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Generalized Frames with C^*-Valued Bounds and their Operator Duals

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Abstract. Certain facts about frames and generalized frames are extended for the new g-frames, referred as $*$-g-frames, in a Hilbert C^*-modules. As a matter of fact, some relations are establish between $*$-frames and $*$-g-frames in a Hilbert C^*-module. Furthermore, the paper studies the operators associated to a given $*$-g-frame, the construction of new $*$-g-frames. Moreover, the operator duals for a $*$-g-frame are introduced and their properties are investigated. Finally, operator duals of a $*$-g-frame are characterized.

1. Introduction

Frame theory is a new and applicable part of harmonic analysis. This theory has been rapidly generalized and various generalizations consisting of vectors in Hilbert spaces or Hilbert C^*-modules have been developed. In 2005, Sun [10] has introduced the notion of g-frames as a generalization of frames for bounded operators on Hilbert spaces. Frank-Larson [4] have extended the theory for the elements of C^*-algebras and (finitely or countably generated) Hilbert C^*-modules. Afterwards, frames with C^*-valued bounds in Hilbert C^*-modules have been considered in [2].

It is well known that Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Also, the theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, and KK-theory. There are some differences between Hilbert C^*-modules and Hilbert spaces. For instance, the Riesz representation theorem for continuous linear functionals on Hilbert spaces can not be extended to Hilbert C^*-modules [9] and there exist closed subspaces in Hilbert C^*-modules that have no orthogonal complement [7]. Moreover, as known, every bounded operator on a Hilbert space has an adjoint whereas there are bounded operators on Hilbert C^*-modules which do not drive this property [8]. So, it is expected that problems about frames and $*$-frames for Hilbert C^*-modules are more complicated than those for Hilbert spaces. This makes the topic of the frames for Hilbert C^*-modules important and absorbing. We would like to point out here that the properties of g-frames for Hilbert C^*-modules have been widely investigated in the literature; for further details see [1], [2], [4], [5], [11] and the references therein. The main purpose of the present paper is to study the subject of g-frames with C^*-valued bounds and their operator duals in a Hilbert C^*-module.

The outline of paper is organized as follows. In the next section, we give a brief survey on some of fundamental definitions and notations of Hilbert C^*-modules, g-frames and $*$-frames in Hilbert C^*-modules.

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Section 3 is devoted to investigating $*$-g-frames with $\mathcal{A}$-valued bounds and analyzing the elementary properties of them. In addition, some nontrivial examples of $*$-g-Bessel sequences and $*$-g-frames are presented which that their $\mathcal{A}$-valued bounds are better than their real valued bounds. That is, we give a tight $*$-g-frame with $\mathcal{A}$-valued bounds which can not be a tight g-frame with real valued bounds. At the end of this section, the relation between g-frames and $*$-g-frames in a Hilbert C^*-module is presented. In Section 4, some the conditions for combination of two $*$-g-frames are obtained. More precisely, new $*$-g-frames and $*$-frames are constructed. The last section contains definition and characterization of the generalized duals of a $*$-g-frame where they are called the operator duals.

2. Preliminaries

In this section, we present a brief account of basic definitions and some properties of Hilbert C^*-modules and their frames. For more information, we refer readers to [6], [9].

Suppose $\mathcal{A}$ is a C^*-algebra. A linear space $\mathcal{H}$ which is also an algebraic (left) $\mathcal{A}$-module together with an $\mathcal{A}$-inner product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \longrightarrow \mathcal{A}$ and possesses the following properties is called a pre-Hilbert C^*-module:

(*i*) $\langle f, f\rangle \geq 0$, for any $f \in \mathcal{H}$.
(*ii*) $\langle f, f\rangle = 0$ if and only if $f = 0$.
(*iii*) $\langle f, g\rangle = \langle g, f\rangle^*$, for any $f, g \in \mathcal{H}$.
(*iv*) $\langle \lambda f, h\rangle = \lambda\langle f, h\rangle$, for any $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}$.
(*v*) $\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$, for any $a, b \in \mathcal{A}$ and $f, g, h \in \mathcal{H}$.

If $\mathcal{H}$ is a Banach space with respect to the induced norm by the $\mathcal{A}$-valued inner product, then $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is called a Hilbert C^*-module over $\mathcal{A}$ or, simply, a Hilbert $\mathcal{A}$-module.

The class of all adjointable maps from Hilbert C^*-module $\mathcal{H}$ into Hilbert C^*-module $\mathcal{K}$ is indicated by $B_*(\mathcal{H},\mathcal{K})$ and the class of all bounded $\mathcal{A}$-module maps from $\mathcal{H}$ into $\mathcal{K}$ is signified by $B_b(\mathcal{H},\mathcal{K})$. It is known that $B_*(\mathcal{H},\mathcal{K}) \subseteq B_b(\mathcal{H},\mathcal{K})$. We denote $B_*(\mathcal{H},\mathcal{H})$ and $B_b(\mathcal{H},\mathcal{H})$ by $B_*(\mathcal{H})$ and $B_b(\mathcal{H})$, respectively.

Throughout the paper, we fix the notations $\mathcal{A}$ and J for a given unital C^*-algebra and a finite or countably infinite index set, respectively. Also, the sets $\mathcal{H}$ and $\mathcal{K}_j$, for all $j \in J$, are finitely or countably generated Hilbert $\mathcal{A}$-modules. The j^{th} projection operator from $\oplus_{j\in J}\mathcal{K}_j$ onto $\mathcal{K}_j$ is represented by π_j.

The notion of a g-frame for a given separable Hilbert space has been introduced by Sun [10]. Then, the authors [5] has defined a g-frame for a Hilbert $\mathcal{A}$-module $\mathcal{H}$, as a family of ordered pairs $\{(\Lambda_j, \mathcal{K}_j) : j \in J\}$ consisting of Hilbert $\mathcal{A}$-modules $\mathcal{K}_j$ and operators $\Lambda_j \in B_*(\mathcal{H},\mathcal{K}_j)$ satisfying

$$A\langle f, f\rangle \leq \sum_{j\in J} < \Lambda_j f, \Lambda_j f > \leq B\langle f, f\rangle,$$

for all $f \in \mathcal{H}$ and some positive constants A and B independent of f.

