GENERALIZED DIMENSION FUNCTION FOR W^*-ALGEBRAS OF INFINITE TYPE

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For the dimension theory of W^*-algebras, many results have been obtained up to present, but the infinite case remains obscure. It is the purpose of the present paper to remove this obscurity by defining a dimension function in a properly infinite W^*-algebra which admits the same properties as in the case of finite W^*-algebra.

1. Considerations on multiplicity theorem. By a W^*-algebra $\mathbf{M}$, we shall always mean a weakly closed self-adjoint operator algebra with the identity operator acting on a Hilbert space $\mathbf{H}$ and denote by $\mathbf{M}^\natural$, Ω the center of $\mathbf{M}$ and its spectrum. For a projection e in $\mathbf{M}$, we denote by $\mathbf{M}_e$ the restricted algebra of $\mathbf{M}$ onto $e\mathbf{H}$. $\Omega_{e\mathrm{H}}$ means the spectrum of $\mathbf{M}_e$. We shall always mean by $z(e)$ the central envelope of a projection e. A word "cardinal number" is used in its infinite case unless otherwise noticed.

LEMMA 1. *Let $\{e_i\}$ $(i \in I)$, $\{f_j\}$ $(j \in J)$ be infinite orthogonal family of countably decomposable projections in a W^*-algebra $\mathbf{M}$, and e a central projection. If $\sum_{i\in I} e_i = e = \sum_{j\in J} f_j$, then the cardinal of I equals the cardinalies of J.*

PROOF. It is sufficient to prove the lemma when $e = 1$. Since $\mathbf{M}_{e_i}$ is countably decomposable, there exists a faithful normal state φ_i on $\mathbf{M}_{e_i}$. Let $K_i = \{j \in J \mid e_i f_j e_i \neq 0\}$; since for any $j \in J$ there exists $i(j) \in I$ such that $e_{i(j)} f_j e_{i(j)} \neq 0$, we have $J = \bigcup_{i\in I} K_i$. From $e_i = \sum_{j\in J} e_i f_j e_i$ we get $\varphi_i(e_i) = \sum_{j\in J} \varphi_i(e_i f_j e_i) < +\infty$, so that K_i is at most countable. Therefore Card. $J \leqq$ (Card. I) $\times \aleph_0 =$ Card. I. By symmetry, we have Card. I = Card. J.

This lemma admits the following

DEFINITION 1. Let $\mathbf{M}$ be a W^*-algebra and α an infinite cardinal. A non-zero central projection e is called *α-dimensional homogeneous projection* if for every central projection f, bounded by e and countably decomposable for $\mathbf{M}^\natural$, there exists a family of orthogonal, equivalent, countably decomposable projection $\{f_j\}$ $(j \in J)$ in $\mathbf{M}$ such that $f = \sum_{j\in J} f_j$, where the cardinal of J is always α. Any projection p in $\mathbf{M}$ is called *α-dimensional homogeneous projection* if it is α-dimensional homogeneous projection as a central proiection in $\mathbf{M}_p$ by the above definition.

Our definition leads us to the multiplicity theorem of W^*-algebra analogous to the usual one in the theory of spatial invariants. We give here only the statements of our lemmas and omit their proofs.

LEMMA 2. *Let* $\mathbf{M}$ *be a properly infinite* W^**-algebra, then* $\mathbf{M}$ *contains a homogeneous central projection.*

LEMMA 3. *For a cardinal number* α *for which there exists an* α*-dimensional homogeneous central projection, we can find the largest* α*-dimensional homogeneous central projection* e_α. *Any central projection contained in* e_α *is* α*-dimensional*

LEMMA 4. *If* $\alpha \neq \beta$, *then* $e_\alpha e_\beta = 0$.

Now we state our multiplicity theorem in the following

THEOREM 1. *Let* $\mathbf{M}$ *be a properly infinite* W^**-algebra on a Hilbert space* $\mathbf{H}$ *and* π *a family of all cardinals for which there are homogeneous central projections. Then there exists a unique central decomposition* $\{e_\alpha\}$ $(\alpha \in \pi)$, *where* e_α *is an* α*-dimensional homogeneous central projection in Lemma* 3.

We call this *a homogeneous decomposition of the identity of* $\mathbf{M}$.

