Generalized Fractional Nonlinear Birth Processes

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Abstract We consider here generalized fractional versions of the difference-differential equation governing the classical nonlinear birth process. Orsingher and Polito (Bernoulli 16(3):858–881, 2010) defined a fractional birth process by replacing, in its governing equation, the first order time derivative with the Caputo fractional derivative of order $\upsilon \in (0, 1]$. We study here a further generalization, obtained by adding in the equation some extra terms; as we shall see, this makes the expression of its solution much more complicated. Moreover we consider also the case $\upsilon \in (1, +\infty)$, as well as $\upsilon \in (0, 1]$, using correspondingly two different definitions of fractional derivative: we apply the fractional Caputo derivative and the right-sided fractional Riemann–Liouville derivative on $\mathbb{R}_+$, for $\upsilon \in (0, 1]$ and $\upsilon \in (1, +\infty)$, respectively. For the two cases, we obtain the exact solutions and prove that they coincide with the distribution of some subordinated stochastic processes, whose random time argument is represented by a stable subordinator (for $\upsilon \in (1, +\infty)$) or its inverse (for $\upsilon \in (0, 1]$).

Keywords Generalized fractional birth process · Fractional Caputo derivative ·
Fractional Riemann–Liouville derivative · Mittag–Leffler functions ·
Stable subordinator

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1 Introduction and Preliminaries

We consider here the following Cauchy problem

$$
\begin{cases}
\dfrac{d^\upsilon}{dt^\upsilon} P_1^\upsilon(t) = -\lambda_1 P_1^\upsilon(t) \\
\dfrac{d^\upsilon}{dt^\upsilon} P_k^\upsilon(t) = -\lambda_k P_k^\upsilon(t) + \displaystyle\sum_{i=1}^{k-1} \lambda_i P_i^\upsilon(t), \quad k = 2, 3, \ldots,
\end{cases}
\tag{1}
$$

for $t \geq 0$, $\upsilon > 0$, with initial conditions

$$
P_k^\upsilon(0) = \begin{cases} 1 & k = 1, \\ 0 & k = 2, 3, \ldots \end{cases},
\tag{2}
$$

where $\lambda_j \in \mathbb{R}$, $j = 1, 2, \ldots$.

We will obtain exact solutions to Eqs. 1 and 2 in both cases $\upsilon \in (0, 1]$ and $\upsilon \in (1, +\infty)$; in the first case they will be expressed in terms of finite sums of the Mittag–Leffler functions and in the second case as finite sums of exponentials.

While the first equation in Eq. 1 is the so-called fractional relaxation equation (see e.g. Uchaikin 2002; Beghin 2012), we can call the second equation in Eq. 1 a generalized fractional birth equation (see e.g. Orsingher and Polito 2010, 2011, 2013; Cahoy and Polito 2012). Indeed, in Orsingher and Polito (2010) and later in Orsingher and Polito (2013), the authors analyze the following Cauchy problem

$$
\frac{d^\upsilon}{dt^\upsilon} P_k^\upsilon(t) = -\lambda_k P_k^\upsilon(t) + \lambda_{k-1} P_{k-1}^\upsilon(t), \quad k \geq 1, \quad \upsilon \in (0, 1],
\tag{3}
$$

(where $P_k^\upsilon(0) = 1_{[k=1]}$ and $\frac{d^\upsilon}{dt^\upsilon}$ denotes the Caputo fractional derivative) and define the corresponding process "fractional birth process". Equation 3 is formally connected to Eq. 1, even though it cannot be obtained as a particular case, since, by assumption, it is $\lambda_j \neq \lambda_i$, for $j \neq i$, in Eq. 1. The extra terms in the second equation of Eq. 1 makes the expression of its solution much more complicated.

We will analyze the problem (1)–(2) for any value of υ: we need to distinguish the two cases $\upsilon \in (0, 1]$ and $\upsilon \in (1, +\infty)$, because we use correspondingly two different definitions of fractional derivative (see, for example, Kilbas et al. 2006):

1. For $\upsilon \in (0, 1]$, we apply the fractional Caputo derivative of order υ, which is defined as follows:

$$
{}_{c}D_{0+,t}^\upsilon \, f(t) := \frac{1}{\Gamma(1-\upsilon)} \int_0^t (t-s)^{-\upsilon} \frac{d}{ds} f(s) ds, \quad t > 0.
$$

2. For $\upsilon \in (1, +\infty)$, we use the right-sided fractional Riemann–Liouville derivative on $\mathbb{R}_+$ of order υ, which is defined as follows:

$$
{}_{RL}D_{-,t}^\upsilon \, f(t) := \frac{1}{\Gamma(m-\upsilon)} \left(-\frac{d}{dt}\right)^m \int_t^{+\infty} \frac{f(s)}{(s-t)^{\upsilon-m+1}} ds, \quad t > 0,
$$

where $m = \lceil \upsilon \rceil$.

Note that, for $\upsilon = 1$, we have $_CD^\upsilon_{0+,t} = \dfrac{d}{dt}$ and $_{RL}D^\upsilon_{-,t} = -\dfrac{d}{dt}$, where $\dfrac{d}{dt}$ is the standard derivative. Throughout this paper we always use the symbol $\dfrac{d^\upsilon}{dt^\upsilon}$ and we set

$$\frac{d^\upsilon}{dt^\upsilon} := \begin{cases} _cD^\upsilon_{0+,t} & \upsilon \in (0,1], \\ -_{RL}D^\upsilon_{-,t} & \upsilon \in (1,+\infty) \end{cases}.$$

The study of fractional generalizations of classical differential equations has attracted a lot of attention over the last decades: see, for example, Mainardi (1996), Angulo et al. (2000) and Orsingher and Beghin (2009) for fractional diffusions; Mainardi et al. (2004), Beghin and Orsingher (2009) and Meerschaert et al. (2011) for equations related to fractional Poisson and renewal processes.

