Generalized Exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions

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Abstract. In this paper we present applications of a special class of homogeneous monogenic polynomials constructed, in the framework of hypercomplex function theory, in order to be an Appell set of polynomials. In particular, we derive important properties of an associated exponential function from $\mathbb{R}^3$ to $\mathbb{R}^3$ and propose a generalization to $\mathbb{R}^{n+1}$.

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1. PRELIMINARIES

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a product according to the multiplication rules $e_k e_l + e_l e_k = -2\delta_{kl},\ k,l = 1,\cdots,n,$ where δ_{kl} is the Kronecker symbol. This non-commutative product generates the 2^n-dimensional Clifford algebra $Cl_{0,n}$ over $\mathbb{R}$ and the set $\{e_A : A \subseteq \{1,\cdots,n\}\}$ with $e_A = e_{h_1}e_{h_2}\cdots e_{h_r}$, $1 \le h_1 \le \cdots \le h_r \le n$, $e_\emptyset = e_0 = 1$, forms a basis of $Cl_{0,n}$. The real vector space $\mathbb{R}^{n+1}$ will be embedded in $Cl_{0,n}$ by identifying the element $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the element $x = x_0 + \underline{x}$ of the algebra, where $\underline{x} = e_1x_1 + \cdots + e_nx_n$.The conjugate of x is $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$. Denote by $\omega(x) = \frac{\underline{x}}{|\underline{x}|} \in S^n$, where S^n is the unit sphere in $\mathbb{R}^n$.

In what follows we consider $Cl_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$, i.e., functions of the form $f(z) = \sum_A f_A(z)e_A$, where $f_A(z)$ are real valued. We suppose that f is hypercomplex differentiable in Ω in the sense of [1], [2], i.e. has a uniquely defined areolar derivative f' in each point of Ω (for the definition of an areolar derivative see [3]). Then f is real differentiable (even real analytic) and f' can be expressed in terms of the partial derivatives with respect to x_k as $f' = 1/2(\partial_0 - \partial_{\underline{x}})f$, where $\partial_0 := \frac{\partial}{\partial x_0}$, $\quad \partial_{\underline{x}} := e_1\frac{\partial}{\partial x_1} + \cdots + e_n\frac{\partial}{\partial x_n}$. If now $D := \partial_0 + \partial_{\underline{x}}$ is the usual generalized Cauchy-Riemann differential operator, then, obviously $f' = 1/2\overline{D}f$. Since in [1] it has been shown that a hypercomplex differentiable function belongs to the kernel of D, i.e. satisfies the property $Df = 0$ (f is a *monogenic function* in the sense of Clifford Analysis), then it follows that in fact $f' = \partial_0 f$ like in the complex case. For our purpose we use monogenic polynomials in terms of the hypercomplex monogenic variables $z_k = x_k - x_0e_k = -\frac{xe_k + e_kx}{2}$, $k = 1,2,\cdots,n$, which leads to *generalized powers* of degree n that are by convention symbolically written as $z_1^{\nu_1} \times \cdots z_n^{\nu_n}$ and defined as an *n-nary* symmetric product by $z_1^{\nu_1} \times \cdots z_n^{\nu_n} = \frac{1}{n!}\sum_{\pi(i_1,\ldots,i_n)} z_{i_1}\cdots z_{i_n}$, where the sum is taken over *all* permutations of $(i_1,\ldots,i_n)$, (see [2] and [3]).

2. THE CASE $n = 2$

As usual, we identify each element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the *paravector* (also called *reduced quaternion*) $z = x_0 + \underline{x} = x_0 + x_1e_1 + x_2e_2$. In this section, we consider a special set of monogenic basis functions defined and studied to some extend with respect to its algebraic properties in [4], namely functions of the form

$$\mathscr{P}_k(x) = \sum_{s=0}^{k} T_s^k x^{k-s}\bar{x}^s, \quad \text{with} \quad T_s^k = \frac{1}{k+1}\frac{\left(\frac{3}{2}\right)_{(k-s)}\left(\frac{1}{2}\right)_{(s)}}{(k-s)!s!}, \tag{1}$$

where $a_{(r)}$ denotes the Pochhammer symbol, i.e. $a_{(r)} := a(a+1)(a+2)\cdots(a+r-1) = \frac{\Gamma(a+r)}{\Gamma(a)}$, for any integer $r > 1$, and $a_{(0)} := 1$. In terms of generalized powers these polynomials are of the form

$$\mathscr{P}_k(x) = \mathbf{P}_k(z_1, z_2) = c_k \sum_{k=0}^{n} z_1^{n-k} \times z_2^k \binom{n}{k} e_1^{n-k} \times e_2^k, \tag{2}$$

where

$$c_k := \sum_{s=0}^{k} T_s^k (-1)^s = \begin{cases} \dfrac{k!!}{(k+1)!!}, & \text{if } k \text{ is odd} \\ \\ c_{k-1}, & \text{if } k \text{ is even} \end{cases} \tag{3}$$