PROOF. The uniqueness is obvious, and the rest is clear by Lemmas 2, 3 and 4.

DEFINITION 2. Let $\mathbf{M}$ be a W^*-algebra on a Hilbert space $\mathbf{H}$ and $\{e_i\}(i \in I)$ any family of orthogonal equivalent projections with Card. $I = \alpha$. (α may be finite in this case.) Set $\lambda = max(\aleph_0, \ sup\ \alpha)$ where α runs over all cardinals of those families as above. In this case we call $\mathbf{M}$ λ*-bounded*. Next, if $\{f_j\}(j \in J)$ is any family of orthogonal projections in $\mathbf{M}$, set $\mu = \max.(\aleph_0, \sup \beta)$ where β runs over such cardinals as those of J. (β may be finite, too.) In this cale $\mathbf{M}$ is said μ*-decomposable* W^**-algebra.*

The $\aleph_0$-decomposabity coincides with the usual countable decomposability.

THEOREM 2. *Suppose* $\mathbf{M}$ *to be a* W^**-algebra on a Hilbert space* $\mathbf{H}$. *If* $\mathbf{M}$ *is* λ*-bounded and* $\mathbf{M}^\natural$ μ*-decomposable, then* $\mathbf{M}$ *itself is* $\lambda\mu$*-decomposable.*

We can prove this theorem even if $\mathbf{M}$ is an AW^*-algebra, but we give here a proof for W^*-algebra.

PROOF. Dropping to direct summand and applying Theorem 1, we may restrict ourself to the following two cases in which we must show $\mathbf{M}$ being at most λ-decomposable. Proving this, we find $\mathbf{M}$ at most $\lambda\mu$-decomposable because the cardinal of a central decomppposition does not exceed μ. Then one verifies easily that $\mathbf{M}$ is $\lambda\mu$-decomposable.

1^0 $\mathbf{M}$ is finite and $\mathbf{M}^\natural$ countably decomposable. In this case, it is clear that $\mathbf{M}$ is countably decomposable, i.e. at most λ-decomposable.

2^0 $\mathbf{M} = \mathbf{M}_0 \otimes \mathbf{B}(\mathbf{K})$, where $\mathbf{M}_0$ is countably decomposable and $\mathbf{B}(\mathbf{K})$ the full operator algebra on some Hilbert space $\mathbf{K}$. In this case, $1 = \sum_{i \in I} e_i$ where $\{e_i\}$

$(i \in I)$ is a family of orthogonal, equivalent, countably decomposable projections. Suppose $\mathbf{M}$ to be λ'-decomposable and for some cardinal α $(\alpha \leqq \lambda')$ there exists a family $\{f_j\}$ $(j \in J)$ of orthogonal projections with Card. $J = \alpha$. Since it might be $1 = \sum_{j \in J} f_j$, we have $\alpha \leqq$ Card. $I \leqq \lambda$, as in the proof of Lemma 1. Therefore $\lambda' \leqq \lambda$; that is, $\mathbf{M}$ is at most λ-decomposable.

2. Existence of dimension functions and additivity property.

LEMMA 5. *Any cardinal valued (order-) continuous function $p(t)$ defined on a dense open set Ω_0 in a Stonean space Ω can be extended to a continuous function on the whole space Ω.*

