Generalized Discrete Fourier Transform With Nonlinear Phase*

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Abstract— **Constant modulus transforms like discrete Fourier transform (DFT), Walsh transform, and Gold codes have been successfully used over several decades in several engineering applications, including discrete multi-tone (DMT), orthogonal frequency division multiplexing (OFDM) and code division multiple access (CDMA) communications systems. In this paper, we present a generalized framework for DFT called Generalized DFT (GDFT) with nonlinear phase by exploiting the phase space. We show that GDFT offers sizable correlation improvements over DFT, Walsh, Oppermann and Gold codes, leading to better performance in all multi-carrier communications scenarios investigated. We also highlight how known constant modulus orthogonal transforms are special solutions of the proposed GDFT framework. Moreover, we introduce practical design methods offering computationally efficient implementations of GDFT as enhancements to DFT. We conclude the paper with examples of communications applications where GDFT is shown to outperform DFT and other known constant modulus bases.**

***Index Terms*— Discrete Fourier Transform, Generalized Discrete Fourier Transform, Walsh Codes, Gold Codes, OFDM, DMT, CDMA, auto-correlation function, cross-correlation function.**

I. INTRODUCTION

Constant modulus function sets have always been of great interest in various applications including communications where efficient implementation is a major concern. Among known binary spreading code families, Gold codes have been successfully used for asynchronous communications in direct sequence CDMA (DS-CDMA) systems due to their low cross-correlation features. Walsh, Gold, Walsh-like [1]-[3] and several other binary spreading code sets are designed to optimize so called even correlation functions. However, the odd correlations are as important as even correlations. Therefore, Fukumasa, Kohno and Imai proposed a new set of complex pseudo-random noise (PN) sequences, called equal odd and even (EOE) sequences, with good odd and even correlations [4]. EOE sequences are generated by using one of the binary code sets like Gold or Walsh.

Spreading codes with non-binary real chip values were also proposed in the literature in order to improve auto- and cross-correlations of the set. More recently, research has refocused on constant modulus spreading codes for radio communications applications due to the efficiency limitations of non-linear gain characteristics of commonly used power amplifiers in transceivers. Hence, the complex roots of unity are widely proposed as complex spreading codes by several authors in the literature. All codes of such a set are placed on the unit circle of the complex z plane. Frank-Zadoff, Chu and Oppermann have forwarded a variety of complex spreading codes [5]-[8]. Moreover, Oppermann has shown that Frank-Zadoff and Chu Sequences are special cases of his family of spreading sequences [9].

This paper introduces Generalized Discrete Fourier Transform (GDFT) with nonlinear phase. GDFT provides a unified theoretical framework where popular constant modulus orthogonal function sets including DFT and others are shown to be special solutions. Therefore, GDFT provides a foundation to exploit the phase space in its entirety in order to improve correlation properties of constant modulus orthogonal codes. We present GDFT and demonstrate its improved correlations over the popular DFT, Gold, Walsh and Oppermann families leading to superior communications performance for the scenarios considered in the paper.

II. MATHEMATICAL PRELIMINARIES

An N^{th} root of unity is a complex number z satisfying the polynomial equation

$$z^N - 1 = 0 \qquad N \in \{1, 2, 3, \ldots\} \tag{1}$$

Moreover, z_p is defined as the pth primitive Nth root of unity if $z_p^m \neq 1 \quad m \in \{1,2,..,N-1\}$, where m and N must be coprime integers. The complex number $z_1 = e^{j(2\pi/N)}$ is the primitive Nth root of unity with the smallest positive argument. There are N distinct Nth roots of unity for any primitive and expressed as

$$z_k = (z_p)^k \quad k = 1,2,3, \ldots, N \tag{2}$$

where z_p is any of the *primitive N^{th} root of unity*. As an example, $z_1 = e^{j\frac{2\pi}{4}}$ and $z_2 = e^{j\frac{3\pi}{2}}$ are the two primitive N^{th} roots of unity for N=4. All *primitive N^{th} roots of unity* satisfy the unique summation property of a geometric series expressed as follows

$$\sum_{n=0}^{N-1} (z_p)^n = \frac{(z_p)^N - 1}{z_p - 1} = \begin{Bmatrix} 1, N = 1 \\ 0, N > 1 \end{Bmatrix} \quad \forall p \tag{3}$$

Now, define a periodic, constant modulus, complex sequence $\{e_r(n)\}$ as the *r^{th} power* of the first *primitive N^{th} root of unity* z_1 raised to the *n^{th} power* as

$$\begin{aligned} &e_r(n) \triangleq (z_1^r)^n = e^{j(2\pi r/N)n} \\ &n = 0,1,2,\ldots, N-1 \text{ and } r = 0,1,2,\ldots, N-1 \end{aligned} \tag{4}$$

The complex sequence (4) over a finite discrete-time interval in a geometric series is expressed according to (3) as follows [10]-[11]

$$\begin{aligned} &\frac{1}{N}\sum_{n=0}^{N-1} e_r(n) = \frac{1}{N}\sum_{n=0}^{N-1} (z_1^r)^n = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi r/N)n} \\ &= \begin{Bmatrix} 1, & r = mN \\ 0, & r \neq mN \\ & m = \text{integer} \end{Bmatrix} \end{aligned} \tag{5}$$

Then, (5) is rewritten as the definition of the discrete Fourier transform (DFT) set $\{e_k(n)\}$ satisfying the orthonormality conditions

$$\begin{aligned} &< e_k(n), e_l^*(n) > = \frac{1}{N}\sum_{n=0}^{N-1} e_k(n) e_l^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi k/N)n} e^{-j(2\pi l/N)n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-l)n} = \begin{Bmatrix} 1, & r = k-l = mN \\ 0, & r = k-l \neq mN \\ & m = \text{integer} \end{Bmatrix} \end{aligned} \tag{6}$$

The notation (*) represents the complex conjugate function of a function. One might rewrite the first primitive N^{th} root of unity as $z_1 = e^{j\omega_0}$ where $\omega_0 = 2\pi/N$, and it is called *the fundamental frequency* defined in radians. We are going to extend the phase functions in (6) in order to define the nonlinear phase GDFT in the following section.