Afterwards, Dehghan-Alijani [2] have developed the following new version of frames for Hilbert $\mathcal{A}$-modules called $*$-frames as the family $\{f_j\}_{j\in J}$ in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ which satisfy

$$A\langle f, f\rangle A^* \leq \sum_{j\in J}\langle f, f_j\rangle\langle f, f_j\rangle^* \leq B\langle f, f\rangle B^*,$$

for all $f \in \mathcal{H}$ and some strictly nonzero elements A and B in $\mathcal{A}$ independent of f.

3. $*$-g-Frames for Hilbert C^*-Modules

In this section, we study the generalized Bessel sequences and the generalized frames with C^*-valued bounds for a Hilbert C^*-module and compare them with the ordinary types.

Definition 3.1. *A $*$-g-frame for $\mathcal{H}$ is a collection of ordered pairs $\{(\Lambda_j, \mathcal{K}_j) : j \in J\}$ such that*

$$A\langle f, f\rangle A^* \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq B\langle f, f\rangle B^*,$$

for all $f \in \mathcal{H}$ and strictly nonzero elements A and B in $\mathcal{A}$.

The numbers A and B are called lower and upper $*$-g-frame bounds, respectively. If $A = B$, the $*$-g-frame is called tight and it is normalized when $A = B$.

The sequence of ordered pairs $\{(\Lambda_j, \mathcal{K}_j) : j \in J\}$ is called to be a $*$-g-Bessel sequence for $\mathcal{H}$ if it has the upper bound condition in the above inequality. In this case, the element B is called the upper $*$-g-Bessel bound.

Since the normalized $*$-g-frames and the normalized g-frames are the same, the definition of a $*$-g-orthonormal basis is the same as the definition of a g-orthonormal basis. Then we can use them.

The sequence $\{(\Lambda_j, \mathcal{K}_j) : j \in J\}$ is said to be a g-orthonormal basis if it is a g-frame for $\mathcal{H}$ and satisfies

i. $\Lambda_i\Lambda_j^* g_j = \delta_{ij} g_j$, for any $i, j \in J$; and

ii. $\sum_{j\in J}\Lambda_j^*\Lambda_j f = f$, for all $j \in J$.

(Throughout the paper, series are assumed to be convergent in the norm sense.)

Remark 3.2. *If $\{(\Lambda_j, \mathcal{K}_j) : j \in J\}$ is a $*$-g-Bessel sequence for the Hilbert $\mathcal{A}$-module $\mathcal{H}$ with a $*$-g-Bessel bound B, then $\{\Lambda_j\}_{j\in J}$ is uniformly bounded by $\|B\|$.*

We mentioned that the set of all of g-frames in a Hilbert $\mathcal{A}$-modules can be considered as a subset of the family of $*$-g-frames. To illustrate this, let $\{(\Lambda_j, \mathcal{K}_j) : j \in J\}$ be a g-frame for the Hilbert $\mathcal{A}$-module $\mathcal{H}$ with real g-frame bounds A and B. Note that for $f \in \mathcal{H}$,

$$(\sqrt{A})1_{\mathcal{A}}\langle f, f\rangle(\sqrt{A})1_{\mathcal{A}} \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq (\sqrt{B})1_{\mathcal{A}}\langle f, f\rangle(\sqrt{B})1_{\mathcal{A}}.$$

Therefore, every g-frame for $\mathcal{H}$ with real bounds A and B is a $*$-g-frame for $\mathcal{H}$ with $\mathcal{A}$-valued $*$-g-frame bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{A}}$.

To throw more light on the subject and understand the use of the concepts, we include some examples of nontrivial $*$-g-Bessel sequences and $*$-g-frames and we show that $\mathcal{A}$-valued bounds are preferred to real-valued bounds in some cases.

Example 3.3. *Let $\mathcal{A}$ be a commutative unital C^*-algebra, $\mathcal{H}$ be the Hilbert $\mathcal{A}^2$-module $\mathcal{A}^2$ and let $J = \mathbb{N}$ and fix nonzero sequences $(a_j)_{j\in J}$ and $(b_j)_{j\in J}$ such that $\sum_{j\in J} a_j a_j^*$ and $\sum_{j\in J} b_j b_j^*$ are invertible elements in $\mathcal{A}$. Define the diagonal operators Λ_j =diag$\{a, b\}$ on $\mathcal{A}^2$ sending (w_1, w_2) to $(a_j w_1, b_j w_2)$. The sequence $\{(\Lambda_j, \mathcal{A}^2) : j \in J\}$ is a tight $*$-g-frame with bound $(\sum_{j\in J} a_j a_j^*, \sum_{j\in J} b_j b_j^*)^{\frac{1}{2}}$. Note that, $\{(\Lambda_j, \mathcal{A}^2)\}_{j\in J}$ is a g-Bessel sequence with real bound $\|(\sum_{j\in J} a_j a_j^*, \sum_{j\in J} b_j b_j^*)\|$ and therefore the $\mathcal{A}^2$-valued bound is optimal rather than the real valued bound.*

Example 3.4. *Let $\mathcal{A} = \ell^\infty$ and let $\mathcal{H} = C_0$, the Hilbert $\mathcal{A}$-module of the set of all null sequences equipped with the $\mathcal{A}$-inner product*

$$\langle (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\rangle = (x_i\overline{y_i})_{i\in\mathbb{N}}.$$

The action of each sequence $(a_i)_{i\in\mathbb{N}} \in \mathcal{A}$ on a sequence $(x_i)_{i\in\mathbb{N}} \in \mathcal{H}$ is implemented as $(a_i)_{i\in\mathbb{N}}(x_i)_{i\in\mathbb{N}} = (a_i x_i)_{i\in\mathbb{N}}$. Let $j \in J = \mathbb{N}$ and $(1 + \frac{1}{i})_{i\in\mathbb{N}} \in \ell^\infty$. Define $\Lambda_j \in B_*(\mathcal{H})$ by

$$\Lambda_j(x_i)_{i\in\mathbb{N}} = (\delta_{ij} a_j x_j)_{i\in\mathbb{N}}, \qquad \forall (x_i)_{i\in\mathbb{N}} \in \mathcal{H}.$$

We observe that

$$\sum_{j\in\mathbb{N}}\langle \Lambda_j x, \Lambda_j x\rangle = ((1 + \frac{1}{i})^2 x_i\overline{x_i})_{i\in\mathbb{N}} = (1 + \frac{1}{i})_{i\in\mathbb{N}}\langle x, x\rangle(1 + \frac{1}{i})_{i\in\mathbb{N}}, \;\; \forall x = (x_i)_{i\in\mathbb{N}} \in \mathcal{H}.$$

Thus $\{(\Lambda_j, \mathcal{H})\}_{j\in J}$ is a tight $*$-g-frame with bounds $(1 + \frac{1}{i})_{i\in\mathbb{N}}$, (The element $(1 + \frac{1}{i})_{i\in\mathbb{N}}$ is strictly nonzero in $\mathcal{A}$). But it is not a tight g-frame for Hilbert l^∞-module C_0. Note that, $\{(\Lambda_j, \mathcal{H})\}_{j\in J}$ is a g-frame with optimal lower and upper real bounds 1 and 2, respectively.