PROOF. Take a point $t_0 \in \Omega - \Omega_0$. We shall show $\bigwedge_U \bigvee p(U \cap \Omega_0) = \bigvee_U \bigwedge p(U \cap \Omega_0)$ where U denotes a neighbourhood of t_0 and $\vee$(resp. $\wedge$) the supremum (resp. infinimum). At first, it is clear that the first member is not less than the second. Therefore, if $\bigwedge_U \bigvee p(U \cap \Omega_0) = \aleph_0$, the above equality is clear, so that we suppose $\bigwedge_U \bigvee p(U \cap \Omega_0) > \aleph_0$. Take a cardinal α such that $\bigwedge_U \bigvee p(U \cap \Omega_0) > \alpha\ (\geqq \aleph_0)$. Then, for any neighbourhood U_0 of t_0 we have $\bigvee p(U_0 \cap \Omega_0) > \alpha$. Set $V = \{t \in U_0 \cap \Omega_0 | p(t) > \alpha\}$; V is open and closed in $U_0 \cap \Omega_0$, for $V = \{t \in U_0 \cap \Omega_0 | p(t) \geqq \alpha'\}$ where α' is the next cardinal to α. Now V is open in Ω, moreover we have $\bar{V} \cap U_0 \cap \Omega_0 = V$ where $\bar{V}$ denotes the closure of V in Ω. Since Ω is Stonean $\bar{V}$ is open and closed in Ω. Suppose $t_0 \notin \bar{V}$, then there exists a neighbourhood U_1 of t_0 such that $U_1 \cap V = \phi$. Take a neighbourhood U_2 of t_0 such as $U_2 \subseteq U_1 \cap U_0$. We have $p(U_2 \cap \Omega_0) \leqq \alpha$, which contradicts our assumption $\bigwedge_U \bigvee p(U \cap \Omega_0) > \alpha$. Hence $t_0 \in \bar{V}$, and as $\bar{V}$ is a neighbourhood of t_0 one can take another neighbourhood U_3 with $U_3 \subseteq \bar{V} \cap U_0$. Thus, $p(U_3 \cap \Omega_0) > \alpha$, i.e. $\wedge p(U_3 \cap \Omega_0) > \alpha$, so that $\bigvee_U \bigwedge p(U \cap \Omega_0) > \alpha$. Therefore $\bigvee_U \bigwedge p(U \cap \Omega_0) \geqq \bigwedge_U \bigvee p(U \cap \Omega_0)$, so that $\bigvee_U \bigwedge p(U \cap \Omega_0) = \bigwedge_U \bigvee p(U \cap \Omega_0)$.

Now we define $p(t_0) = \bigvee_U \bigwedge p(U \cap \Omega_0) = \bigwedge_U \bigvee p(U \cap \Omega_0)$ for $t_0 \in \Omega - \Omega_0$, then $p(t)$ is defined for all $t \in \Omega$.

We shall prove that $p(t)$ is also continuous on $\Omega - \Omega_0$. For the proof, we may assume $\bigwedge_U \bigvee p(U) > \aleph_0$. Take a cardinal α for which $\bigwedge_U \bigvee p(U) > \alpha$ $(\geqq \aleph_0)$. We shall show $p(t_0) > \alpha$. Suppose $p(t_0) \leqq \alpha$, then there exists a neighbourhood U_0 of t_0 such that $p(U_0 \cap \Omega_0) \leqq \alpha$. Hence $p(U_0) \leqq \alpha$; for if there exists $t \in U_0$ with $p(t) > \alpha$ we can take a neighbourhood $U(t)$ of t with $p(U(t)$

$\cap \Omega_0) > \alpha$, which contradicts $p(U_0 \cap \Omega_0) \leqq \alpha$. Therefore $\bigwedge_U \vee p(U') \leqq \alpha$. However this contradicts our assumption. We have $p(t_0) > \alpha$. Now there exists a neighbourhood U_1 of t_0 such that $p(U_1 \cap \Omega_0) > \alpha$, then the same argument used above yieds $p(U_1) > \alpha$ which implies $\wedge p(U_1) > \alpha$. Hence $\bigvee_U \wedge p(U) > \alpha$, and $\bigwedge_U \vee p(U) = p(t_0) = \bigvee_U \wedge p(U)$. Therefore $p(t)$ is continuous on $\Omega - \Omega_0$, so on the whole Ω. This completes the proof.

LEMMA 6. *For any properly infinite projection e of a W^*-algebra $\mathbf{M}$ on a Hilbert space $\mathbf{H}$ there corresponds a cardinal valued (including zero) continuous function $D(e)(t)$ on Ω such that $1^0, 0 \leqq D(e)(t) \leqq dim(\mathbf{H})$ and $D(e)(t) = 0$ if and only if $e = 0$; 2^0, for any central projection z, $D(ze)(t) = z(t)D(e)(t)$ in the obvious sense.*

PROOF. Consider a homogeneous decomposition of e and let $\{K_\alpha\}$ $(\alpha \in \pi)$ be a family of mutually disjoint open and closed sets in $\Omega_{e\mathrm{H}}$ corresponding to the homogeneous decomposition of e. Set $\Omega_0 = \bigcup_{\alpha \in \pi} K_\alpha$; Ω_0 is a dense open set in $\Omega_{e\mathrm{H}}$. By the isomorphism between $\mathbf{M}_e^\natural$ and $\mathbf{M}_{z(e)}^\natural$ one can consider Ω_0 as a dense open subset of $\Omega_{z(e)\mathrm{H}}$, a subset of Ω.

