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Research Article

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Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function

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Abstract: The primary objective of this research is to establish the generalized fractional integral inequalities of Hermite-Hadamard-type for MT-convex functions and to explore some new Hermite-Hadamard-type inequalities in a form of Riemann-Liouville fractional integrals as well as classical integrals. It is worth mentioning that our work generalizes and extends the results appeared in the literature.

Keywords: Riemann-Liouville fractional integral, MT-convex function, integral inequalities

MSC 2010: 26D07, 26D10, 26D15, 26A33

1 Introduction

Integral inequality plays a critical role in both fields of pure and applied mathematics; see e.g. [1–14]. It is clear that mathematical methods are useless without inequalities. For this reason, there is a present-day need for accurate inequalities in proving the existence and uniqueness of the mathematical methods. Besides, convexity plays a strong role in the field of inequalities due to the behaviour of its definition.

Let $\mathcal{I} \subseteq \mathbb{R}$ and denote $\mathcal{I}^{\circ}$ by the interior of $\mathcal{I}$. A function $g : \mathcal{I} \to \mathbb{R}$ is said to be convex on $\mathcal{I}$, if the inequality

$$g(\zeta a + (1 - \zeta)b) \le \zeta g(a) + (1 - \zeta)g(b) \tag{1}$$

holds for all $a, b \in \mathcal{I}$ and $\zeta \in [0, 1]$. We say that g is concave, if $-g$ is convex.

Recently, a great number of equalities or inequalities for convex functions have been established by many authors. The representative results include Ostrowski-type inequality [15], Hardy-type inequality [16], Olsen-type inequality [17], Gagliardo-Nirenberg-type inequality [18], and the most well-known inequality of, namely, the Hermite-Hadamard-type inequality [19]. Here, we focus on it, which is formulated for a convex function $g : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ by:

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_{a}^{b} g(\zeta)\mathrm{d}\zeta \le \frac{g(a)+g(b)}{2}, \tag{2}$$

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with $a < b$ and $a, b \in \mathcal{I}$, which can be a significant tool to obtain various priori estimates. Because of its importance, a number of scholars in the field of pure and applied mathematics have dedicated their efforts to extend, generalize, counterpart, and refine the Hermite-Hadamard inequality (2) for different classes of convex functions and mappings, see [20–28].

Definition 1.1. [29] A function $g : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be MT-convex on $\mathcal{I}$, if it is nonnegative and satisfies the following inequality:

$$g(\zeta a + (1-\zeta)b) \le \frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}} g(a) + \frac{\sqrt{1-\zeta}}{2\sqrt{\zeta}} g(b) \tag{3}$$

for all $a, b \in \mathcal{I}$ and $\zeta \in (0, 1)$.

By virtue of the concept of *MT*-convexity, Park in [29] proved the following Hermite-Hadamard-type inequalities.

Lemma 1.1. [29, Lemma 2] *Let $g : \mathcal{I} \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$, where $a, b \in \mathcal{I}$ is such that $a < b$. If $g' \in L^1[a, b]$, then for all $h \in (0, 1)$ and $\mu > 0$ we have*

$$\begin{aligned} &\frac{(1-h)^\mu + h^\mu}{(b-a)^{\mu-1}} g(w) - \frac{\Gamma(\mu+1)}{(b-a)} [J_{w^-}^\mu g(a) + J_{w^+}^\mu g(b)] \\ &= (b-a)^\mu \left\{ (1-h)^{\mu+1} \int_0^1 \zeta^\mu g'(\zeta w + (1-\zeta)a) \mathrm{d}\zeta - h^{\mu+1} \int_0^1 \zeta^\mu g'(\zeta w + (1-\zeta)b) \mathrm{d}\zeta \right\}, \end{aligned} \tag{4}$$

where $w = ha + (1-h)b$ and

$$\begin{aligned} J_{w^-}^\mu g(a) &= \frac{1}{\Gamma(\mu)} \int_a^w (t-a)^{\mu-1} g(t) \mathrm{d}t, \\ J_{w^+}^\mu g(b) &= \frac{1}{\Gamma(\mu)} \int_w^b (b-t)^{\mu-1} g(t) \mathrm{d}t. \end{aligned}$$

Theorem 1.1. [29, Theorem 3.1] *Let $g : \mathcal{I} \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ such that $g' \in L^1[a, b]$, where $a, b \in \mathcal{I}$ is such that $a < b$. If $|g'|$ is an MT-convex function on $[a, b]$ such that $|g'(x)| \le M$ for each $x \in [a, b]$ with some $M > 0$. Then, for any $h \in (0, 1)$ and $\mu > 0$, the following inequality holds*

$$\left| \frac{\{(1-h)^\mu + h^\mu\}}{(b-a)^{1-\mu}} g(w) - \frac{\Gamma(\mu+1)}{b-a} [J_{w^-}^\mu g(a) + J_{w^+}^\mu g(b)] \right| \le \frac{(b-a)^\mu}{2} \beta\left(\frac{1}{2}, \mu + \frac{1}{2}\right) M, \tag{5}$$

where β stands for the beta function of Euler type defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{for all } x, y > 0.$$