Finally we prove that the solution to Eq. 1 can be expressed as the distribution of the stochastic process defined as

$$N_\upsilon(t) := N_1(H_\upsilon(t)),$$

where $N_1(t), t \geq 0$ is the process governed by Eq. 1 with $\upsilon = 1$, and

$$H_\upsilon(t) := \begin{cases} L^\upsilon(t), & \upsilon \in (0,1) \\ A^{\frac{1}{\upsilon}}(t), & \upsilon \in (1,+\infty) \end{cases}, \quad t \geq 0, \tag{4}$$

with $H_\upsilon(t) := t$ for $\upsilon = 1$. The processes $A^{\frac{1}{\upsilon}}(t)$ and $L^\upsilon(t)$ in Eq. 4 are respectively a stable subordinator of index $\frac{1}{\upsilon}$ and the inverse of a stable subordinator of index υ (see Theorem 5 below for their exact definitions).

Due to the different properties of the sample paths between the processes given in Eq. 4, the two forms of fractional birth processes presented here display a very different behavior, in the two cases $\upsilon \in (0,1]$ and $\upsilon \in (1,+\infty)$. Indeed in the first case the process can perform only unitary jumps in an interval of infinitesimal length (as in the standard case $\upsilon = 1$), while in the latter the number of jumps can be of any integer size.

2 Main Results

We start by considering the case where $\upsilon \in (0,1]$ and deriving the Laplace transform of the solution to Eqs. 1 and 2. We use the following notation:

$$L[f(t); \mu] := \int_0^\infty e^{-\mu t} f(t)\, dt.$$

Theorem 1 *For the fractional equations*

$$\begin{cases} \dfrac{d^\upsilon}{dt^\upsilon} P^\upsilon_1(t) = -\lambda_1 P^\upsilon_1(t) \\ \dfrac{d^\upsilon}{dt^\upsilon} P^\upsilon_k(t) = -\lambda_k P^\upsilon_k(t) + \displaystyle\sum_{i=1}^{k-1} \lambda_i P^\upsilon_i(t), \quad k = 2,3,\dots, \end{cases} \quad t > 0, \quad 0 < \upsilon \leq 1, \tag{5}$$

with initial conditions

$$P_k^v(0) = \begin{cases} 1 & k = 1, \\ 0 & k = 2, 3, \ldots \end{cases},$$ (6)

for $\lambda_i \in \mathbb{R}$ ($i = 1, 2, \ldots$), we have

$$L\left[P_k^v(t); \mu\right] = \begin{cases} \dfrac{\mu^{v-1}}{\mu^v + \lambda_1}, & k = 1 \\ \dfrac{\lambda_1 \mu^{v-1}}{(\mu^v + \lambda_1)(\mu^v + \lambda_2)}, & k = 2 \\ \dfrac{\lambda_1 \mu^{v-1}}{(\mu^v + \lambda_1)(\mu^v + \lambda_k)} \displaystyle\sum_{(q_2, \ldots, q_{k-1}) \in A_k} \prod_{i=2}^{k-1} \left(\dfrac{\lambda_i}{\mu^v + \lambda_i}\right)^{q_i}, & k = 3, 4, \ldots \end{cases},$$ (7)

where

$$A_k = \{(q_2, q_2, \ldots, q_{k-1}) \,|\, q_i \in \{0, 1\}, \ i = 2, \ldots, k-1\}.$$ (8)

Proof We prove the result (7) by induction.

For $k = 1$,

$$\frac{d^v}{dt^v} P_1^v(t) = -\lambda_1 P_1^v(t), \quad P_1^v(0) = 1,$$

by the Laplace transform we get

$$\mu^v L\left[P_1^v(t); \mu\right] - \mu^{v-1} P_1^v(0) = -\lambda_1 L\left[P_1^v(t); \mu\right], \quad \Rightarrow \quad L\left[P_1^v(t); \mu\right] = \frac{\mu^{v-1}}{\mu^v + \lambda_1}.$$

For $k = 2$, Eq. 5 becomes

$$\frac{d^v}{dt^v} P_2^v(t) = -\lambda_2 P_2^v(t) + \lambda_1 P_1^v(t), \quad P_2^v(0) = 0.$$

By taking the Laplace transform, we have

$$\mu^v L\left[P_2^v(t); \mu\right] - \mu^{v-1} P_2^v(0) = -\lambda_2 L\left[P_2^v(t); \mu\right] + \lambda_1 \frac{\mu^{v-1}}{\mu^v + \lambda_1},$$

$$\Rightarrow L\left[P_2^v(t); \mu\right] = \frac{\lambda_1 \mu^{v-1}}{(\mu^v + \lambda_1)(\mu^v + \lambda_2)}.$$

For $k = 3$, Eq. 5 becomes

$$\frac{d^v}{dt^v} P_3^v(t) = -\lambda_3 P_3^v(t) + \lambda_2 P_2^v(t) + \lambda_1 P_1^v(t), \quad P_3^v(0) = 0.$$

By taking the Laplace transform, we get

$$\mu^v L\left[P_3^v(t); \mu\right] - \mu^{v-1} P_3^v(0) = -\lambda_3 L\left[P_3^v(t); \mu\right]$$

$$+ \lambda_2 \frac{\lambda_1 \mu^{v-1}}{(\mu^v + \lambda_1)(\mu^v + \lambda_2)} + \lambda_1 \frac{\mu^{v-1}}{(\mu^v + \lambda_1)}.$$

Therefore, we obtain

$$L\left[P_3^\upsilon(t); \mu\right] = \lambda_2 \frac{\lambda_1 \mu^{\upsilon-1}}{(\mu^\upsilon + \lambda_1)(\mu^\upsilon + \lambda_2)(\mu^\upsilon + \lambda_3)} + \lambda_1 \frac{\mu^{\upsilon-1}}{(\mu^\upsilon + \lambda_1)(\mu^\upsilon + \lambda_3)}$$

$$= \frac{\lambda_1 \mu^{\upsilon-1}}{(\mu^\upsilon + \lambda_1)(\mu^\upsilon + \lambda_3)} \left(\frac{\lambda_2}{\mu^\upsilon + \lambda_2} + 1\right).$$

Similarly, for $k = 4$, we get

$$L\left[P_4^\upsilon(t); \mu\right] = \frac{\lambda_1 \mu^{\upsilon-1}}{(\mu^\upsilon + \lambda_1)(\mu^\upsilon + \lambda_4)}$$

$$\times \left(\frac{\lambda_2\lambda_3}{(\mu^\upsilon + \lambda_2)(\mu^\upsilon + \lambda_3)} + \frac{\lambda_2}{(\mu^\upsilon + \lambda_2)} + \frac{\lambda_3}{(\mu^\upsilon + \lambda_3)} + 1\right).$$

Then, we can see that Eq. 7 holds for $k = 1, 2, 3, 4$.

