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# Generalized Free Fields and the AdS-CFT Correspondence

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**Abstract.** Motivated by structural issues in the AdS-CFT correspondence, the theory of generalized free fields is reconsidered. A stress-energy tensor for the generalized free field is constructed as a limit of Wightman fields. Although this limit is singular, it fulfills the requirements of a conserved local density for the Poincaré generators. An explicit "holographic" formula relating the Klein-Gordon field on AdS to generalized free fields on Minkowski space-time is provided, and contrasted with the "algebraic" notion of holography. A simple relation between the singular stress-energy tensor and the canonical AdS stress-energy tensor is exhibited.

## 1 Introduction

According to Maldacena's conjecture [31], type IIB string theory on 5-dimensional asymptotically Anti-deSitter (AdS) backgrounds with five compactified dimensions is equivalent to a maximally supersymmetric Yang-Mills theory in physical Minkowski space-time. In the limit of large number of colors $N \to \infty$ and large 't Hooft coupling $\theta = Ng^2 \to \infty$, it is conjectured that the string may be replaced by classical supergravity on AdS. Roughly speaking, $1/N$ corrections correspond to quantum corrections, and the strong coupling expansion in $1/\theta$ corresponds to the perturbative incorporation of string corrections measured by the string tension $\alpha'$.

The "dual" correspondence between fields on AdS and conformal fields on Minkowski space-time was made concrete by a proposal in [22, 36]. As a scalar model, the conformal field dual to the free Klein-Gordon field on AdS has been considered as an exercise in [36]. It is a generalized free field [20, 27, 29]. It follows that the perturbative treatment of the AdS-CFT correspondence amounts to an expansion around a generalized free field.

For this reason, we believe it worthwhile to reconsider the properties of generalized free fields. Generalized free fields have been introduced by Greenberg [20] as a new class of models for local quantum fields, and have been further studied by Licht [29] as candidates for more general asymptotic fields as required by the LSZ asymptotic condition. They can be characterized in several equivalent ways: the commutator is a numerical distribution; the truncated (connected) $n$-point functions vanish for $n \neq 2$; the correlation functions factorize into 2-point functions; the generating functional for the correlation functions is a Gaussian. But in distinction from a canonical free field, a generalized free field is not the solution to an equation of motion, and its 2-point function is not supported on a mass shell. It rather has the form of a superposition

$$\langle\Omega,\ \varphi(x)\varphi(x')\Omega\rangle = \int_{\mathbb{R}_+} d\rho(m^2)\ W_m(x-x') \tag{1.1}$$

where $W_m$ are the 2-point functions of the Klein-Gordon fields of mass $m$ in $d$-dimensional Minkowski space-time. The (positive and polynomially bounded) weight $d\rho(m^2)$ occurring in the superposition is known as the Källen-Lehmann weight [28].

In fact, this is the most general form of the 2-point function of any scalar quantum field [28], and a general theorem [21] states that a field whose 2-point function is supported within a finite interval of masses, is necessarily a (generalized) free field.

There are several ways by which generalized free fields arise. E.g., from any Wightman field $\phi$ on a Hilbert space $\mathcal{H}$, one can obtain Wightman fields $\phi^{(N)}$, $N \in \mathbb{N}$, as the normalized sum of replicas of $\phi$ on $\mathcal{H}^{\otimes N}$. This suppresses the truncated $n$-point functions with a factor $N^{(2-n)/2}$. In the "central" limit $N \to \infty$, one obtains a generalized free field which has the same 2-point function as the original (interacting) field. Similarly, in the large $N$ limit of $O(N)$ or $U(N)$ symmetric theories, all truncated functions of gauge invariant (composite) fields exhibit a leading factor of $N$, so that if the 2-point function is normalized, the higher truncated functions are also suppressed by inverse powers of $N$, and the limit is again a generalized free field [26].[1]

Another obvious way to obtain a generalized free field is to restrict a free Klein-Gordon field of mass $M$ in 1+4 dimensions to the 1+3-dimensional hypersurface $x^4 = 0$. The resulting Källen-Lehmann weight is $d\rho(m^2) = dm^2/\sqrt{m^2 - M^2}$, supported at $m^2 \geq M^2$.

Finally, the AdS-CFT correspondence associates with the free Klein-Gordon field on $d+1$-dimensional Anti-deSitter space-time a Gaussian conformal field in $d$-dimensional Minkowski space-time whose scaling dimension $\Delta = \frac{d}{2} + \nu$ depends on the Klein-Gordon mass $M$ through the parameter $\nu = \frac{1}{2}\sqrt{d^2 + 4M^2}$. Its 2-point function proportional to $(-(x-x')^2)^{-\Delta}$ is a superposition of *all* masses with Källen-Lehmann weight $d\rho(m^2) = dm^2 m^{2\nu}$ (cf. Sect. 4).

As a consequence of the continuous superposition of masses, there is no Lagrangean and no canonical stress-energy tensor associated with a generalized free field. The first purpose of this article (entirely unrelated to AdS) is the construction of a non-canonical stress-energy tensor (Sect. 3). This stress-energy tensor turns out to be more singular than a Wightman field, but it fulfills the requirements as a density for the generators of (global) space-time symmetries. If smeared with a

[1] On the other hand, the limit $W$ of $1/N$ times the connected functional for finite $N$, $W_N = \log Z_N$, is finite and non-Gaussian. But $W$ does not define a quantum field theory of its own because it violates positivity (unitarity). The significance of this quantity is that $N \cdot W$ gives the asymptotic behaviour of the large $N$ expansion of $W_N$.

test function, it has finite expectation values but infinite fluctuations in almost every state of the Hilbert space (including the vacuum). Technically speaking, it is a quadratic form on the Wightman domain, rather than an unbounded operator. Its commutator with field operators, however, is well defined as an operator.

Our construction of the stress-energy tensor relies on the fact that on the vacuum Hilbert space of a generalized free field $\varphi$ there exists a large class of mutually local Wightman fields (including $\varphi$ itself, hence relatively local w.r.t. $\varphi$) which is much larger [11, 20, 29] than the class of Wick polynomials of $\varphi$. They form what is known as the Borchers class [5, 27] of $\varphi$. These fields can be expressed in terms of $\varphi$ (and its Wick polynomials) by the use of highly non-local (pseudo) differential operators or convolutions, while still satisfying local commutativity with $\varphi$ (i.e., the commutator vanishes at spacelike separation). This fact illustrates the distinction between the algebraic notion of local commutativity (Einstein locality) underlying the concept of the Borchers class, and a notion of "expressibility in terms of local operations". E.g., in perturbation theory the Lagrangean interaction density should be local in the former sense in order to ensure locality of the interacting field.

