose now that $\dim V_D \leq 2$. In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By [10] (Lemma 2), it follows that there exists a suitable field E such that $R \subseteq M_k(E)$, the ring of all $k \times k$ matrices over E, and moreover $M_k(E)$ satisfies the same generalized polynomial identities of R. In particular $M_k(E)$ satisfies $(a[x_1, x_2] + [[q, x_1], x_2] + [x_1, [q, x_2]])^n - [x_1, x_2]$. If $k \geq 3$, by the same above argument we get $q \in C$. Obviously if $k = 1$, then R is commutative.

Thus we may assume that $k = 2$, i.e., $R \subseteq M_2(E)$, where $M_2(E)$ satisfies $(a[x_1, x_2] + [[q, x_1], x_2] + [x_1, [q, x_2]])^n - [x_1, x_2]$.

Denote e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. Let $[x_1, x_2] = [e_{21}, e_{11}] = e_{21}$. In this case we have $(ae_{21} + qe_{21} - e_{21}q)^n = e_{21}$. Right multiplying by e_{21}, we get

$$(-1)^n(e_{21}q)^n e_{21} = (ae_{21} + qe_{21} - e_{21}q)^n e_{21} = e_{21}e_{21} = 0.$$

Set $q = \sum_{i,j=1}^{2} q_{ij}e_{ij}$, with $q_{ij} \in E$. By calculation we find that $(-1)^n q_{12}^n e_{21} = 0$, which implies that $q_{12} = 0$. Similarly we can see that $q_{21} = 0$. Therefore q is diagonal in $M_2(E)$. Let f be any automorphism of $M_2(E)$ and notice that $(f(a)[f(x), f(y)]+[[f(q), f(x)], f(y)]+[f(x), [f(q), f(y)]])^n = [f(x), f(y)]$. Thus the same above argument shows that $f(q)$ is a diagonal matrix in $M_2(E)$. In particular, let $f(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$, then $f(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies again that q is central in $M_2(E)$. Therefore $d = 0$ and $M_2(E)$ satisfies the generalized identity $(a[x_1, x_2])^n - [x_1, x_2]$. Let $[x_1, x_2] = e_{21}$. Thus we have $(ae_{21})^n = e_{21}$. Analogously, for $[x_1, x_2] = e_{12}$ we have that $(ae_{12})^n = e_{12}$. As above we obtain that a is a diagonal matrix and using the same above argument, we conclude that a is a central matrix. Thus $M_2(E)$ satisfies $a^n[x_1, x_2]^n - [x_1, x_2]$. In this case we notice that, for $[x_1, x_2] = e_{12}$, the contradiction $0 = e_{12}$ follows.

Assume now that d is not an inner derivation of U. Hence, by (1) and the Kharchenko's result in [9], it follows that U satisfies the generalized polynomial identity

$$(a[x_1, x_2] + [y_1, x_2] + [x_1, y_2])^n - [x_1, x_2]. \tag{2}$$

As above, we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite, and assume that R is centrally closed over C. Thus R satisfies (2) and in particular, R satisfies the blended component $([x_1, y_2])^n$, that is, R is a ring satisfying a polynomial identity. Hence there exists a suitable field E such that $R \subseteq M_k(E)$, the ring of all $k \times k$ matrices over E, and moreover $M_k(E)$ satisfies the same polynomial identities of R. In particular $M_k(E)$ satisfies $[x_1, x_2]^n$. If $k \geq 2$, for $x_1 = e_{12}$ and $x_2 = e_{21}$, we get the contradiction $(e_{11} - e_{22})^n = 0$. Thus $k = 1$ and then R is commutative. $\square$

The following example shows that R to be prime is essential in the hypothesis.

Example. Let S be any ring and let $R = \{\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \mid a, b \in S\}$ and let $I = \{\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \mid a \in S\}$ be a nonzero ideal of R. We define a map $F : R \to R$ by $F(x) = 2e_{11}x - xe_{11}$. Then it is easy to see that F is a generalized derivation associated with a nonzero derivation $d(x) = [e_{11}, x]$. It is straightforward to check that F satisfies the property: $(F([x, y]))^n = [x, y]$ for all $x, y \in I$. However, R is not commutative.