Now, we suppose that Eq. 7 holds for $L[P_i^\upsilon(t); \mu]$, $i = 1, \ldots, k-1$. Then we will prove that Eq. 7 is true for $L[P_k^\upsilon(t); \mu]$.

Thus, by taking the Laplace transform of Eq. 5 for $k \geq 2$, we can write

$$\mu^\upsilon L\left[P_k^\upsilon(t); \mu\right] - \mu^{\upsilon-1} P_k^\upsilon(0) = -\lambda_k L\left[P_k^\upsilon(t); \mu\right] + \sum_{i=1}^{k-1} \lambda_i L\left[P_i^\upsilon(t); \mu\right].$$

Therefore,

$$(\mu^\upsilon + \lambda_k) L\left[P_k^\upsilon(t); \mu\right] = \sum_{i=3}^{k-1} \lambda_i \frac{\lambda_1 \mu^{\upsilon-1}}{(\mu^\upsilon + \lambda_1)(\mu^\upsilon + \lambda_i)} \sum_{(q_2,\ldots,q_{i-1})\in A_i} \prod_{j=2}^{i-1} \left(\frac{\lambda_j}{\mu^\upsilon + \lambda_j}\right)^{q_j}$$

$$+ \lambda_2 \frac{\lambda_1 \mu^{\upsilon-1}}{(\mu^\upsilon + \lambda_1)(\mu^\upsilon + \lambda_2)} + \lambda_1 \frac{\mu^{\upsilon-1}}{\mu^\upsilon + \lambda_1}$$

and

$$L\left[P_k^\upsilon(t); \mu\right] = \frac{\lambda_1 \mu^{\upsilon-1}}{(\mu^\upsilon + \lambda_1)(\mu^\upsilon + \lambda_k)}$$

$$\times \left(\sum_{i=3}^{k-1} \frac{\lambda_i}{(\mu^\upsilon + \lambda_i)} \sum_{(q_2,\ldots,q_{i-1})\in A_i} \prod_{j=2}^{i-1} \left(\frac{\lambda_j}{\mu^\upsilon + \lambda_j}\right)^{q_j} + \frac{\lambda_2}{(\mu^\upsilon + \lambda_2)} + 1\right)$$

Note that

$$\sum_{i=3}^{k-1} \frac{\lambda_i}{(\mu^\upsilon + \lambda_i)} \sum_{(q_2,\ldots,q_{i-1})\in A_i} \prod_{j=2}^{i-1} \left(\frac{\lambda_j}{\mu^\upsilon + \lambda_j}\right)^{q_j} + \frac{\lambda_2}{(\mu^\upsilon + \lambda_2)} + 1$$

$$= \sum_{(q_2,\ldots,q_{k-1})\in A_k} \prod_{j=2}^{k-1} \left(\frac{\lambda_j}{\mu^\upsilon + \lambda_j}\right)^{q_j},$$

so that we get formula (7). □

Lemma 2 *Let $x \in \mathbb{R}$, $a_i \in \mathbb{R}$ ($i = 1, \ldots n$) and $a_i \neq a_j$ for $i \neq j$, then*

$$\frac{1}{\prod\limits_{i=1}^{n}(x+a_i)} = \sum_{i=1}^{n}\frac{1}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n}(a_j-a_i)} \times \frac{1}{(x+a_i)} \tag{9}$$

Proof We prove Eq. 9 by induction.

For $n = 2$, it is clear. We suppose that it is true for $n = k - 1$, then we will show that it is true for $n = k$.

$$\frac{1}{\prod\limits_{i=1}^{k}(x+a_i)} = \frac{1}{(x+a_k) \times \prod\limits_{i=1}^{k-1}(x+a_i)} = \frac{1}{(x+a_k)} \times \sum_{i=1}^{k-1}\frac{1}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{k-1}(a_j-a_i)} \times \frac{1}{(x+a_i)}$$

$$= \sum_{i=1}^{k-1}\frac{1}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{k-1}(a_j-a_i)} \times \left(\frac{1}{(a_k-a_i)} \times \frac{1}{(x+a_i)} + \frac{1}{(a_i-a_k)} \times \frac{1}{(x+a_k)}\right)$$

$$= \sum_{i=1}^{k}\frac{1}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{k}(a_j-a_i)} \times \frac{1}{(x+a_i)}.$$

□

In the next theorem, we will obtain the exact solutions to Eqs. 1 and 2, for $\upsilon \in (0, 1]$ which will be expressed in terms of finite sums of the Mittag–Leffler function, i.e.

$$E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty}\frac{x^j}{\Gamma(\alpha j+\beta)}, \quad x \in \mathbb{R}, \quad \alpha, \beta \in \mathbb{C}, \quad \Re(\alpha), \Re(\beta) > 0.$$

Theorem 3 *The solution to the problem (1)–(2) for $\upsilon \in (0, 1]$, is given by*

$$P_k^{\upsilon}(t) = \begin{cases} E_{\upsilon,1}(-\lambda_1 t^{\upsilon}), & k=1 \\ \sum\limits_{j=1}^{k}\left(\sum\limits_{(q_1,q_2,\ldots,q_k)\in\tilde{A}_k}q_j\left(\prod\limits_{r=1}^{k-1}(\lambda_r)^{q_r}\right) \times \left(\prod\limits_{\substack{i=1 \\ i\neq j}}^{k}\frac{1}{(\lambda_i-\lambda_j)^{q_i}}\right)\right)E_{\upsilon,1}\left(-\lambda_j t^{\upsilon}\right), & k=2,3,\ldots \end{cases}$$

$$\tag{10}$$

where

$$\tilde{A}_k = \{(q_1, q_2, \ldots, q_k)\,|(q_2, \ldots, q_{k-1}) \in A_k \quad and \quad q_1 = q_k = 1\}. \tag{11}$$