The second prominent issue of this article is the "holographic" identification of a quantum field on AdS (the free Klein-Gordon field $\phi$) in the Borchers class of the boundary generalized free field $\varphi$ (Sect. 4). More precisely, AdS is regarded conveniently as a warped product of Minkowski space-time with $\mathbb{R}_+$ (whose coordinate we call $z > 0$). Then for every fixed value $z$, $\phi(z, \cdot)$ is a Wightman field $\varphi_z(\cdot)$ on Minkowski space-time in the Borchers class of $\varphi$, obtained from $\varphi$ by a non-local (pseudo) differential operator involving a $z$-dependent Bessel function.

Local commutativity of the Klein-Gordon field $\phi$ on AdS implies that the fields $\varphi_z$ and $\varphi_{z'}$ in Minkowski space-time satisfy a certain "bonus locality" (local commutativity at finite timelike distance). We shall explicitly derive this property as a consequence of the specific non-local operations relating the AdS field to the boundary field, invoking a nontrivial identity for Bessel functions.

The canonical stress-energy tensor of the Klein-Gordon field on AdS is identified as a $z$-dependent generalized Wick product of the boundary field. Integrating this field over $z$, yields the singular stress-energy tensor of the generalized free field mentioned above. This complies with the fact that the canonical AdS stress-energy tensor is a density in a Cauchy surface of AdS, while the stress-energy tensor for the generalized free field on the boundary is a density in a time zero plane of Minkowski space.

With these findings, we want to point out that generalized free fields are rather well behaved Wightman fields, which moreover are "closer" to interacting quantum fields than free Klein-Gordon fields. It might be advantageous to perform a perturbation around a generalized free field, which has already the correct 2-point function of the interacting field, while the perturbation only affects the higher truncated correlations.

As was noticed implicitly, e.g., in [4], and systematically analyzed in [15], the perturbative approach to the AdS-CFT correspondence may be understood as a perturbation around a canonical field on AdS with subsequent restriction to the boundary. The interaction part of the action is an integral over AdS of some Wick polynomial in the AdS field. Expressing the latter in terms of the limiting generalized free field on the boundary, and performing the (regularized) $z$-integral, one obtains (at least formally) a Lagrangean density on the boundary which is a generalized Wick polynomial of the boundary generalized free field.

In view of this observation, the present work is also considered as a starting point for a perturbation theory of the generalized free field with generalized Wick polynomials as interactions, which includes the perturbative AdS-CFT correspondence as a special case.

In the last, somewhat tentative section, we point out the relation between the existence of relatively local fields beyond the Wick polynomials, and the violation of the time-slice property (primitive causality [24]) and "Haag duality" for generalized free fields in Minkowski space-time. These issues are discussed in terms of the von Neumann algebras of localized observables associated with a quantum field [23]. Although logically unrelated to AdS-CFT, they may be nicely understood in terms of geometric properties of AdS and its boundary, using the above holographic interpretation. The discussion also exhibits a slight but important difference between the present holographic picture and the "algebraic" notion of holography [33].

## 2 Generalized free fields

Let $\mathbb{M}^d = (\mathbb{R}^d, \eta_{\mu\nu})$ denote $d \geq 2$-dimensional Minkowski space-time, and $V_+$ the open forward light-cone (in momentum space).

We consider a hermitian scalar generalized free field [27, Chap. 2.6] on $\mathbb{M}^d$ with Källen-Lehmann weight

$$d\rho(m^2) = dm^2 \quad \text{on} \quad \mathbb{R}_+.$$

It has the form

$$\varphi(x) = \int_{V_+} d^dk \, [a(k)e^{-ikx} + a^+(k)e^{ikx}] \tag{2.1}$$

in terms of creation and annihilation operators

$$[a(k), a^+(k')] = (2\pi)^{-(d-1)}\delta^d(k-k'), \qquad [a,a] = 0 = [a^+, a^+]. \tag{2.2}$$

It is defined on the Fock space $\mathcal{H}$ over the 1-particle space $\mathcal{H}_1 = L^2(V_+, d^dk)$, identifying

$$L^2(V_+, d^dk) \ni f \quad \equiv \quad (2\pi)^{\frac{d-1}{2}} \int_{V_+} d^dk \; f(k)a^+(k)\Omega \in \mathcal{H}_1. \tag{2.3}$$

$\mathcal{H}_1$, and hence $\mathcal{H}$, is equipped with the obvious unitary positive-energy representation of the Poincaré group with generators[2]

$$P_\mu = (2\pi)^{d-1} \int_{V_+} d^dk \; a^+(k) \; k_\mu \; a(k), \tag{2.4}$$

$$M_{\mu\nu} = (2\pi)^{d-1} \int_{V_+} d^dk \; a^+(k) \; i(k_\nu \frac{\partial}{\partial k^\mu} - k_\mu \frac{\partial}{\partial k^\nu}) \; a(k), \tag{2.5}$$

such that

$$i[P_\mu, \varphi(x)] = \partial_\mu \varphi(x), \tag{2.6}$$

$$i[M_{\mu\nu}, \varphi(x)] = (x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi(x). \tag{2.7}$$

The generalized free field $\varphi$ is a *local* field because its commutator reads

$$\begin{aligned}[]
[\varphi(x), \varphi(x')] &= (2\pi)^{-(d-1)} \int_{V_+} d^dk \; \left(e^{-ik(x-x')} - e^{ik(x-x')}\right) = \\
&= \int_{\mathbb{R}_+} dm^2 \; (2\pi)^{-(d-1)} \int_{V_+} d^dk \; \delta(k^2 - m^2) \; \left(e^{-ik(x-x')} - e^{ik(x-x')}\right) \\
&= \int_{\mathbb{R}_+} dm^2 \Delta_m(x - x')
\end{aligned} \tag{2.8}$$

where $\Delta_m$ is the commutator function of the free Klein-Gordon field of mass $m$.[3]

## 2.1 Relatively local generalized free fields and generalized Wick products

Our first observation is that on the same Hilbert space, we can define

$$\varphi_h(x) = \int_{V_+} d^dk \; h(k^2) \; [a(k)e^{-ikx} + a^+(k)e^{ikx}] \tag{2.9}$$

with $h$ any smooth polynomially bounded real function on $\mathbb{R}_+$ (called "weight function"). These are again hermitian scalar fields on $\mathcal{H}$, satisfying (2.6) and (2.7). Moreover, all $\varphi_h$ are local and mutually local fields, because their commutators

$$[\varphi_{h_1}(x), \varphi_{h_2}(x')] = \int_{\mathbb{R}_+} dm^2 \; h_1(m^2)h_2(m^2) \; \Delta_m(x - x') \tag{2.10}$$

[2] The reader might be worried about the meaning of derivatives of $a(k)$. The expressions (2.4), (2.5) as well as (2.17), (2.22) below are understood in the distributional sense, i.e., after application of an integral $\int d^dka^+(k)Xa(k)$ to a 1-particle vector of the form (2.3) the differential operator $X$ is found acting on the smearing function $f \in L^2(V_+, d^dk)$, and likewise for $n$-particle vectors. Thus, (2.4), (2.5), (2.17), (2.22) are the "second quantizations" of the corresponding differential operators on $L^2(V_+, d^dk)$. We shall not discuss here the precise domains on which these hermitian generators are (essentially) self-adjoint.

