GENERALIZED FOURIER-FEYNMAN TRANSFORM AND SEQUENTIAL TRANSFORMS ON FUNCTION SPACE

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Abstract. In this paper we first investigate the existence of the generalized Fourier-Feynman transform of the functional F given by

$$F(x) = \hat{\nu}((e_1, x)^{\sim}, \ldots, (e_n, x)^{\sim}),$$

where $(e, x)^{\sim}$ denotes the Paley-Wiener-Zygmund stochastic integral with x in a very general function space $C_{a,b}[0, T]$ and $\hat{\nu}$ is the Fourier transform of complex measure ν on $\mathcal{B}(\mathbb{R}^n)$ with finite total variation. We then define two sequential transforms. Finally, we establish that the one is to identify the generalized Fourier-Feynman transform and the another transform acts like an inverse generalized Fourier-Feynman transform.

1. Introduction

Let $C_0[0, T]$ denote one-parameter Wiener space. The concept of an L_1 analytic Fourier-Feynman transform(FFT) for functionals on Wiener space $C_0[0, T]$ was introduced by Brue in [1]. This transform and its properties are similar in many respects to the ordinary Fourier function transform. Further works involving the L_2–L_2 theory and the L_p–$L_{p'}$ theory, $1/p + 1/p' = 1$, includes [2, 14]. In [10], Huffman, Park and Skoug defined a convolution product for functionals on Wiener space, and they obtained various results for the analytic FFT and the convolution product [10], [11], [12]. Also see the references in [16] for further informations and results.

In [9], Chang and Skoug defined a generalized analytic Fourier-Feynman transform(GFFT) for functionals on very general function space $C_{a,b}[0, T]$. The function space $C_{a,b}[0, T]$, induced by generalized Brownian motion process, was introduced by J. Yeh [17, 18] and was used extensively by Chang and Chung [7], Chang and Skoug [9], and Chang, Chung and Skoug [8]. The GFFT theory on $C_{a,b}[0, T]$ have been developed in many papers including [3, 5, 6, 9].

In this paper we investigate the existence of the GFFT for bounded cylinder functionals of the form

$$F(x) = \hat{\nu}((e_1, x)^\sim, \ldots, (e_n, x)^\sim),$$

where $(e, x)^\sim$ denotes the Paley-Wiener-Zygmund stochastic integral with x in function space $C_{a,b}[0, T]$ and $\hat{\nu}$ is the Fourier transform of complex measure ν on $\mathcal{B}(\mathbb{R}^n)$ with finite total variation. We define sequential $\mathcal{P}$ and $\mathcal{N}$ function space transforms of such cylinder functionals. We then study relationships between the sequential $\mathcal{P}$-transform and the GFFT of the functionals on $C_{a,b}[0, T]$ and present a version of the inverse GFFT via the sequential $\mathcal{N}$-transform.

The Wiener process used in [1], [2], [10], [11], [12], [14] is free of drift and stationary in time while the stochastic process used in this paper, as well as in [3], [4], [5], [6], [7], [8], [9] is nonstationary in time and is subject to a drift $a(t)$. It turns out, as noted in Remark 4.1 below, that including a drift term $a(t)$ makes establishing the existence of the GFFT of functionals on $C_{a,b}[0, T]$ very difficult. However, when $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, the general function space $C_{a,b}[0, T]$ reduces to the Wiener space $C_0[0, T]$.

2. Preliminaries

Let $D = [0, T]$ and let $(\Omega, \mathcal{W}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{W}, P)$ and D is called a generalized Brownian motion process if $Y(0, \omega) = 0$ almost everywhere and for $0 = t_1 < t_2 < \cdots < t_n \leq T$, the n-dimensional random vector $(Y(t_1, \omega), \ldots, Y(t_n, \omega))$ is normally distributed with density function

$$K_n(\vec{t}, \vec{u}) = \left(\prod_{j=1}^n 2\pi \big(b(t_j) - b(t_{j-1})\big) \right)^{-1/2}$$
$$\times \exp\left\{ -\frac{1}{2} \sum_{j=1}^n \frac{\big[(u_j - a(t_j)) - (u_{j-1} - a(t_{j-1}))\big]^2}{b(t_j) - b(t_{j-1})} \right\},$$

where $\vec{u} = (u_1, \ldots, u_n)$, $u_0 = 0$, $\vec{t} = (t_1, \ldots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$.

In [18], Yeh shows that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$, and that the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous functions x on $[0, T]$ with $x(0) = 0$ under the sup norm). Hence, $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0, T])$ is the Borel σ-algebra of $C_{a,b}[0, T]$. We then complete this function space to obtain $(C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)$ where $\mathcal{W}(C_{a,b}[0, T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0, T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable [13] provided $\rho B \in \mathcal{W}(C_{a,b}[0,T])$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to be hold scale-invariant almost everywhere(s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

In this paper, as in [3], [4], [5], [6], [8], [9], we consider the incomplete function space $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ and we denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$E[F] = \int_{C_{a,b}[0,T]} F(x) d\mu(x)$$

whenever the integral exists.

Let $L_{a,b}^2[0,T]$ be the separable Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $b(t)$ and $a(t)$: i.e.,

$$L_{a,b}^2[0,T] = \left\{ v : \int_0^T v^2(s) db(s) < +\infty \text{ and } \int_0^T v^2(s) d|a|(s) < +\infty \right\},$$

where $|a|(t)$ denote the total variation function of $a(t)$. The inner product on $L_{a,b}^2[0,T]$ is defined by $(u,v)_{a,b} = \int_0^T u(t)v(t) d[b(t) + |a|(t)]$. Note that $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$ and that all functions of bounded variation on $[0,T]$ are elements of $L_{a,b}^2[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$, then $L_{a,b}^2[0,T] = L^2[0,T]$. In fact,

$$(L_{a,b}^2[0,T], \|\cdot\|_{a,b}) \subset (L_{0,b}^2[0,T], \|\cdot\|_{0,b}) = (L^2[0,T], \|\cdot\|_2)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

For each $v \in L_{a,b}^2[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral $\langle v, x \rangle$ is given by the formula

$$\langle v, x \rangle = \lim_{n \to \infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_{a,b} \phi_j(t) dx(t)$$

for μ-a.e. $x \in C_{a,b}[0,T]$, where $\{\phi_j\}_{j=1}^\infty$ is a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that $(\phi_j, \phi_k)_{a,b} = \delta_{jk}$ (the Kronecker delta). If v is of bounded variation on $[0,T]$, then the PWZ stochastic integral $\langle v, x \rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t) dx(t)$ for s-a.e. $x \in C_{a,b}[0,T]$.