Proof For $k = 1$, by Theorem 1, taking the inverse of Laplace transform and by noting that

$$L\left[E_{v,1}\left(-\lambda t^v\right); \mu\right] = \frac{\mu^{v-1}}{(\mu^v + \lambda)}, \tag{12}$$

we have

$$P_1^v(t) = E_{v,1}\left(-\lambda_1 t^v\right).$$

For $k = 2$, from Theorem 1,

$$L\left[P_2^v(t); \mu\right] = \frac{\lambda_1 \mu^{v-1}}{(\mu^v + \lambda_1)(\mu^v + \lambda_2)}.$$

By using Lemma 2, we can get

$$L\left[P_2^v(t); \mu\right] = \lambda_1 \mu^{v-1}\left(\frac{1}{(\lambda_2 - \lambda_1)} \times \frac{1}{(\mu^v + \lambda_1)} + \frac{1}{(\lambda_1 - \lambda_2)} \times \frac{1}{(\mu^v + \lambda_2)}\right)$$

$$= \frac{\lambda_1}{(\lambda_2 - \lambda_1)}\left(\frac{\mu^{v-1}}{\mu^v + \lambda_1} - \frac{\mu^{v-1}}{\mu^v + \lambda_2}\right).$$

By taking the inverse of Laplace transform and considering Eq. 12, we obtain

$$P_2^v(t) = \frac{\lambda_1}{(\lambda_2 - \lambda_1)}\left(E_{v,1}\left(-\lambda_1 t^v\right) - E_{v,1}\left(-\lambda_2 t^v\right)\right).$$

Then, Eq. 10 is proved for $k = 2$.

For $k \geq 3$, from Theorem 1, we have

$$L\left[P_k^v(t); \mu\right] = \frac{\lambda_1 \mu^{v-1}}{(\mu^v + \lambda_1)(\mu^v + \lambda_k)} \sum_{(q_2,\ldots,q_{k-1})\in A_k} \prod_{j=2}^{k-1}\left(\frac{\lambda_j}{\mu^v + \lambda_j}\right)^{q_j}.$$

By definitions of the sets A_k and $\tilde{A}_k$ in Eqs. 8 and 11, respectively, we can write

$$L\left[P_k^v(t); \mu\right] = \mu^{v-1} \sum_{(q_1,\ldots,q_k)\in\tilde{A}_k} \prod_{r=1}^{k-1}(\lambda_r)^{q_r}\left(\frac{1}{\prod\limits_{j=1}^{k}(\mu^v + \lambda_j)^{q_j}}\right).$$
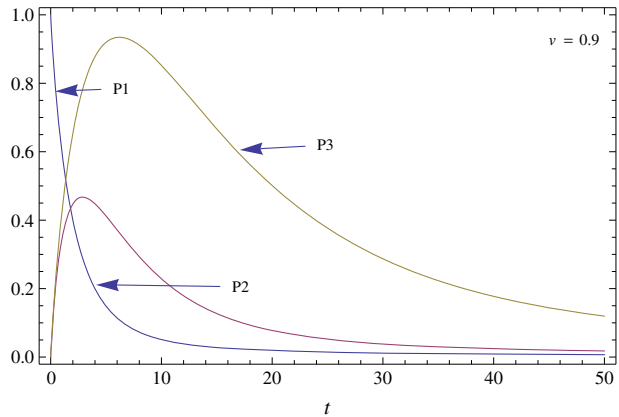
On the other hand, by Lemma 2, we have

$$\frac{1}{\prod\limits_{j=1}^{k}(\mu^v + \lambda_j)^{q_j}} = \sum_{j=1}^{k} q_j\left(\prod_{\substack{i=1\\i\neq j}}^{k}\frac{1}{(\lambda_i - \lambda_j)^{q_i}}\right) \times \frac{1}{(\mu^v + \lambda_j)}, \quad \forall (q_1,\ldots,q_k) \in \tilde{A}_k.$$

Therefore, we get

$$L\left[P_k^v(t); \mu\right] = \sum_{(q_1,\ldots,q_k)\in\tilde{A}_k} \prod_{r=1}^{k-1}(\lambda_r)^{q_r} \sum_{j=1}^{k}\left(q_j\left(\prod_{\substack{i=1\\i\neq j}}^{k}\frac{1}{(\lambda_i - \lambda_j)^{q_i}}\right) \times \frac{\mu^{v-1}}{(\mu^v + \lambda_j)}\right).$$

Fig. 1 Plot of $P_k^v(t)$ for $v = 0.9$, $k = 1, 2, 3$ and $\lambda_i = 2^{-i}$ ($i \in \mathbb{N}$)

So, by taking the inverse Laplace transform and by Eq. 12, we can obtain

$$P_k^v(t) = \sum_{(q_1,\ldots,q_k)\in \tilde{A}_k} \prod_{r=1}^{k-1} (\lambda_r)^{q_r} \sum_{j=1}^{k} \left(q_j \left(\prod_{\substack{i=1\\i\neq j}}^{k} \frac{1}{(\lambda_i - \lambda_j)^{q_i}} \right) \times \mathrm{E}_{v,1}\left(-\lambda_j t^v\right) \right).$$
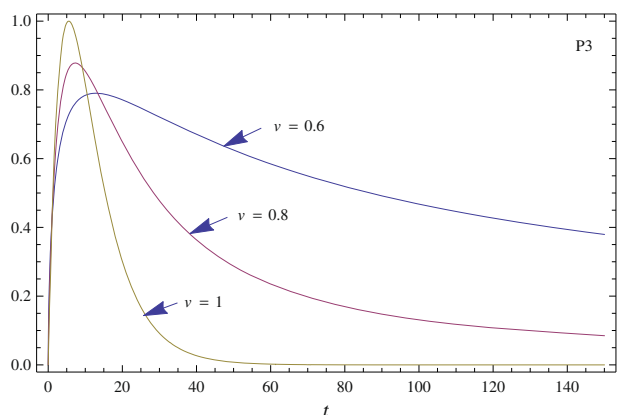
Finally, by interchanging the two sums, we get formula (10) for $k = 2, 3, \ldots$ □

Let $\lambda_i = 2^{-i}$ ($i \in \mathrm{N}$), we can see the plot of $P_k^{0.9}(t)$ for $k = 1, 2, 3$ and $P_3^v(t)$ for $v = 0.6, 0.8, 1$, in Figs. 1 and 2, respectively.