Theorem 1.2. [29, Theorem 3.2] *Let $g : \mathcal{I} \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ such that $g' \in L^1[a, b]$, where $a, b \in \mathcal{I}$ is such that $a < b$. If $|g'|^q$ is an MT-convex function on $[a, b]$ with $q > 1$, then we have*

$$\left| \frac{\{(1-h)^\mu + h^\mu\}}{(b-a)^{1-\mu}} g(w) - \frac{\Gamma(\mu+1)}{(b-a)} [J_{w^-}^\mu g(a) + J_{w^+}^\mu g(b)] \right| \le \left(\frac{1}{\mu p + 1}\right)^{\frac{1}{p}} \left(\frac{\pi}{2}\right)^{\frac{1}{q}} M(b-a)^\mu, \tag{6}$$

where $p > 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.3. [29, Theorem 3.3] *Let $g : \mathcal{I} \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ such that $g' \in L^1[a, b]$, where $a, b \in \mathcal{I}$ is such that $a < b$. Assume that $|g'|^q$ is an MT-convex function on $[a, b]$ with $q > 1$ such that $|g'(t)| \leq M$ for all $t \in [a, b]$ with some $M > 0$. Then, for any $h \in (0,1)$ and $\mu > 0$, the following inequality holds:*

$$\left| \frac{\{(1-h)^\mu + h^\mu\}}{(b-a)^{1-\mu}} g(w) - \frac{\Gamma(\mu+1)}{(b-a)} [J_{w^-}^\mu g(a) + J_{w^+}^\mu g(b)] \right| \leq \left(\frac{1}{\mu+1}\right)^{1-\frac{1}{q}} \left(\frac{\beta\left(\frac{1}{2}, \mu + \frac{1}{2}\right)}{2}\right)^{\frac{1}{q}} M(b-a)^\mu. \tag{7}$$

For more details and interesting applications on Hermite-Hadamard-type inequalities for *MT*-convex functions, the reader is welcome to consult [30–32] and references therein.

Furthermore, we recall the definition of generalized fractional integral operators by Sarikaya and Ertugral [33]. Let $\rho : [0, \infty) \to [0, \infty)$ be such that

$$\int_0^1 \frac{\rho(\zeta)}{\zeta} \mathrm{d}\zeta < \infty.$$

The left- and right-sided generalized fractional integral operators are defined as follows:

$$_{a^+}I_\rho g(x) = \int_a^x \frac{\rho(x-\zeta)}{x-\zeta} g(\zeta)\mathrm{d}\zeta, \quad x > a, \tag{8}$$

$$_{b^-}I_\rho g(x) = \int_x^b \frac{\rho(\zeta-x)}{\zeta-x} g(\zeta)\mathrm{d}\zeta, \quad x < b, \tag{9}$$

respectively.

In order to highlight the generality of the fractional integrals under consideration, we have to mention that in the case:

(i) If $\rho(\zeta) = \zeta$, then (8) and (9) reduce to the usual Riemann integral

$$I_{a^+}g(x) = \int_a^x g(\zeta)\mathrm{d}\zeta, \quad x > a,$$
$$I_{b^-}g(x) = \int_x^b g(\zeta)\mathrm{d}\zeta, \quad x < b.$$

(ii) When ρ is specialized by $\rho(\zeta) = \frac{\zeta^\mu}{\Gamma(\mu)}$, then (8) and (9) convert to the Riemann-Liouville fractional integral [34,35]

$$I_{a^+}^\mu g(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-\zeta)^{\mu-1} g(\zeta)\mathrm{d}\zeta, \quad x > a,$$
$$I_{b^-}^\mu g(x) = \frac{1}{\Gamma(\mu)} \int_x^b (\zeta-x)^{\mu-1} g(\zeta)\mathrm{d}\zeta, \quad x < b.$$

(iii) While ρ is formulated by $\rho(\zeta) = \frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then (8) and (9) convert to the k-Riemann-Liouville fractional integral [36]

$$I_{a^+,k}^\mu g(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-\zeta)^{\frac{\mu}{k}-1} g(\zeta)\mathrm{d}\zeta, \quad x > a,$$
$$I_{b^-,k}^\mu g(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (\zeta-x)^{\frac{\mu}{k}-1} g(\zeta)\mathrm{d}\zeta, \quad x < b,$$

where

$$\Gamma_k(\mu) = \int_0^\infty \zeta^{\mu-1} e^{-\frac{\zeta^k}{k}} \mathrm{d}\zeta, \quad \mathbb{R}(\mu) > 0$$

and

$$\Gamma_k(\mu) = k^{\frac{\mu}{k}-1}\Gamma\left(\frac{\mu}{k}\right), \quad \mathbb{R}(\mu) > 0;\; k > 0.$$

For other special cases, the reader can consult [1,37–39].

More recently, Qi et al. [40] established some inequalities of Hermite-Hadamard-type for (α, m)-convex functions by using generalized fractional integral operators (8) and (9). However, the main objective of this article is to explore several new and generalized fractional integral inequalities of Hermite-Hadamard-type for *MT*-convex functions involving generalized fractional integral operators (8) and (9).

2 Main results

At the beginning of the section, we first deliver an identity, which will play a significant role in the proof of our main results.

Lemma 2.1. *Let $g : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ such that $g \in L^1[a, b]$. Then, for each $h \in (0, 1)$ we have*

$$\begin{aligned}&\frac{(1-h)^\mu \Omega(1) + h^\mu \nabla(1)}{b-a} g(w) - \frac{1}{b-a}[(1-h)^\mu ({}_{w^-}I_\rho g(a)) + h^\mu ({}_{w^+}I_\rho g(b))]\\ &= (1-h)^{\mu+1} \int_0^1 \Omega(\zeta) g'(\zeta w + (1-\zeta)a)\mathrm{d}\zeta - h^{\mu+1} \int_0^1 \nabla(\zeta) g'(\zeta w + (1-\zeta)b)\mathrm{d}\zeta,\end{aligned} \tag{10}$$

where $w = ha + (1-h)b$, and $\Omega(\zeta)$ and $\nabla(\zeta)$ are defined by

$$\Omega(\zeta) = \int_0^\zeta \frac{\rho((w-a)u)}{u}\mathrm{d}u < \infty, \quad \nabla(\zeta) = \int_0^\zeta \frac{\rho((b-w)u)}{u}\mathrm{d}u < \infty. \tag{11}$$