III. GENERALIZED DISCRETE FOURIER TRANSFORM

Let's generalize (5) by rewriting the phase as the difference of two functions $\varphi_{kl}(n) = \varphi_k(n) - \varphi_l(n) = r \ \forall n$, and expressing a constant modulus orthogonal set as follows,

$$\begin{aligned} &\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi r/N)n} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi\varphi_{kl}(n)/N]n} \\ &= \begin{Bmatrix} 1, & \varphi_{kl}(n) = \varphi_k(n) - \varphi_l(n) = k - l = r = mN \\ 0, & \varphi_{kl}(n) = \varphi_k(n) - \varphi_l(n) = k - l = r \neq mN \\ & m = \text{integer } 0 \le k,l,n \le N-1 \end{Bmatrix} \end{aligned} \tag{7.a}$$

Therefore, by inspection

$$\begin{aligned} &\frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi(\varphi_k(n)-\varphi_l(n))/N)]n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi\varphi_k(n)/N]n} e^{-j[2\pi\varphi_l(n)/N]n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} e_k(n) e_l^*(n) = < e_k(n), e_l^*(n) > \end{aligned} \tag{7.b}$$

Hence, the basis functions of the new set are defined as

$$\{e_k(n)\} \triangleq e^{j(2\pi/N)\varphi_k(n)n} \quad k,n = 0,1,\ldots,N-1 \tag{8}$$

We call this orthogonal function set as the *Generalized Discrete Fourier Transform* (GDFT). It is observed from (7) and (8) that there are infinitely many sets of constant modulus and *nonlinear phase functions* available. Therefore, we might methodically design such functions. As an example, one can define the discrete time function $\varphi_k(n)$ in (8) as the ratio of two polynomials,

$$\varphi_k(n) = \frac{N_k(n)}{D_k(n)} = \frac{\sum_{j=1}^{N} a_{kj} n^{b_{kj}}}{\sum_{j=1}^{M} c_{kj} n^{d_{kj}}} \quad N \le M; \ k \in \{0,1,\ldots,N-1\} \tag{9}$$

Note that any function pair $\{\varphi_k(n), \varphi_l(n)\}$ must satisfy the orthogonality requirements of $\varphi_k(n) - \varphi_l(n) = k - l = mN$ $m = \text{integer}, \{k,l,n\} \in \{0,1,..,N-1\}$ as stated in (7).

Let's assume that the denominator polynomial $D_k(n) = 1$ and the numerator polynomial is defined as follows

$$\varphi_k(n) = N_k(n) = \sum_{j=1}^{N} a_{kj} n^{b_{kj}} = a_{k1} n^{b_{k1}} + a_{k2} n^{b_{k2}} + a_{k3} n^{b_{k3}} + \ldots + a_{kN} n^{b_{kN}} \tag{10}$$

In general, the polynomial coefficients $\{a_{kj}\}$ and $\{c_{kj}\}$ are complex, the powers $\{b_{kj}\}$ and $\{d_{kj}\}$ are real numbers. Now, we make several remarks that link the proposed GDFT framework to the other known transforms in the literature, and its potential impact on a multicarrier communications system.

Remark 1: DFT is a special solution of GDFT where $\varphi_k(n) = a_{k1} = k$ and $a_{k2} = a_{k3} = ... = a_{kN} = 0$, and $b_{k1} = b_{k2} = ... = b_{kN} = 0$ in (9) and (10) for all k. Note that having constant valued $\{\varphi_k(n)\}$ functions for all k makes DFT a linear-phase transform where $\{e_k(n)\} \triangleq e^{j(2\pi/N)kn} \quad k,n = 0,1,...,N-1$.

Remark 2: There are infinitely many possible GDFT sets with constant modulus available due to the introduction of the nonlinear phase as formulated in (10) and later in (30). Therefore, one can design the optimal basis (phase) for the desired figure of merit. As an example, if the application at hand requires a function set with minimized correlation and does not mind the linearity of phase, naturally, DFT is not the optimal solution for this specs. One might exploit the phase to design various GDFT's where CDMA and OFDM performances in a multicarrier communications system might be improved over the existing solutions like discrete-time Walsh transform and DFT.

Remark 3: Since DFT is a special solution of GDFT, it offers only one unique set to be used in a multicarrier communications system. Therefore, the carrier level (physical layer) security is vulnerable to a potential intrusion to the system. In contrast, the proposed GDFT provides many possible constant modulus carrier sets of various lengths with comparable or better correlation performance than the DFT. The additional flexibility of changing phase functions of the GDFT set allows us to design adaptive systems where carrier allocations and basis assignments (basis hopping) are made dynamically in order to better match channel spectrum as well as for improved physical layer security. The phase shaping function $\psi(n)$ of (30) behaves as the security key of the system and is known in advance by the receiver.