[3] The Klein-Gordon fields themselves are not present in the theory, though, because square integrable functions in $\mathcal{H}_1$ cannot have sharp mass.

vanish at spacelike distance irrespective of the functions $h_i$. By inspection of the 2-point functions

$$\langle \Omega, \varphi_{h_1}(x)\varphi_{h_2}(x')\Omega\rangle = \int_{\mathbb{R}_+} dm^2\; h_1(m^2)h_2(m^2)\; W_m(x-x'), \tag{2.11}$$

one sees that each $\varphi_h$ is a generalized free field with Källen-Lehmann weight $dm^2 h(m^2)^2$. In fact, the weight function $h$ need not be smooth as long as $dm^2 h(m^2)^2$ is a polynomially bounded measure.

If the weight function $h$ is a polynomial, then

$$\varphi_h = h(-\Box)\varphi \tag{2.12}$$

is just a derivative of $\varphi$, and $\varphi_h(f) = \varphi(h(-\Box)f)$ where the support of $h(-\Box)f$ equals (a subset of) the support of $f$. Hence $h(-\Box)$ is a local operation. But if $h$ is not a polynomial, then $h(-\Box)$ may be tentatively defined on $f$ by multiplication of the Fourier transform $\hat{f}(k)$ with any function of $k^2$ which coincides with $h$ on $\mathbb{R}_+$ (all giving the same field operator $\varphi(h(-\Box)f)$). This is a highly non-local operation which does not preserve supports. Likewise, one may formally read (2.9) as a convolution in $x$-space [20]

$$\varphi_h(x) = \int_{\mathbb{M}^d} d^dx\; \check{H}(x-y)\varphi(y) \tag{2.13}$$

with the distributional inverse Fourier transform of any function $H(k)$ which equals $h(k^2)$ on $V_+$. E.g., if $h$ is analytic, $h(-\Box)$ and $H(k)$ may be defined as power series; but as the example of $h(z) = \cos\sqrt{z}$ exemplifies, the inverse Fourier transform of $h(k^2)\hat{f}(k)$ or $H(k)$ may not exist due to the rapid growth of $h(k^2)$ at negative $k^2$. Therefore, expressions like (2.12) or (2.13) in the general case should not be taken literally. These are suggestive ways of rewriting the definition (2.9), indicating a non-local operation on $\varphi$ which yet gives rise to a local field.

The above construction of fields satisfying local commutativity with $\varphi$ and among themselves can be extended to Wick products [11, 20, 29]. The expressions

$$\begin{aligned}(:\varphi^2:)_h(x) = \int_{V_+} d^dk_1 \int_{V_+} d^dk_2\; h(k_1^2, k_2^2)& \\ :[a(k_1)e^{-ik_1x} + h.c.][a(k_2)e^{-ik_2x} + h.c.]:&\end{aligned} \tag{2.14}$$

define Wightman fields, relatively local with respect to $\varphi$ and $\varphi_{h'}$ and mutually local among each other, for every (smooth) polynomially bounded real symmetric function $h$ on $\mathbb{R}_+ \times \mathbb{R}_+$. Formally, they may be represented as point-split limits of the form

$$(:\varphi^2:)_h(x) = \lim_{x'\to x} h(-\Box, -\Box')\big(\varphi(x)\varphi(x') - \langle \Omega,\; \varphi(x)\varphi(x')\Omega\rangle\big). \tag{2.15}$$

The smoothness of the functions $h$ may be considerably relaxed. While we refer to [11] for details, we point out that for $(:\varphi^2:)_h$ to be a Wightman field, $h^2$

ought to be at least a measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$: otherwise $(:\varphi^2:)_h(f)$ fails to be an operator with the vacuum vector in its domain. This can be seen easily from the 2-point function

$$\begin{aligned}&\langle\Omega, (:\varphi^2:)_h(x)(:\varphi^2:)_h(x')\Omega\rangle = \\ &\quad = 2\int_{\mathbb{R}_+} dm_1^2 dm_2^2\; h(m_1^2, m_2^2)^2\; W_{m_1}(x-x')W_{m_2}(x-x'). \end{aligned} \tag{2.16}$$

It is clear how this construction generalizes to higher Wick polynomials, and also to multi-local fields such as $(:\varphi(x_1)\varphi(x_2):)_h$. All these fields satisfy local commutativity among each other with respect to their arguments in spite of the non-local operations involved. It is crucial that the weight functions $h$ depend only on the squares of the four-momenta, since general functions of the components $k^\mu$ would spoil local commutativity. It is also clear that the construction can be as well applied to Wick polynomials of derivatives $\partial_\mu \dots \partial_\nu \varphi$ of the generalized free field.

Generalized Wick polynomials belong to the Borchers class of the generalized free field consisting of the relatively local Wightman fields defined on the same Hilbert space. With suitable specifications of the functions $h$ involved, they exhaust the Borchers class [29, 11]. They are natural candidates for perturbative interactions, e.g., in causal perturbation theory [17, 9, 14].

## 2.2 Conformal symmetry

The 1-particle Hilbert space $\mathcal{H}_1$, and hence the Fock space $\mathcal{H}$, carry also a natural representation of the group of dilations with generator

$$D = (2\pi)^{d-1}\int_{V_+} d^dk\; a^+(k)\; \tfrac{i}{2}\big((k\cdot\partial_k) + (\partial_k\cdot k)\big)\; a(k). \tag{2.17}$$

Under this representation, the generalized free fields $\varphi_h$ transform according to

$$U(\lambda)\varphi_h(x)U(\lambda)^* = \varphi_{h_\lambda}(\lambda x) \tag{2.18}$$

where $h_\lambda(m^2) = \lambda^{\frac{d}{2}} h(\lambda^2 m^2)$. In particular, the generalized free fields with homogeneous weight functions $m^\nu$,

$$\varphi^{(\Delta)} = (-\Box)^{\nu/2}\varphi \qquad \text{with} \quad \nu \equiv \Delta - \tfrac{d}{2} \tag{2.19}$$

transform like scale-invariant fields of scaling dimensions $\Delta = \frac{d}{2} + \nu$:

$$U(\lambda)\varphi^{(\Delta)}(x)U(\lambda)^* = \lambda^\Delta \varphi^{(\Delta)}(\lambda x), \tag{2.20}$$

or in infinitesimal form

$$i[D, \varphi^{(\Delta)}(x)] = (x^\mu\partial_\mu + \Delta)\varphi^{(\Delta)}(x). \tag{2.21}$$

$U(\lambda)$ scales the momenta (2.4) and commutes with the Lorentz transformations (2.5). Hence it extends the representation of the Poincaré group to a representation of the Poincaré-dilation group, also denoted by $U$.