Proof. Integrating by parts and then using the equality $x = \zeta w + (1-\zeta)a$, it yields

$$\begin{aligned} I_1 &= (1-h)^{\mu+1} \int_0^1 \Omega(\zeta) g'(\zeta w + (1-\zeta)a)\mathrm{d}\zeta \\ &= \frac{(1-h)^{\mu+1}}{w-a}\left[\Omega(\zeta) g(\zeta w + (1-\zeta)a)|_0^1 - \int_0^1 \frac{\rho((w-a)\zeta)}{\zeta} g(\zeta w + (1-\zeta)a)\mathrm{d}\zeta\right] \\ &= \frac{(1-h)^{\mu+1}}{w-a}\left[\Omega(1) g(w) - \int_a^w \frac{\rho(x-a)}{x-a} g(x)\mathrm{d}x\right] \\ &= \frac{(1-h)^\mu}{b-a}[\Omega(1) g(w) - {}_{w^-}I_\rho g(a)]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &= -h^{\mu+1}\int_0^1 \nabla(\zeta)g'(\zeta w + (1-\zeta)b)\mathrm{d}\zeta \\ &= \frac{-h^{\mu+1}}{w-b}\left[\nabla(\zeta)g(\zeta w + (1-\zeta)b)|_0^1 - \int_0^1 \frac{\rho((b-w)\zeta)}{\zeta}g(\zeta w + (1-\zeta)b)\mathrm{d}\zeta\right] \\ &= \frac{h^{\mu+1}}{b-w}\left[\nabla(1)g(w) - \int_w^b \frac{\rho(b-x)}{b-x}g(x)\mathrm{d}x\right] \\ &= \frac{h^{\mu}}{b-a}[\nabla(1)g(w) - {}_{w^+}I_{\rho}g(b)]. \end{aligned}$$

Adding I_1 and I_2, it easily implies identity (10). ☐

Remark 2.1. Indeed, it is not difficult to see that (10) can be rewritten equivalently to the equality

$$\begin{aligned} &\frac{(1-h)^{\mu}\Omega(1) + h^{\mu}\nabla(1)}{b-a}g(w) - \frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a)) + h^{\mu}({}_{w^+}I_{\rho}g(b))] \\ &= (1-h)^{\mu+1}\int_0^1 \Omega(\zeta)g'(\zeta w + (1-\zeta)a)\mathrm{d}\zeta + h^{\mu+1}\int_0^1 \nabla(1-\zeta)g'(\zeta b + (1-\zeta)w)\mathrm{d}\zeta. \end{aligned}$$

In particular, if $\rho(\zeta) = \zeta$, then we have the following corollary, which has been proved by Park [29].

Corollary 2.1. *Under the assumptions of* Lemma 2.1, *the identity holds*

$$g(w) - \frac{1}{b-a}\int_a^b g(u)\mathrm{d}u = (b-a)\left[(1-h)^2\int_0^1 \zeta g'(\zeta w + (1-\zeta)a)\mathrm{d}\zeta + h^2\int_0^1 (\zeta-1)g'(\zeta w + (1-\zeta)b)\mathrm{d}\zeta\right].$$

Proof. The desired conclusion is a direct consequence of Remark 2.1 by taking $\rho(\zeta) = \zeta$. ☐

Remark 2.2. Under the assumptions of Lemma 2.1, we could observe that identity (4) is valid, when $\rho(\zeta) = \frac{\zeta^{\mu}}{\Gamma(\mu)}$.

Additionally, if $\rho(\zeta) = \frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then we have the following result for k-fractional integral.

Corollary 2.2. *Under the assumptions of* Lemma 2.1, *if* $\rho(\zeta) = \frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, *then the identity is available*

$$\begin{aligned} &\frac{\left\{(1-h)^{\frac{\mu}{k}} + h^{\frac{\mu}{k}}\right\}}{(b-a)^{\frac{\mu}{k}-1}}g(w) - \frac{\Gamma_k(\mu+k)}{(b-a)}[I^{\mu}_{w^-,k}g(a) + I^{\mu}_{w^+,k}g(b)] \\ &= (b-a)^{\frac{\mu}{k}}\left\{(1-h)^{\frac{\mu}{k}+1}\int_0^1 \zeta^{\frac{\mu}{k}}g'(\zeta w + (1-\zeta)a)\mathrm{d}\zeta - h^{\frac{\mu}{k}+1}\int_0^1 \zeta^{\frac{\mu}{k}}g'(\zeta w + (1-\zeta)b)\mathrm{d}\zeta\right\}. \end{aligned}$$