IV. GDFT DESIGN METHODS

Let's define the DFT matrix of size NxN as

$$A_{DFT} = [A_{DFT_{(k,n)}}] = [e^{j(2\pi/N)kn}] \quad k,n = 0,1,2,...,N-1 \tag{11}$$

Now, we will define a GDFT by relaxing the linear phase property of DFT without compromising the orthogonality. This is a marked departure from the traditional Fourier analysis including DFT where any set regardless continuous or discrete in time has its linear phase functions. Hence, we express the square GDFT matrix as a product of the three orthogonal matrices as follows

$$\begin{aligned} A_{GDFT} &= G_1 A_{DFT} G_2 \\ A_{GDFT} A_{GDFT}^{-1} &= I \\ A_{GDFT}^{-1} &= A_{GDFT}^{*T} \\ G_1 G_1^{*T} = I \quad & G_2 G_2^{*T} = I \end{aligned} \tag{12}$$

where G_1 and G_2 are constant modulus diagonal matrices and written as follows

$$G_1(k,n) = \begin{Bmatrix} e^{j\theta_{kk}}, & k=n \\ 0, & k \neq n \\ & k,n = 0,1,...N \end{Bmatrix} \tag{13.a}$$

and

$$G_2(k,n) = \begin{Bmatrix} e^{j\gamma_{nn}}, & n=k \\ 0, & n \neq k \\ & k,n = 0,1,...N \end{Bmatrix} \tag{13.b}$$

The notation ($*T$) indicates that conjugate and transpose operations applied to the matrix, and I is the *identity matrix*. Therefore, the transform kernel generating A_{GDFT} matrix through this methodology is expressed as follows

$$\{e_k(n)\} \triangleq e^{j[(2\pi k/N)n + \theta_{kk} + \gamma_{nn}]} \quad k,n = 0,1, \ldots, N-1 \tag{14}$$

G_1 and G_2 are the diagonal complex orthogonal *generalization matrices* yielding A_{GDFT} in (12) with the desired time and frequency domain features. Note that G_1 allows us to phase shift k^{th} DFT basis function by $e^{j\theta_{kk}}$ while G_2 adds the nonlinear phase terms, $e^{j\gamma_{nn}} \quad n = 0,1,..,N-1$, to the linear phase functions of the DFT set. The latter is independent of the function index k and will be discussed in detail in Sec. V. We are going to introduce several generalization matrix families that are useful to design A_{GDFT} out of A_{DFT}. GDFT extensions with diagonal generalization matrices are efficient to implement with simple modifications to the celebrated fast Fourier transform (FFT) algorithms.

Single Diagonal G Matrix:

The diagonal G matrix is constant modulus and defined from (12)

$$\begin{aligned} G_1 &= I \\ G &= G_2 \end{aligned} \tag{15}$$

offering two types of GDFT matrices as follows.

a. Constant Valued Diagonal Elements:

The elements of this diagonal matrix are the same constant modulus complex number as expressed in

$$G(k,n) = \begin{Bmatrix} e^{j\theta}, & k=n \\ 0, & k \neq n \\ & k,n = 0,1,...N-1 \end{Bmatrix} \tag{16}$$

This type of G matrix generates θ radians phase shifted version of the DFT matrix. Moreover, the linear phase property of A_{DFT} is still preserved in this case. This is a well known property of traditional Fourier analysis and included here for the completeness of presentation [12]-[13].

b. Non-Constant Diagonal Elements:

The non-zero, non-constant and constant modulus diagonal elements of G matrix are defined as

$$G(k,n)=\begin{cases} e^{j\theta_{nn}}, & k=n \\ 0, & k\neq n \\ & k,n=0,1,...,N-1 \end{cases} \tag{17}$$

The rows--basis functions-- of A_{GDFT} in (12) are obtained as the element by element products of A_{DFT} rows with the elements of diagonal G matrix in this scenario. Each sample of a basis function in A_{DFT} is phase shifted independently of the other samples. Therefore, the resulting basis function set is entirely different from DFT set. One might design a linear phase--phase shifted version of DFT-- or nonlinear phase GDFT by designing elements of the diagonal G matrix, $\{e^{j\theta_{nn}}\}$. This form of GDFT is further studied in Sec. V for the design of phase shaping function where $\psi(n)=\theta_{nn}$.

The overall computational burden of A_{GDFT} is the combined implementation cost of the A_{DFT} and the G matrices. Since DFT has its efficient fast algorithms, FFT, the complexity of G matrix indicates the required additional computational resources to implement GDFT. Therefore, this point needs to be further studied in applications where one might generalize DFT into GDFT as presented in this section.

Remark 4: Oppermann forwarded a new group of constant modulus orthogonal spreading codes, and also showed in [9] that the well-known Frank-Zadoff and Chu Sequences [5]-[7] are the special cases of his code family. It is shown below that the orthogonal Oppermann codes are also special solutions to the proposed GDFT framework. The Oppermann codes in an *NxN* square matrix notation are defined as [8], [9]

$$A_{OPP}(k,i)=(-1)^{ki}\exp\left(\frac{j\pi(k^m i^p+i^n)}{N}\right) \quad k,i=1,2,....,N \tag{18}$$

In [8], it was proven that Oppermann codes are orthogonal only for the case of $p=1$ and m is any positive nonzero integer. Note that if one defines the parameters of (10) for this case as $a_j=0 \;\; j=3,4,.....,N$ and

$$a_1=\frac{k^m+kN}{2}; \quad b_1=0$$
$$a_2=\frac{1}{2}; \quad b_2=n-1 \tag{19}$$

Then, we obtain the equality $A_{GDFT}=A_{OPP}=A_{DFT}G_{OPP}$. As an example, G_{OPP} for the Oppermann set of N=7 with the parameters $\{N=7,\ m=1,\ p=1,\ n=2.98\}$ is expressed as

$$G_{OPP}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & e^{-j\pi 0.87} \\ 0 & 0 & 0 & 0 & e^{-j\pi} & 0 & 0 \\ 0 & 0 & e^{-j\pi 0.8} & 0 & 0 & 0 & 0 \\ e^{-j\pi 0.71} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{j\pi 0.63} & 0 \\ 0 & 0 & 0 & e^{-j\pi 0.54} & 0 & 0 & 0 \\ 0 & e^{-j\pi 0.59} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

Remark 5: The term *Generalized DFT* was also used by other authors [14]-[19]. Their focus was only on linear phase sets. Therefore, the non-linear phase GDFT is the superset of those techniques. Note that any linear phase extensions of DFT yield the same auto- and cross correlation performance as DFT, and hence provide no enhancements of these metrics.