Let

$$C_{a,b}'[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s) db(s) \text{ for some } z \in L_{a,b}^2[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D_t : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$D_t w = z(t) = \frac{w'(t)}{b'(t)}.$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T D_t w_1 D_t w_2 db(t) = \int_0^T z_1(t)z_2(t)db(t)$$

is a separable Hilbert space. Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are homeomorphic.

Throughout this paper we assume $a \in C'_{a,b}[0,T]$ and, for notational convenience, we use the notation $(w,x)^\sim$ instead of $\langle D_t w, x \rangle$. Then we see that for each w in $C'_{a,b}[0,T]$, the random variable $x \mapsto (w,x)^\sim$ is Gaussian with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$, and that $(w, \alpha x)^\sim = \alpha(w,x)^\sim = (\alpha w, x)^\sim$ for any real number α, $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$. We also see that if $\{w_1, \ldots, w_n\}$ is an orthonormal set in $C'_{a,b}[0,T]$, then the random variables $(w_i, x)^\sim$'s are independent. From these we obtain the following integration formula for the function space $C_{a,b}[0,T]$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in $C'_{a,b}[0,T]$, let $g : \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function and let $G : C_{a,b}[0,T] \to \mathbb{C}$ be given by equation

$$G(x) = g((e_1,x)^\sim, \ldots, (e_n,x)^\sim).$$

Then

$$E[G] = \int_{C_{a,b}[0,T]} g((e_1,x)^\sim, \ldots, (e_n,x)^\sim)d\mu(x)$$

$$(2.1) \qquad = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(u_1, \ldots, u_n)$$

$$\times \exp\left\{ -\sum_{j=1}^n \frac{[u_j - (e_j,a)_{C'_{a,b}}]^2}{2} \right\} du_1 \cdots du_n$$

in the sense that if either side of equation (2.1) exists, both sides exist and equality holds.

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\tilde{\mathbb{C}}_+$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively. For each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{-1/2}$ (or $\lambda^{1/2}$) is always chosen to have positive real part. The following integration formula is used several times in this paper:

$$(2.2) \qquad \int_{\mathbb{R}} \exp\{-\alpha u^2 + \beta u\}du = \left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left\{\frac{\beta^2}{4\alpha}\right\}$$

for complex numbers α and β with $\mathrm{Re}(\alpha) > 0$.

3. Bounded cylinder functionals

In this section, we introduce a class of certain bounded cylinder functionals and state the concepts of the generalized analytic Feynman integral and the GFFT on function space $C_{a,b}[0,T]$. We then establish the existence of the GFFT of such cylinder functionals.

A functional F is called a cylinder functional on $C_{a,b}[0,T]$ if there exists a finite subset $\{h_1,\ldots,h_m\}$ of $C'_{a,b}[0,T]$ such that

$$(3.1) \qquad F(x) = \phi((h_1,x)^\sim,\ldots,(h_m,x)^\sim), \quad x \in C_{a,b}[0,T],$$

where ϕ is a complex-valued Borel measurable function on $\mathbb{R}^m$. It is easy to show that for given cylinder functional F of the form (3.1) there exists an orthonormal subset $\{e_1,\ldots,e_n\}$ of $C'_{a,b}[0,T]$ such that F is expressed as

$$(3.2) \qquad F(x) = f((e_1,x)^\sim,\ldots,(e_n,x)^\sim), \quad x \in C_{a,b}[0,T],$$

where f is a complex-valued Borel measurable function on $\mathbb{R}^n$. Thus we lose no generality in assuming that every cylinder functional on $C_{a,b}[0,T]$ is of the form (3.2).

Definition 3.1. Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of complex-valued Borel measures on $\mathcal{B}(\mathbb{R}^n)$. It is well known that a complex-valued Borel measure ν necessarily has a finite total variation $\|\nu\|$, and $\mathcal{M}(\mathbb{R}^n)$ is a Banach algebra under the norm $\|\cdot\|$ and with convolution as multiplication.

For $\nu \in \mathcal{M}(\mathbb{R}^n)$, the Fourier transform $\hat{\nu}$ of ν is a complex-valued function defined on $\mathbb{R}^n$ by the formula

$$(3.3) \qquad \hat{\nu}(\vec{u}) = \int_{\mathbb{R}^n} \exp\left\{i\sum_{j=1}^n u_j v_j\right\} d\nu(\vec{v}),$$

where $\vec{u} = (u_1,\ldots,u_n)$ and $\vec{v} = (v_1,\ldots,v_n)$ are in $\mathbb{R}^n$.

Let $\{e_1,\ldots,e_n\}$ be an orthonormal subset of $C'_{a,b}[0,T]$. Define the functional $F : C_{a,b}[0,T] \to \mathbb{C}$ by

$$(3.4) \qquad F(x) = \hat{\nu}((e_1,x)^\sim,\ldots,(e_n,x)^\sim), \quad x \in C_{a,b}[0,T],$$

where $\hat{\nu}$ is the Fourier transform of ν in $\mathcal{M}(\mathbb{R}^n)$. Then F is a bounded cylinder functional and $|\hat{\nu}(\vec{u})| \leq \|\nu\| < +\infty$. Throughout this paper, we fix the positive integer n.

Next (see [9, 5, 6, 4]) we state the definition of the generalized analytic Feynman integral and the GFFT.

Definition 3.2. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be a scale-invariant measurable functional such that the function space integral $J(\lambda) = E[F(\lambda^{-1/2}\cdot)]$ exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic

function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$E^{\mathrm{an}_\lambda}[F] \equiv E_x^{\mathrm{an}_\lambda}[F(x)] = J^*(\lambda).$$

Let $q \neq 0$ be a real number and let F be a functional such that $E^{\mathrm{an}_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$E^{\mathrm{anf}_q}[F] \equiv E_x^{\mathrm{anf}_q}[F(x)] = \lim_{\lambda \to -iq} E^{\mathrm{an}_\lambda}[F],$$

where $\lambda \to -iq$ through values in $\mathbb{C}_+$.