Theorem 2.1. *Let* $g : [a, b] \to \mathbb{R}$ *be a differentiable function on* (a, b) *and* $g' \in L^1[a, b]$ *with* $0 \le a < b$ *and* $\mu > 0$. *If* $|g'|$ *is MT-convex on* $[a, b]$, *then for every* $h \in (0, 1)$ *we have*

$$\begin{aligned} &\left|\frac{(1-h)^{\mu}\Omega(1) + h^{\mu}\nabla(1)}{b-a}g(w) - \frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a)) + h^{\mu}({}_{w^+}I_{\rho}g(b))]\right| \\ &\le \frac{(1-h)^{\mu+1}}{2}[A_1|g'(w)| + B_1|g'(a)|] + \frac{h^{\mu+1}}{2}[A_1|g'(w)| + B_1|g'(b)|], \end{aligned} \tag{12}$$

where the constants A_1, A_2, B_1, and B_2 are given by

$$A_1 = \int_0^1 \sqrt{\frac{\zeta}{1-\zeta}}|\Omega(\zeta)|\mathrm{d}\zeta, \quad A_2 = \int_0^1 \sqrt{\frac{\zeta}{1-\zeta}}|\nabla(\zeta)|\mathrm{d}\zeta,$$
$$B_1 = \int_0^1 \sqrt{\frac{1-\zeta}{\zeta}}|\Omega(\zeta)|\mathrm{d}\zeta, \quad B_2 = \int_0^1 \sqrt{\frac{1-\zeta}{\zeta}}|\nabla(\zeta)|\mathrm{d}\zeta.$$

Proof. It follows from Lemma 2.1 and *MT*-convexity of $|g'|$ that

$$\begin{aligned}
&\left|\frac{(1-h)^\mu\Omega(1)+h^\mu\nabla(1)}{b-a}g(w)-\frac{1}{b-a}[(1-h)^\mu({}_{w^-}I_\rho g(a))+h^\mu({}_{w^+}I_\rho g(b))]\right| \\
&\le (1-h)^{\mu+1}\int_0^1 |\Omega(\zeta)||g'(\zeta w+(1-\zeta)a)|\mathrm{d}\zeta + h^{\mu+1}\int_0^1 |\nabla(\zeta)||g'(\zeta w+(1-\zeta)b)|\mathrm{d}\zeta \\
&\le (1-h)^{\mu+1}\int_0^1 |\Omega(\zeta)|\left[\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}|g'(w)|+\frac{\sqrt{1-\zeta}}{2\sqrt{\zeta}}|g'(a)|\right]\mathrm{d}\zeta \\
&\quad + h^{\mu+1}\int_0^1 |\nabla(\zeta)|\left[\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}|g'(w)|+\frac{\sqrt{1-\zeta}}{2\sqrt{\zeta}}|g'(b)|\right]\mathrm{d}\zeta \\
&= \frac{(1-h)^{\mu+1}}{2}A_1|g'(w)| + \frac{h^{\mu+1}}{2}A_2|g'(w)| + \frac{(1-h)^{\mu+1}}{2}B_1|g'(a)| + \frac{h^{\mu+1}}{2}B_2|g'(b)|,
\end{aligned}$$

which completes the proof of the theorem. □

Remark 2.3. From Theorem 2.1, we can see that if $|g'(x)| < M$ for each $x \in [a,b]$, and ρ is specialized by

(i) $\rho(\zeta) = \zeta$, then the following inequality is true

$$\left| g(w) - \frac{1}{b-a}\int_a^b g(u)\mathrm{d}u \right| \le \frac{\pi}{4}M(b-a),$$

which is obtained by Park [29];

(ii) $\rho(\zeta) = \frac{\zeta^\mu}{\Gamma(\mu)}$, then inequality (12) reduces to (5);

(iii) $\rho(\zeta) = \frac{\zeta^{\mu/k}}{k\Gamma_k(\mu)}$, then it holds

$$\left|\frac{\{(1-h)^{\frac{\mu}{k}}+h^{\frac{\mu}{k}}\}}{(b-a)^{\frac{\mu}{k}-1}}g(w)-\frac{\Gamma_k(\mu+k)}{(b-a)}[I^\mu_{w^-,k}g(a)+I^\mu_{w^+,k}g(b)]\right| \le \frac{(b-a)^{\frac{\mu}{k}}}{2}\beta\left(\frac{1}{2},\frac{2\mu+k}{2k}\right)M.$$

Theorem 2.2. *Let $g : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) and $g' \in L^1[a,b]$ with $0 \le a < b$ and $\mu > 0$. If $|g'|^q$ is MT-convex on with $q > 1$, then for each $h \in (0,1)$ the inequality holds*

$$\begin{aligned}
&\left|\frac{(1-h)^\mu\Omega(1)+h^\mu\nabla(1)}{b-a}g(w)-\frac{1}{b-a}[(1-h)^\mu({}_{w^-}I_\rho g(a))+h^\mu({}_{w^+}I_\rho g(b))]\right| \\
&\le \left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[(1-h)^\mu\left(\int_0^1 |\Omega(\zeta)|^p\mathrm{d}\zeta\right)^{\frac{1}{p}}(|g'(w)|^q+|g'(a)|^q)^{\frac{1}{q}} + h^\mu\left(\int_0^1 |\nabla(\zeta)|^p\mathrm{d}\zeta\right)^{\frac{1}{p}}(|g'(w)|^q+|g'(b)|^q)^{\frac{1}{q}}\right].
\end{aligned} \tag{13}$$