It is well known, that the scale-invariant 2-point functions of these fields are in fact conformally covariant, extending the fields to a suitable covering of Minkowski space-time $\mathbb{CM}^d$ [30]. This means in particular that the unitary representation $U$ of the Poincaré-dilation group extends to a unitary representation $U^{(\Delta)}$ of the conformal covering group. Under $U^{(\Delta)}$, the field $\varphi^{(\Delta)}$ transforms as a conformally covariant scalar field of scaling dimension $\Delta$. Unlike those of the Poincaré-dilation group, the generators of the special conformal transformations depend on the parameter $\Delta = \frac{d}{2} + \nu$ and are explicitly given on $\mathcal{H}_1$ by

$$K_\mu^{(\Delta)} = (2\pi)^{d-1} \int_{V_+} d^d k$$
$$a^+(k) \left( \frac{\partial}{\partial k^\alpha} k_\mu \frac{\partial}{\partial k_\alpha} - (k \cdot \partial_k) \frac{\partial}{\partial k^\mu} - \frac{\partial}{\partial k^\mu} (\partial_k \cdot k) + \nu^2 \frac{k_\mu}{k^2} \right) a(k), \tag{2.22}$$

such that

$$i[K_\mu^{(\Delta)}, \varphi^{(\Delta)}(x)] = \left(2x_\mu (x \cdot \partial) - x^2 \partial_\mu + 2\Delta x_\mu\right) \varphi^{(\Delta)}(x). \tag{2.23}$$

We emphasize that although the fields $\varphi^{(\Delta)}$ are for all values of $\Delta$ defined on the same Hilbert space (the common Fock space $\mathcal{H}$ for all generalized free fields $\varphi_h$), they are conformally covariant with respect to different representations $U^{(\Delta)}$ of the conformal group on the same Hilbert space. These representations coincide only on the Poincaré-dilation subgroup. $U^{(\Delta)}$ does not implement a geometrical point transformation of $\varphi^{(\Delta')}$, $\Delta' \neq \Delta$, nor of $\varphi_h$ in general.

## 3 The stress-energy tensor

The purpose of this section is to find a stress-energy tensor $\Theta_{\mu\nu}(x)$ for the generalized free field (2.1) which has the properties of a local and covariant conserved tensor density for the generators of the Poincaré group. It should thus satisfy

$$\partial^\mu \Theta_{\mu\nu} = 0, \tag{3.1}$$

$$\int d^{d-1}\vec{x}\ \Theta_{0\nu} = P_\nu, \tag{3.2}$$

$$\int d^{d-1}\vec{x}\ (x_\mu \Theta_{0\nu} - x_\nu \Theta_{0\mu}) = M_{\mu\nu}, \tag{3.3}$$

$$\Theta_{\mu\nu}(x) = \Theta_{\nu\mu}(x), \tag{3.4}$$

$$[\Theta_{\mu\nu}(x), \varphi_h(x')] = 0 \qquad ((x-x')^2 < 0). \tag{3.5}$$

With an ansatz of the form (2.14), including derivatives of $\varphi$, we find the solution

$$\Theta_{\mu\nu} = \big( : \partial_\mu\varphi \partial_\nu\varphi - \tfrac{1}{2}\eta_{\mu\nu} \left(\partial_\alpha\varphi\partial^\alpha\varphi + \varphi\Box\varphi\right) : \big)_\delta. \tag{3.6}$$

The generalized Wick product $(:\cdots:)_\delta$ is understood as $(:\cdots:)_h$ with the choice of the weight "function"

$$h(m_1^2, m_2^2) = \delta(m_1^2, m_2^2) \equiv \delta(m_1^2 - m_2^2)\theta(m_1^2). \tag{3.7}$$

This singular choice cannot be avoided due to the requirement (3.2), since the spatial integral over a generalized Wick square such as (2.14) can only enforce equality of the spatial components of the momenta of the creation and annihilation operators, $\vec{k}_1 = \vec{k}_2$, while the representation (2.4) of the total momentum operator requires $k_1^\mu = k_2^\mu$. But with this singular weight function $h$, the 2-point function (2.16) involving $h^2$ becomes highly divergent, hence the stress-energy tensor has infinite fluctuations in the vacuum state. Smearing $\Theta_{\mu\nu}$ with a test function does not give an operator whose domain contains the vacuum vector. Thus the stress-energy tensor is not a Wightman field.

But $\Theta_{\mu\nu}(f)$ is a quadratic form on the Wightman domain of $\varphi_h$, i.e., its matrix elements with vectors from that domain are finite (more precisely: are continuous functionals of the test function $f$). To prove this, only the finiteness of

$$\begin{gathered}\langle\Omega,\ \Theta_{\mu\nu}(f)\varphi_{h_1}(f_1)\varphi_{h_2}(f_2)\Omega\rangle,\\ \langle\Omega,\ \varphi_{h_1}(f_1)\Theta_{\mu\nu}(f)\varphi_{h_2}(f_2)\Omega\rangle,\\ \langle\Omega,\ \varphi_{h_1}(f_1)\varphi_{h_2}(f_2)\Theta_{\mu\nu}(f)\Omega\rangle\end{gathered} \tag{3.8}$$

needs to be checked since every matrix element of $\Theta_{\mu\nu}(f)$ on this domain is a sum of terms of either of these forms with finite coefficients. Explicit evaluation of the above matrix elements, which are all of the form

$$\begin{aligned}\int_{V_+} d^dk_1\ \hat{f}_1(\pm k_1)h_1(k_1^2)\int_{V_+} d^dk_2\ \hat{f}_2(\pm k_2)h_2(k_2^2)\ \delta(k_1^2 - k_2^2)\\ \times P(k_1,k_2)\hat{f}(\,\pm k_1 \pm k_2)\end{aligned} \tag{3.9}$$

with $P$ some polynomial, exhibits their finiteness and continuity with respect to $f$, for arbitrary test functions $f_i$ and arbitrary weight functions $h_i$. $\Theta_{\mu\nu}(f)$ being a quadratic form on the Wightman domain of $\varphi_h$, its commutator with $\varphi_h(g)$ is a priori well defined as a quadratic form. It turns out to be in fact an operator on the Wightman domain. It vanishes if the supports of $f$ and $g$ are spacelike separated.

The formula (3.6) for $\Theta_{\mu\nu}$ is uniquely determined by the requirements (3.1–5), up to addition of a multiple of $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)(:\varphi^2:)_\delta$. We note that $\Theta_{\mu\nu}$ is not traceless, nor can it be made traceless by such an addition. Thus it does not provide a density for the generator of the dilations (2.17), nor for the conformal transformations (2.22).

## 4 Application to AdS-CFT

The conformal group $SO(2,d)$ of $d$-dimensional Minkowski space-time coincides with the group of isometries of $d+1$-dimensional anti-deSitter space-time $\mathbb{AdS}_{d+1}$.

In fact, $\mathbb{AdS}_{d+1}$ has a conformal boundary which is a twofold covering[4] of the Dirac compactification $\mathbb{CM}^d$ of Minkowski space-time, such that the AdS group restricted to the boundary acts like the conformal group on $\mathbb{CM}^d$. One-parameter subgroups with future-directed timelike tangent vectors in AdS ("time evolutions") have future-directed timelike tangent vectors in $\mathbb{CM}^d$. Hence, the respective (AdS and conformal) notions of "positive energy" for the unitary representations of $SO(2,d)$ coincide.