Remark 1 In formula (10), if we just consider the case $q_i = 1$ for any i, we have

$$P_k^v(t) = \begin{cases} \mathrm{E}_{v,1}\left(-\lambda_1 t^v\right), & k = 1 \\ \prod_{r=1}^{k-1}(\lambda_r) \sum_{j=1}^{k} \left(\prod_{\substack{i=1\\i\neq j}}^{k} \frac{1}{(\lambda_i-\lambda_j)} \right) \mathrm{E}_{v,1}\left(-\lambda_j t^v\right), & k = 2, 3, \ldots \end{cases},$$

Fig. 2 Plot of $P_3^v(t)$ for $v = 0.6, 0.8, 1$ and $\lambda_i = 2^{-i}$ ($i \in \mathbb{N}$)

that is the exact solution of the fractional birth equation proposed by Orsingher and Polito (2010).

Remark 2 By considering the special case $\upsilon = 1$ in Theorem 3 and writing $P_k^1 = P_k$, we can conclude that the solution to the integer-order fractional equations

$$\begin{cases} \dfrac{d}{dt} P_1(t) = -\lambda_1 P_1(t) \\ \dfrac{d}{dt} P_k(t) = -\lambda_k P_k(t) + \displaystyle\sum_{i=1}^{k-1} \lambda_i P_i(t), \quad k = 2, 3, \ldots, \end{cases} \quad t \geq 0, \qquad (13)$$

with

$$P_k(0) = \begin{cases} 1 & k = 1, \\ 0 & k = 2, 3, \ldots \end{cases}. \qquad (14)$$

is given by

$$P_k(t) = \begin{cases} e^{-\lambda_1 t}, & k = 1 \\ \displaystyle\sum_{j=1}^{k} \left(\sum_{(q_1, q_2, \ldots, q_k) \in \tilde{A}_k} q_j \left(\prod_{r=1}^{k-1} (\lambda_r)^{q_r} \right) \times \left(\prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{1}{(\lambda_i - \lambda_j)^{q_i}} \right) \right) e^{-\lambda_j t}, & k = 2, 3, \ldots \end{cases}. \qquad (15)$$

We prove now that the previous expression represents a proper probability distribution, for any $t \geq 0$, under the condition that $\lambda_i \geq 0$, $(i = 1, 2, \ldots)$ and thus we can define $P_k(t) = \Pr\{N_1(t) = k\}$, for the random process $N_1(t), t \geq 0$.

Theorem 4 *If $\lambda_i \geq 0$, $(i = 1, 2, \ldots)$, then*

(i) $P_k(t) \geq 0, t \geq 0, k = 1, 2, \ldots$.

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty \Rightarrow \sum_{k=1}^{\infty} P_k(t) = 1, \forall t \geq 0.$

Proof

(i)   We prove it by induction. For $k = 1$ have that $P_1(t) = e^{-\lambda_1 t} > 0$. For $k = 2$, by Eqs. 13 and 14, we get $\frac{d}{dt} P_2(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t)$ and $P_2(0) = 0$. Therefore

$$P_2(t) = e^{-\lambda_2 t} \int_0^t \lambda_1 e^{(\lambda_2 - \lambda_1)t} dt \overset{\lambda_1 \geq 0}{\Longrightarrow} P_2(t) \geq 0.$$

If $P_i(t) \geq 0$ for $i = 1, 2, \ldots, k-1$, we show that $P_k(t) \geq 0$ as follows. By solving Eqs. 13 and 14 with respect to $P_i(t)$ $(i = 1, 2, \ldots, k-1)$, we obtain

$$P_k(t) = e^{-\lambda_k t} \int_0^t e^{\lambda_k t} \sum_{i=1}^{k-1} \lambda_i P_i(t) dt \overset{\lambda_i, P_i(t) \geq 0, i=1,\ldots,k-1}{\Longrightarrow} P_k(t) \geq 0.$$

(ii) Let $S_n(t) = \displaystyle\sum_{k=1}^{n} P_k(t)$ and $S(t) = \sum_{k=1}^{\infty} P_k(t)$. Then, from Eq. 14 we have $S_n(0) = 1$, $\forall n \geq 1$.

By summing the differential Eq. 13 for $k = 1, \ldots, n$ and $n \geq 3$ we have

$$\sum_{k=1}^{n} \frac{d}{dt} P_k(t) = -\lambda_n P_n(t) + \sum_{j=3}^{n} \sum_{i=1}^{j-2} \lambda_i P_i(t) \quad \Rightarrow \quad \frac{d}{dt} S_n(t) = -\lambda_n P_n(t) + \sum_{j=3}^{n} \sum_{i=1}^{j-2} \lambda_i P_i(t)$$

$$\int_0^t \frac{d}{dt} S_n(t) dt = -\lambda_n \int_0^t P_n(t) dt + \sum_{j=3}^{n} \sum_{i=1}^{j-2} \lambda_i \int_0^t P_i(t) dt$$

$$S_n(t) - 1 = -\lambda_n \int_0^t P_n(t) dt + \sum_{j=3}^{n} \sum_{i=1}^{j-2} \lambda_i \int_0^t P_i(t) dt.$$

Since $\sum_{j=3}^{n} \sum_{i=1}^{j-2} \lambda_i \int_0^t P_i(t) dt \geq 0$, we can write

$$1 - S(t) \leq 1 - S_n(t) \leq \lambda_n \int_0^t P_n(t) dt \leq \lambda_n \int_0^{+\infty} P_n(t) dt, \quad \forall n \geq 2.$$

On the other hand, from Eq. 15, we have

$$P_k(t) = \frac{1}{\lambda_k} \sum_{(q_1, \ldots, q_k) \in \tilde{A}_k} \sum_{j=1}^{k} q_j \lambda_j \prod_{\substack{i=1 \\ i \neq j}}^{k} \left(\frac{\lambda_i}{\lambda_i - \lambda_j} \right)^{q_i} e^{-\lambda_j t}, \quad k = 2, 3, \ldots.$$

$$\int_0^{+\infty} P_k(t) dt = \frac{1}{\lambda_k} \sum_{(q_1, \ldots, q_k) \in \tilde{A}_k} \sum_{j=1}^{k} q_j \lambda_j \prod_{\substack{i=1 \\ i \neq j}}^{k} \left(\frac{\lambda_i}{\lambda_i - \lambda_j} \right)^{q_i} \int_0^{+\infty} e^{-\lambda_j t} dt, \quad k = 2, 3, \ldots.$$

$$\int_0^{+\infty} P_k(t) dt = \frac{1}{\lambda_k} \sum_{(q_1, \ldots, q_k) \in \tilde{A}_k} \sum_{j=1}^{k} q_j \prod_{\substack{i=1 \\ i \neq j}}^{k} \left(\frac{\lambda_i}{\lambda_i - \lambda_j} \right)^{q_i}, \quad k = 2, 3, \ldots.$$