In the frame theory, operators play an important role. for example, by the *pre-∗-frame operator*, duals of g-frames are characterized and the *frame operator* is used to give the reconstruction formula. The definitions of pre-∗-frame operator and frame operator are similar to ordinary types in Hilbert C^*-modules.

Definition 3.5. *Given a ∗-g-Bessel sequence* $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ *in a Hilbert* $\mathcal{A}$*-module* $\mathcal{H}$ *with bound B, its corresponding pre-∗-g-frame operator is an operator* Θ *from* $\mathcal{H}$ *into* $\oplus_{j\in J}\mathcal{K}_j$ *by* $\Theta f = (\Lambda_j f)_{j\in J}$.

It is easily to see that the pre-∗-frame operator is adjointable and then we can characterize ∗-g-Bessel sequences with respect to the adjointable $\mathcal{A}$-module maps.

Theorem 3.6. *The set of all ∗-g-Bessel sequences for* $\mathcal{H}$ *with respect to* $\{\mathcal{K}_j\}_{j\in J}$ *is precisely*

$$\{(\pi_j\Theta)_{j\in J} : \Theta \in B_*(\mathcal{H}, \oplus_{j\in J}\mathcal{K}_j)\}.$$

Definition 3.7. *Given a ∗-g-frame* $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ *in* $\mathcal{H}$ *with bounds A and B. The ∗-g-frame operator of* $\{\Lambda_j\}_{j\in J}$ *is an operator S by* $Sf = \sum_{j\in J}\Lambda_j^*\Lambda_j f$ *for all* $f \in \mathcal{H}$.

In this case, the ∗-g-frame operator has some properties similar to g-frame operator and some others is not similar.

Theorem 3.8. *Let* $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ *be a ∗-g-frame for* $\mathcal{H}$ *with ∗-g-frame operator S and lower and upper ∗-g-frame bounds A and B, respectively. Then S is positive, invertible and adjointable. Also,*

$$\|A^{-1}\|^{-2} \le \|S\| \le \|B\|^2 \ , \ f = \sum_{j\in J}\Lambda_j^*\Lambda_j S^{-1}f,$$

are valid for $f \in \mathcal{H}$.

Proof. Since $\langle Sf, f\rangle = \sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle$, for $f \in \mathcal{H}$, and the set of positive elements of $\mathcal{A}$ is closed, S is a positive element in C^*-algebra $B_*(\mathcal{H})$. We show that S is invertible . For see this, we use an other operator. By positivity of S, there is a positive element G in $B_*(\mathcal{H})$ such that $S = G^*G$. Let $\{Gf_n\}_{n\in\mathbb{N}}$ be a sequence in R_G such that $Gf_n \longrightarrow g$ as $n \to \infty$. For $n, m \in \mathbb{N}$,

$$\|A\langle f_n - f_m, f_n - f_m\rangle A^*\| \le \|\langle S(f_n - f_m), f_n - f_m\rangle\| = \|G(f_n - f_m)\|^2.$$

Since $\{Gf_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$,

$$\|A\langle f_n - f_m, f_n - _m\rangle A^*\| \longrightarrow 0 \quad as \quad n, m \to \infty.$$

Note that for $n, m \in \mathbb{N}$,

$$\|\langle f_n - f_m, f_n - f_m\rangle\| = \|A^{-1}A\langle f_n - f_m, f_n - f_m\rangle A^*(A^*)^{-1}\| \le \|A^{-1}\|^2\|A\langle f_n - f_m, f_n - f_m\rangle A^*\|.$$

Therefore the sequence $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy and hence there exists $f \in \mathcal{H}$ such that $f_n \longrightarrow f$ as $n \to \infty$. Again by the definition of ∗-g-frames, the following inequality holds,

$$\|G(f_n - f)\|^2 \le \|B\|^2\|\langle f_n - f, f_n - f\rangle\|.$$

Thus $\|Gf_n - Gf\| \longrightarrow 0$ as $n \to \infty$ implies that $Gf = g$. It concludes that R_G is closed.

By the like proof, G is injective. Therefore G is injective, closed range and self-adjoint and hence S is invertible. For the rest of the proof, we show the inequality. The definition of ∗-g-frames implies that $\langle f, f\rangle \le A^{-1}\langle Sf, f\rangle(A^*)^{-1}$ and $\langle Sf, f\rangle \le B\langle f, f\rangle B^*$, and then

$$\|A^{-1}\|^{-2}\|\langle f, f\rangle\| \le \|\langle Sf, f\rangle\| \le \|B\|^2\|\langle f, f\rangle\|, \qquad \forall f \in \mathcal{H}.$$

If we take supremum on all $f \in \mathcal{H}$, where $\|f\| \le 1$, then $\|A^{-1}\|^{-2} \le \|S\| \le \|B\|^2$. In the end, for $f \in \mathcal{H}$, we obtain

$$f = SS^{-1}f = \sum_{j\in J}\Lambda_j^*\Lambda_j S^{-1}f.$$

□

Finding optimal bounds plays an important role to study of g-frames and $*$-g-frames. As we saw in the examples that their $\mathcal{A}$-valued bounds may be more suitable than real valued bounds for a $*$-g-frame. In addition, there were tight $*$-g-frames that they are not tight g-frames. At the end of the section, we introduce lower and upper real bounds for every $*$-g-frame and we see that $*$-g-frames can be studied as g-frames with different bounds.