It was shown in [19] that the scalar Klein-Gordon field on $\mathbb{AdS}_{d+1}$ can be canonically quantized on a 1-particle space which carries a positive-energy representation of the AdS group.

Our aim is to find the explicit relation between these scalar Klein-Gordon fields $\phi$ on $\mathbb{AdS}_{d+1}$ (parameterized by a parameter $\nu > -1$ such that the Klein-Gordon mass equals $M^2 = \nu^2 - \frac{d^2}{4}$) and the generalized free field on Minkowski space-time $\mathbb{M}^d$ (characterized by its scaling dimension $\Delta = \frac{d}{2} + \nu$). Both fields are defined on the same Fock space $\mathcal{H}$ over the 1-particle space $\mathcal{H}_1$ (2.3), carrying the same unitary positive-energy representation $U^{(\Delta)}$ of $SO(2,d)$ under which both fields transform covariantly in the respective (AdS or conformal) sense.

We shall work in the convenient chart of AdS given by Poincaré coordinates $x^M \equiv (z \in \mathbb{R}_+, x^\mu \in \mathbb{M}^d)$, in which the metric takes the form

$$ds^2 = g_{MN}\, dx^M dx^N = z^{-2} \cdot (\eta_{\mu\nu}\, dx^\mu dx^\nu - dz^2), \tag{4.1}$$

i.e., it is a "warped product" of $d$-dimensional Minkowski space-time $\mathbb{M}^d$ by $\mathbb{R}_+$, or in the terminology of [7], AdS has a foliation by $\mathbb{M}^d$. The chart is given as follows. We fix a pair $e_\pm$ of lightlike vectors in $\mathbb{R}^{2,d}$, $e_+ \cdot e_- = \frac{1}{2}$, and a basis $e_\mu$ of the subspace orthogonal to $e_\pm$, with $e_\mu \cdot e_\nu = \eta_{\mu\nu}$. Then

$$\xi = z^{-1} \cdot (x^\mu e_\mu + e_- + (z^2 - x_\mu x^\mu) e_+) \tag{4.2}$$

fulfills $\xi \cdot \xi = 1$. This chart covers $\mathrm{P}\mathbb{AdS}_{d+1}$ except for the hypersurface $\xi \cdot e_+ = 0$ which formally corresponds to $z = \infty$.

The corresponding chart $(x^\mu \in \mathbb{M}^d)$ of $\mathbb{CM}^d$, parameterizing the lightlike rays in $\mathbb{R}^{2,d}$ by

$$\zeta = \mathbb{R} \cdot (x^\mu e_\mu + e_- - x_\mu x^\mu e_+), \tag{4.3}$$

is Minkowski space-time $\mathbb{M}^d \subset \mathbb{CM}^d$. This chart misses out the hypersurface of compactification points "at infinity" of $\mathbb{M}^d$, consisting of the lightlike rays orthogonal to $e_+$, namely the rays $\mathbb{R} \cdot (\lambda e_+ + x^\mu e_\mu)$, $x_\mu x^\mu = 0$. In $\mathbb{CM}^d$, these are the points at lightlike distance from $\omega \equiv \mathbb{R} \cdot e_+$ (the compactification point "at spacelike infinity" of $\mathbb{M}^d$).

[4] We denote by $\mathbb{AdS}_{d+1}$ the quadric $\xi \cdot \xi = 1$ in $\mathbb{R}^{2,d}$ (signature $(+,+,-,\dots,-)$), and by $\mathrm{P}\mathbb{AdS}_{d+1}$ its quotient by the antipodal identification $\xi \leftrightarrow -\xi$. The conformal boundary of this quotient is the Dirac compactification $\mathbb{CM}^d$ of Minkowski space-time whose points are the lightlike rays $\zeta = \mathbb{R} \cdot n$, $n \cdot n = 0$, in $\mathbb{R}^{2,d}$. The field theories discussed below in general are defined on covering spaces of the respective manifolds.

$\zeta(x^\mu)$ is the boundary point approached by $\xi(z, x^\mu)$ as $z \to 0$. Thus the Poincaré chart meets the boundary exactly in Minkowski space-time.

From this discussion, we conclude that the chart $(z, x^\mu)$ is mapped onto itself only by the stabilizer subgroup in $SO(2, d)$ of the ray $\mathbb{R}_+ \cdot e_+$ in $\mathbb{R}^{2,d}$. This subgroup has the form

$$(SO(1, d-1) \times \mathbb{R}_+) \ltimes \mathbb{R}^d. \tag{4.4}$$

Here, $SO(1, d-1) \ltimes \mathbb{R}^d$ is the stabilizer group of the vector $e_+$. $SO(1, d-1)$ preserves also $e_-$ and transforms the basis $e_\mu$ like the Lorentz group, while $\mathbb{R}^d$ takes $e_- \mapsto e_- + a^\mu e_\mu - a_\mu a^\mu e_+$, $e_\mu \mapsto e_\mu - 2a_\mu e_+$. Hence the stabilizer subgroup of $e_+$ preserves the coordinate $z = (2\xi \cdot e_+)^{-1}$ and acts on $x^\mu$ like the Poincaré group. The remaining factor $\mathbb{R}_+$ in (4.4) scales $e_\pm \mapsto \lambda^{\pm 1} e_\pm$ and preserves $e_\mu$, hence it takes $(z, x^\mu)$ to $(\lambda z, \lambda x^\mu)$ and acts on the boundary $z = 0$ like the dilations. We shall refer to these subgroups of $SO(2, d)$ as Poincaré and dilation subgroups also in the AdS context. Thus, the Poincaré chart of AdS is preserved by the Poincaré-dilation group of Minkowski space-time. The remaining elements of $SO(2, d)$ induce rational transformations of the coordinates $(z, x^\mu)$, such as $(z, x^\mu) \mapsto (z, x^\mu - b^\mu(x^2 - z^2))/(1 - 2(b \cdot x) + b^2(x^2 - z^2))$, $b \in \mathbb{M}^d$, restricting to the special conformal transformations of the boundary $z = 0$.

### 4.1 The Klein-Gordon field on AdS

We fix any value $\nu > -1$ and set $\Delta = \frac{d}{2} + \nu$ and $M^2 = \Delta(\Delta - d) = \nu^2 - \frac{d^2}{4}$.