We define a cardinal valued continuous function $\bar{D}(e)(t)$ on Ω_0 such as $\bar{D}(e)(t) = \alpha$ for $t \in K_\alpha$. By Lemma 5 we can extend $\bar{D}(e)(t)$ to a continuous function $\bar{D}(e)(t)$ on $\Omega_{z(e)\mathrm{H}}$. Next, we set a function on Ω;

$$\begin{aligned} D(e)(t) &= \bar{D}(e)(t) \quad \text{for} \quad t \in \Omega_{z(e)\mathrm{H}} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Clearly $D(e)(t)$ is a continuous function on Ω and satisfies the conditions 1^0 and 2^0.

LEMMA 7. *Let e, f be properly infinite projections of $\mathbf{M}$, then $D(e)(t) \leqq D(f)(t)$ if and only if $e \lesssim f$.*

PROOF. Since "*if*" part is clear we shall show "*only if*" part alone. As we may work in each direct summand we can assume $z(e) = z(f) = 1$ and further the central decomposition makes us possible to consider $D(e)(t) = \alpha$, $D(f)(t) = \beta$. Moreover, we may assume that $\mathbf{M}^\natural$ is countably decomposable. Then, $e = \sum_{i \in I} e_i$, $f = \sum_{j \in J} f_j$ where $\{e_i\}$ $(i \in I)$ and $\{f_j\}$ $(j \in J)$ are families of orthogonal, equivalent, countably decomposable projections with Card. $I = \alpha$ and Card. $J = \beta$. Without changing cardinals of I and J, we may assume that e_i and f_j are all properly infinite. Thus, e_i s equivalent to f_j (cf. [8; Lemma 2]). Therefore $e = \sum_{i \in I} e_i \lesssim \sum_{j \in J} f_j = f$.

LEMMA 8. *If e, f are orthogonal properly infinite projections,*

$$D(e+f)(t) = D(e)(t) + D(f)(t).$$

PROOF. If $z(e)z(f) = 0$ this identity is clear, so that we assume $z(e)z(f) \neq 0$. Since it is sufficient to prove this equality on a dense open set in Ω, we can suppose, as in the proof of Lemma 7, $D(e)(t) = \alpha$, $D(f)(t) = \beta$ and $e = \sum_{i\in I} e_i$, $f = \sum_{j\in J} f_j$. Without loss of generality, e_i and f_j are supposed to be properly infinite for all $i \in I$ and $j \in J$. We have $e_i \sim f_j$, for $z(e) = z(f) = 1$. Thus $e + f$ is an $(\alpha + \beta)$-dimensional homogeneous projection. Hence $D(e + f)(t) = \alpha + \beta = D(e)(t) + D(f)(t)$. This completes the proof.

THEOREM 3. *For every properly infinite W^*-algebra $\mathbf{M}$ on $\mathbf{H}$, there exists a function $D(e)(t)$ on the projections in $\mathbf{M}$ to the continuous functions on Ω, with the following properties;*

1^0 $0 \leqq D(e)(t) \leqq$ *dim* $(\mathbf{H})$ *for any projection e and $D(e)(t) = 0$ if and only if $e = 0$,*[1)]

2^0 *$D(e)(t) \leqq D(f)(t)$ if and only if $e \precsim f$;*

3^0 *if e and f are mutually orthogonal projections in $\mathbf{M}$ then $D(e + f)(t) = D(e)(t) + D(f)(t)$;*

4^0 *if z is a central projection, then*

$$D(ze)(t) = z(t)D(e)(t);$$

5^0 *$D(e)(t)$ is minimal in its infinite part among the functions satisfying the conditions 1^0—4^0; that is, if $\overline{D}(e)(t)$ is such a function on $\mathbf{M}$ then $D(e)(t) \leqq \overline{D}(e)(t)$ for any properly infinite projection e.*

Notices that $D(e)(t)$ is numerical valued except on a dense open set of Ω if and only if e is finite.