Theorem 3.9. *Let $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ be a $*$-g-frame for $\mathcal{H}$ with pre-$*$-g-frame operator Θ and lower and upper $*$-g-frame bounds A and B, respectively. Then $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{H}$ with lower and upper frame bounds $\|(\Theta^*\Theta)^{-1}\|^{-1}$ and $\|\Theta\|^2$, respectively.*

Proof. By Theorem 3.8, Θ is injective and has closed range and obtain

$$\|(\Theta^*\Theta)^{-1}\|^{-1}\langle f, f\rangle \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq \|\Theta\|^2\langle f, f\rangle, \quad \forall f \in \mathcal{H},$$

by Lemma 2.7 [1]. Then $\{\Lambda_j\}_{j\in J}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $\|(\Theta^*\Theta)^{-1}\|^{-1}$ and $\|\Theta\|^2$, respectively. □

In the reminder of the paper, the given results are valid for g-frames in Hilbert C^*-modules by Theorem 3.9.

Remark 3.10. *Suppose $\mathcal{A}$ is the self-dual Hilbert $\mathcal{A}$-module $\mathcal{A}$ when $\mathcal{A}$ is a commutative C^*-algebra. Then for every $*$-g-frame $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$, there exists the sequence $\{f_j\}_{j\in J}$ in $\mathcal{A}$ such that*

$$\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle = \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle, \ \ \forall f \in \mathcal{H}.$$

In [2], we shown that $\sum_{j\in J}|f_j|^2$ is invertible and then every $*$-frame in the Hilbert $\mathcal{A}$-module $\mathcal{A}$ is tight $*$-frame. By the equality and the invertibility of $\sum_{j\in J}|f_j|^2$, the every $*$-g-frame in $\mathcal{A}$ is tight.

4. The New $*$-g-Frames and Frames

In this section, we consider some conditions for the composition of two $*$-g-frames. Also, the new $*$-g-frames are given with the other $*$-g-frames, the $*$-frames, an element of $\mathcal{H}$, and the $\mathcal{A}$-valued multiples of a $*$-g-frame.

Theorem 4.1. *Assume that $\Lambda = \{(\Lambda_j, \mathcal{K}_j) : j \in J\}$ and $\Gamma = \{(\Gamma_j, \mathcal{K}_j) : j \in J\}$ are $*$-g-Bessel sequences for Hilbert C^*-modules $\mathcal{H}_1$ and $\mathcal{H}_2$ with $*$-g-Bessel bounds B_Λ and B_Γ, respectively. Then $\Omega = \{(\Lambda_j^*\Gamma_j, \mathcal{H}_1) : j \in J\}$ is a $*$-g-Bessel sequence for $\mathcal{H}_2$ with $*$-g-Bessel bound $\|B_\Lambda\|B_\Gamma$ and the pre-$*$-g-frame operator of Ω is a bounded operator Θ_Ω from $\mathcal{H}_2$ into $\oplus_{j\in J}\mathcal{H}_1$ by $\Theta_\Omega f = (\Lambda_j^*\Gamma_j f)_{j\in J}$.*

Proof. By the properties of adjointable operators and the definition of $*$-g-Bessel sequence Γ, we obtain for $f \in \mathcal{H}_2$,

$$\sum_{j\in J}\langle \Lambda_j^*\Gamma_j f, \Lambda_j^*\Gamma_j f\rangle \leq \sum_{j\in J}\|\Lambda_j^*\|^2\langle \Gamma_j f, \Gamma_j f\rangle \leq \|B_\Lambda\|^2\sum_{j\in J}\langle \Gamma_j f, \Gamma_j f\rangle \leq \|B_\Lambda\|B_\Gamma\langle f, f\rangle\|B_\Lambda\|B_\Gamma^*.$$

Then $\{\Lambda_j^*\Gamma_j\}_{j\in J}$ is a $*$-g-Bessel sequence with bound $\|B_\Lambda\|B_\Gamma$. The pre-$*$-g-frame operator of Ω is $\Theta_\Omega f = (\Lambda_j^*\Gamma_j f)_{j\in J}$ for all $f \in \mathcal{H}_2$, clearly. □

The following example illustrates this fact that Theorem 4.1 is not valid for the composition of two $*$-g-frames.

Example 4.2. *Let T be the right shift operator in $B_*(l^2(\mathcal{A}))$ and let α be an element in the center of $\mathcal{A}$. Assume that Λ is defined by $\Lambda := \alpha T$. Since $\langle \Lambda(a_i)_{i\in\mathbb{N}}, \Lambda(a_i)_{i\in\mathbb{N}}\rangle = \alpha\langle (a_i)_{i\in\mathbb{N}}, (a_i)_{i\in\mathbb{N}}\rangle\alpha^*$ on $l^2(\mathcal{A})$. The single set $\{\Lambda\}$ is an α-tight $*$-g-frame for $l^2(\mathcal{A})$, but the single set $\{\Lambda^*\}$ is not a $*$-g-frame. To see this, we choose the subsequence $\{(n,1,0,0,...) : n \in \mathbb{N}\}$ in $l^2(\mathcal{A})$. There dose not exist $A > 0$ such that*

$$A\langle (n,1,0,0,...),(n,1,0,0,...)\rangle A^* \leq \langle \Lambda^*(n,1,0,0,...), \Lambda^*(n,1,0,0,...)\rangle,$$

$$\|A(n^2+1)A^*\|^2 \leq \|\alpha\|^2, \quad \forall n \in \mathbb{N}.$$

Then $\{\Lambda^*\}$ has not lower bound condition and is not a $*$-g-frame, whereas $\{\Lambda^*\} = \{\Lambda^* I\}$ is the composition of two $*$-g-frames $\{\Lambda\}$ and $\{I\}$.

Now, we characterize the class of all of $*$-g-frames by $*$-g-orthonormal bases and the composition of $*$-g-frames. The following theorem illustrates that the lower bound condition is preserved in the composition of some $*$-g-frames.