The Klein-Gordon field on AdS

$$(\Box_g + M^2)\phi = (-z^{1+d}\partial_z z^{1-d}\partial_z + z^2\Box_\eta + M^2)\phi = 0 \tag{4.5}$$

has been quantized with an AdS-invariant vacuum state, e.g., in [3, 19]. Its 2-point function can be displayed in the form [7]

$$\begin{aligned}\langle \Omega,\ \phi(z,x)\phi(z',x')\Omega\rangle &= \tfrac{1}{2}(zz')^{\frac{d}{2}} \int_{\mathbb{R}_+} dm^2\ J_\nu(zm)J_\nu(z'm)\ W_m(x-x') \\ &= (2\pi)^{-(d-1)}\tfrac{1}{2}(zz')^{\frac{d}{2}} \int_{V_+} d^dk\ J_\nu(z\sqrt{k^2})J_\nu(z'\sqrt{k^2})\ e^{-ik(x-x')}. \end{aligned} \tag{4.6}$$

Here, $J_\nu$ is the Bessel function solving Bessel's differential equation

$$((u\partial_u)^2 + u^2)J_\nu(u) = \nu^2 J_\nu(u), \tag{4.7}$$

and $z^{\frac{d}{2}} J_\nu(z\sqrt{k^2})e^{\pm ikx}$ ($k \in V_+$) are the plane-wave solutions of the Klein-Gordon equation (4.5).

We note that, depending on the integrality of the parameter $\Delta = \frac{d}{2} + \nu$, the quantum field $\phi$ is in general defined on a covering space of $\mathbb{AdS}_{d+1}$ [7, 19]. This complicates the analysis, but the complications precisely match the complications arising in the corresponding conformal QFT which is defined on a covering space of $\mathbb{CM}^d$. We shall limit ourselves to the Klein-Gordon field on the Poincaré chart $(z, x^\mu)$, and correspondingly to boundary fields in Minkowski space-time.

We also note that for $|\nu| < 1$, the two possible signs of $\nu$ give rise to inequivalent covariant quantizations [8] (in fact, an interpolating one-parameter family [7]) of the Klein-Gordon field of the same mass. The commutator functions derived from (4.6) are the same for both signs of $\nu$ [7], hence both quantum field theories have the same *local* structure.

$\phi$ satisfies the canonical equal-time commutation relation between the field and its canonical momentum $\pi = z^{1-d}\partial_0\phi$

$$[\phi(z,x), \pi(z',x')]|_{x^0=x'^0} = i\delta^{d-1}(\vec{x}-\vec{x}')\delta(z-z'), \tag{4.8}$$

as can be verified from (4.6), using the fact that $W_m(x-x')$ satisfies canonical commutation relations on $\mathbb{M}^d$, and using Hankel's identity

$$\int_0^\infty t\,dt\; J_\nu(tu)J_\nu(tu') = u^{-1}\;\delta(u-u'), \tag{4.9}$$

which expresses the completeness of the plane-wave solutions involved in the integral (4.6).

## 4.2 Expression in terms of generalized free fields

By comparison of (4.6) with (2.11), we conclude that $\phi(z,x)$ can be identified with

$$\phi(z,x) = \tfrac{1}{\sqrt{2}}\, z^{\frac{d}{2}} \int_{V_+} d^dk\; J_\nu(z\sqrt{k^2})[a(k)e^{-ikx} + a^+(k)e^{ikx}]. \tag{4.10}$$

This is of the form

$$\phi(z,x) = \varphi_{h_z}(x) \tag{4.11}$$

with

$$h_z(m^2) = \tfrac{1}{\sqrt{2}}\, z^{\frac{d}{2}} J_\nu(zm) \tag{4.12}$$

i.e., for each value of $z$, $\phi(z,\cdot)$ is one of the generalized free fields considered in Sect. 2.1.

From the power law behaviour of $J_\nu(u) \approx \frac{2^{-\nu}}{\Gamma(\nu+1)}\; u^\nu(1+O(u^2))$ at small arguments, one obtains

$$\lim_{z\to 0} z^{-\Delta}\phi(z,x) = \tfrac{2^{-\nu-\frac{1}{2}}}{\Gamma(\nu+1)}\; \varphi^{(\Delta)}(x), \tag{4.13}$$

i.e., the generalized free field $\varphi^{(\Delta)}$ is the boundary limit of the Klein-Gordon field on AdS. This agrees with the results discussed in [7] and [15].

### 4.3 Identification of the representations of $SO(2,d)$

The 2-point function (4.6) is AdS-invariant, i.e., it depends only on the AdS-invariant "chordal distance" (the distance within $\mathbb{R}^{2,d}$) $(-(x-x')^2+(z-z')^2)/2zz'$. It follows that the representation of the AdS group defined by

$$U^{(\mathrm{KG})}(g)\ \phi(z,x)\Omega := \phi(g(z,x))\Omega \qquad (g \in SO(2,d)) \tag{4.14}$$

is unitary on the 1-particle space $\mathcal{H}_1$, and hence on $\mathcal{H}$. It implements the covariant transformation law

$$U^{(\mathrm{KG})}(g)\phi(z,x)U^{(\mathrm{KG})}(g)^* = \phi(g(z,x)). \tag{4.15}$$

We want to show that this representation of $SO(2,d)$ coincides with the representation $U^{(\Delta)}$ of the conformal group, constructed on the same Hilbert space by the extension of the scale-invariant field $\varphi^{(\Delta)}$ on $\mathbb{M}^d$ to the conformally invariant field on $\mathbb{CM}^d$ (cf. Sect. 2.2).

Eq. (4.15), restricted to the boundary $z = 0$ where $g$ acts like the conformal group on $x$, shows that $U^{(\mathrm{KG})}(g)$ implements the same conformal point transformation of the limiting field $\varphi^{(\Delta)}$ as $U^{(\Delta)}(g)$. More specifically, consider the infinitesimal form of the AdS transformation law (4.15) for the relevant subgroups,

$$\begin{aligned} i[P_\mu^{(\mathrm{KG})}, \phi(z,x)] &= \partial_\mu \phi(z,x), \\ i[M_{\mu\nu}^{(\mathrm{KG})}, \phi(z,x)] &= (x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(z,x), \\ i[D^{(\mathrm{KG})}, \phi(z,x)] &= (z\partial_z + x^\mu \partial_\mu)\phi(z,x), \\ i[K_\mu^{(\mathrm{KG})}, \phi(z,x)] &= \big(2x_\mu(z\partial_z + (x\cdot\partial)) + (z^2 - x^2)\partial_\mu\big)\phi(z,x). \end{aligned} \tag{4.16}$$

In the limit $z \to 0$ according to (4.13), the right-hand sides of (4.16) turn into (2.6), (2.7), (2.21), (2.23), respectively. Thus, the infinitesimal generators of the respective subgroups coincide. We conclude that the two representations $U^{(\mathrm{KG})}$ and $U^{(\Delta)}$ of $SO(2,d)$ coincide, cf. [12].

With the help of the representation $U^{(\mathrm{KG})} = U^{(\Delta)}$, the AdS field $\phi$ and the boundary field $\varphi^{(\Delta)}$ extend to the respective covering spaces of $\mathrm{P}\mathbb{A}\mathbb{D}\mathbb{S}_{d+1}$ and $\mathbb{CM}^d$. We emphasize once more that this does not apply for the other boundary fields $\varphi_h$.