In [5], it was proved that the sequence of such polynomials forms a *quaternionic Appell sequence*. More precisely, these polynomials behave like monomial functions in the sense of the complex powers $z^k = (x_0 + ix_1)^k;\ k = 1, 2, \cdots$ and the power rule of complex differentiation in the form $\frac{d}{dz} z^k = k z^{k-1}$ is extended with respect to the hypercomplex derivative of these special monomial functions, since $P_k' = kP_{k-1}$. Moreover

$$\mathscr{P}_k(x_0) = x_0^k, \quad \mathscr{P}_k(\underline{x}) = c_k \underline{x}^k$$

and

$$\mathscr{P}_k(x_0 + \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} \mathscr{P}_s(\underline{x}).$$

For the above reasons, it is natural to propose the following monogenic function (see [4]),

$$\mathrm{Exp}(x) := \sum_{k=0}^{\infty} \frac{\mathscr{P}_k(x)}{k!}, \tag{4}$$

as a *generalized exponential* function in $\mathbb{R}^3$. This Exp-function, as in the real and complex case, is a solution of the differential equation

$$f' = f,\ f(0) = 1 \tag{5}$$

and satisfies also

$$\mathrm{Exp}'(\lambda x) = \lambda \mathrm{Exp}(\lambda x),\ \lambda \in \mathbb{R} \tag{6}$$

and

$$\mathrm{Exp}(x_0 + \underline{x}) = e^{x_0} \mathrm{Exp}(\underline{x}). \tag{7}$$

Another important property of the real and complex exponential function is that it should be different from zero everywhere. We will prove this property for the exponential function (4) by first writing it in closed form.

Theorem 1 *The* Exp-*function* (4) *can be written in terms of Bessel functions of the first kind, $J_a(x)$, for integer orders $a = 0, 1$ as*

$$\mathrm{Exp}(x_0 + \underline{x}) = e^{x_0} (J_0(|\underline{x}|) + \omega(x) J_1(|\underline{x}|). \tag{8}$$

Proof. We start by first considering the expression of $\mathrm{Exp}(x_i e_i)$, for $i = 1, 2$, i.e.

$$\mathrm{Exp}(x_i e_i) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{k} T_s^k (x_i e_i)^{k-s} (-x_i e_i)^s = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{k} T_s^k (-1)^s x_i^k e_i^k$$

and applying relation (3) to obtain

$$\mathrm{Exp}(x_i e_i) = \sum_{k=0}^{\infty} \frac{1}{k!} c_k x_i^k e_i^k = \sum_{m=0}^{\infty} \frac{1}{(2m)!} c_{2m} x_i^{2m} (-1)^m + e_i \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} c_{2m+1} x_i^{2m+1} (-1)^m.$$

After some calculation we end up with

$$\mathrm{Exp}(x_i e_i) = \sum_{m=0}^{\infty} a_m (-1)^m x_i^{2m} + e_i \frac{x_i}{2} \sum_{m=0}^{\infty} b_m (-1)^m x_i^{2m},$$

where

$$a_m = \frac{(2m-1)!!}{(2m)!(2m)!!} = \frac{1}{2^{2m}(m!)^2} \qquad \text{and} \qquad b_m = \frac{2(2m+1)!!}{(2m+1)!(2m+2)!!} = \frac{1}{2^{2m}m!(m+1)!}.$$

Therefore,

$$\mathrm{Exp}(x_i e_i) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \frac{x_i}{2}^{2m} + e_i \frac{x_i}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \frac{x_i}{2}^{2m}. \tag{9}$$

Recalling the expression of the Bessel functions of the first kind J_0 and J_1 we can conclude that

$$\mathrm{Exp}(x_i e_i) = J_0(x_i) + e_i J_1(x_i);\ i = 1,2. \tag{10}$$

We note that it is possible to express $\mathrm{Exp}(\underline{x})$ almost in the same way. In fact, let

$$\underline{x} = x_1 e_1 + x_2 e_2 = \frac{x_1 e_1 + x_2 e_2}{|\underline{x}|} |\underline{x}| = \omega(x) |\underline{x}|.$$

Since $\omega(x)^2 = -1$, then it follows from (9) and (10):

$$\mathrm{Exp}(\underline{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \frac{|\underline{x}|}{2}^{2m} + \omega(x) \frac{|\underline{x}|}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \frac{|\underline{x}|}{2}^{2m} = J_0(|\underline{x}|) + \omega(x) J_1(|\underline{x}|).$$

The final result follows at once from (7). □

Straightforward calculations lead to the following important properties of the exponential function (4).