Theorem 4.3. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ and K_j, for $j \in J$, be Hilbert C^*-modules. Let $\Lambda = \{(\Lambda_j, \mathcal{K}_j) : j \in J\}$ be a g-orthonormal basis for $\mathcal{H}_1$ and $\Gamma = \{(\Gamma_j, \mathcal{K}_j) : j \in J\}$. Then $\Omega = \{(\Lambda_j^*\Gamma_j, \mathcal{H}_1) : j \in J\}$ is a $*$-g-frame for $\mathcal{H}_2$ if and only if Γ is a $*$-g-frame for $\mathcal{H}_2$. Moreover, $S_\Omega = S_\Gamma$ where S_Ω and S_Γ are $*$-g-frame operators for Ω and Γ, respectively.*

Proof. By the definition of $*$-g-orthonormal basis Λ, we have

$$\sum_{j\in J}\langle \Lambda_j^*\Gamma_j f, \Lambda_j^*\Gamma_j f\rangle = \sum_{j\in J}\langle \Gamma_j f, \Gamma_j f\rangle, \qquad \forall f \in \mathcal{H}_2.$$

So $\{\Lambda_j^*\Gamma_j\}_{j\in J}$ is a $*$-g-frame if and only if the sequence $\{\Gamma_j\}_{j\in J}$ is a $*$-g-frame. By the above equality, obtain

$$\langle S_\Omega f, f\rangle = \langle \sum_{j\in J}\Gamma_j^*\Lambda_j\Lambda_j^*\Gamma_j f, f\rangle = \sum_{j\in J}\langle \Lambda_j^*\Gamma_j f, \Lambda_j^*\Gamma_j f\rangle = \sum_{j\in J}\langle \Gamma_j f, \Gamma_j f\rangle = \langle \sum_{j\in J}\Gamma_j^*\Gamma_j f, f\rangle = \langle S_\Gamma f, f\rangle,$$

for all $f \in \mathcal{H}_2$, then it concludes that $S_\Omega = S_\Gamma$ on $\mathcal{H}_2$. □

The following proposition illustrates the properties of $\mathcal{A}$-valued multiples of a $*$-g-frame.

Proposition 4.4. *If $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ is a $*$-g-frame for $\mathcal{H}$ with bounds A, B, and α is a strictly positive element in the center of $\mathcal{A}$, then $\{\alpha\Lambda_j\}_{j\in J}$ is a $*$-g-frame for $\mathcal{H}$ with bounds $\alpha A, \alpha B$.*

Proof. For $f \in \mathcal{H}$, we have

$$\sum_{j\in J}\langle \alpha\Lambda_j f, \alpha\Lambda_j f\rangle = \sum_{j\in J}\alpha\langle \Lambda_j f, \Lambda_j f\rangle\alpha^*.$$

By the definition of $*$-g-frame $\{\Lambda_j\}_{j\in J}$ and the properties of the inequalities in C^*-algebras, for $f \in \mathcal{H}$

$$\alpha A\langle f, f\rangle(\alpha A)^* \leq \sum_{j\in J}\langle \alpha\Lambda_j f, \alpha\Lambda_j f\rangle \leq \alpha B\langle f, f\rangle(\alpha B)^*.$$

It completes the proof. □

Later, some relations between $*$-frames and $*$-g-frames are considered. First step studies the image of elements of a $*$-g-frame on an element of $\mathcal{H}$. And second step considers the image of elements of a $*$-g-frame on elements of a $*$-frame.

Theorem 4.5. *Let $\{(\Lambda_j, \mathcal{H})\}_{j\in J}$ be a $*$-g-frame for $\mathcal{H}$ and let g be an element of $\mathcal{H}$ such that the series $\sum_{j\in J}\|\Lambda_j g\|^2$ is convergent and*

$$\{\alpha\Lambda_j g : \alpha \in \mathcal{A}\} = \mathcal{H},$$

for all $j \in J$. Then the sequence $\{\Lambda_j g\}_{j\in J}$ is a frame for $\mathcal{H}$.

Proof. For $j \in J$, suppose that the operator θ_j from $\mathcal{H}$ into $\mathcal{A}$ is defined by $\theta_j(f) = \langle f, \Lambda_j g\rangle$. It is bounded $\mathcal{A}$-module map, $\|\theta_j\| = \|\Lambda_j g\|$, and adjointable with the adjoint $\theta_j^*(\alpha) = \alpha\Lambda_j g$, for all $\alpha \in \mathcal{A}$. For $j \in J$ and $f \in \mathcal{H}$, we have

$$\sum_{j\in J}\langle f, \Lambda_j g\rangle\langle\Lambda_j g, f\rangle = \sum_{j\in J}\langle\theta_j f, \theta_j f\rangle \le \sum_{j\in J}\|\theta_j\|^2\langle f, f\rangle = \sum_{j\in J}\|\Lambda_j g\|^2\langle f, f\rangle.$$

Then $\{\Lambda_j g\}_{j\in J}$ has an upper bound condition with the upper bound $\sum_{j\in J}\|\Lambda_j g\|^2$. For the lower bound condition, we must use the equality $\{\alpha\Lambda_j g : \alpha \in \mathcal{A}\} = \mathcal{H}$, for all $j \in J$. It concludes that every θ_j^* is surjective and by Lemma 2.7 [1], the operator $\theta_j^*\theta_j$ is invertible and

$$\sum_{j\in J}\langle f, \Lambda_j g\rangle\langle\Lambda_j g, f\rangle = \sum_{j\in J}\langle\theta_j f, \theta_j f\rangle = \sum_{j\in J}\langle\theta_j^*\theta_j f, f\rangle \ge \sum_{j\in J}\|(\theta_j^*\theta_j)^{-1}\|^{-1}\langle f, f\rangle, \quad \forall f \in \mathcal{H}.$$

These show that $\{\Lambda_j g\}_{j\in J}$ is a frame for $\mathcal{H}$. □

Theorem 4.6. *Let $\{(\Lambda_j, \mathcal{H})\}_{j\in J}$ be a $*$-g-frame for $\mathcal{H}$ with bounds A_Λ and B_Λ, and let $\{f_i\}_{i\in I}$ be a $*$-frame for $\mathcal{H}$ with bounds A and B. Then the sequence $\{\Lambda_j^* f_i\}_{i\in I, j\in J}$ is a $*$-frame for $\mathcal{H}$ with bounds AA_Λ and BB_Λ.*

Proof. Assume that $f \in \mathcal{H}$. Then

$$\sum_{j\in J}\sum_{i\in I}\langle f, \Lambda_j^* f_i\rangle\langle\Lambda_j^* f_i, f\rangle = \sum_{j\in J}\sum_{i\in I}\langle\Lambda_j f, f_i\rangle\langle f_i, \Lambda_j f\rangle \le B\sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle B^* \le BB_\Lambda\langle f, f\rangle(BB_\Lambda)^*.$$

It shows that the sequence $\{\Lambda_j^* f_i\}_{i\in I, j\in J}$ has the upper bound condition. The proof of the lower bound condition is similar. □