V. DESIGN AND PERFORMANCE OF GENERALIZED DFT

A. Performance Metrics:

In order to compare performance of code families, several objective performance metrics were used in the literature. All the metrics used in this study depend on *aperiodic correlation functions (ACF)* of the spreading code set. The ACF metric $d_{k,l}(m)$ is defined for the complex sequences $\{e_k(n)\}$ and $\{e_l(n)\}$ [18],

$$d_{k,l}(m)=\begin{cases} \frac{1}{N}\sum_{n=0}^{N-1-m} e_k(n)e_l^*(n+m), & 0<m\le N-1 \\ \frac{1}{N}\sum_{n=0}^{N-1+m} e_k(n-m)e_l^*(n), & 1-N<m\le 0 \\ 0 \qquad , & |m|\ge N \end{cases} \tag{21}$$

Out-of-phase autocorrelation sequence of the complex sequence $e_k(n)$ is defined also from (21) as the absolute sequence of $d_{k,k}(m)$ as $|d_{k,k}(m)|$. Similarly, out-of-phase cross-correlation of two complex sequences $e_k(n)$ and $e_l(n)$ is defined as the absolute function of $d_{k,l}(m)$ and expressed with $|d_{k,l}(m)|$.

In this paper, the following correlation metrics are used for optimal GDFT design and performance comparisons of various code families [20] where M is the set size and N is the length of each spreading code.

a. Maximum Value of Out-of-Phase Auto-correlation, d_{am}:

$$d_{am}=\max_{\substack{0\le k<M \\ 1\le m<N}}\left\{\left|d_{k,k}(m)\right|\right\} \tag{22}$$

b. Maximum Value of Out-of-Phase Cross-correlation, d_{cm}:

$$d_{cm}=\max_{\substack{0\le k,l<M \;\; k\neq l \\ 0\le m<N}}\left\{\left|d_{k,l}(m)\right|\right\} \tag{23}$$

$$d_{\max}=\max\left\{d_{am},d_{cm}\right\} \tag{24}$$

The relationship between the maximum out-of-phase auto-correlation d_{am} and the maximum out-of-phase cross-correlation d_{cm} was shown by Sarwate in [21],

$$(2N-1){d_{cm}}^2 + \frac{(2N-1)}{(M-1)}{d_{am}}^2 \geq 1 \tag{25}$$

and leading to the Welch bound for complex spreading sequences expressed as [22],

$$d_{\max} = \max\{d_{am}, d_{cm}\} = \sqrt{\frac{M-1}{M(2N-1)-1}} \tag{26}$$

c. Mean Square Value of Auto-correlation, R_{AC} :

$$R_{AC} = \frac{1}{M}\sum_{k=1}^{M}\sum_{\substack{m=1-N \\ m\neq 0}}^{N-1}\left|d_{k,k}(m)\right|^2 \tag{27}$$

d. Mean Square Value of Cross-correlation, R_{CC} :

$$R_{CC} = \frac{1}{M(M-1)}\sum_{k=1}^{M}\sum_{\substack{l=1 \\ l\neq k}}^{M}\sum_{m=1-N}^{N-1}\left|d_{k,l}(m)\right|^2 \tag{28}$$

e. The merit factor (*F*): The merit factor for the k^{th} code is the ratio of the energy in the main lobe of the autocorrelation function over the total energy in its side lobes and mathematically expressed as [23]

$$F_k = \frac{d_{k,k}(0)}{2\sum_{m=1}^{N-1}\left|d_{k,k}(m)\right|^2} \tag{29}$$

In CDMA communications systems, merit factor is desired to be as large as possible in order to improve the code synchronization and amiability [23].

B. Phase Shaping Function and Optimal Design:

The phase shaping function $\psi(n)$ is formally defined, and its optimal design based on a performance metric is presented in this section. The phase function $\{\varphi_k(n)\,n\}$ of (8) is now decomposed into two functions in the time variable *n* as follows

$$\begin{aligned} &\hat{\varphi}_k(n) = \varphi_k(n)n = kn + \psi(n) \\ &\psi(n) = \hat{\varphi}_k(n) - kn = [\varphi_k(n) - k]n \\ &k \in \{0,1,...,N-1\},\ n \in \{1,...,N-1\};\ \psi(0) \in \mathbb{R} \quad \hat{\varphi}_k(0) = \psi(0) \end{aligned} \tag{30}$$

The linear term kn of phase function in (30) is highlighted due to its significance in the orthogonality requirements of (7). Note that there are infinitely many possibilities for $\{\psi(n)\} \in \mathbb{R}$ in (30) and, generally speaking, it might be linear or non-linear function of time. The GDFT framework offers us the flexibility to define the phase shaping function $\psi(n)$ according to the design requirements. Note that any $\psi(n)$ function will give us an orthogonal GDFT set as stated in (8) and (30).

The cross-correlation sequence of a GDFT *basis function pair (k, l)* with length N is defined as

$$\begin{aligned} R_{\hat{\varphi}_k\hat{\varphi}_l}(m) &= \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})\hat{\varphi}_k(n)}\, e^{-j(\frac{2\pi}{N})\hat{\varphi}_l(n+m)} \\ &= \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})[-lm+(k-l)n+\psi(n)-\psi(n+m)]} \end{aligned} \tag{31}$$

where $R_{\hat{\varphi}_k\hat{\varphi}_l}(m) = 0;\ \forall m$ for the ideal case, and $R_{\hat{\varphi}_k\hat{\varphi}_l}(0) = 0$ implies the orthogonality of the function pair. Similarly, we can define the auto-correlation function of a GDFT basis function as

$$\begin{aligned} R_{\hat{\varphi}_k\hat{\varphi}_k}(m) &= \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})\hat{\varphi}_k(n)}\, e^{-j(\frac{2\pi}{N})\hat{\varphi}_k(n+m)} \\ &= \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})[-km+\psi(n)-\psi(n+m)]} \end{aligned} \tag{32}$$

where $R_{\hat{\varphi}_k\hat{\varphi}_k}(m) = \delta(m)$ for the ideal case. The correlation sequence of a basis pair is incorporated in R_{AC} and R_{CC} metrics of (27) and (28), respectively, in the following design example.