Moreover, from Eq. 3.12 in Orsingher et al. (2010), we have that

$$\sum_{j=1}^{k} \prod_{\substack{i=1 \\ i \neq j}}^{k} \left(\frac{\lambda_i}{\lambda_i - \lambda_j} \right) = 1, \quad \forall \lambda_i \ (i = 1, 2, \ldots, k) \ and \ \lambda_i \neq \lambda_j \ for \ i \neq j.$$

Therefore, we can conclude that

$$\sum_{j=1}^{k} q_j \prod_{\substack{i=1 \\ i \neq j}}^{k} \left(\frac{\lambda_i}{\lambda_i - \lambda_j} \right)^{q_i} = 1, \quad \forall (q_1, \ldots, q_k) \in \tilde{A}_k,$$

and that

$$\int_0^{+\infty} P_k(t)dt = \frac{1}{\lambda_k} \sum_{(q_1,\ldots,q_k)\in\tilde{A}_k} 1 = \frac{2^{k-2}}{\lambda_k}, \quad k = 2, 3, \ldots.$$

Thus we can write

$$1 - S(t) \leq \lambda_n \int_0^t P_n(t)dt \leq 2^{n-2}, \quad \forall n \geq 3$$

$$\Rightarrow \frac{(1 - S(t))}{\lambda_n} \leq \int_0^t P_n(t)dt \leq \frac{2^{n-2}}{\lambda_n}, \quad \forall n \geq 3. \tag{16}$$

Moreover we have that

$$\int_0^{+\infty} P_1(t)dt = \frac{1}{\lambda_1} \quad \text{and} \quad \int_0^{+\infty} P_2(t)dt = \frac{1}{\lambda_2},$$

so that we get

$$\frac{(1 - S(t))}{\lambda_n} \leq \int_0^t P_n(t)dt \leq \frac{1}{\lambda_n}, \quad \forall n = 1, 2. \tag{17}$$

By Eqs. 16, 17 and summing for $n = 1, \ldots m, \forall m = 3, 4, \ldots$, we have

$$(1 - S(t)) \sum_{n=1}^m \frac{1}{\lambda_n} \leq \int_0^t S_m(t)dt \leq \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \sum_{n=3}^m \frac{2^{n-2}}{\lambda_n}, \quad \forall m = 1, 2, \ldots,$$

$$\Rightarrow (1 - S(t)) \sum_{n=1}^m \frac{1}{\lambda_n} \leq \int_0^t S_m(t)dt \leq \sum_{n=1}^m \frac{2^n}{\lambda_n}, \quad \forall m = 1, 2, \ldots. \tag{18}$$

Therefore, using the comparison test for series, we have

$$\sum_{n=1}^{\infty} \frac{2^n}{\lambda_n} < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{2^n}{\lambda_n} = \infty. \tag{19}$$

Finally, similarly to the proof at page 452 of Feller (1968), we can conclude from Eqs. 18 and 19 that (ii) holds. □

Turning back to the fractional case, we prove the following result, which holds for any $\upsilon > 0$. Let $A^\alpha(t)$ be the stable subordinator of index α with parameters $\mu = 0$, $\beta = 1$ and $\sigma = \left(t\cos\left(\frac{\pi\alpha}{2}\right)\right)^{1/\alpha}$, for $t \geq 0$, in the notation of Samorodnitsky and Taqqu (1994); thus its Laplace transform can be written as $L\left[h_\alpha(x, t); \theta\right] = \exp\left\{-\theta^\alpha t \cos\left(\frac{\pi\alpha}{2}\right)\right\}$.

Furthermore its inverse (or hitting time) process is defined as $L^\alpha(t) = \inf\{z \mid z > 0, \ A^\alpha(z) > t\}$ for all $t \geq 0$.

Theorem 5 *Let $N_\upsilon(t)$ be the process defined as $N_1(H_\upsilon(t))$, $t \geq 0$, where*

$$H_\upsilon(t) = \begin{cases} L^\upsilon(t), & \upsilon \in (0, 1), \\ A^{\frac{1}{\upsilon}}(t), & \upsilon \in (1, +\infty), \end{cases}$$

and $H_v(t) = t$ for $v = 1$, *under the assumption that N_1 and H_v are independent. Then the distribution $P_k^v(t) = \Pr\{N_v(t) = k\}$, $k \geq 1$, satisfies the problem* (1)–(2) *for any $v > 0$.*

Proof By definition, we can write

$$
P_k^v(t) =
\begin{cases}
\displaystyle\int_0^{+\infty} P_k(z)\, l_v(z,t)\, dz, & v \in (0,1], \\[3ex]
\displaystyle\int_0^{+\infty} P_k(z)\, h_{\frac{1}{v}}(z,t)\, dz, & v \in (1,+\infty),
\end{cases}
$$

where $l_v(z,t)$ and $h_{\frac{1}{v}}(z,t)$ are the densities of $L^v(t)$ and $A^{\frac{1}{v}}(t)$, respectively.