Definition 3.3. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$T_\lambda(F)(y) = E_x^{\mathrm{an}_\lambda}[F(y+x)].$$

For $p \in (1,2]$, we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, by the formula $(\lambda \in \mathbb{C}_+)$,

$$T_q^{(p)}(F)(y) = \underset{\lambda \to -iq}{\mathrm{l.i.m.}}\, T_\lambda(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left| T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y) \right|^{p'} d\mu(y) = 0,$$

where $1/p + 1/p' = 1$. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, by the formula $(\lambda \in \mathbb{C}_+)$

$$(3.5) \qquad\qquad T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y)$$

if it exists.

We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$. From the definition of the L_1 analytic GFFT, we can see that for $q \in \mathbb{R} - \{0\}$,

$$(3.6) \qquad\qquad E^{\mathrm{anf}_q}[F] = T_q^{(1)}(F)(0)$$

if it exists [5, 9].

Lemma 3.4. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal subset of $C_{a,b}[0,T]$. Then for all $\zeta \in \mathbb{C}_+$, the function space integral*

$$E_x\left[\exp\left\{ i\zeta \sum_{j=1}^n (e_j, x)^\sim v_j \right\} \right]$$

exists and is given by the formula

$$(3.7)\ \ E_x\left[\exp\left\{ i\zeta \sum_{j=1}^n (e_j, x)^\sim v_j \right\} \right] = \exp\left\{ -\frac{\zeta^2}{2} \sum_{j=1}^n v_j^2 + i\zeta \sum_{j=1}^n (e_j, a)_{C'_{a,b}} v_j \right\}.$$

Proof. Using (2.1), the Fubini theorem, and (2.2), it follows that the equation (3.7) holds for all $\zeta \in \mathbb{C}_+$. $\qquad\square$

For convenience we use the following notation throughout this paper:

$$\psi_{\vec{e}}(\lambda; \vec{v}) \equiv \psi_{e_1,\ldots,e_n}(\lambda; v_1, \ldots, v_n)$$

$$(3.8) \qquad = \exp\left\{ -\frac{1}{2\lambda}\sum_{j=1}^{n} v_j^2 + i\lambda^{-1/2}\sum_{j=1}^{n}(e_j, a)_{C'_{a,b}} v_j \right\}$$

for an orthonormal subset $\{e_1, \ldots, e_n\}$ of $C'_{a,b}[0,T]$, $\lambda \in \tilde{\mathbb{C}}_+$ and $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

In next theorem, we show that the analytic function space integral $T_\lambda(F)(y) = E_x^{\mathrm{an}\lambda}[F(y+x)]$ of the functional F given by equation (3.4) exists.

Theorem 3.5. *Let F be given by the equation (3.4). Then for all $\lambda \in \mathbb{C}_+$, $T_\lambda(F)$ exists and is given by the formula*

$$(3.9) \qquad T_\lambda(F)(y) = \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \right\} \psi_{\vec{e}}(\lambda; \vec{v}) d\nu(\vec{v}),$$

where $\psi_{\vec{e}}(\lambda; \vec{v})$ is given by the equation (3.8).

Proof. By (3.4), (3.3), the Fubini theorem, (3.7) with ζ replaced with $\lambda^{-1/2}$, and (3.8), we have that for all $\lambda > 0$,

$$J(\lambda) = E_x[F(y + \lambda^{-1/2} x)] = \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \right\} \psi_{\vec{e}}(\lambda; \vec{v}) d\nu(\vec{v}).$$

Now let

$$J^*(\lambda) = \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \right\} \psi_{\vec{e}}(\lambda; \vec{v}) d\nu(\vec{v})$$

for $\lambda \in \mathbb{C}_+$. Then $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$. We will use the Morera theorem to show that $J^*(\lambda)$ is analytic on $\mathbb{C}_+$. Let $\{\lambda_l\}_{l=1}^{\infty}$ be a sequence in $\mathbb{C}_+$ such that $\lambda_l \to \lambda$ through $\mathbb{C}_+$. Then $\lambda_l^{-1/2} \to \lambda^{-1/2}$ and $\mathrm{Re}(\lambda_l) \neq 0$ for all $l \in \mathbb{N}$. Thus we have that for each $l \in \mathbb{N}$,

$$(3.10) \qquad \left| \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \right\} \psi_{\vec{e}}(\lambda_l; \vec{v}) \right| = \left| \psi_{\vec{e}}(\lambda_l; \vec{v}) \right|$$

$$= \left| \exp\left\{ -\frac{1}{2\lambda_l}\sum_{j=1}^{n} v_j^2 + i\lambda_l^{-1/2}\sum_{j=1}^{n}(e_j, a)_{C'_{a,b}} v_j \right\} \right|$$

$$= \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n}\left(\frac{\sqrt{\mathrm{Re}(\lambda_l)}}{|\lambda_l|} v_j + \frac{|\lambda_l|\mathrm{Im}(\lambda_l^{-1/2})}{\sqrt{\mathrm{Re}(\lambda_l)}}(e_j, a)_{C'_{a,b}} \right)^2 \right.$$

$$\left. + \frac{1}{2}\sum_{j=1}^{n}\frac{|\lambda_l|^2(\mathrm{Im}(\lambda_l^{-1/2}))^2}{\mathrm{Re}(\lambda_l)}(e_j, a)_{C'_{a,b}}^2 \right\}$$

$$\leq \exp\left\{\frac{|\lambda_l|^2(\text{Im}(\lambda_l^{-1/2}))^2}{2\text{Re}(\lambda_l)}\sum_{j=1}^n (e_j,a)^2_{C'_{a,b}}\right\}.$$