Now, we focus on the optimal design of the phase shaping function, $\psi(n)$, by utilizing a performance metric. Since there is no optimal closed form solution for $\psi(n)$, we used the numerical optimization software tools in Mathematica and MATLAB to obtain optimal phase shaping functions of (30) with respect to the metrics of (27) and (28). Note that the entire set uses the same phase shaping function and there are $P_N = \binom{N}{2} = \frac{N!}{(N-2)!2!}$ basis function pairs in a set of size N to be considered in the design of an optimum $\psi(n)$ yielding the best cross-correlation performance. Similarly, any basis function might have its own optimal phase shaping function optimizing the auto-correlation sequence. Therefore, one needs to pick the sequence providing the best auto-correlation for the entire set as the phase shaping function.

Fig. 1a and Fig. 1b display the optimal phase shaping functions which minimize the correlation metrics R_{AC} and R_{CC} , respectively, for the first two functions of the GDFT with size N=8. The resulting R_{AC} and R_{CC} values for these optimal GDFT designs are tabulated in Table I along with their DFT counterparts for comparison purposes.

Note the consistency of the two numerical search tools, and the improvements in correlation metrics R_{AC} and R_{CC} are shown in this table. The optimal design method explained in this section can be generalized for any performance metric and for any size GDFT. In Sec. V.E, we will present examples of communications applications employing the proposed GDFT framework where correlations dictate the system performance.

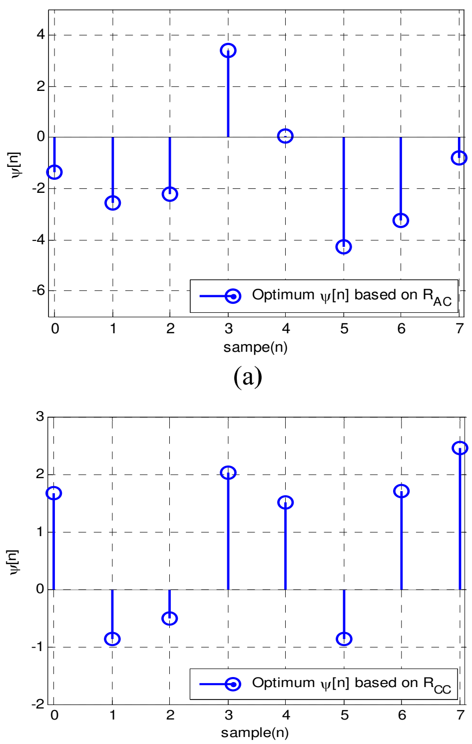

Fig. 1. Optimal phase shaping functions minimizing the correlation metrics a) R_{AC} and b) R_{CC} for the first two functions of GDFT set with N=8.

TABLE I

R_{AC} AND R_{CC} VALUES FOR THE FIRST TWO FUNCTIONS OF OPTIMAL GDFT SETS WITH N=8 ALONG WITH THEIR DFT COUNTERPARTS

Numerical Search Tool and Optimal Phase Shaping Function	OPTIMIZATION METRIC (N=8)	
	R_{AC}	R_{CC}
GDFT (Mathematica, FindMin)	0.0877	0.4219
$\psi(n)$	{ -1.37 -2.53 -2.21 3.39 0.0 -4.21 -3.19 -0.83 }	{ 1.637 -0.79 -0.54 2.01 1.59 -0.83 1.73 2.44}
GDFT (MATLAB, fminsearch)	0.086	0.4205
$\psi(n)$	{ -1.38 -2.56 -2.24 3.42 0.07 -4.27 -3.27 -0.80 }	{ 1.673 -0.87 -0.51 2.02 1.51 -0.86 1.70 2.46 }
DFT	4.375	0.8536

C. Design of Optimal GDFT Based on Brute Force Search:

We use two terms in (10)

$$\varphi_k(n) = a_{k1}n^{b_{k1}} + a_{k2}n^{b_{k2}} \tag{33}$$

with $a_{k1}=k,\ b_{k1}=0$ leading to $\varphi_k(n)=k+a_{k2}n^{b_{k2}}$. Therefore, the basis functions of the set are defined according to (8) as follows

$$\begin{aligned} e_k(n) &= e^{j(2\pi/N)\varphi_k(n)n} \\ e_k(n) &= e^{j(2\pi/N)(k+a_{k2}n^{b_{k2}})n} = e^{j(2\pi/N)[kn+a_{k2}n^{(b_{k2}+1)}]} \\ e_k(n) &= e^{j(2\pi/N)kn}e^{j(2\pi/N)[a_{k2}n^{(b_{k2}+1)}]} \quad k,n=0,1,\ldots,N\text{-}1 \end{aligned} \tag{34}$$

Note that the first exponential term in (34) is the DFT kernel with linear phase while the second defines the G matrix, and $\{e_k(n)\}$ are the row sequences of A_{GDFT} as expressed in the matrix form

$$A_{GDFT} = A_{DFT}G \tag{35}$$

In this representation, varying the values of the a_{k2} and b_{k2} coefficients generate a medley of GDFT sets along with their associated nonlinear phase functions and auto- and cross-correlation properties. This suggests that a brute force search algorithm can scan the phase space with various grid resolutions to find optimum G matrices with respect to given performance metric.

The search grid resolution is defined by the binary values of coefficients $a_2 = a_{k2}$ and $b_2 = b_{k2}$ for all *k*. Table II tabulates the optimal values of the metric $d_{\max}$ along with other performance metrics for various search grid resolutions defined as $\Delta_{a_2,b_2} = (N-1)/2^b$ where *b* is in bits per coefficient and $0 < a_2, b_2 \le 7$ for the code length of $N = 8$. Note that as the search grid resolution is improves, the search yields better performance with respect to design metric $d_{\max}$.