### 4.4 "Holographic" interpretation

Combining (4.11) and (2.19), we find

$$\phi(z,x) = \tfrac{1}{\sqrt{2}}\ z^\Delta\ j_\nu(-z^2\Box)\ \varphi^{(\Delta)}(x), \tag{4.17}$$

where

$$j_\nu(u^2) = u^{-\nu} J_\nu(u) \tag{4.18}$$

is a (polynomially bounded) convergent power series in $u^2$. This is an explicit expression for the Klein-Gordon field on AdS in terms of its limiting generalized free field on the boundary. This "holographic" relation[5] is possible with the help of non-local (pseudo) differential operators of the kind discussed in Sect. 2.

Obviously, the fact that $\varphi_h = h(-\Box)\varphi$ are mutually local fields for arbitrary functions $h$, ensures that $\phi(z,x)$ and $\phi(z',x')$ written in the form (4.11) or (4.17) commute if $(x-x')^2 < 0$. But this is less than locality on AdS, which requires that the commutator must vanish even if $(x-x')^2 < (z-z')^2$. Of course we *know* that $\phi$ is a local field on AdS, so the stronger local commutativity of generalized free fields $\varphi_{h_z}(x)$ and $\varphi_{h_{z'}}(x')$ at finite timelike Minkowski distance must be true.

One can understand the origin of this "bonus locality" for the generalized free fields involved. Evaluating the commutator function according to (2.10), gives an integral over three Bessel functions (because $\Delta_m$ at timelike distance $(x-x')^2 = \tau^2$ is also given by a Bessel function $(m/\tau)^{\frac{d-2}{2}} J_{\frac{2-d}{2}}(m\tau)$) of the form

$$I(a,b,c) = \int_0^\infty u^{1-\mu}\, du\; J_\mu(au) J_\nu(bu) J_\nu(cu) \tag{4.19}$$

with $a^2 = (x-x')^2$, $b = z$, $c = z'$ and $\mu = \frac{2-d}{2}$. This integral can be found, e.g., in [35, Sect. 13·46(1)], where it is shown to vanish if $a^2 < (b-c)^2$. Thus we precisely find local commutativity on AdS. It is the specific form of the Bessel functions in (4.17), (4.18) which is able to ensure locality in a higher-dimensional space-time.

This remark should make it clear that defining $\phi(z,x) = \varphi_{h_z}(x)$ with any suitable family of functions $h_z(m^2)$, depending on a parameter $z$ and solving a suitable differential equation with respect to $z$, may well produce a quantum field on a higher-dimensional space-time solving some equation of motion, but this field will in general not satisfy local commutativity.[6]

### 4.5 The stress-energy tensor

The Klein-Gordon field on AdS has a canonical covariantly conserved stress-energy tensor given by

$$\Theta^{\rm KG}_{MN}(z,x) = \; : D_M\phi D_N\phi - \tfrac{1}{2} g_{MN}(g^{AB} D_A\phi D_B\phi - M^2\phi^2) : . \tag{4.20}$$

Because of the special form of the AdS metric, the *covariant* tensor continuity equation gives rise to *ordinary* continuity equations for the Minkowski components $\Theta^{\rm KG}_{M\nu}$, $\nu \neq z$,

$$g^{MN}\partial_N (z^{1-d}\Theta^{\rm KG}_{M\nu}) = 0. \tag{4.21}$$

[5]Our use of the term "holography" does not quite match the one originally suggested by 't Hooft [25], namely the *reduction in the bulk* of degrees of freedom of a QFT, ascribed to gravitational effects in the presence of a horizon. We rather allude to the *enhancement on the boundary* of degrees of freedom necessary and sufficient to "encode" a non-gravitational QFT in the bulk.

[6]Such constructions were proposed in [32].

Thus $z^{1-d}\Theta^{\mathrm{KG}}_{0\nu}$ are densities of conserved quantities, which are the generators of the Poincaré subgroup of $SO(2,d)$,

$$P_\mu = \int_0^\infty dz z^{1-d} \int d^{d-1}\vec{x}\ \Theta^{\mathrm{KG}}_{0\mu}(z,0,\vec{x}), \tag{4.22}$$

$$M_{\mu\nu} = \int_0^\infty dz z^{1-d} \int d^{d-1}\vec{x}\ (x_\mu \Theta^{\mathrm{KG}}_{0\nu}(z,0,\vec{x}) - x_\nu \Theta^{\mathrm{KG}}_{0\mu}(z,0,\vec{x})). \tag{4.23}$$

In these integrals, we may express $\phi$ in terms of the generalized free field $\varphi$ by means of (4.11), thus introducing two $z$-dependent (Bessel) weight functions, and perform the $z$-integration. Because of the term $g^{MN}\partial_M\phi\partial_N\phi$ involving $z$-derivatives, a partial integration becomes necessary after which Bessel's differential equation can be used to eliminate all derivatives of the Bessel functions.

After performing these steps on the Minkowski components of $\Theta^{\mathrm{KG}}_{\mu\nu}$, one ends up with a generalized Wick product of (derivatives of) $\varphi$, whose weight function is the result of the $z$-integration over the Bessel functions:

$$\int_0^\infty dz\, z^{1-d}\ \Theta^{\mathrm{KG}}_{\mu\nu}(z,x) = \big(:\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}\eta_{\mu\nu}\,(\partial_\alpha\varphi\partial^\alpha\varphi + \varphi\Box\varphi):\big)_h \tag{4.24}$$

where

$$h(m_1^2, m_2^2) = \tfrac{1}{2}\int_0^\infty z\, dz\ J_\nu(zm)J_\nu(zm'). \tag{4.25}$$

Once more using Hankel's identity (4.9) which in this case plays the role of an orthonormality relation, the integral can be performed, giving $h(m_1^2, m_2^2) = \delta(m_1^2 - m_2^2)$.

We have thus exactly reproduced the singular stress-energy tensor found in Sect. 3,

$$\Theta_{\mu\nu}(x) = \int_0^\infty dz\, z^{1-d}\ \Theta^{\mathrm{KG}}_{\mu\nu}(z,x). \tag{4.26}$$

But the origin of its singular weight function appears in an entirely new light: it is the result of the "holographic" projection of AdS onto its boundary. Some cutoff in the $z$-integral would smoothen the resulting weight function (4.25). The smoothened stress-energy tensor would still act as a density for generators which generate the correct transformation laws on those AdS fields which are causally disconnected from the AdS region where the cutoff is effective. In terms of the corresponding generalized free fields on the boundary, these are generalized free fields within a restricted region *and* with a restricted set of weight functions $h_z$.

## 5 Local algebras

At first sight, our "holographic" result of Sect. 4.4 does not quite agree with the algebraic analysis of AdS-CFT in [33]. Let us sketch the situation.