Since $\nu \in \mathcal{M}(\mathbb{R}^n)$, we see that

$$\left|\int_{\mathbb{R}^n}\exp\left\{\frac{|\lambda_l|^2(\text{Im}(\lambda_l^{-1/2}))^2}{2\text{Re}(\lambda_l)}\sum_{j=1}^n (e_j,a)^2_{C'_{a,b}}\right\}d\nu(\vec{v})\right|$$

$$\leq \int_{\mathbb{R}^n}\exp\left\{\frac{|\lambda_l|^2(\text{Im}(\lambda_l^{-1/2}))^2}{2\text{Re}(\lambda_l)}\sum_{j=1}^n (e_j,a)^2_{C'_{a,b}}\right\}d|\nu|(\vec{v})$$

$$= \exp\left\{\frac{|\lambda_l|^2(\text{Im}(\lambda_l^{-1/2}))^2}{2\text{Re}(\lambda_l)}\sum_{j=1}^n (e_j,a)^2_{C'_{a,b}}\right\}\|\nu\| < +\infty$$

for each $l \in \mathbb{N}$. Furthermore we have that

$$\lim_{l\to\infty}\int_{\mathbb{R}^n}\exp\left\{\frac{|\lambda_l|^2(\text{Im}(\lambda_l^{-1/2}))^2}{2\text{Re}(\lambda_l)}\sum_{j=1}^n (e_j,a)^2_{C'_{a,b}}\right\}d|\nu|(\vec{v})$$

$$= \int_{\mathbb{R}^n}\exp\left\{\frac{|\lambda|^2(\text{Im}(\lambda^{-1/2}))^2}{2\text{Re}(\lambda)}\sum_{j=1}^n (e_j,a)^2_{C'_{a,b}}\right\}d|\nu|(\vec{v}).$$

Thus, by Theorem 4.17 in [15, p. 92], $J^*(\lambda)$ is continuous on $\mathbb{C}_+$. Since $k(\lambda) \equiv \exp\{i\sum_{j=1}^n (e_j,y)^\sim v_j\}\psi_{\vec{e}}(\lambda;\vec{v})$ is analytic on $\mathbb{C}_+$, applying the Fubini theorem, we have

$$\int_\triangle J^*(\lambda)d\lambda = \int_{\mathbb{R}^n}\int_\triangle k(\lambda)d\lambda d\nu(\vec{v}) = 0$$

for all rectifiable simple closed curve $\triangle$ lying in $\mathbb{C}_+$. Thus by the Morera theorem, $J^*(\lambda)$ is analytic on $\mathbb{C}_+$. Therefore the analytic function space integral $J^*(\lambda) = E_x^{\text{an}_\lambda}[F(y+x)] \equiv T_\lambda(F)(y)$ exists and is given by equation (3.9). $\square$

4. Generalized Fourier-Feynman transform

The remark below will be very useful in the development of our results for the GFFT of functionals given by equation (3.4).

Remark 4.1. If $a(t) \equiv 0$ on $[0,T]$, then for all F given by equation (3.4), the L_1 analytic GFFT $T_q^{(1)}(F)$ will always exist for all real $q \neq 0$ and be given by the formula

$$T_q^{(1)}(F)(y) = \int_{\mathbb{R}^n}\exp\left\{i\sum_{j=1}^n (e_j,y)^\sim v_j\right\}\psi_{\vec{e}}(-iq;\vec{v})d\nu(\vec{v})$$

$$= \int_{\mathbb{R}^n}\exp\left\{i\sum_{j=1}^n (e_j,y)^\sim v_j - \frac{i}{2q}\sum_{j=1}^n v_j^2\right\}d\nu(\vec{v}).$$

However for $a(t)$ as in Section 2, and proceeding formally using equations (3.4) and (3.9), we see that $T_q^{(1)}(F)(y)$ will be given by equation (4.2) below if it

exists. But the integral on the right-hand side of (4.2) might not exist if the real part of

$$\ln \psi_{\vec{e}}(-iq; \vec{v}) = \left\{ -\frac{i}{2q} \sum_{j=1}^{n} v_j^2 + i(-iq)^{-1/2} \sum_{j=1}^{n} (e_j, a)_{C'_{a,b}} v_j \right\}$$

is positive. However, by Cauchy-Schwartz inequality,

$$\left| \psi_{\vec{e}}(-iq; \vec{v}) \right| \le \exp \left\{ \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}} \sum_{j=1}^{n} |v_j| \right\},$$

and so the L_1 GFFT $T_q^{(1)}(F)$ of F will certainly exist provided the associated measure ν of F satisfies the condition

$$(4.1) \qquad \int_{\mathbb{R}^n} \exp \left\{ \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}} \sum_{j=1}^{n} |v_j| \right\} d|\nu|(\vec{v}) < +\infty.$$

Note that in case $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, the function space $C_{a,b}[0, T]$ reduces to the classical Wiener space $C_0[0, T]$ and $(e_j, a)_{C'_{a,b}} = 0$ for all $j = 1, \ldots, n$. Hence for all $\lambda \in \tilde{\mathbb{C}}_+$,

$$\left| \psi_{\vec{e}}(\lambda; \vec{v}) \right| = \left| \exp \left\{ -\frac{1}{2\lambda} \sum_{j=1}^{n} v_j^2 \right\} \right| = \exp \left\{ -\frac{\text{Re}(\lambda)}{2|\lambda|^2} \sum_{j=1}^{n} v_j^2 \right\} \le 1.$$

Theorem 4.2. *Let F be given by the equation (3.4). Let q_0 be a positive real number. Suppose that the associated measure ν of F satisfies condition (4.1) with q replaced with q_0. Then for all real q with $|q| > q_0$, the L_1 GFFT of F, $T_q^{(1)}(F)$ exists and is given by the formula*

$$(4.2) \qquad T_q^{(1)}(F)(y) = \int_{\mathbb{R}^n} \exp \left\{ i \sum_{j=1}^{n} (e_j, y)^{\sim} v_j \right\} \psi_{\vec{e}}(-iq; \vec{v}) d\nu(\vec{v}),$$

where $\psi_{\vec{e}}(-iq; \vec{v})$ is given by the equation (3.8).