TABLE II

CORRELATION PERFORMANCE OF OPTIMAL GDFT DESIGNS BASED ON $d_{\max}$ FOR $N = 8$.

Search Grid Resolution (bits/c)	d_{am}	d_{cm}	$d_{\max}$ ***(OPT)***	R_{AC}	R_{CC}	F
4	0.301	0.442	0.442	0.526	0.925	1.900
6	0.376	0.409	0.409	0.854	0.878	1.171
8	0.341	0.387	0.387	0.576	0.918	1.738
9	0.376	0.387	0.387	1.095	0.843	0.912
Achievable Welch Bound [8x8] [22]	-----	-----	0.243	-----	-----	----

Table III compares the A_{GDFT} with other known constant modulus code sets on the basis of the metrics indicated. In this case, A_{GDFT} is obtained through a brute force search based on minimization of $d_{\max}$ for *N=8*.

TABLE III
CORRELATION PERFORMANCE COMPARISONS OF VARIOUS CODE FAMILIES FOR N=7 OR 8.

Code	d_{am}	d_{cm}	$d_{\max}$	R_{AC}	R_{CC}	F
Walsh [8x8]	0.875	0.875	0.875	2.375	0.661	0.421
Walsh-like [8x8], [3]	0.625	0.625	0.625	0.875	0.875	1.143
DFT [8x8]	0.875	0.327	0.875	4.375	0.375	0.220
7/8 Gold	0.714	0.714	0.714	0.857	0.878	1.167
Oppermann Set, [8] (*opt* $d_{\max}$) (m=1, p=1, n=2.98, N=7)	0.425	0.419	0.465	1.278	0.787	0.783
A_{GDFT} [8x8] (*opt* $d_{\max}$)	0.376	0.387	0.387	1.095	0.843	0.912
Achievable Welch Bound [8x8] [22]	-----	-----	0.243	-----	-----	-----

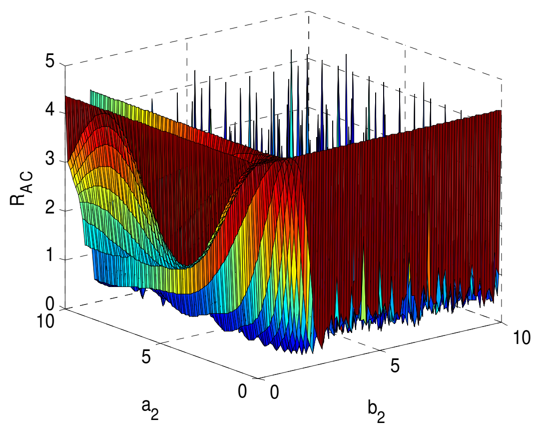

Fig. 2.a. Variation of the auto-correlation metric R_{AC} as a function of the design parameters a_2 and b_2 , N=8.

Figures 2.a and 2.b demonstrate the interdependence of the auto- and cross-correlation metrics R_{AC} and R_{CC}, respectively, on design parameters a_2 and b_2 defined in (33). Note that in a multi-carrier communications application, be they OFDM or CDMA based, one can choose values of a_2 and b_2 to realize desired values of auto- and cross-correlation metrics, R_{AC} and R_{CC}, respectively. In OFDM systems, frequency localization is more important and the optimization on R_{CC} parameters is emphasized whereas in a DS-CDMA system, R_{AC} and R_{CC} are each equally significant. The low values of R_{AC} is desirable for synchronization of the system. In contrast, the low values of R_{CC} is required to minimize multi-user interference (MUI) that dictates BER performance of the system. However, these two parameters cannot be minimized at the same time due to their interaction as reported in [24];

$$R_{CC}(M-1)+R_{AC}\geq M-1 \tag{36}$$

where M is the number of simultaneous users or codes in the multiuser communications channel.

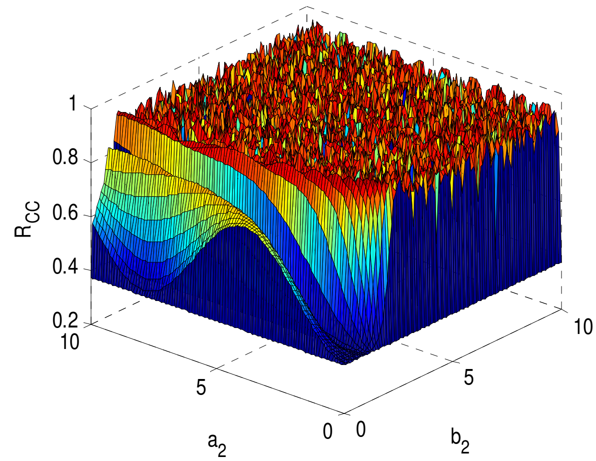

Fig. 2.b. Variation of the cross-correlation metric R_{CC} as a function of the design parameters a_2 and b_2 , N=8.

D. A Closed Form Phase Shaping Function for GDFT:

We would like to define a closed form expression for the *phase shaping function (PSF)* $\psi(n)$ of (30) which approximates the brute force based optimal GDFT solutions obtained in the previous section via curve fitting. We used the signal processing software tool *Table Curve 2D* and fitted the following *PSF* to the nonlinear phase function of the optimal GDFT previously obtained via the brute force search

$$\psi(n)=a_1\exp\left(-\left[\frac{n-b_1}{c_1}\right]^2\right)+a_2\exp\left(-\left[\frac{n-b_2}{c_2}\right]^2\right) \qquad n=0,1,.....,N-1 \tag{37}$$

This is a second degree *Gaussian function* defined with six parameters $\{a_1,b_1,c_1,a_2,b_2,c_2\}$. The values of the parameter set, $\{a_1,b_1,c_1,a_2,b_2,c_2\}$ are then chosen to fit the brute force based optimal solution for a given criterion. Our simulation studies showed that the phase shaping function of (37) provides low values of d_{cm} and R_{AC} regardless of the GDFT size *N*.