Proof. Employing Lemma 2.1, Hölder's inequality and the MT-convexity of $|g'|^q$ implies

$$\left|\frac{(1-h)^{\mu}\Omega(1)+h^{\mu}\nabla(1)}{b-a}g(w)-\frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a))+h^{\mu}({}_{w^+}I_{\rho}g(b))]\right|$$
$$\leq(1-h)^{\mu+1}\left(\int_0^1|\Omega(\zeta)|^p\mathrm{d}\zeta\right)^{\frac{1}{p}}\left(\int_0^1|g'(\zeta w+(1-\zeta)a)|^q\mathrm{d}\zeta\right)^{\frac{1}{q}}+h^{\mu+1}\left(\int_0^1|\nabla(\zeta)|^p\mathrm{d}\zeta\right)^{\frac{1}{p}}\left(\int_0^1|g'(\zeta w+(1-\zeta)b)|^q\mathrm{d}\zeta\right)^{\frac{1}{q}}$$
$$\leq(1-h)^{\mu+1}\left(\int_0^1|\Omega(\zeta)|^p\mathrm{d}\zeta\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}|g'(w)|^q+\frac{\sqrt{1-\zeta}}{2\sqrt{\zeta}}|g'(a)|^q\right]\mathrm{d}\zeta\right)^{\frac{1}{q}}$$
$$+h^{\mu+1}\left(\int_0^1|\nabla(\zeta)|^p\mathrm{d}\zeta\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}|g'(w)|^q+\frac{\sqrt{1-\zeta}}{2\sqrt{\zeta}}|g'(b)|^q\right]\mathrm{d}\zeta\right)^{\frac{1}{q}}$$
$$=(1-h)^{\mu+1}\left(\int_0^1|\Omega(\zeta)|^p\mathrm{d}\zeta\right)^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}(|g'(w)|^q+|g'(a)|^q)^{\frac{1}{q}}+h^{\mu+1}\left(\int_0^1|\nabla(\zeta)|^p d\zeta\right)^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}(|g'(w)|^q+|g'(b)|^q)^{\frac{1}{q}},$$

where we have used the following identity

$$\int_0^1\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}\mathrm{d}\zeta=\int_0^1\frac{\sqrt{1-\zeta}}{2\sqrt{\zeta}}\mathrm{d}\zeta=\frac{\pi}{4}.$$

This completes the proof of the theorem. □

Remark 2.4. Let $M>0$ be such that $|g'(x)|<M$ for each $x\in[a,b]$. Invoking Theorem 2.2, we conclude that

(i) if $\rho(\zeta)=\zeta$, then the following inequality holds

$$\left|g(w)-\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u\right|\leq\left(\frac{\beta\left(\frac{1}{2},q+\frac{1}{2}\right)}{2}\right)^{\frac{1}{q}}M(b-a),$$

which has been discussed by Park [29];

(ii) if $\rho(\zeta)=\frac{\zeta^{\mu}}{\Gamma(\mu)}$, then (13) converts to (6);

(iii) if $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then we have

$$\left|\frac{\left\{(1-h)^{\frac{\mu}{k}}+h^{\frac{\mu}{k}}\right\}}{(b-a)^{\frac{\mu}{k}-1}}g(w)-\frac{\Gamma_k(\mu+k)}{(b-a)}[I^{\mu}_{w^-,k}g(a)+I^{\mu}_{w^+,k}g(b)]\right|\leq\left(\frac{k}{\mu p+k}\right)^{\frac{1}{p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}M(b-a)^{\frac{\mu}{k}}.$$

Theorem 2.3. *Let $g:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and $g'\in L^1[a,b]$ with $0\leq a<b$ and $\mu>0$. If the mapping $|g'|^q$ is MT-convex on $[a,b]$ for $q>1$, then for each $h\in(0,1)$, we have*

$$\left|\frac{(1-h)^{\mu}\Omega(1)+h^{\mu}\nabla(1)}{b-a}g(w)-\frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a))+h^{\mu}({}_{w^+}I_{\rho}g(b))]\right|$$
$$\leq\frac{(1-h)^{\mu+1}}{2}\left(\int_0^1|\Omega(\zeta)|d\zeta\right)^{1-\frac{1}{q}}(A_1|g'(w)|^q+B_1|g'(a)|^q)^{\frac{1}{q}}$$
$$+\frac{h^{\mu+1}}{2}\left(\int_0^1|\nabla(\zeta)|d\zeta\right)^{1-\frac{1}{q}}(A_2|g'(w)|^q+B_2|g'(b)|^q)^{\frac{1}{q}}. \tag{14}$$

Proof. From Lemma 2.1, power-mean integral inequality and the MT-convexity of $|g'|^q$ on $[a, b]$, we conclude

$$\begin{aligned}
&\left|\frac{(1-h)^{\mu}\Omega(1)+h^{\mu}\nabla(1)}{b-a}g(w)-\frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a))+h^{\mu}({}_{w^+}I_{\rho}g(b))]\right|\\
&\leq(1-h)^{\mu+1}\left(\int_0^1|\Omega(\zeta)|d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1|\Omega(\zeta)||g'(\zeta w+(1-\zeta)a)|^q d\zeta\right)^{\frac{1}{q}}\\
&\quad+h^{\mu+1}\left(\int_0^1|\nabla(\zeta)|d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1|\nabla(\zeta)||g'(\zeta w+(1-\zeta)b)|^q d\zeta\right)^{\frac{1}{q}}\\
&\leq(1-h)^{\mu+1}\left(\int_0^1|\Omega(\zeta)|d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1|\Omega(\zeta)|\left[\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}|g'(w)|^q+\frac{\sqrt{1-\zeta}}{2\sqrt{\zeta}}|g'(a)|^q\right]d\zeta\right)^{\frac{1}{q}}\\
&\quad+h^{\mu+1}\left(\int_0^1|\nabla(\zeta)|d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1|\nabla(\zeta)|\left[\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}|g'(w)|^q+\frac{\sqrt{1-\zeta}}{2\sqrt{\zeta}}|g'(b)|^q\right]d\zeta\right)^{\frac{1}{q}}\\
&=\frac{1}{\Omega(1)}\left[\frac{(1-h)^{\mu+1}}{2}\left(\int_0^1|\Omega(\zeta)|d\zeta\right)^{1-\frac{1}{q}}(A_1|g'(w)|^q+B_1|g'(a)|^q)^{\frac{1}{q}}\right.\\
&\quad\left.+\frac{h^{\mu+1}}{2}\left(\int_0^1|\nabla(\zeta)|d\zeta\right)^{1-\frac{1}{q}}(A_2|g'(w)|^q+B_2|g'(b)|^q)^{\frac{1}{q}}\right],
\end{aligned}$$

which completes the proof of the theorem. □

Remark 2.5. Let $M > 0$ be such that $|g'(x)| < M$ for each $x \in [a, b]$. Applying Theorem 2.3, it concludes that