Theorem 4.7. *Let $\{g_{ij}\}_{i\in I_j}$ be a $*$-frame for $\mathcal{K}_j$ with bounds A_j and B_j, for all $j \in J$, and let $\{\Lambda_j \in B_*(\mathcal{H}, \mathcal{K}_j)\}_{j\in J}$ be a sequence such that $\{\langle\Lambda_j f, \Lambda_j f\rangle; j \in J, f \in \mathcal{H}\}$ is a subset of the center of $\mathcal{A}$. If there exist two strictly positive elements C and D in $\mathcal{A}$ by the properties $C \le A_j A_j^*$ and $B_j B_j^* \le D$, then $\{\Lambda_j^* g_{ij}\}_{i\in I_j, j\in J}$ is a $*$-frame for $\mathcal{H}$ if and only if $\{\Lambda_j\}_{j\in J}$ is a $*$-g-frame for $\mathcal{H}$.*

Proof. Since C and D are strictly positive, there exist A and B strictly nonzero elements in $\mathcal{A}$ such that $C = AA^*$ and BB^*. Now, assume that $\{\Lambda_j^* g_{ij}\}_{i\in I_j, j\in J}$ is a $*$-frame with bounds α and β. For $f \in \mathcal{H}$, obtain

$$\begin{aligned}\alpha\langle f, f\rangle\alpha^* \le \sum_{j\in J}\sum_{i\in I_j}\langle f, \Lambda_j^* g_{ij}\rangle\langle\Lambda_j^* g_{ij}, f\rangle &= \sum_{j\in J}\sum_{i\in I_j}\langle\Lambda_j f, g_{ij}\rangle\langle g_{ij}, \Lambda_j f\rangle \\ &\le \sum_{j\in J} B_j\langle\Lambda_j f, \Lambda_j f\rangle B_j^* \le D\sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle = B\sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle B^*,\end{aligned}$$

then

$$B^{-1}\alpha\langle f, f\rangle(B^{-1}\alpha)^* \le \sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle.$$

So, $\{\Lambda_j\}_{j\in J}$ has a lower bound $B^{-1}\alpha$ in $\mathcal{A}$. Similarly, $A^{-1}\beta$ is an upper bound for $\{\Lambda_j\}_{j\in J}$. Conversely, let $\{\Lambda_j\}_{j\in J}$ be a $*$-g-frame with bounds A_Λ and B_Λ. Suppose $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{j\in J}\sum_{i\in I_j}\langle f, \Lambda_j^* g_{ij}\rangle\langle\Lambda_j^* g_{ij}, f\rangle &= \sum_{j\in J}\sum_{i\in I_j}\langle\Lambda_j f, g_{ij}\rangle\langle g_{ij}, \Lambda_j f\rangle \\ &\le \sum_{j\in J} B_j\langle\Lambda_j f, \Lambda_j f\rangle B_j^* \\ &= \sum_{j\in J} B_j B_j^*\langle\Lambda_j f, \Lambda_j f\rangle \le \sum_{j\in J} D\langle\Lambda_j f, \Lambda_j f\rangle \le BB_\Lambda\langle f, f\rangle(BB_\Lambda)^*.\end{aligned}$$

Similarly, for $f \in \mathcal{H}$

$$AA_\Lambda \langle f, f \rangle (AA_\Lambda)^* \leq \sum_{j\in J} \sum_{i \in I_j} \langle f, \Lambda_j^* g_{ij} \rangle \langle \Lambda_j^* g_{ij}, f \rangle.$$

Then $\{\Lambda_j^* g_{ij}\}_{i\in I_j, j\in J}$ is a $*$-frame and the proof is complete. □

5. The Operator Duals of $*$-g-Frames

In the frame theory, a collection of frames corresponding to a given frame that have a special relation with respect to first frame is defined. They are called dual frames. Afterwards, generalized duals have been introduced [3]. Here, the ordinary duals of a given $*$-g-frame are defined and these concepts are generalized. Then we consider their properties and characterize all of dual $*$-g-frames associated to a given $*$-g-frame in a Hilbert C^*-module. These facts are valid for g-frames in Hilbert spaces because of Hilbert C^*-modules are extended of Hilbert spaces.

Definition 5.1. *A $*$-g-frame $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ is a dual $*$-g-frame for a given $*$-g-frame $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ if $\sum_{j\in J} \Lambda_j^* \Gamma_j = I$. In particular, the $*$-g-frame $\{(\widetilde{\Lambda}_j, \mathcal{K}_j)\}_{j\in J} := \{(\Lambda_j S^{-1}, \mathcal{K}_j)\}_{j\in J}$ is called the canonical dual $*$-g-frame.*

Here, we extend this type of duals to larger than the family which are called operator duals.

Definition 5.2. *Let $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ and $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ be two the $*$-g-frames for $\mathcal{H}$. If there exists an invertible adjointable $\mathcal{A}$-module map Υ on $\mathcal{H}$ such that*

$$f = \sum_{j\in J} \Lambda_j^* \Gamma_j \Upsilon(f), \quad \forall f \in \mathcal{H},$$

then $\{\Gamma_j\}_{j\in J}$ is called to be an operator dual of $\{\Lambda_j\}_{j\in J}$.

Remark 5.3. *Every $*$-g-frame $\{\Lambda_j\}_{j\in J}$ with the frame operator S is an operator dual for itself. For see this, set $\Upsilon := S^{-1}$ and use Theorem 3.8.*

Remark 5.4. *Let $\Gamma = \{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ be an operator dual of the $*$-g-frame $\Lambda = \{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ in $\mathcal{H}$. Then for some invertible adjointable map $\Upsilon \in B_*(\mathcal{H})$,*

$$f = \sum_{j\in J} \Lambda_j^* \Gamma_j \Upsilon(f), \quad \forall f \in \mathcal{H}.$$

The equality shows that $I = (\Theta_\Lambda^* \Theta_\Gamma)\Upsilon$ where I is the identity map on $\mathcal{H}$, and Θ_Γ and Θ_Λ are the pre-$*$-g-frame operators of Γ and Λ, respectively. Therefore, the operator Υ is unique and $\Upsilon^{-1} = \Theta_\Lambda^* \Theta_\Gamma$.

By Remark 5.4, we say that $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ is an operator dual of $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ with the corresponding invertible operator Υ.