DEFINITION 3. A function $D(\ )(t)$ in the above theorem is callec a *dimension function* of a properlyinfinite W^*-algebra $\mathbf{M}$.

PROOF OF THEOREM 3. Take a normal, faithful, semi-finite pseudo-application $\natural$, Φ_0, on the semi-finite part of $\mathbf{M}$. For any projection $e \in \mathbf{M}$ there exists a central decomposition $e = e_1 + e_2$ where e_1 is finite and e_2 properly infinite. Consider the central decomposition $\mathbf{M} = \mathbf{M}z + \mathbf{M}(1 - z)$ where $\mathbf{M}z$ is semi-finite and $\mathbf{M}(1 - z)$ purely infinite. Then $\Phi_0(e_1)$ is a continuous function on $\Omega_{z\mathrm{H}}$, a compact open subset of Ω. Hence if we set $\Phi_0(e_1)(t) = 0$ for $t \in \Omega_{(1-z)\mathrm{H}}$, $\Phi_0(e_1)(t)$ is considered a continuous function on Ω. Moreover we assume $\Phi_0(e_1)(t) = \aleph_0$ if $\Phi_0(e_1)(t)$ is infinite. Now we define $D(e)(t) = \Phi_0(e_1)(t) + D(e_2)(t)$ for $t \in \Omega$ in which $D(e_2)(t)$ is a cardinal-valued continuous function described in Lemma 6. Clearly $D(e)(t)$ is a continuous function on Ω and one can easily verify, together with Lemmas 6, 7, 8, and the property of Φ_0, that this function $D(e)(t)$ satisfies the conditions 1^0, 4^0 and 2^0, 3^0 if e and f are both finite or properly infinite. On the other hand, if e is finite and f properly infinite, one verifies easily that the conditions 2^0, 3^0 also hold recalling the convention for this theorem. Therefore all the conditions, except 5^0, hold

1) Here we need some conventions to prevent us from unnecessary confusions. Let α be a cardinal number and s a positive number, then we write that $s<\alpha$ and $s+\alpha=\alpha$ even if s is not an integer.

for this function $D(e)(t)$.

Now suppose $\bar{D}(e)(t)$ to be another function satisfying the conditions 1^0—4^0 and let e be a properly infinite projection. By 4^0, we may assume $D(e)(t) = \alpha$ and $\mathbf{M}^\natural$ being countably decomposable. Set $\{e_i\}(i \in I)$ for a family of projections such that $e_i \leqq e$, $D(e_i)(t) = \alpha_i \leqq \alpha$ and $D(e_i)(t) \leqq \bar{D}(e_i)(t)$ where $\{\alpha_i\}(i \in I)$ are cardinals. By Zorn's lemma we have a maximal family of such projections. Set $\beta = \vee_{i\in I}\alpha_i \leqq \alpha$, then there exists a projection $\bar{e}$ such that $\bar{e} \leqq e$ and $D(\bar{e})(t) = \beta$. Suppose $\beta < \alpha$. Take the cardinal β' next to β and a projection $\bar{\bar{e}}$ such as $\bar{\bar{e}} \leqq e$ and $D(\bar{\bar{e}})(t) = \beta'$. Then from the maximality of $\{e_i\}(i \in I)$, $D(\bar{\bar{e}})(t) > \bar{D}(\bar{\bar{e}})(t)$ on some open and closed set Ω_0. Now $D(e_i)(t) \leqq D(\bar{e})(t)$ implies $e_i \precsim \bar{e}$, whence $\bar{D}(e_i)(t) \leqq \bar{D}(\bar{e})(t)$ for all t. And $\beta = D(\bar{e})(t) = \vee_{i\in I} D(e_i)(t) \leqq \vee_{i\in I}\bar{D}(e_i)(t) \leqq \bar{D}(\bar{e})(t)$. Therefore $D(\bar{e})(t) = \bar{D}(\bar{e})(t) = \beta$ on Ω_0 and taking a central projection z corresponding to Ω_0 we have $z\bar{e} \sim z\bar{\bar{e}}$ which contradicts to $D(\bar{e})(t) < D(\bar{\bar{e}})(t)$. Hence $\beta = \alpha$; we have $D(e)(t) = D(\bar{e})(t) \leqq \bar{D}(\bar{e})(t) = \bar{D}(e)(t)$. Thus, the condition 5^0 also holds.