(i) if $\rho(\zeta) = \zeta$, the following inequality is true

$$\left|g(w)-\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u\right|\leq\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\pi^{1/q}M(b-a),$$

which has been obtained by Park [29];

(ii) if $\rho(\zeta) = \frac{\zeta^{\mu}}{\Gamma(\mu)}$, then (14) converts to (7);

(iii) if $\rho(\zeta) = \frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then we have

$$\left|\frac{\left\{(1-h)^{\frac{\mu}{k}}+h^{\frac{\mu}{k}}\right\}}{(b-a)^{\frac{\mu}{k}-1}}g(w)-\frac{\Gamma_k(\mu+k)}{(b-a)}[I^{\mu}_{w^-,k}g(a)+I^{\mu}_{w^+,k}g(b)]\right|\leq\left(\frac{k}{\mu+k}\right)^{1-\frac{1}{q}}\left(\frac{\beta\left(\frac{1}{2},\frac{2\mu+k}{2k}\right)}{2}\right)^{\frac{1}{q}}M(b-a)^{\frac{\mu}{k}}.$$

3 Applications

3.1 Moment of random variables

Distribution functions and density functions provide complete descriptions of the distribution of probability for a given random variable. However, they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterize the distribution under reasonable conditions are useful in making comparisons. Knowing the probability function, we can determine moments if they exist. Applying the mathematical inequalities, some estimations for the moments of random variables were recently studied by many authors, for more details see [41–44].

Let χ be a random variable whose probability function is $g:\mathfrak{I}\subset\mathbb{R}\to\mathbb{R}_+$, where g is a convex function on the interval of real numbers $\mathfrak{I}$ such that $a,b\in\mathfrak{I}$ with $a<b$. Denote by $M_r(x)$ the rth moment about any arbitrary point x of the random variable χ, $r\geq 0$, defined as

$$M_r(x)=\int_a^b (z-x)^r g(z)\mathrm{d}z,\quad r=0,1,2,\ldots \tag{15}$$

In view of (19), we obtain the following propositions.

Proposition 3.1. *Let χ be a random variable whose probability function is $g:\mathfrak{I}\subset\mathbb{R}\to\mathbb{R}_+$, where g is a convex function on the interval of real numbers $\mathfrak{I}$ such that $a,b\in\mathfrak{I}$ with $a<b$. Then, for any $r\geq 1$, we have*

$$\begin{aligned}
&\frac{(1-h)^{\mu}\Omega(1)+h^{\mu}\nabla(1)}{b-a}M_r(w)-\frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}M_r(a))+h^{\mu}({}_{w^+}I_{\rho}M_r(b))]\\
&=-r(1-h)^{\mu+1}\int_0^1\Omega(\zeta)M_{r-1}(\zeta w+(1-\zeta)a)\mathrm{d}\zeta+rh^{\mu+1}\int_0^1\nabla(\zeta)M_{r-1}(\zeta w+(1-\zeta)b)\mathrm{d}\zeta\\
&=-r(1-h)^{\mu+1}\int_0^1\Omega(\zeta)\left(\int_a^b(z-(\zeta w+(1-\zeta)a))^{r-1}g(z)\mathrm{d}z\right)\mathrm{d}\zeta\\
&\quad+rh^{\mu+1}\int_0^1\nabla(\zeta)\left(\int_a^b(z-(\zeta w+(1-\zeta)b))^{r-1}g(z)\mathrm{d}z\right)\mathrm{d}\zeta,
\end{aligned}\tag{16}$$

where $w=ha+(1-h)b$.

Proof. The proof of this proposition follows by applying Lemma 2.1 with $g(x)=M_r(x)$. □

Proposition 3.2. *Let χ be a random variable whose probability function is $g:\mathfrak{I}\subset\mathbb{R}\to\mathbb{R}_+$, where g is a convex function on the interval of real numbers $\mathfrak{I}$ such that $a,b\in\mathfrak{I}$ with $a<b$. If the function $|g|$ is bounded, then for any $r\geq 1$, we have*

$$\begin{aligned}
&\left|\frac{(1-h)^{\mu}\Omega(1)+h^{\mu}\nabla(1)}{b-a}M_r(w)-\frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}M_r(a))+h^{\mu}({}_{w^+}I_{\rho}M_r(b))]\right|\\
&\leq\frac{\kappa}{r+1}(b-a)^r\{[(1-h^{r+1})(1-h)^{\mu+1}+(-1)^{r+1}(1-h)^{r+\mu+2}]\Omega(1)\\
&\quad+[h^{r+\mu+2}+(-1)^{r+1}h^{\mu+1}(1-(1-h)^{r+1})]\nabla(1)\}.
\end{aligned}\tag{17}$$

In particular, for $h=\frac{1}{2}$ and any odd number r, we have

$$\left|\frac{2\nabla(1)}{b-a}M_r\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\left[\left({}_{\left(\frac{a+b}{2}\right)^-}I_{\rho}M_r(a)\right)+\left({}_{\left(\frac{a+b}{2}\right)^+}I_{\rho}M_r(b)\right)\right]\right|\leq\frac{2\nabla(1)\kappa}{r+1}(b-a)^r,\tag{18}$$

where $\kappa=\sup_{z\in[a,b]}|g(z)|$.