Proposition 5.5. *Let $\Gamma = \{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ and $\Lambda = \{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ be $*$-g-Bessel sequences for $\mathcal{H}$ with pre-$*$-g-frame operators Θ_Γ and Θ_Λ, respectively. If there exists an adjointable and invertible operator Υ on $\mathcal{H}$ such that*

$$f = \sum_{j\in J} \Lambda_j^* \Gamma_j \Upsilon(f), \quad \forall f \in \mathcal{H},$$

then Γ and Λ are the operator duals to each other.

Proof. By the invertibility of Υ, for $f \in \mathcal{H}$, there is a $g \in \mathcal{H}$ such that $\Upsilon g = f$. So

$$\langle g, g\rangle = \langle \Theta_\Lambda^* \Theta_\Gamma \Upsilon g, \Theta_\Lambda^* \Theta_\Gamma \Upsilon g\rangle \leq \|\Theta_\Lambda\|^2 \langle \Theta_\Gamma f, \Theta_\Gamma f\rangle.$$

On the other hand,

$$\langle g, g\rangle = \langle \Upsilon^{-1} f, \Upsilon^{-1} f\rangle \geq \|\Upsilon\|^{-2} \langle f, f\rangle.$$

Therefore, for $f \in \mathcal{H}$

$$(\|\Theta_\Lambda\| \|\Upsilon\|)^{-2} \langle f, f\rangle \leq \langle \Theta_\Gamma f, \Theta_\Gamma f\rangle,$$

and Γ has the lower bound condition. Then it is a $*$-g-frame. Similarly, Λ is a $*$-g-frame and then are the operator duals to each other by Remark 5.7. □

Now, we can obtain a collection of operator duals with respect to a given operator dual for a $*$-g-frame. The following proposition illustrates this subject.

Proposition 5.6. *Let $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ be an operator dual of the $*$-g-frame $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ in $\mathcal{H}$ with the corresponding invertible operator Υ, and let $\{\widetilde{\Lambda}_j\}$ be the canonical dual $*$-g-frame of $\{\Lambda_j\}_{j\in J}$. If u is a strictly nonzero element in the center of $\mathcal{A}$ and $\Omega_j = u\Gamma_j + u\widetilde{\Lambda}_j\Upsilon^{-1}$ for $j \in J$, then $\{\Omega_j\}_{j\in J}$ is an operator dual of $\{\Lambda_j\}_{j\in J}$ with the corresponding invertible operator $\frac{1}{2}u^{-1}\Upsilon$. Also, The sequence $\{u\Gamma_j\}$ is an operator dual of $\{\Lambda_j\}_{j\in J}$ with the corresponding invertible operator $u^{-1}\Upsilon$.*

Proof. By the properties of operator duality of $\{\Gamma_j\}_{j\in J}$ and the canonical dual $*$-g-frame, we have for $f \in \mathcal{H}$

$$\sum_{j\in J} \Lambda_j^* \Omega_j (\frac{1}{2}u^{-1}\Upsilon) f = \sum_{j\in J} [\Lambda_j^* u\Gamma_j (\frac{1}{2}u^{-1}\Upsilon) + \Lambda_j^* u\widetilde{\Lambda}_j \Upsilon^{-1} (\frac{1}{2}u^{-1}\Upsilon)] f = \frac{1}{2} f + \frac{1}{2} f = f.$$

The equality shows that $\{\Omega_j\}_{j\in J}$ is an operator dual with the corresponding invertible operator $\frac{1}{2}u^{-1}\Upsilon$. The proof of the last part is similarly. □

In more, we mention that the operator duality relation of $*$-g-frames is symmetric. It is considered in the next remark.

Remark 5.7. *If $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ is an operator dual for $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ with the corresponding invertible operator Υ, then $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ is an operator dual for $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ with the corresponding invertible operator Υ^*. For see this, assume that Θ_Λ and Θ_Γ are the pre-$*$-g-frame operators of $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ and $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$, respectively, and I is identity operator on $\mathcal{H}$. By the definition of operator dual,*

$$f = \sum_{j\in J} \Lambda_j^* \Gamma_j \Upsilon f, \ \ \forall f \in \mathcal{H}, \Longrightarrow I = (\Theta_\Lambda^* \Theta_\Gamma)\Upsilon$$

Since Υ is invertible, $\Upsilon^{-1} = \Theta_\Lambda^* \Theta_\Gamma$ and

$$I = \Upsilon(\Theta_\Lambda^* \Theta_\Gamma) = (\Theta_\Gamma^* \Theta_\Lambda)\Upsilon \Longrightarrow f = \sum_{j\in J} \Gamma_j^* \Lambda_j \Upsilon^* f \ \ \forall f \in \mathcal{H}.$$

The last remark concludes $f = \sum_{j\in J} \Gamma_j^* \Lambda_j \Upsilon f = \sum_{j\in J} \Lambda_j^* \Gamma_j \Upsilon f$, for $f \in \mathcal{H}$. Now, if $\{\Gamma_j\}_{j\in J}$ is a $*$-g-frame with bounds A and B and Υ is an invertible and adjointable operator on $\mathcal{H}$, then $\{\Gamma_j \Upsilon\}_{j\in J}$ is a $*$-g-frame because

$$\sum_{j\in J} \langle \Gamma_j \Upsilon f, \Gamma_j \Upsilon f\rangle \leq B\|\Upsilon\| \langle f, f\rangle B^* \|\Upsilon\|,$$

and

$$\sum_{j\in J} \langle \Gamma_j \Upsilon f, \Gamma_j \Upsilon f\rangle \geq A\langle \Upsilon^* \Upsilon f, f\rangle A^* \geq A\|(\Upsilon^* \Upsilon)^{-1}\|^{-1/2} \langle f, f\rangle A^* \|(\Upsilon^* \Upsilon)^{-1}\|^{-1/2}.$$

Therefore, $\{\Gamma_j\Upsilon\}_{j\in J}$ is an ordinary dual for $\{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$, and it seems that generalized duals of $\{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$ are not different with ordinary duals. But since the form of them are different, we characterize the all of generalized duals of a given $*$-g-frame. For ordinary case, it is enough that $\Upsilon = I$ in the following results. Later, the operator duals of a given $*$-g-frame are studied. By Remark 5.7, we have $I = \Theta_\Lambda^*\Theta_\Gamma\Upsilon = \Theta_\Gamma^*\Theta_\Lambda\Upsilon^*$. Then $\{(\Gamma_j,\mathcal{K}_j)\}_{j\in J}$ is an operator dual of $\{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$ if and only if Θ_Γ is a right inverse of $\Upsilon\Theta_\Lambda^*$. Therefore, to characterize all of the operator duals of $\{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$, we must study all of the right inverses of $\Upsilon\Theta_\Lambda^*$. The following proposition considers this subject.