Next, we shall consider the normality property of our dimension function.

THEOREM 4. *Let* $\mathbf{M}$ *be a properly infinite* W^**-algebra on* $\mathbf{H}$. *Suppose* $e = \sum_{i\in I} e_i$ *where* e *and* e_i *are all properly infinite projections. Then for any central projection* z *which is countably decomposable for* $\mathbf{M}^\natural$, *we have*

$$D(ze)(t) = \bigwedge_U \sum_{i\in I} \vee D(ze_i)(U),$$

where U *is any neighbourhood of* $t \in \Omega$.

PROOF. We may assume that $\mathbf{M}^\natural$ is countably decomposable. We must show $D(e)(t) = \bigwedge_U \sum_{i\in I} \vee D(e_i)(U)$. At first, we have $\vee D(e_i)(U) = \vee D(e_i)(U \cap \Omega_i)$, where Ω_i is an original definition domain of $D(e_i)(t)$ (*cf. Lemma* 6). We have also $\vee D(e)(U) = \vee D(e)(U \cap \Omega_0)$ for an original definition domain Ω_0 of $D(e)(t)$.

Now let $\Omega_i = \bigcup_\alpha K^i_\alpha$, $\Omega_0 = \bigcup_\beta K_\beta$ be the decompositions corresponding to the homogenous decompositions of e_i and e. Set $\{K^i_{\alpha_j}\}$ for a sub-family of K^i_α which meets U; then $\vee D(e_i)(U) = \vee D(e_i)(U \cap \Omega_i) = \vee\{\alpha^i_1, \alpha^i_2, \ldots\} = \alpha_1^i + \alpha^i + \ldots$.. Hence $\sum_{i\in I} \vee D(e_i)(U\cap\Omega_i) = \sum_{i\in I}\sum_j \alpha^i_j$. On the other hand, let $\{K_{\alpha_s}\}$ be the sub-family of K_α with $U \cap K_\alpha \neq \phi$. Then $\vee D(e)(U \cap \Omega_0) = \sum_s \alpha_s$. Put $I_s = \{\alpha^i_j | U \cap K_{\alpha_s} \cap K^i_{\alpha_j} \neq \phi\}$, then we have, by Lemma 1, $\alpha_s = \sum_{I_s} \alpha^i_j$. Since $\{s\}$ is countable, we get $\sum_s \alpha_s = \sum_s \sum_{I_s} \alpha^i_j \leqq \sum_{i\in I}\sum_j \alpha^i_j$. However for any α^i_j such a $K^i_{\alpha_j} \cap U \neq \phi$ there exists a set $I_s(i, j)$ containing α^i_j, so that $\sum_s \alpha_s \geqq$

$\sum_{i\in I}\sum_j\alpha_j^i$, whence $\sum_s\alpha_s = \sum_{i\in I}\sum_j\alpha_j^i$.

Thus $D(e)(t) = \bigwedge_U \vee D(e)(U\cap\Omega_0) = \bigwedge_U \sum_{i\in I} \vee D(e_i)(U\cap\Omega_i) = \bigwedge_U \sum_{i\in I} \vee D(e_i)(U)$.

COROLLARY. If $\mathbf{M}$ *is an infinite factor*, $D(e)(t)$ *is completely additive over infinite projections.*

Now we show some examples which illustrate the situation of Theorem 4.

EXAMPLE 1. From the definition of $D(\ )(t)$, it is normal in finite projections. And yet $D(\ )(t)$ is not normal in properly infinite projections even if they form a linearly ordered family. Let $\mathbf{H}$ be an $\aleph_1$-dimensional Hilbert space and $\mathbf{M} = \mathbf{B}(\mathbf{H})$, the full operator algebra over $\mathbf{H}$. Let ω_1 be the first ordinal of $\aleph_1$, then $1 = \sum_{\lambda<\omega_1} e_\lambda$, where e_λ is a minimal projection of $\mathbf{M}$. Set $\bar{e}_\lambda = \sum_{\mu<\lambda} e_\mu$ and denote by ω the first ordinal of $\aleph_0$. We get a linearly ordered increasing family of properly infinite projections $\{\bar{e}_\lambda\}_{\omega\leqq\lambda<\omega_1}$ and $1 = \bigvee_{\omega\leqq\lambda<\omega_1}\bar{e}_\lambda$. However, $D(1)(t) = \aleph_1$ and $D(\bar{e}_\lambda)(t) = \aleph_0$ where $\omega\leqq\lambda<\omega_1$. Thus $D(1)(t) > \bigvee_{\omega\leqq\lambda<\omega_1} D(\bar{e}_\lambda)(t)$.