Proof. Let $r\geq 1$. By using proposition 3.1, $\Omega(\zeta)\leq\Omega(1)$, $\nabla(\zeta)\leq\nabla(1)$ for each $\zeta\in[0,1]$, and boundedness of $|g|$, we have

$$\left|\frac{(1-h)^{\mu}\Omega(1)+h^{\mu}\nabla(1)}{b-a}M_r(w)-\frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}M_r(a))+h^{\mu}({}_{w^+}I_{\rho}M_r(b))]\right|$$

$$\le r(1-h)^{\mu+1}\int_0^1 |\Omega(\zeta)|\left(\int_a^b (z-(\zeta w+(1-\zeta)a))^{r-1}|g(z)|\mathrm{d}z\right)\mathrm{d}\zeta$$
$$+ rh^{\mu+1}\int_0^1 |\nabla(\zeta)|\left(\int_a^b (z-(\zeta w+(1-\zeta)b))^{r-1}|g(z)|\mathrm{d}z\right)\mathrm{d}\zeta$$
$$\le (1-h)^{\mu+1}\Omega(1)\kappa\int_0^1\left(\int_a^b (z-(\zeta w+(1-\zeta)a))^{r-1}\mathrm{d}z\right)d\zeta + h^{\mu+1}\nabla(1)\kappa\int_0^1\left(\int_a^b (z-(\zeta w+(1-\zeta)b))^{r-1}\mathrm{d}z\right)\mathrm{d}\zeta.$$

Then, by using very simple integrations, we get

$$\left|\frac{(1-h)^{\mu}\Omega(1)+h^{\mu}\nabla(1)}{b-a}M_r(w)-\frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_\rho M_r(a))+h^{\mu}({}_{w^+}I_\rho M_r(b))]\right|$$
$$\le (1-h)^{\mu+1}(b-a)^r\Omega(1)\kappa\int_0^1 [(1-(1-h)\zeta)^r+(-1)^{r+1}(1-h)^r\zeta^r]\mathrm{d}\zeta$$
$$+ h^{\mu+1}(b-a)^r\nabla(1)\kappa\int_0^1 [h^r\zeta^r+(-1)^{r+1}(1-h\zeta)^r]\mathrm{d}\zeta$$
$$\le \frac{\kappa}{r+1}(b-a)^r\{[(1-h^{r+1})(1-h)^{\mu+1}+(-1)^{r+1}(1-h)^{r+\mu+2}]\Omega(1)$$
$$+[h^{r+\mu+2}+(-1)^{r+1}h^{\mu+1}(1-(1-h)^{r+1})]\nabla(1)\}.$$

This rearranges to the desired inequality (17).

On the other hand, for $h=\frac{1}{2}$, we see that $w=\frac{a+b}{2}$ and $\Omega(1)=\nabla(1)$. Since $(-1)^{r+1}$ either equals to -1 or 1 for an even value r and an odd value r, respectively, we need to split into two cases according to the parity of r. It is easy to see that for an even r, the result vanishes. Thus, for any odd number r, we have

$$\left|\frac{2\nabla(1)}{b-a}M_r\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\left[\left({}_{\left(\frac{a+b}{2}\right)^-}I_\rho M_r(a)\right)+\left({}_{\left(\frac{a+b}{2}\right)^+}I_\rho M_r(b)\right)\right]\right| \le \frac{2\kappa}{r+1}(b-a)^r\nabla(1).$$

This rearranges to the desired inequality (18). Thus, our proof is done. □

Corollary 3.1. *With the similar assumptions of Proposition 3.2, if $h=\frac{1}{2}$ and $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$, we have*

$$\left|M_r\left(\frac{a+b}{2}\right)-\frac{\Gamma(\mu+1)}{b-a}\left[I_{\left(\frac{a+b}{2}\right)^-}M_r(a)+I_{\left(\frac{a+b}{2}\right)^+}M_r(b)\right]\right| \le \frac{2^{r+1}\kappa}{(r+1)\Gamma(\mu+1)}\left(\frac{b-a}{2}\right)^{r+\mu}.$$

3.2 Volterra integral equation

Consider the Volterra integral equation of the first kind [45]:

$$g(x)=\int_a^b K_r(x,z)g(z)\mathrm{d}z, \tag{19}$$

where

$$k_r(x,z)=\begin{cases}(x-z)^r & a\le z\le w,\\ (z-x)^r & w\le z\le b,\end{cases}$$

for $r=0,1,2,\ldots$.

In view of (19), we obtain the following propositions.

Proposition 3.3. *Let $g : \mathfrak{I} \subset \mathbb{R} \to \mathbb{R}_+$ be an MT-convex function on the interval of real numbers $\mathfrak{I}$ such that $a, b \in \mathfrak{I}$ with $a < b$. Then, for any $r \geq 1$, we have*

$$\begin{aligned}&\frac{(1-h)^{\mu}\Omega(1) + h^{\mu}\nabla(1)}{b-a}g(w) - \frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a)) + h^{\mu}({}_{w^+}I_{\rho}g(b))]\\&= (1-h)^{\mu+1}\int_0^1 \Omega(\zeta)g(\zeta w + (1-\zeta)a)\mathrm{d}\zeta - h^{\mu+1}\int_0^1 \nabla(\zeta)(\zeta w + (1-\zeta)b)\mathrm{d}\zeta\\&= r(1-h)^{\mu+1}\int_0^1 \Omega(\zeta)\left(\int_a^w ((\zeta w + (1-\zeta)a) - z)^{r-1}g(z)\mathrm{d}z\right)\mathrm{d}\zeta\\&\quad + rh^{\mu+1}\int_0^1 \nabla(\zeta)\left(\int_w^b (z - (\zeta w + (1-\zeta)b))^{r-1}g(z)\mathrm{d}z\right)\mathrm{d}\zeta.\end{aligned} \tag{20}$$