Proposition 5.8. *Let $\Lambda = \{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$ be a $*$-g-frame for $\mathcal{H}$ with the pre-$*$-frame operator Θ_Λ and the $*$-g-frame operator S. If Υ is an invertible element in $B_*(\mathcal{H})$, the set of all of right inverses of $\Upsilon\Theta_\Lambda^*$ is*

$$\{\Theta_\Lambda S^{-1}\Upsilon^{-1} + (I - \Theta_\Lambda S^{-1}\Theta_\Lambda^*)\xi \ ; \ \xi \in B_*(\mathcal{H}, \oplus_{j\in J}\mathcal{K}_j)\}.$$

Proof. Assume that ξ is an arbitrary element in $B_*(\mathcal{H}, \oplus_{j\in J}\mathcal{K}_j)$. We have

$$\begin{aligned}\Upsilon\Theta_\Lambda^*[\Theta_\Lambda S^{-1}\Upsilon^{-1} + (I - \Theta_\Lambda S^{-1}\Theta_\Lambda^*)\xi] &= \Upsilon\Theta_\Lambda^*\Theta_\Lambda S^{-1}\Upsilon^{-1} + \Upsilon\Theta_\Lambda^*\xi - \Upsilon\Theta_\Lambda^*\Theta_\Lambda S^{-1}\Theta_\Lambda^*\xi \\ &= \Upsilon SS^{-1}\Upsilon^{-1} + \Upsilon\Theta_\Lambda^*\xi - \Upsilon SS^{-1}\Theta_\Lambda^*\xi = I + \Upsilon\Theta_\Lambda^*\xi - \Upsilon\Theta_\Lambda^*\xi = I.\end{aligned}$$

Now, if Φ is an arbitrary right inverse of $\Upsilon\Theta_\Lambda^*$, then it is enough that set $\xi = \Phi$ and the proof of the proposition is complete. □

Considering an arbitrary right inverse of the operator $\Upsilon\Theta_\Lambda^*$, we obtain an operator dual corresponding it. The following proposition illustrates this fact.

Proposition 5.9. *Let $\Lambda = \{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$ be a $*$-g-frame in $\mathcal{H}$ with the pre-$*$-g-frame operator Θ_Λ. If $\Phi : \mathcal{H} \to \oplus_{j\in J}\mathcal{K}_j$ is any adjointable right inverse of $\Upsilon\Theta_\Lambda^*$, then $\{(\pi_j\Phi,\mathcal{K}_j)\}_{j\in J}$ is an operator dual of $\{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$ with the corresponding invertible operator Υ.*

Proof. By Proposition 3.6, the sequence $\{(\pi_j\Phi)\}_{j\in J}$ is a $*$-g-Bessel sequence in $\mathcal{H}$. Also, since $\Phi^*(\Upsilon\Theta_\Lambda^*)^* = I$, Φ^* is surjective and for $f \in \mathcal{H}$,

$$\|(\Phi^*\Phi)^{-1}\|^{-1}\langle f, f\rangle \le \langle \Phi f, \Phi f\rangle = \sum_{j\in J}\langle(\pi_j\Phi)f, (\pi_j\Phi)f\rangle,$$

and we have $\{(\pi_j\Phi,\mathcal{K}_j)\}_{j\in J}$ is a $*$-g-frame for $\mathcal{H}$ with pre-$*$-g-frame operator Φ. Moreover, from $I = \Phi^*(\Theta_\Lambda\Upsilon^*)$ obtain $f = \sum_{j\in J}(\pi_j\Phi)\Lambda_j\Upsilon^*(f)$, for $f \in \mathcal{H}$. It means that $\{(\pi_j\Phi,\mathcal{K}_j)\}_{j\in J}$ is an operator dual for $\{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$ with the corresponding invertible operator Υ^*. □

We can summarize the results in this section in the following theorem about to characterize of the all of operator duals for a given $*$-g-frame.

Theorem 5.10. *Let $\{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$ be a $*$-g-frame in $\mathcal{H}$ with the pre-$*$-g-frame operator Θ, the $*$-g-frame operator S and the canonical dual $*$-g-frame $\{(\widetilde{\Lambda}_j,\mathcal{K}_j)\}_{j\in J}$. Then the set of all of operator duals for $\{(\Lambda_j,\mathcal{K}_j)\}_{j\in J}$ is of the form*

$$\widetilde{\Lambda}_j\Upsilon + \Delta_j - \sum_{k\in J}\widetilde{\Lambda}_j\Lambda_k^*\Delta_k,$$

such that the sequence $\{(\Delta_j,\mathcal{K}_j)\}_{j\in J}$ is a $*$-g-Bessel sequence and Υ is an invertible operator in $B_*(\mathcal{H})$.

Proof. Let $\{(\Delta_j,\mathcal{K}_j)\}_{j\in J}$ be a $*$-g-Bessel sequence in $\mathcal{H}$ with the pre-$*$-g-frame operator Φ and let Υ is an invertible operator in $B_*(\mathcal{H})$. Set

$$\xi_j = \widetilde{\Lambda}_j\Upsilon + \Delta_j - \sum_{k\in J}\widetilde{\Lambda}_j\Lambda_k^*\Delta_k,$$

for $j \in J$, and define the linear operator

$$\Xi : \mathcal{H} \to \oplus_{j\in J}\mathcal{K}_j, \ \ by \ \ \Xi f = (\xi_j f)_{j\in J}.$$

Clearly, Ξ is adjointable. For every $j \in J$, we have

$$\pi_j \Xi = \Lambda_j S^{-1}\Upsilon + \Delta_j - \Lambda_j S^{-1}\sum_{k\in J}\Lambda_k^*\Delta_k = \pi_j(\Theta S^{-1}\Upsilon + \Phi - \Theta S^{-1}\Theta^*\Phi).$$

Then $\Xi = \Theta S^{-1}\Upsilon + (I - \Theta S^{-1}\Theta^*)\Phi$. By Proposition 5.8 and Proposition 5.9, $\{(\xi_j, \mathcal{K}_j)\}_{j\in J}$ becomes an operator dual $*$-g-frame of $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ with the corresponding invertible operator Υ^{-1}. □

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