Fig. 3 displays $\psi(n)$ of (37) which approximates the brute force GDFT designs based on d_{cm}. The resulting parameter values are $\{a_1=1, b_1=1.75, c_1=3.75, a_2=1.75, b_2=6, c_2=0.5\}$. Moreover, Fig. 4 displays the nonlinear phase functions of GDFT basis generated by using this $\psi(n)$ along with linear phase functions of DFT for N=8. Relaxing the linear phase condition of DFT offers GDFT solutions with significantly improved correlation properties. Hence, one may obtain many orthogonal GDFT bases by changing the set of 6 parameters in (37) according to the design metric of interest.

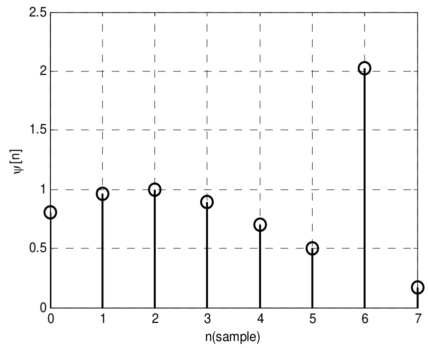

Fig. 3. Closed form phase shaping function $\psi(n)$ of (37) for N = 8.

In this section, we limit our discussions to the optimal GDFT designs based on d_{cm} and R_{AC}. It is observed that BER performance of a multiuser system with small number of users and additive white Gaussian noise (AWGN) channel model is closely coupled with the d_{cm} metric. In contrast, its BER performance for multipath fading channel model is related to the R_{AC} metric. The goal of an engineer is to define the parameters of GDFT kernel in (37) yielding BER performance improvements for both AWGN and multipath channel models. These observations are quantified by several system performance simulations presented in the following section.

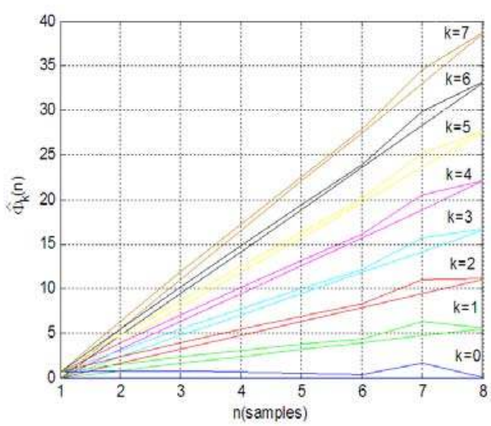

Fig. 4. The linear phase functions of DFT along with the non-linear phase functions of GDFT set obtained from (37) for N=8.

We used Genetic Search Algorithm [25] to find optimum PSFs for different sizes and design types. The number of initial population, the number of population, the probability of crossover, and the probability of mutation for the search algorithm were chosen as 1000, 100, 0.9 and 0.1, respectively. The phase shaping function of (37) is used in the GDFT phase function defined in (30) for different values of parameters $\{a_1, b_1, c_1, a_2, b_2, c_2\}$. They are obtained through the search with the parameter value resolution of 0.25 for the given transform size N. The algorithm is run for the three different transform sizes, N=8, 16, 32. The resulting correlation performances are displayed in Tables IV and V.

It is observed from Table IV that d_{cm} of GDFT decreases faster than DFT as code size increases. R_{AC} of the DFT increases significantly faster than the GDFT for higher values of *N* as tabulated in Table V. These correlation improvements lead us to superior BER performance of GDFT in multicarrier communications that will be highlighted in the following section.

The main advantage of the proposed method is the ability to design a wide collection of constant modulus orthogonal code sets based on the desired correlation performance mimicking the specs of interest. Moreover, the proposed GDFT technique can also be considered as a natural enhancement to DFT to obtain improved performance. Note that the auto-correlation magnitude functions of the codes in any GDFT set are the same. Fig. 5 displays auto-correlation function of a size 16 GDFT code optimized based on R_{AC} along with size 16 DFT set for comparison purposes.

TABLE IV

CORRELATION PERFORMANCES OF DFT AND OPTIMAL GDFT BASED ON d_{cm} FOR N = 8, 16, AND 32.

Size (N)	Corresponding correlation metrics optimized based on d_{cm} along with DFT				
	d_{am}	d_{cm}	$d_{\max}$	R_{AC}	R_{CC}
8 GDFT 8 DFT	0.703 0.875	0.288 0.327	0.703 0.875	3.261 4.375	0.534 0.375
16 GDFT 16 DFT	0.744 0.938	0.248 0.321	0.744 0.938	6.653 9.688	0.557 0.354
32 GDFT 32 DFT	0.794 0.969	0.233 0.319	0.794 0.969	13.68 20.34	0.559 0.344

TABLE V

CORRELATION PERFORMANCES OF DFT AND OPTIMAL GDFT BASED ON R_{AC} FOR N = 8, 16, AND 32.

Size (N)	Corresponding correlation metrics optimized based on R_{AC} along with DFT				
	d_{am}	d_{cm}	$d_{\max}$	R_{AC}	R_{CC}
8 GDFT 8 DFT	0.125 0.875	0.679 0.327	0.679 0.875	0.089 4.375	0.987 0.375
16 GDFT 16 DFT	0.242 0.938	0.700 0.321	0.700 0.938	0.234 9.688	0.984 0.354
32 GDFT 32 DFT	0.604 0.969	0.746 0.319	0.746 0.969	1.165 20.34	0.962 0.344

Similarly, cross-correlation functions (CCF) of the first and second codes of optimal GDFT design based on d_{cm} metric and DFT set for N=16 are displayed in Fig. 6. These figures highlight the merit of the proposed GDFT framework over the traditional DFT.