Proof. Let $\{\lambda_l\}_{l=1}^{\infty}$ be a sequence in $\mathbb{C}_+$ such that $\lambda_l \to -iq$ through $\mathbb{C}_+$. Then $\psi_{\vec{e}}(\lambda_l; \vec{v})$ converges to $\psi_{\vec{e}}(-iq; \vec{v})$. By Theorem 3.5,

$$T_{\lambda_l}(F)(y) = \int_{\mathbb{R}^n} \exp \left\{ i \sum_{j=1}^{n} (e_j, y)^{\sim} \right\} \psi_{\vec{e}}(\lambda_l; \vec{v}) d\nu(\vec{v})$$

exists for all $l \in \mathbb{N}$. Since $|\arg(\lambda_l^{-1/2})| < \pi/4$ for every $l \in \mathbb{N}$ and $\lambda_l^{-1/2} = \text{Re}(\lambda_l^{-1/2}) + i\text{Im}(\lambda_l^{-1/2}) \to (-iq)^{-1/2} = 1/\sqrt{|2q|} + i\text{sign}(q)/\sqrt{|2q|}$, we see that $\text{Re}(\lambda_l^{-1/2}) > |\text{Im}(\lambda_l^{-1/2})|$ for every $l \in \mathbb{N}$ and there exists a sufficiently large

$k \in \mathbb{N}$ such that $|\text{Im}(\lambda_l^{-1/2})| < 1/\sqrt{|q_0|}$ for every $l \geq k$. Thus for each $l \geq k$,

$$
\begin{aligned}
|\psi_{\vec{e}}(\lambda_l; \vec{v})| = \Bigg| \exp \Bigg\{ &- \frac{1}{2} \Big([\text{Re}(\lambda_l^{-1/2})]^2 - [\text{Im}(\lambda_l^{-1/2})]^2 \\
&+ i\text{Re}(\lambda_l^{-1/2})\text{Im}(\lambda_l^{-1/2}) \Big) \sum_{j=1}^{n} v_j^2 \\
&+ i\Big(\text{Re}(\lambda_l^{-1/2}) + i\text{Im}(\lambda_l^{-1/2}) \Big) \sum_{j=1}^{n} (e_j, a)_{C'_{a,b}} v_j \Bigg\} \Bigg| \\
< \exp \Bigg\{ &\frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q_0|}} \sum_{j=1}^{n} |v_j| \Bigg\}
\end{aligned}
$$

and so, by condition (4.1) with q replaced with q_0,

$$
\begin{aligned}
\big| T_{\lambda_l}(F)(y) \big| &\leq \int_{\mathbb{R}^n} \big| \psi_{\vec{e}}(\lambda_l; \vec{v}) \big| d|\nu|(\vec{v}) \\
&< \int_{\mathbb{R}^n} \exp \Bigg\{ \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q_0|}} \sum_{j=1}^{n} |v_j| \Bigg\} d|\nu|(\vec{v}) < +\infty.
\end{aligned}
$$

Also, by condition (4.1) with q replaced with q_0, we have

$$
\begin{aligned}
\Bigg| \int_{\mathbb{R}^n} \exp \Bigg\{ i \sum_{j=1}^{n} (e_j, y)^\sim v_j \Bigg\} &\psi_{\vec{e}}(-iq; \vec{v}) d\nu(\vec{v}) \Bigg| \\
&\leq \int_{\mathbb{R}^n} \big| \psi_{\vec{e}}(-iq; \vec{v}) \big| d|\nu|(\vec{v}) \\
&< \int_{\mathbb{R}^n} \exp \Bigg\{ \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q_0|}} \sum_{j=1}^{n} |v_j| \Bigg\} d|\nu|(\vec{v}) < +\infty.
\end{aligned}
$$

Hence by (3.5), (3.9) and the dominated convergence theorem, we have the equation (4.2). $\qquad \square$

The following corollary follows from equations (3.6) and (4.2).

Corollary 4.3. *Let F and q_0 be as in Theorem 4.2. Then for all real q with $|q| > q_0$, the generalized analytic Feynman integral of F, $E^{\text{anf}_q}[F]$ exists and is given by the formula*

$$
E^{\text{anf}_q}[F] = \int_{\mathbb{R}^n} \psi_{\vec{e}}(-iq; \vec{v}) d\nu(\vec{v}),
$$

where $\psi_{\vec{e}}(-iq; \vec{v})$ is given by the equation (3.8).

Theorem 4.4. *Let F and q_0 be as in Theorem 4.2. Then for all $p \in (1, 2]$ and all real q with $|q| > q_0$, the L_p GFFT of F, $T_q^{(p)}(F)$ exists and is given by the right hand side of equation (4.2).*

Proof. It was shown in the proof of Theorem 3.5 that $T_\lambda(F)(y)$ given by equation (3.9) is an analytic function of λ throughout $\mathbb{C}_+$. In view of the definition of the L_p analytic GFFT, it suffices to show that for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left| T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y) \right|^{p'} d\mu(y) = 0.$$

Given positive real number $q_0 > 0$, let $\Gamma_{q_0} = \{\lambda \in \tilde{\mathbb{C}}_+ : |\text{Im}(\lambda^{-1/2})| < (2q_0)^{-1/2}\}$. Then for all real q with $|q| > q_0$,

$$(-iq)^{-1/2} = 1/\sqrt{2|q|} + i\text{sign}(q)/\sqrt{2|q|} \in \Gamma_{q_0}$$

and for all $\lambda \in \Gamma_{q_0}$,

(4.3) $$\left| \psi_{\vec{e}}(\lambda; \vec{v}) \right| < \exp\left\{ \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q_0|}} \sum_{j=1}^n |v_j| \right\}.$$

Fixing $p \in (1, 2]$ and using inequalities (4.3) and (4.1), we obtain that for all $\rho > 0$ and all $\lambda \in \Gamma_{q_0}$,

$$\left| T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y) \right|^{p'}$$

$$\leq \left(\int_{\mathbb{R}^n} \left| \exp\left\{ i\rho \sum_{j=1}^n (e_j, y)^\sim \right\} \right| \left\{ \left| \psi_{\vec{e}}(\lambda; \vec{v}) \right| + \left| \psi_{\vec{e}}(-iq; \vec{v}) \right| \right\} d\nu(\vec{v}) \right)^{p'}$$

$$\leq \left(2 \int_{\mathbb{R}^n} \exp\left\{ \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q_0|}} \sum_{j=1}^n |v_j| \right\} d|\nu|(\vec{v}) \right)^{p'} < +\infty.$$

Hence by the dominated convergence theorem, we see that for all $p \in (1, 2]$ and all $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left| T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y) \right|^{p'} d\mu(y)$$

$$= \int_{C_{a,b}[0,T]} \left| \int_{\mathbb{R}^n} \exp\left\{ i\rho \sum_{j=1}^n (e_j, y)^\sim \right\} \lim_{\lambda \to -iq} \left\{ \psi_{\vec{e}}(\lambda; \vec{v}) - \psi_{\vec{e}}(-iq; \vec{v}) \right\} d\nu(\vec{v}) \right|^{p'} d\mu(y)$$

$$= 0$$

which concludes the proof of Theorem 4.4. $\qquad \square$

5. Sequential function space transforms

In this section, we introduce two (non-analytic) sequential transforms of cylinder functionals discussed in Sections 3 and 4. We establish that the one is to identify the GFFT and the another transform acts like an inverse GFFT.