Finally we extend the above result to semiprime rings:

Theorem 2. *Let R be a semiprime ring and n a fixed positive integer. If R admits a generalized derivation F associated with a derivation d such that $(F([x, y])^n = [x, y]$ for all $x, y \in R$, then either R is commutative or $n = 1$, $d(R) \subseteq Z(R)$, R contains a non-zero central ideal and $F(x) - x \in Z(R)$ for all $x \in R$.*

Proof. First consider $n = 1$. Let P be a prime ideal of R such that $[R, R] \not\subseteq P$ and set $\overline{R} = R/P$. Assume first that $d(P) \not\subseteq P$. Let p be any element of P. Since for all $y \in R$, $a[p, y] + [d(p), y] + [p, d(y)] = [p, y] \in P$, then $[d(p), y] \in P$, that is $[d(P), R] \subseteq P$. Thus $[d(PR), R] \subseteq P$ and by calculations we get $d(P)[R, R] \subseteq P$. So $d(P)[R^2, R] \subseteq P$ which implies that $d(P)R[R, R] \subseteq P$. By the primeness of P and since $d(P) \not\subseteq P$, it follows that $[R, R] \subseteq P$, a contradiction.

Hence we may assume that $d(P) \subseteq P$, then d induces a canonical derivation $\overline{d}$ on $\overline{R}$. By the assumption we have $\overline{a}[\overline{x}, \overline{y}] + \overline{d}([\overline{x}, \overline{y}]) = [\overline{x}, \overline{y}]$ for all $x, y \in R$. It follows from the prime case that one of the following holds:

1. either $[\overline{R}, \overline{R}] = \overline{0}$, that is $[R, R] \subseteq P$, a contradiction; or
2. $\overline{d}(\overline{R}) = (\overline{0})$ and $\overline{ax} - \overline{x} = \overline{0}$ for all $x \in R$, that is $d(P) \subseteq P$ and $ax - x \in P$ for all $x \in R$.

In light of previous argument we have that both $d(R)[R, R] \subseteq \bigcap_i P_i = (0)$ and $(ax - x)[R, R] \subseteq \bigcap_i P_i = (0)$ for all $x \in R$ (where P_i are all prime ideals of R). Starting from $d(R)[R, R] = 0$, we have $0 = d(R^2)[d(R), R] = d(R)R[d(R), R]$, in particular we have both $d(R)R^2[d(R), R] = 0$ and

$$Rd(R)R[d(R), R] = 0.$$

Therefore $[d(R), R]R[d(R), R] = 0$ and by the semiprimeness of R, $[d(R), R] = 0$, that is $d(R) \subseteq Z(R)$.

Now consider also $(ax - x)[R, R] = 0$. A result of Zalar [15, Lemma 1.3] says that in this case there exists a non-zero central ideal of R. Moreover we have that $0 = (ax - x)[R^2, R] = (ax - x)R[R, R]$, in particular we have both $(ax - x)R^2[R, R] = 0$ and $R(ax-x)R[R, R] = 0$. Therefore $[ax-x, R]R[R, R] = 0$ and a fortiori $[ax-x, R]R[ax-x, R] = 0$. By the semiprimeness of R, $[ax-x, R] = 0$, that is $ax - x \in Z(R)$ for all $x \in R$. Thus, for all $x \in R$, $ax = x + \alpha_x$, where $\alpha_x \in Z(R)$ is depending on the choice of x; hence $F(x) = ax + \beta_x$, where $\beta_x = d(x) \in Z(R)$, that is $F(x) = x + \gamma_x$ for $\gamma_x = \alpha_x + \beta_x \in Z(R)$.

Let now $n \geq 2$. As above let P be a prime ideal of R, and set $\overline{R} = R/P$. Assume first that $d(P) \not\subseteq P$. Let p be any element of P. Since for all $y \in R$, $(a[p, y] + [d(p), y] + [p, d(y)])^n - [p, y] = 0$, then $[\overline{d(p)}, \overline{y}] = \overline{0}$ in $\overline{R}$ for all $\overline{y} \in \overline{R}$. Since $\overline{R}$ is a prime ring, by a result of Giambruno and Herstein [7] (Theorem 1) either $\overline{R}$ is commutative, that is $[R, R] \subseteq P$, or $\overline{d(p)}$ is central in $\overline{R}$, that is $[d(P), R] \subseteq P$. In this last case, by using the same above argument, one can see that again $[R, R] \subseteq P$. Hence we may assume that $d(P) \subseteq P$, then d induces a canonical derivation $\overline{d}$ on $\overline{R}$. By the assumption we have $(\overline{a}[\overline{x}, \overline{y}] + \overline{d}([\overline{x}, \overline{y}]))^n - [\overline{x}, \overline{y}] = \overline{0}$ for all $x, y \in R$. It follows from the prime case that $\overline{R}$ is commutative, that is $[R, R] \subseteq P$. In light of previous argument we have that in any case $[R, R] \subseteq P$. So $[R, R] \subseteq \bigcap_i P_i = (0)$ (where P_i are all prime ideals of R) and R is commutative. $\square$

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