Case 1 $v \in (0,1]$:
  We set

$$
a_j^k =
\begin{cases}
1 & k = 1,\ j = 1, \\[2ex]
\displaystyle\sum_{(q_1,\ldots,q_k)\in \tilde{A}_k} q_j \prod_{r=1}^{k-1} \lambda_r^{q_r} \times \prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{1}{(\lambda_i - \lambda_j)^{q_i}} & k \geq 2,\ j = 1,\ldots,k,\ ,
\end{cases}
$$

then formula (10) can be written as $P_k^v(t) = \sum_{j=1}^{k} a_j^k E_{v,1}\left(-\lambda_j t^v\right)$. Let $G_v(u,t) = \sum_{k=1}^{\infty} u^k P_k^v(t)$ be the probability generating function, then we can write

$$
\int_0^{+\infty} e^{-\mu t} G_v(u,t)\, dt = \int_0^{+\infty} e^{-\mu t} \sum_{k=1}^{\infty} u^k P_k^v(t)\, dt
$$

$$
= \int_0^{+\infty} e^{-\mu t} \sum_{k=1}^{\infty} u^k \sum_{j=}^{k} a_j^k E_{v,1}\left(-\lambda_j t^v\right) dt
$$

$$
= \sum_{k=1}^{\infty} u^k \sum_{j=1}^{k} a_j^k \frac{\mu^{v-1}}{\mu^v + \lambda_j} = \sum_{k=1}^{\infty} u^k \mu^{v-1} \sum_{j=1}^{k} a_j^k \int_0^{+\infty} e^{-s(\mu^v + \lambda_j)}\, ds
$$

$$
= \int_0^{+\infty} \mu^{v-1} e^{-s\mu^v} \sum_{k=1}^{\infty} u^k \sum_{j=1}^{k} a_j^k e^{-s\lambda_j}\, ds
$$

$$
= \int_0^{+\infty} \mu^{v-1} e^{-s\mu^v} \sum_{k=1}^{\infty} u^k P_k(s)\, ds
$$

$$
= \int_0^{+\infty} \mu^{v-1} e^{-s\mu^v} G_1(u,s)\, ds
$$

$$
= \int_0^{+\infty} e^{-\mu t} \int_0^{+\infty} G_1(u,s)\, l_v(s,t)\, ds\, dt,
$$

since $\mu^{\upsilon-1}e^{-s\mu^{\upsilon}} = \int_0^{+\infty} e^{-\mu t}l_{\upsilon}(s,t)dt$ (see Hahn et al. 2001). Thus, we get

$$G_{\upsilon}(u,t) = \int_0^{+\infty} G_1(u,s)\,l_{\upsilon}(s,t)ds \Rightarrow P_k^{\upsilon}(t) = \int_0^{+\infty} P_k(s)\,l_{\upsilon}(s,t)ds.$$

Case 2 $\upsilon \in (1,+\infty)$:

For $P_k^{\upsilon}(t) = \int_0^{+\infty} P_k(z)\,h_{\frac{1}{\upsilon}}(z,t)dz$, we can write

$$_{RL}D_{-,t}^{\upsilon}P_k^{\upsilon}(t) = \int_0^{+\infty} P_k(z)\,_{RL}D_{-,t}^{\upsilon}h_{\frac{1}{\upsilon}}(z,t)dz = \int_0^{+\infty} P_k(z)\,\frac{\partial}{\partial z}h_{\frac{1}{\upsilon}}(z,t)dz$$

$$= \left[P_k(z)\,h_{\frac{1}{\upsilon}}(z,t)\right]_{z=0}^{+\infty} - \int_0^{+\infty} \frac{d}{dz}P_k(z)\,h_{\frac{1}{\upsilon}}(z,t)dz$$

$$= -\int_0^{+\infty}\left(-\lambda_k P_k(z) + \sum_{i=1}^{k-1}\lambda_i P_i(z)\right)h_{\frac{1}{\upsilon}}(z,t)dz$$

$$= \int_0^{+\infty}\lambda_k P_k(z)\,h_{\frac{1}{\upsilon}}(z,t)dz - \sum_{i=1}^{k-1}\lambda_i\int_0^{+\infty}P_i(z)h_{\frac{1}{\upsilon}}(z,t)dz$$

$$= \lambda_k P_k^{\upsilon}(t) - \sum_{i=1}^{k-1}\lambda_i P_i^{\upsilon}(t).$$

It is well-known that $\lim_{z\to\infty} h_{\frac{1}{\upsilon}}(z,t) = 0$ (see Uchaikin and Zolotarev 1999). Moreover the law of $A^{1/\upsilon}(t)$ is governed by the following equation:

$$_{RL}D_{-,t}^{\upsilon}h_{\frac{1}{\upsilon}}(z,t) = \frac{\partial}{\partial z}h_{\frac{1}{\upsilon}}(z,t), \quad z,t>0, \ \upsilon\in(1,+\infty),$$

with the following conditions

$$\begin{cases} h_{\frac{1}{\upsilon}}(0,t) = 0, \\ h_{\frac{1}{\upsilon}}(z,0) = \delta(z). \end{cases}$$

The previous result is proved, for $\upsilon = n \in \mathbb{N}$, in D'Ovidio (2011) and later generalized to any $\upsilon > 1$ in Beghin and Macci (2012) (see formula (5.17), with $\gamma = 1/\upsilon$, in the last reference). The proof is complete by considering the definition $\frac{d^{\upsilon}}{dt^{\upsilon}} := -_{RL}D_{-,t}^{\upsilon}, \upsilon \in (1,+\infty)$. $\square$

Remark 3 The processes representing the random-time arguments in Theorem 5 exhibit different properties, in the two cases $\upsilon \in (0,1]$ and $\upsilon \in (1,+\infty)$: the stable subordinator $A^{\frac{1}{\upsilon}}(t)$ is strictly increasing and jumps with non-negative probability, while its inverse $L^{\upsilon}(t)$ is continuous, non-decreasing and it is not a Lévy process. As a consequence, the sample paths of the two forms of fractional birth processes, i.e. $N_1\left(A^{\frac{1}{\upsilon}}(t)\right)$ and $N_1(L^{\upsilon}(t))$, display a very different behavior. Indeed in the first case

the process can perform only unitary jumps in an interval of infinitesimal length (as in the standard case $\upsilon = 1$), while in the latter the number of jumps can be of any integer size.

We are now ready to obtain an explicit expression for the solution to Eqs. 1 and 2 in the case $\upsilon \in (1, +\infty)$.