For an orthonormal subset $\{e_1, \ldots, e_n\}$ of $C'_{a,b}[0,T]$, $\lambda \in \mathbb{C}_+$ and $x \in C_{a,b}[0,T]$, let

$$
(5.1) \quad
\begin{aligned}
W_{\vec{e}}^+(\lambda; x) &\equiv \lambda^{n/2} \exp \left\{ \left(\frac{1-\lambda}{2} \right) \sum_{j=1}^n [(e_j, x)^\sim]^2 \right. \\
&\qquad \left. + (\lambda^{1/2} - 1) \sum_{j=1}^n (e_j, a)_{C'_{a,b}} (e_j, x)^\sim \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
W_{\vec{e}}^-(\lambda; x) &\equiv \lambda^{n/2} \exp \left\{ \left(\frac{1-\lambda}{2} \right) \sum_{j=1}^n [(e_j, x)^\sim]^2 \right. \\
&\quad \left. + ((-\lambda)^{1/2} - 1) \sum_{j=1}^n (e_j, a)_{C'_{a,b}} (e_j, x)^\sim + \sum_{j=1}^n (e_j, a)^2_{C'_{a,b}} \right\}.
\end{aligned}
$$

Definition 5.1. Let F and $\{e_1, \ldots, e_n\}$ be related by the equation (3.2) and let q be a nonzero real number. We define the sequential $\mathcal{P}$-transform of F with parameter q (if it exists) by

$$
(5.2) \qquad \mathcal{P}_{\vec{e},q}(F)(y) = \lim_{l \to \infty} \int_{C_{a,b}[0,T]} F(y+x) W_{\vec{e}}^+(\lambda_l; x) d\mu(x),
$$

where $\{\lambda_l\}_{l=1}^\infty$ is any sequence in $\mathbb{C}_+$ with $\lambda_l \to -iq$.

We also define the sequential $\mathcal{N}$-transform of F with parameter q (if it exists) by

$$
\mathcal{N}_{\vec{e},q}(F)(y) = \lim_{l \to \infty} \int_{C_{a,b}[0,T]} F(y+x) W_{\vec{e}}^-(\lambda_l; x) d\mu(x),
$$

where $\{\lambda_l\}_{l=1}^\infty$ is any sequence in $\mathbb{C}_+$ with $\lambda_l \to -iq$.

Remark 5.2. Let $\{e_1, \ldots, e_n\}$ be an orthonormal subset $C'_{a,b}[0,T]$. Then applying equations (2.1) and (2.2), we can see that for all $\lambda \in \mathbb{C}_+$,

$$
\int_{C_{a,b}[0,T]} W_{\vec{e}}^+(\lambda; x) d\mu(x) = \int_{C_{a,b}[0,T]} W_{\vec{e}}^-(\lambda; x) d\mu(x) = 1.
$$

In our next theorems we establish the existence of the sequential $\mathcal{P}$ and $\mathcal{N}$-transforms of functional F given by equation (3.4).

Theorem 5.3. *Let F and $\{e_1, \ldots, e_n\}$ be related by the equation (3.4). Let q_0 be a positive real number. Suppose that the associated measure ν of F satisfies condition (4.1) with q replaced with q_0. Then for all real q with $|q| > q_0$, the sequential $\mathcal{P}$-transform of F, $\mathcal{P}_{\vec{e},q}(F)$ exists and is given by the right hand side of equation (4.2).*

Proof. Let $\{\lambda_l\}_{l=1}^{\infty}$ be any sequence in $\mathbb{C}_+$ with $\lambda_l \to -iq$. By (3.4), (5.1), the Fubini theorem, (2.1) and (2.2), we have that for every $l \in \mathbb{N}$,

(5.3)

$$
\int_{C_{a,b}[0,T]} F(y+x)W_{\vec{e}}^{+}(\lambda_l; x)d\mu(x)
$$

$$
= \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \right\} \left(\frac{\lambda_l}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} \exp\left\{ -\frac{\lambda_l}{2}\sum_{j=1}^{n} u_j^2 \right.
$$

$$
\left. + \sum_{j=1}^{n}\left(\lambda_l^{1/2}(e_j, a)_{C'_{a,b}} + iv_j\right)u_j - \frac{1}{2}\sum_{j=1}^{n}(e_j, a)_{C'_{a,b}}^{2} \right\} d\vec{u}\, d\nu(\vec{v})
$$

$$
= \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j - \frac{1}{2\lambda_l}\sum_{j=1}^{n} v_j^2 + i\lambda_l^{-1/2}\sum_{j=1}^{n}(e_j, a)_{C'_{a,b}} v_j \right\} d\nu(\vec{v})
$$

$$
= \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \right\} \psi_{\vec{e}}(\lambda_l; \vec{v}) d\nu(\vec{v}).
$$

Applying (3.10), we see that the absolute value of the last expression of (5.3) is less than

$$
\exp\left\{ \frac{|\lambda_l|^2 \left(\text{Im}(\lambda_l^{-1/2})\right)^2}{2\text{Re}(\lambda_l)} \sum_{j=1}^{n}(e_j, a)_{C'_{a,b}}^{2} \right\} \|\nu\|,
$$

and so the function space integral given by (5.3) exists for each $l \in \mathbb{N}$. Using equations (5.2), (5.3) and the dominated convergence theorem with the techniques similar to those used in the proof of Theorem 4.2, we have the desired result. $\qquad\square$

By similar methods, we obtain the following theorem.

Theorem 5.4. *Let F, $\{e_1, \ldots, e_n\}$ and q_0 be as in Theorem 5.3. Then for all real q with $|q| > q_0$, the sequential $\mathcal{N}$-transform of F, $\mathcal{N}_{\vec{e},q}(F)$ exists and is given by the formula*

(5.4)
$$
\mathcal{N}_{\vec{e},q}(F)(y) = \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j - \frac{i}{2q}\sum_{j=1}^{n} v_j^2 \right.
$$
$$
\left. - i(iq)^{-1/2}\sum_{j=1}^{n}(e_j, a)_{C'_{a,b}} v_j \right\} d\nu(\vec{v}).
$$

Next we present concluding remarks.