EXAMPLE 2. So far as it is concerned with our dimension function, the normality is not equivalent to the complete additivity over properly infinite projections. It is hopeful that $D(\ )(t)$ has this property, but this is not the case as shown by the next example.

Let $\{\mathbf{H}_i \mid i = 1,2,3,\dots\}$ be a family of separable Hilbert spaces and consider $\mathbf{M} = \prod_{i=1}^{\infty}\mathbf{B}(\mathbf{H}_i)$, the product algebra of $\mathbf{B}(\mathbf{H}_i)$. Then we can express $\Omega = \{t_1, t_2, t_3, \dots t_\infty\}$. Since the identity is an $\aleph_0$-dimensional homogeneous projection, we have $D(1)(t_\infty) = \aleph_0$. Put $1 = \sum_{i=1}^{\infty} e_i$ where e_i is the identity operators on $\mathbf{H}_i$. We have $D(e_i)(t_\infty) = 0$ for all i. Hence $D(1)(t_\infty) \neq \sum_{i=1}^{\infty} D(e_i)(t_\infty)$.

EXAMPLE 3. If z is a central projection not countably decomposable for $\mathbf{M}^\natural$, Theorem 4 does not necessarily hold.

Take the first ordinal ω_1 of $\aleph_1$ and $\{\mathbf{H}_\lambda \mid \lambda<\omega_1\}$ be a family of separable Hilbert spaces. Put $\mathbf{M} = \prod_{\lambda<\omega_1}\mathbf{B}(\mathbf{H}_\lambda)$, the product algebra of $\mathbf{B}(\mathbf{H}_\lambda)$, then $1 = \sum_{\lambda<\omega_1} e_\lambda$ where e_λ is the identity operator on $\mathbf{H}_\lambda$. We may express $\Omega = \{t_1, t_2, \dots t_\lambda, \dots t_{\omega_1} \mid \lambda<\omega_1\}$. Let U_0 be a neighbourhood of t_{ω_1}, then $U_0 \cap$ (support of $D(e_\lambda)(t)) \neq \phi$ for uncountable numbers of λ. Hence $\sum_{\lambda<\omega_1} \vee D(e_\lambda)(U_0) = \aleph_1$. Since U_0 is arbitrary, we have $\bigwedge_U \sum_{\lambda<\omega_1} \vee D(e_\lambda)(U) = \aleph_1 > D(1)(t_{\omega_1})$

$= \aleph_0$ where U runs over the neighbourhoods of t_{ω_1}.

3. **Some applications.** At first we apply the above dimension theory to the spatial invariants.

THEOREM 5. *Let $\mathbf{M}_1, \mathbf{M}_2$ be properly infinite W^*-algebras not containing the coupled component such as (properly infinite, finite). Suppose $\mathbf{M}_1$ is isomorphic to $\mathbf{M}_2$ by an isomorphism θ and $\mathbf{M}_1$ and $\mathbf{M}_2$ has the same invariant function $C(t)$ on the common spectrum of $\mathbf{M}_1^\natural$ and $\mathbf{M}_2^\natural$, then θ is spatial.*

$C(t)$ is used here in the sense of E. L. Griffin [6] and J. Tomiyama [12]; further references R. Pallu de la Barrière [9], E. L. Griffin [5] and N. Suzuki [11].