Corollary 1

1. $\mathrm{Exp}(x)$ *has no zeros.*
2. *If* $\mathfrak{J}(\mathrm{Exp}(x))$ *denotes the jacobian of the function* $\mathrm{Exp}(x)$*, then*

$$\mathfrak{J}(\mathrm{Exp}(x_0 + \underline{x})) = \begin{cases} -\frac{e^{3x_0}}{|\underline{x}|^2} J_1(|\underline{x}|) \left(J_0(|\underline{x}|) J_2(|\underline{x}|) |\underline{x}| - J_0(|\underline{x}|) J_1(|\underline{x}|) - J_1(|\underline{x}|)^2 |\underline{x}| \right), & \text{if } \underline{x} \neq 0 \\ \frac{1}{4} e^{3x_0}, & \text{if } \underline{x} = 0 \end{cases}$$

We underline that the zeros of the jacobian of $\mathrm{Exp}(x)$ are exactly the zeros of $J_1(|\underline{x}|)$ and therefore they are all laid on concentric cylinders with radius corresponding to de zeros of $J_1(|\underline{x}|)$. In particular, the first positive zero is $\zeta_1 \approx 3.831705970$. Thus, we can conclude, for example, that on the cylinder $\mathscr{C} := \{(x_0, x_1, x_2) \in \mathbb{R}^3 : x_1^2 + x_2^2 < \zeta_1^2\}$, the Exp-function can be treated as a mapping function from $\mathbb{R}^3$ to $\mathbb{R}^3$. Images of some domains in $\mathbb{R}^3$ by this exponential function can be found in [4].

3. A GENERALIZED EXPONENTIAL FUNCTION FROM $\mathbb{R}^{n+1}$ TO $\mathbb{R}^{n+1}$

In the case $x = x_0 + \underline{x} = x_0 + x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$, we denote by $\mathscr{P}_k^n(x)$ the homogeneous monogenic polynomial of degree k, generalizing the particular $n = 2$ case in (1). It can be proved that for arbitrary $n \geq 1$, the polynomials

$$\mathscr{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n) x^{k-s} \overline{x}^s, \quad \text{with} \quad T_s^k(n) = \frac{n!}{(n)_k} \frac{(\frac{n+1}{2})_{(k-s)} (\frac{n-1}{2})_{(s)}}{(k-s)!s!}, \tag{11}$$

form an Appell sequence. In terms of generalized powers these polynomials are of the form

$$\mathscr{P}_k^n(x) = \mathbf{P}_k(z_1, \cdots, z_n) = c_k(n) \sum_{|\nu|=n} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \binom{n}{\nu} e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}, \tag{12}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ is a multiindex and

$$c_k(n) := \frac{\sum_{s=0}^{k} (-1)^s T_s^k(n)}{\sum_{s=0}^{k} T_s^k(n)} = \begin{cases} \dfrac{k!}{n_{(k)}} \left(\dfrac{n+1}{2}\right)_{\left(\frac{k-1}{2}\right)} \dfrac{1}{\left(\frac{k-1}{2}\right)!}, & \text{if } k \text{ is odd} \\ \dfrac{k!}{n_{(k)}} \left(\dfrac{n+1}{2}\right)_{\left(\frac{k}{2}\right)} \dfrac{1}{\left(\frac{k}{2}\right)!}, & \text{if } k \text{ is even} \end{cases} \tag{13}$$

Now if we define

$$\mathrm{Exp}_n(x) := \sum_{k=0}^{\infty} \frac{\mathscr{P}_k^n(x)}{k!} \tag{14}$$

then we can prove the following result.

Theorem 2 *The* Exp_n*-function* (14) *can be written in terms of Bessel functions of the first kind,* $J_a(x)$*, for orders* $a = \frac{n}{2} - 1, \frac{n}{2}$ *as*

$$\mathrm{Exp}_n(x_0 + \underline{x}) = e^{x_0} \Gamma(\frac{n}{2}) \left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1} \left(J_{\frac{n}{2}-1}(|\underline{x}|) + \omega(x) J_{\frac{n}{2}}(|\underline{x}|)\right). \tag{15}$$

For $n = 1$ and $n = 2$, (15) gives the ordinary complex exponential and the reduced quaternion valued Exp-function (4), respectively. Moreover, it is easy to conclude that this function is different from zero everywhere and fulfills properties (5)-(7), for any arbitrary $n \geq 1$.

For the particular case $n = 3$, (15) leads to

$$\mathrm{Exp}_3(x_0 + \underline{x}) = e^{x_0} \left(\frac{\sin(|\underline{x}|)}{|\underline{x}|} + \omega(x) \frac{\sin(|\underline{x}|) - |\underline{x}| \cos(|\underline{x}|)}{|\underline{x}|}\right). \tag{16}$$

This Exp_3-function coincides with the one referred by Sprössig in [6], based on results of [7, 8, 9, 10] which are generalizations of Fueter's approach.

We would like to point out that the homogeneous monogenic polynomials (1) constructed so that they form an Appell sequence, are particularly easy to handle and can play an important role in 3D-mapping problems. For a case study of such approach, see for example [11].

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