(1) Let F, $\{e_1, \ldots, e_n\}$ and q_0 be as in Theorem 5.3. Then by Theorems 4.2 and 4.4, we have that for all real q with $|q| > q_0$, the GFFT $T_q^{(p)}(F)$ of F is given by the right hand side of equation (4.2), and by Theorem 5.3, we have

(5.5)
$$
T_q^{(p)}(F)(y) = \mathcal{P}_{\vec{e},q}(F)(y).
$$

For nonzero real number q with $|q| > q_0$, define a set function $\nu_q : \mathcal{B}(\mathbb{R}^n) \to \mathbb{C}$ by

$$\nu_q(B) = \int_{\mathbb{R}^n} \psi_{\vec{e}}(-iq; \vec{v}) d\nu(\vec{v}), \quad B \in \mathcal{B}(\mathbb{R}^n),$$

where ν is the associated measure of F and $\psi_{\vec{e}}(-iq; \vec{v})$ is given by the equation (3.8). Then using condition (4.1) with q replaced with q_0, we see that ν_q is a complex measure on $\mathcal{B}(\mathbb{R}^n)$ with finite total variation, and $T_q^{(p)}(F)$ and $\mathcal{P}_{\vec{e},q}(F)$ can be expressed as

$$T_q^{(p)}(F)(y) = \mathcal{P}_{\vec{e},q}(F)(y) = \int_{\mathbb{R}^n} \exp\left\{i \sum_{j=1}^n (e_j, y)^\sim v_j\right\} d\nu_q(\vec{v})$$

for all $p \in [1, 2]$.

(2) For positive real number q and complex measure ν_q discussed above, we assume

(5.6)
$$\int_{\mathbb{R}^n} \exp\left\{\frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}} \sum_{j=1}^n |v_j|\right\} d|\nu_q|(\vec{v})$$
$$\leq \int_{\mathbb{R}^n} \exp\left\{\frac{2\|a\|_{C'_{a,b}}}{\sqrt{|2q|}} \sum_{j=1}^n |v_j|\right\} d|\nu|(\vec{v}) < +\infty.$$

Then by equations (5.4) and (5.5), we have that for all $p \in [1, 2]$ and all nonzero real number q,

(5.7)
$$\mathcal{N}_{\vec{e},-q}(T_q^{(p)}(F))(y) = \mathcal{N}_{\vec{e},-q}(\mathcal{P}_{\vec{e},q}(F))(y)$$
$$= \int_{\mathbb{R}^n} \exp\left\{i \sum_{j=1}^n (e_j, y)^\sim v_j + \frac{i}{2q} \sum_{j=1}^n v_j^2\right.$$
$$\left. - i(-iq)^{-1/2} \sum_{j=1}^n (e_j, a)_{C'_{a,b}} v_j\right\} d\nu_q(\vec{v})$$
$$= \int_{\mathbb{R}^n} \exp\left\{i \sum_{j=1}^n (e_j, y)^\sim v_j + \frac{i}{2q} \sum_{j=1}^n v_j^2\right.$$
$$\left. - i(-iq)^{-1/2} \sum_{j=1}^n (e_j, a)_{C'_{a,b}} v_j\right\} \psi_{\vec{e}}(-iq; \vec{v}) d\nu_q(\vec{v})$$
$$= F(y).$$

By similar arguments we can see that $T_{-q}^{(p)}(\mathcal{N}_{\vec{e},q}(F))(y) = F(y)$. That is to say, the equation (5.7) tells us that sequential transform $\mathcal{N}_{\vec{e},-q}$ plays a prominent role as an inverse transform of GFFT $T_q^{(p)}$ for functionals F given by equation (3.4).

6. Examples

In this section we present several interesting functionals to apply our results in previous sections.

Example 6.1. For given $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ and $\vec{\sigma^2} = (\sigma_1^2, \ldots, \sigma_n^2) \in \mathbb{R}^n$ with $\sigma_j^2 > 0$, $j = 1, \ldots, n$, let $\nu_{\vec{m},\vec{\sigma^2}}$ be the Gaussian measure given by

(6.1)
$$\nu_{\vec{m},\vec{\sigma^2}}(B) = \left(\prod_{j=1}^{n} 2\pi\sigma_j^2 \right)^{-1/2} \int_B \exp\left\{ -\sum_{j=1}^{n} \frac{(u_j - m_j)^2}{2\sigma_j^2} \right\} d\vec{u}, \quad B \in \mathcal{B}(\mathbb{R}^n).$$

Then $\nu_{\vec{m},\vec{\sigma^2}} \in \mathcal{M}(\mathbb{R}^n)$ and

$$\widehat{\nu_{\vec{m},\vec{\sigma^2}}}(\vec{u}) = \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \sigma_j^2 u_j^2 + i\sum_{j=1}^{n} m_j u_j \right\}.$$

Using the equation (6.1), the Fubini theorem and equation (2.2), we have that for all nonzero real number q,

$$\int_{\mathbb{R}^n} \exp\left\{ \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}} \sum_{j=1}^{n} |v_j| \right\} d|\nu_{\vec{m},\vec{\sigma^2}}|(\vec{v})$$

$$< \prod_{j=1}^{n} \left[\exp\left\{ \frac{\sigma_j^2\|a\|_{C'_{a,b}}^2}{2|2q|} - \frac{m_j\|a\|_{C'_{a,b}}}{\sqrt{|2q|}} \right\} + \exp\left\{ \frac{\sigma_j^2\|a\|_{C'_{a,b}}^2}{2|2q|} + \frac{m_j\|a\|_{C'_{a,b}}}{\sqrt{|2q|}} \right\} \right]$$

and the Gaussian measure $\nu_{\vec{m},\vec{\sigma^2}}$ satisfies conditions (4.1) and (5.6) for all $q \in \mathbb{R} - \{0\}$. Thus we can apply our results in previous sections to the functionals of the form

(6.2)
$$F_1(x) = \widehat{\nu_{\vec{m},\vec{\sigma^2}}}((e_1, x)^\sim, \ldots, (e_n, x)^\sim)$$
$$= \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \sigma_j^2[(e_j, x)^\sim]^2 + i\sum_{j=1}^{n} m_j(e_j, x)^\sim \right\},$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal set in $C'_{a,b}[0, T]$.