Theorem 6 *The solution to problem* (1) *and* (2) *is given, for* $\upsilon \in (1, +\infty)$, *by*

$$
P_k^\upsilon(t) = \begin{cases} e^{-\lambda_1^{\frac{1}{\upsilon}} t} & k = 1, \\ \displaystyle\sum_{j=1}^{k} \left(\sum_{(q_1,\ldots,q_k)\in\tilde{A}_k} q_j \left(\prod_{r=1}^{k-1} (\lambda_r)^{q_r} \right) \times \left(\prod_{\substack{i=1 \\ i\neq j}}^{k} \frac{1}{(\lambda_i - \lambda_j)^{q_i}} \right) \right) e^{-\lambda_j^{\frac{1}{\upsilon}} t} & k \geq 2, \end{cases}
$$

Proof From Theorem 5, for $\upsilon \in (1, +\infty)$, by considering formula (15), we can get

$$
P_k^\upsilon(t) = \int_0^{+\infty} P_k(z)\, h_{\frac{1}{\upsilon}}(z, t)\, dz
$$

$$
= \begin{cases} \displaystyle\int_0^{+\infty} e^{-\lambda_1 z} h_{\frac{1}{\upsilon}}(z, t)\, dz & k = 1, \\ \displaystyle\sum_{j=1}^{k} \left(\sum_{(q_1,\ldots,q_k)\in\tilde{A}_k} q_j \left(\prod_{r=1}^{k-1} (\lambda_r)^{q_r} \right) \times \left(\prod_{\substack{i=1 \\ i\neq j}}^{k} \frac{1}{(\lambda_i - \lambda_j)^{q_i}} \right) \right) \int_0^{+\infty} e^{-\lambda_j z} h_{\frac{1}{\upsilon}}(z, t)\, dz & k \geq 2, \end{cases}
$$

$$
= \begin{cases} e^{-\lambda_1^{\frac{1}{\upsilon}} t} & k = 1, \\ \displaystyle\sum_{j=1}^{k} \left(\sum_{(q_1,\ldots,q_k)\in\tilde{A}_k} q_j \left(\prod_{r=1}^{k-1} (\lambda_r)^{q_r} \right) \times \left(\prod_{\substack{i=1 \\ i\neq j}}^{k} \frac{1}{(\lambda_i - \lambda_j)^{q_i}} \right) \right) e^{-\lambda_j^{\frac{1}{\upsilon}} t} & k \geq 2, \end{cases}
$$

since $\int_0^{+\infty} e^{-\lambda_j z} h_{\frac{1}{\upsilon}}(z, t)\, dz = e^{-\lambda_j^{\frac{1}{\upsilon}} t}$ (see Proposition 1.2.12 in Samorodnitsky and Taqqu 1994).

Fig. 3 Plot of $P_k^\upsilon(t)$ for $\upsilon = 2.75$, $k = 1, 2, 3$ and $\lambda_i = 2^{-i}$ $(i \in \mathbb{N})$
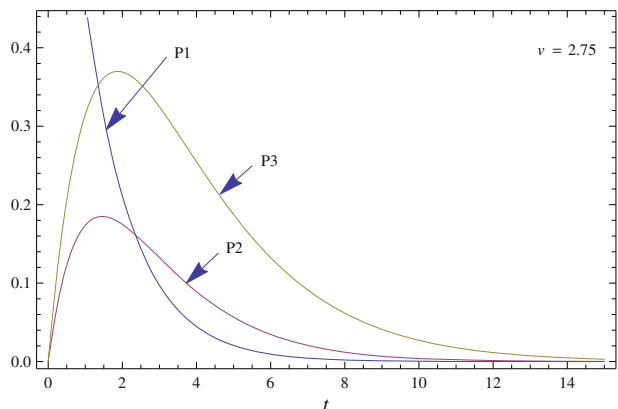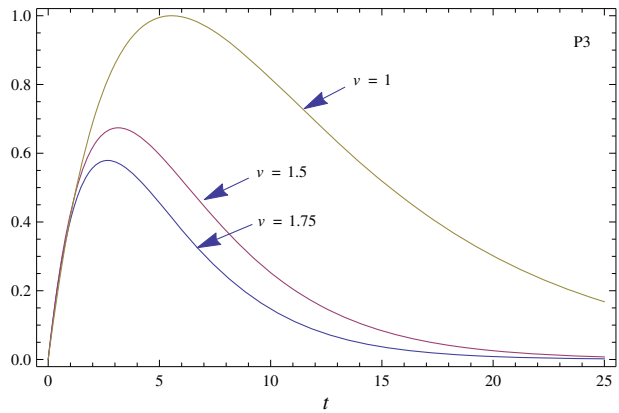
Fig. 4 Plot of $P_3^v(t)$ for $v = 1$, 1.5, 1.75 and $\lambda_i = 2^{-i}$ ($i \in \mathbb{N}$)

In order to check that the initial conditions in Eq. 2 are satisfied we write:

$$P_k^v(0) = \int_0^{+\infty} P_k(z) h_{\frac{1}{v}}(z, 0)\, dz$$

$$= P_1^k(0) = \begin{cases} 1, & k = 1 \\ 0, & k = 2, 3, \ldots \end{cases},$$

since it is well-known that $h_{\frac{1}{v}}(z, 0) = \delta(z)$.   □

Let $\lambda_i = 2^{-i}$ ($i \in \mathbb{N}$), then we can see the behavior of $P_k^{2.75}(t)$ for $k = 1, 2, 3$ and $P_3^v(t)$ for $v = 1, 1.5, 1.75$, in Figs. 3 and 4, respectively.

We check now the conditions under which the solution to problem (1) and (2) represents a proper probability distribution.

Corollary 7 *The following implications hold:*

$$P_k(t) \geq 0\ \forall k \geq 1,\ t > 0 \qquad \Rightarrow \qquad P_k^v(t) \geq 0\ \forall k \geq 1,\ t > 0,\ v > 0,$$

$$\sum_{k=1}^{\infty} P_k(t) = 1\ \forall t > 0 \qquad \Rightarrow \qquad \sum_{k=1}^{\infty} P_k^v(t) = 1\ \forall t > 0,\ v > 0,$$

where $P_k(t) = \Pr\{N_1(t) = k\}\ \forall k \geq 1,\ t > 0$.

Proof It easily follows from Theorem 5.   □

Corollary 8 *If $\lambda_i \geq 0$, ($i = 1, 2, \ldots$), then*

(i) $P_k^v(t) \geq 0, t \geq 0, k = 1, 2, \ldots$.

(ii) $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty \Rightarrow \sum_{k=1}^{\infty} P_k^v(t) = 1, \forall t \geq 0$.

Proof It follows immediately from Corollary 7 and Theorem 4.   □

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