PROOF. Let $D_i(\ )(t)$ and $D_i'(\ )(t)$ be dimension functions of $\mathbf{M}_i$ and $\mathbf{M}_i'$ $(i = 1, 2)$. Then an easy computation shows $C_i(t) = (D_i(1)(t), D_i'(1)(t))$ $(i = 1, 2)$, where $C_i(t)$ denote the invariant functions of $\mathbf{M}_i$. By the result easily verified from the theorem due to J. Dixmier [3] there exists a W^*-algebra $\mathbf{N}$ and two projections e, f in $\mathbf{N}'$ with $z(e) = z(f) = 1$ such that θ may be spatially identified with the isomorphism $\theta : a_e \rightarrow a_f$ for all $a \in \mathbf{N}$. Hence to prove θ being spatial is reduced to prove that $e \sim f$ mod $\mathbf{N}'$. Take a dimension function $D'(\ )(t)$ of $\mathbf{N}'$. From the assumption, we get $D'(e)(t) = D'(f)(t)$. This implies that e is equivalent to f mod $\mathbf{N}'$. That is, θ is spatial.

THEOREM 6. *If $\mathbf{M}$ is a λ-bounded properly infinite W^*-algebra, then $\lambda = \bigvee_{\alpha\in\pi} \alpha$ where π is a family of cardinals used in Theorem 1.*

PROOF. Suppose there exists a family $\{e_i\}(i \in I)$ of orthogonal, equivalent projections with Card. $I = \beta \leqq \lambda$. Take a countably decomposable projection $\overline{e_{i_0}}(\leqq e_{i_0})$, then this induces a family $\{\overline{e_i}\}$ $(i \in I)$ of orthogonal, equivalent, countably decomposable projections. Set $e = \sum_{i\in I} \overline{e_i}$, which becomes an β-dimensional homogeneous projection. And $D(e)(t) \leqq D(1)(t) \leqq \bigvee_{\alpha\in\pi} \alpha$. That is, $\beta \leqq \bigvee_{\alpha\in\pi} \alpha$ which implies $\lambda \leqq \bigvee_{\alpha\in\pi} \alpha$. Since the inverse inequality is clear, we have $\lambda = \bigvee_{\alpha\in\pi} \alpha$.

Lastly, we consider a result by R. V. Kadision (cf. [7: Theorem 13]). Terms are used in the same sense as used there.

THEOREM 7. *Factors of type I_∞ and III have extreme point classes corresponding to each cardinal number in the range of their dimension functions. Each such class, other than unitary operators, consists of all those proper semi-unitary operators in the factor whose initial or final spaces (whichever is not the whole space) have their complementary manifolds of the same dimension (the cardinal corresponding to the extreme point class). In factors of type II_∞ each extreme point class which consists of all those proper semi-unitary operator whose initial or final spaces (whichever is not whole space) have their complementary manifolds of infinite dimension, has the same correspondence as described above. And those for which the complementary projections have finite* *dimension form other extreme point classes (not necessarily one extreme point class).*

PROOF. It is clear from the proof of [7: Theorem 13] that the set of all unitary operators forms an extreme point class.

Suppose u and v are semi-unitary operators in the factor $\mathbf{M}$ with $uu^* = e$, $vv^* = f$, e and f different from 1 and $1-e$ equivalent to $1-f$. Then v can be reached from u as in the proof of [7: Theorem 13].

Now if v can be reached from u by an isometry ρ we can express $\rho = t\cdot\sigma$ where t is unitary in $\mathbf{M}$ and σ either a $*$-automorphism or a $*$-anti-automorphism of $\mathbf{M}$. Then $\sigma(u) = t^*v$. But if σ is a $*$-anti-automorphism of $\mathbf{M}$, we have $\sigma(uu^*) = \sigma(u)^*\sigma(u) = v^*tt^*v = v^*v = 1$ so that $uu^* = e = 1$ which contradicts our assumption. Hence σ is a $*$-automorphism. We get $\sigma(uu^*) = \sigma(e) = t^*vv^*t = t^*ft$, whence $\sigma(1-e) = t^*(1-f)t$. Then one easily verifies that $D(1-e) = D(1-f)$ if $1-e$ has infinite dimension in the sense of our dimension function. In the case of $\mathbf{I}_\infty$, a simple consideration shows $D(1-e) = D(1-f)$ even if they are finite. Now $1-e$ is infinite if and only if $1-f$ is infinite from the relation $\sigma(1-e) = t^*(1-f)t$. Thus the proof is completed.

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