By Theorems 4.2, 4.4, 5.3 and 5.4, for all $q \in \mathbb{R} - \{0\}$ and all $p \in [1, 2]$, the L_p GFFT $T_q^{(p)}(F_1)$, the sequential $\mathcal{P}$-transform $\mathcal{P}_{\vec{e},q}(F_1)$ and the sequential $\mathcal{N}$-transform $\mathcal{N}_{\vec{e},q}(F_1)$ exist and are given by

$$T_q^{(p)}(F_1)(y) = \mathcal{P}_{\vec{e},q}(F_1)$$
$$= \left(\prod_{j=1}^{n} 2\pi\sigma_j^2 \right)^{-1/2} \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^\sim v_j - \frac{i}{2q}\sum_{j=1}^{n} v_j^2 \right.$$
$$\left. + i(-iq)^{-1/2}\sum_{j=1}^{n}(e_j, a)_{C'_{a,b}} v_j - \sum_{j=1}^{n} \frac{(v_j - m_j)^2}{2\sigma_j^2} \right\} d\vec{v}$$

$$= \Big(\prod_{j=1}^{n} \frac{q}{q + i\sigma_j^2} \Big)^{1/2} \exp \Big\{ \sum_{j=1}^{n} \frac{q\sigma_j^2}{2(q + i\sigma_j^2)} \Big(i(e_j, y)^{\sim}$$

$$+ i(-iq)^{-1/2}(e_j, a)_{C'_{a,b}} + \frac{m_j}{\sigma_j^2} \Big)^2 - \frac{m^2}{2\sigma_j^2} \Big\}$$

and

$$\mathcal{N}_{\vec{e},q}(F_1) = \Big(\prod_{j=1}^{n} \frac{q}{q + i\sigma_j^2} \Big)^{1/2} \exp \Big\{ \sum_{j=1}^{n} \frac{q\sigma_j^2}{2(q + i\sigma_j^2)} \Big(i(e_j, y)^{\sim}$$

$$- i(iq)^{-1/2}(e_j, a)_{C'_{a,b}} + \frac{m_j}{\sigma_j^2} \Big)^2 - \frac{m^2}{2\sigma_j^2} \Big\},$$

respectively. Furthermore,

$$\mathcal{N}_{\vec{e},-q}(T_q^{(p)}(F_1))(y) = \mathcal{N}_{\vec{e},-q}(\mathcal{P}_{\vec{e},q}(F_1))(y)$$

$$= T_{-q}^{(p)}(\mathcal{N}_{\vec{e},q}(F_1))(y) = \mathcal{P}_{\vec{e},-q}(\mathcal{N}_{\vec{e},q}(F_1))(y) = F_1(y)$$

for all $q \in \mathbb{R} - \{0\}$.

Example 6.2. Let $S : C'_{a,b}[0,T] \to C'_{a,b}[0,T]$ be the linear operator defined by $Sw(t) = \int_0^t w(s)db(s)$. Then the adjoint operator S^* of S is given by

$$S^* w(t) = \int_0^t \big(w(T) - w(s) \big) db(s).$$

It is easily shown that S is injective. Let

(6.3) $\gamma(t) = \sqrt{3} b^{-3/2}(T) b(t).$

Using an integration by parts formula, we see that $\{S^*\gamma\}$ is an orthonormal set in $C'_{a,b}[0,T]$ and

$$\frac{1}{\sqrt{3}} b^{3/2}(T)(S^*\gamma, x)^{\sim} = (S^*b, x)^{\sim} = \int_0^T x(t)db(t).$$

In the case $n = 1$, $e_1 = S^*\gamma$, $\vec{m} = m_1 = 0$ and $\vec{\sigma}^2 = \sigma_1^2 = 2b^3(T)/3$, the functional F_1 given by equation (6.2) can be rewritten by

(6.4) $F_2(x) = \exp \Big\{ - \Big(\int_0^T x(t)db(t) \Big)^2 \Big\}.$

We give the transforms studied in this paper for functional F_2.

$$T_q^{(p)}(F_2)(y)$$

$$= \mathcal{P}_{\vec{S}^*\gamma,q}(F_2)$$

$$= \Big(\frac{3q}{3q + i2b^3(T)} \Big)^{1/2} \exp \Big\{ \frac{3q}{3q + i2b^3(T)}$$

$$\times \left(i \int_0^T y(t)db(t) + i(-iq)^{-1/2} \int_0^T a(t)db(t) \right)^2 \Bigg\}$$

and

$$\mathcal{N}_{\bar{S}^*\gamma,q}(F_2) = \left(\frac{3q}{3q + i2b^3(T)} \right)^{1/2} \exp\left\{ \frac{3q}{3q + i2b^3(T)} \right.$$
$$\left. \times \left(i \int_0^T y(t)db(t) - i(iq)^{-1/2} \int_0^T a(t)db(t) \right)^2 \right\}.$$

Example 6.3. The functional

$$(6.5) \qquad\qquad F_3(x) = \exp\left\{ i \int_0^T x(t)db(s) \right\}$$

also is a functional under our consideration because

$$F_3(x) = \exp\{i(S^*b, x)^\sim\} = \exp\left\{ \frac{i}{\sqrt{3}} b^{3/2}(T)(S^*\gamma, x)^\sim \right\}$$
$$= \int_{\mathbb{R}} \exp\{i(S^*\gamma, x)^\sim v\} d\delta_1(v) = \widehat{\delta_1}((S^*\gamma, x)^\sim),$$

where γ is given by equation (6.3) and δ_1 is the Dirac measure concentrated at $v = b^{3/2}(T)/\sqrt{3}$ in $\mathbb{R}$. Clearly, δ_1 satisfies condition (4.1) and (5.6) for all $q \in \mathbb{R} - \{0\}$.

We finally present the two transforms of F_3.

$$T_q^{(p)}(F_3)(y) = \mathcal{P}_{S^*\gamma,q}(F_3)$$
$$= \exp\left\{ i \int_0^T y(t)db(t) - \frac{i}{6q}b^3(T) + i(-iq)^{1/2} \int_0^T a(t)db(t) \right\}$$

and

$$\mathcal{N}_{S^*\gamma,q}(F_3) = \exp\left\{ i \int_0^T y(t)db(t) - \frac{i}{6q}b^3(T) - i(iq)^{1/2} \int_0^T a(t)db(t) \right\}.$$

Remark 6.4. The functionals given by equations (6.4) and (6.5) arise naturally in quantum mechanics.

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