Proof. The proof of this proposition follows by applying Lemma 2.1 with $g(x)$ defined in (19). □

Proposition 3.4. *Let $g : \mathfrak{I} \subset \mathbb{R} \to \mathbb{R}_+$ be an MT-convex function on the interval of real numbers $\mathfrak{I}$ such that $a, b \in \mathfrak{I}$ with $a < b$. If the function $|g|$ is bounded, then, for any odd number r, we have*

$$\begin{aligned}&\left|\frac{(1-h)^{\mu}\Omega(1) + h^{\mu}\nabla(1)}{b-a}g(w) - \frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a)) + h^{\mu}({}_{w^+}I_{\rho}g(b))]\right|\\&\leq \frac{2(b-a)^r\kappa}{r+1}[(1-h)^{r+\mu+1}\Omega(1) + h^{r+\mu+1}\nabla(1)].\end{aligned}$$

Proof. By using Proposition 3.3 and boundedness of $|g|$, we have

$$\begin{aligned}&\left|\frac{(1-h)^{\mu}\Omega(1) + h^{\mu}\nabla(1)}{b-a}g(w) - \frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a)) + h^{\mu}({}_{w^+}I_{\rho}g(b))]\right|\\&\leq r(1-h)^{\mu+1}\int_0^1 \Omega(\zeta)\left(\int_a^w ((\zeta w + (1-\zeta)a) - z)^{r-1}|g(z)|\mathrm{d}z\right)\mathrm{d}\zeta + rh^{\mu+1}\int_0^1 \nabla(\zeta)\left(\int_w^b (z - (\zeta w + (1-\zeta)b))^{r-1}|g(z)|\mathrm{d}z\right)\mathrm{d}\zeta\\&\leq r(1-h)^{\mu+1}\Omega(1)\kappa\int_0^1\left(\int_a^w ((\zeta w + (1-\zeta)a) - z)^{r-1}\mathrm{d}z\right)\mathrm{d}\zeta + rh^{\mu+1}\nabla(1)\kappa\int_0^1\left(\int_w^b (z - (\zeta w + (1-\zeta)b))^{r-1}\mathrm{d}z\right)\mathrm{d}\zeta.\end{aligned}$$

Then, by calculating the two simple integrations, we have for any odd value r:

$$\begin{aligned}&\left|\frac{(1-h)^{\mu}\Omega(1) + h^{\mu}\nabla(1)}{b-a}g(w) - \frac{1}{b-a}[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a)) + h^{\mu}({}_{w^+}I_{\rho}g(b))]\right|\\&\leq r(1-h)^{\mu+1}\Omega(1)\kappa\int_0^1\left(\frac{(b-a)^r(1-h)^r(\zeta^r + (-1)^{r+1}(1-\zeta)^r)}{r}\right)\mathrm{d}\zeta\\&\quad + rh^{\mu+1}\nabla(1)\kappa\int_0^1\left(\frac{(b-a)^r h^r(\zeta^r + (-1)^{r+1}(1-\zeta)^r)}{r}\right)\mathrm{d}\zeta\\&\leq (1-h)^{r+\mu+1}(b-a)^r\Omega(1)\kappa\int_0^1 (\zeta^r + (-1)^{r+1}(1-\zeta)^r)\mathrm{d}\zeta + h^{r+\mu+1}(b-a)^r\nabla(1)\kappa\int_0^1 (\zeta^r + (-1)^{r+1}(1-\zeta)^r))\mathrm{d}\zeta\\&= \frac{2(b-a)^r\kappa}{r+1}[(1-h)^{r+\mu+1}\Omega(1) + h^{r+\mu+1}\nabla(1)],\end{aligned}$$

which completes the proof. □

Corollary 3.2. *With the similar assumptions of* Proposition 3.4,

1. *if* $h = \frac{1}{2}$, *we have*

$$\left| \frac{2\Omega(1)}{b-a} g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\left[{}_{\left(\frac{a+b}{2}\right)^-} I_\rho g(a) + {}_{\left(\frac{a+b}{2}\right)^+} I_\rho g(b) \right] \right| \le \frac{\Omega(1)\kappa}{r+1}\left(\frac{b-a}{2}\right)^r;$$

2. *if* $h = \frac{1}{2}$ *and* $\rho(\zeta) = \frac{\zeta^\mu}{\Gamma(\mu)}$, *we have*

$$\left| g\left(\frac{a+b}{2}\right) - \frac{\Gamma(\mu+1)}{b-a}\left[I_{\left(\frac{a+b}{2}\right)^-} g(a) + I_{\left(\frac{a+b}{2}\right)^+} g(b) \right] \right| \le \frac{\kappa}{(r+1)\Gamma(\mu+1)}\left(\frac{b-a}{2}\right)^{r+\mu}.$$

4 Conclusion

In this work, we establish new generalized fractional integral inequalities of Hermite-Hadamard-type for *MT*-convex functions. In view of this, we obtained some new inequalities of Hermite-Hadamard-type which are related to Riemann-Liouville fractional integrals and classical integrals. The results presented in this article would provide generalizations and extension of those given in earlier works.

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