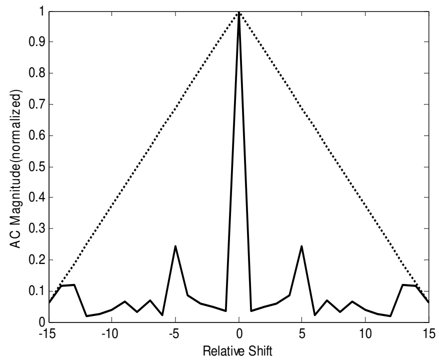

Fig. 5. Auto-correlation magnitude functions (ACF) of optimal GDFT design based on R_{AC} (solid line) and DFT (dashed line) for N=16.

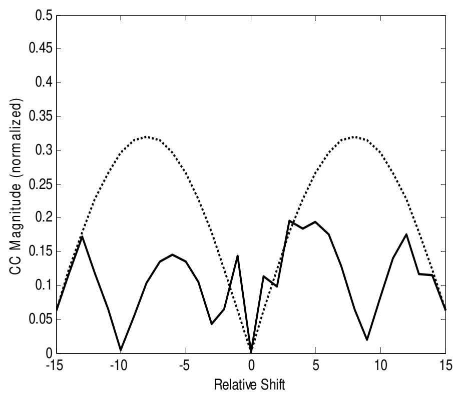

Fig. 6. Cross-correlation magnitude functions (CCF) of optimal GDFT design based on d_{cm} (solid line) and DFT (dashed line) for N=16.

E. BER on AWGN and Multipath Fading Channels:

BER performance of a 2 user asynchronous CDMA communications system with AWGN channel model employing d_{cm} based optimal GDFT of various lengths and size 8 DFT are displayed in Fig. 7.

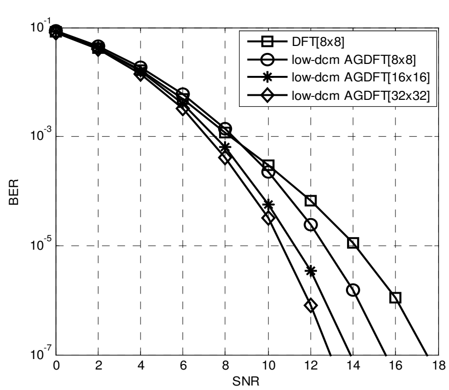

Fig. 7. BER Performance of a 2 user asynchronous CDMA communications system with AWGN channel model employing d_{cm} based optimal GDFT of various sizes and size 8 DFT.

Similarly, Fig. 8 displays BER performance of a 2 user asynchronous CDMA communications system with AWGN channel model employing various known code families and d_{cm} based optimal GDFT of length 8. Note that in CDMA communications, the multiuser interference (MUI) becomes the dominant factor defining BER performance as the number of users in the system increases.

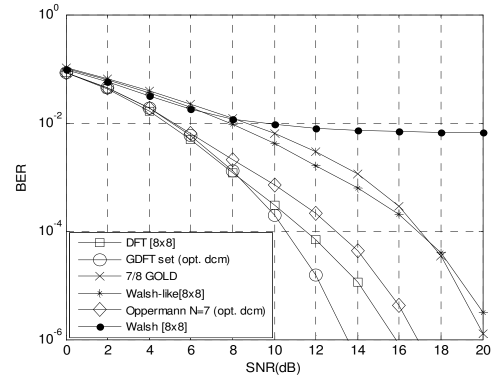

Fig. 8. BER performance of a 2 user asynchronous CDMA communications system with AWGN channel model employing various code sets of length 8 (N=7 for Gold and Oppermann codes).

We also simulated BER performance of a wireless CDMA communications system with Rayleigh channel model employing various code families. We assumed a two-ray "multipath channel" with the impulse response of [26]

$$h(t) = \beta_0\delta(t) + \beta_1\delta(t-\tau) \qquad (38)$$

In BER simulations, the parameters $\{\beta_0, \beta_1\}$ are the Rayleigh distributed random variables defining power of the desired and interfering paths, respectively. The sum of $E\left[\beta_0^2\right]$ and $E\left[\beta_1^2\right]$ is set to be equal to one. Fig. 9 displays BER performance of a 2 user CDMA communications system with Rayleigh multipath channel model employing size 8 DFT and GDFT codes where the power of interfering path is 3dB ($\frac{D}{I} = 3dB$) and 5dB ($\frac{D}{I} = 5dB$) less than the power of the desired path and the path delay, τ , is set to be equal to T/8. In this example, we used GDFT set designed by using *PSF* of (37) optimized based on minimization of R_{AC}, and the resulting R_{AC} values are 4.375 and 0.089 for DFT and GDFT, respectively. For the Rayleigh multipath channel model BER curves show that GDFT set significantly outperforms DFT due to its superior R_{AC} characteristics.

VI. CONCLUSIONS

In this paper, we introduced a theoretical framework and several methods exploring the entire phase space for optimal design of constant modulus transforms. Correlations performance of the proposed GDFT is compared with the industry standard DFT and other code families. Superior correlations of GDFT result in improved BER performance over the known code families in CDMA communications scenarios considered herein. Some of the DFT based engineering applications might benefit from the proposed nonlinear phase GDFT in the future.

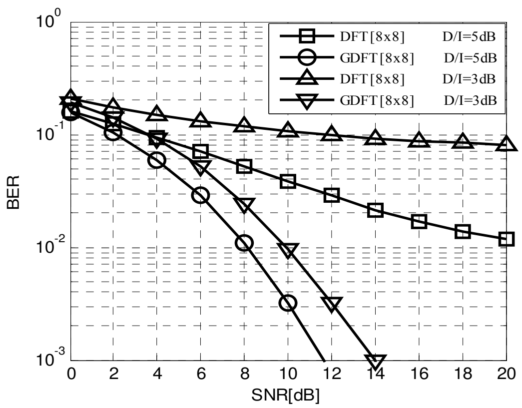

Fig. 9. BER Performance of a 2 user asynchronous CDMA communications system with Rayleigh multipath channel model employing DFT and GDFT of length 8 with $D/I = 5dB$ and $D/I = 3dB$.

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