According to (4.11), the AdS fields at the point $(z, x)$ are expressed in terms of Minkowski space-time fields at the point $x$. Hence boundary observables (smeared fields) localized in a region $K \subset \mathbb{M}^d$ encode all AdS observables localized in the region $V(K) = \pi^{-1}(K) \subset \mathbb{AdS}_{d+1}$ where $\pi$ denotes the projection $(z, x) \to x$. We may call this feature "projective holography".

On the other hand, the analysis called "algebraic holography" in [33] is based on the identification

$$A_{\text{CFT}}(K) = A_{\text{AdS}}(W) \qquad (W = W(K)) \tag{5.1}$$

of the local algebras $A_{\text{CFT}}(K)$ of boundary observables localized in "double-cones" $K \subset \widetilde{\mathbb{CM}}^d$ (the conformal transforms of $K_0 = \{(x^\mu) : |\vec{x}| < 1-|x^0|\}$ in the covering space of the Dirac compactification of Minkowski space-time), with local algebras $A_{\text{KG}}(W)$ of AdS observables localized in the "wedge" regions $W = W(K) \subset \widetilde{\mathbb{AdS}}_{d+1}$ (the AdS transforms of $W_0 = \{(z, x^\mu) : \sqrt{z^2 + \vec{x}^2} < 1 - |x^0|\}$ in the covering space of AdS). The wedge $W(K)$ is the causal completion of the boundary region $K$. The map $W = W(K)$ is a bijection which preserves inclusions, takes causal complements in $\widetilde{\mathbb{CM}}^d$ into causal complements in $\widetilde{\mathbb{AdS}}_{d+1}$, and is compatible with the respective actions of the covering group of $SO(2, d)$.

For $K \subset \mathbb{M}^d$, the wedge region $W(K)$ extends only to finite "depth" $z$ into AdS and is strictly smaller than $V(K)$, which extends to $z = \infty$ and contains points causally disconnected from $W(K)$. Hence $A_{\text{AdS}}(V(K))$ is strictly larger than $A_{\text{AdS}}(W(K))$, and the two notions of "holography" cannot be equivalent.

We shall show how this apparent conflict is resolved, although the discussion should by no means be considered as rigorous. We shall deliberately ignore most of the technical subtleties involved in the passage between the Wightman axiomatic formulation of QFT (in terms of fields, which are unbounded-operator valued distributions) and the Haag-Kastler [23] algebraic formulation (in terms of localized observables, which are bounded operators). But we are confident that our argument captures correctly the essential features concerning the "size" of von Neumann algebras of local observables associated with generalized free fields and with free fields on AdS.

The general idea for the passage from fields to local algebras is to define for any open space-time region $O$ the von Neumann algebra

$$A(O) := \{\phi(f) : \text{supp } f \subset O\}'' \tag{5.2}$$

where $X'$ stands for the algebra of bounded operators on the given Hilbert space which commute with (the closures of) all elements of $X$. By von Neumann's density theorem, the double commutant $A(O)$ is the weak closure of the bounded functions of the unbounded smeared field operators (such as $\exp i\phi(f)$ if $\phi(f)$ is self-adjoint). Obviously, the algebras increase as the regions increase ("isotony"). Although the underlying fields are local, it is less trivial [13] that local algebras of the form (5.2) associated with spacelike separated regions mutually commute ("locality"). A covariant transformation law of the fields such as (4.15) involves the corresponding transformation of the support of test functions, hence the local algebras (5.2) transform in the obvious sense under conjugation with the unitary representatives of the group ("covariance"). One may imagine that the fields can be recovered from the algebras by taking suitably regularized limits of elements of algebras associated with regions shrinking to a point, see e.g., [18]. Together with the group of covariance which involves the time evolution, the algebraic data determine the quantum field theory.

### 5.1 "Algebraic" vs. "projective holography"

Applying the prescription (5.2) to the free Klein-Gordon field of mass $M$ on AdS, we obtain local algebras

$$A_{\mathrm{KG}}(O), \qquad (O \subset \widetilde{\mathbb{AdS}}_{d+1}). \tag{5.3}$$

Applying the same prescription to the single generalized free field $\varphi^{(\Delta)}$ on the conformal completion of Minkowski space-time, we obtain local algebras

$$A_{\Delta}(K), \qquad (K \subset \widetilde{\mathbb{CM}}^d). \tag{5.4}$$

Applying it to the the entire family of generalized free fields $\varphi_h$ with arbitrary weight functions $h$ on Minkowski space-time, we obtain local algebras

$$A_{\mathrm{tot}}(K), \qquad (K \subset \mathbb{M}^d). \tag{5.5}$$

We have the rather obvious inclusions for $K \subset \mathbb{M}^d$

$$A_{\Delta}(K) \subset A_{\mathrm{KG}}(W(K)) \subset A_{\mathrm{KG}}(V(K)) \subset A_{\mathrm{tot}}(K), \tag{5.6}$$

of which the first reflects the limit (4.13), the second is isotony, and the last reflects (4.11). We shall show that in fact

$$A_{\Delta}(K) = A_{\mathrm{KG}}(W(K)) \tag{5.7}$$

are proper subalgebras of

$$A_{\mathrm{KG}}(V(K)) = A_{\mathrm{tot}}(K) = A_{\Delta}^{\mathrm{dual}}(K). \tag{5.8}$$

In this formula, the "dual completion" [34] is defined as

$$A_{\Delta}^{\mathrm{dual}}(K) := A_{\Delta}(K^c)' \tag{5.9}$$

where $K^c$ is the causal complement of $K$ within $\mathbb{M}^d$. Fields which are relatively local to the given field, are among the generators of the dual completion. Roughly speaking, the dual completion is the algebraic counterpart of the Borchers class in

Wightman quantum field theory. By a general theorem [10], conformal invariance ensures "conformal duality"

$$A_\Delta(K) = A_\Delta(K')' \tag{5.10}$$

where $K'$ is the causal complement of $K$ within $\widetilde{\mathbb{CM}}^d$; but if $K$ is a double-cone within $\mathbb{M}^d$, then $K^c$ is strictly smaller than $K'$, and hence $A_\Delta^{\rm dual}(K)$ is expected to be strictly larger than $A_\Delta(K)$ (violation of Haag duality).

(5.7) is the holographic identification of $A_{\rm KG}(W)$ with $A_\Delta(K)$ in the sense of (5.1). It is defined globally, and covariant with respect to $SO(2,d)$. On the other hand, (5.8) shows that the projective notion of holography pertains to $A_{\rm tot}(K) = A_\Delta^{\rm dual}(K)$ instead. Projective holography is defined only with respect to the chosen chart $(z, x^\mu)$, and is covariant only under the Poincaré-dilation subgroup.

Before we prove (5.7) and (5.8), we note that $A_{\rm tot}(K)$ does not depend on the parameter $\nu$ specifying the scaling dimension of the field $\varphi^{(\Delta)}$ and the mass of the corresponding Klein-Gordon field $\phi$. Hence, (5.7) can only be true, if $A_{\rm KG}(V(K))$ does not change if the generating Klein-Gordon field $\phi$ of mass $M^2$ is replaced by $\phi'$ of mass $M'^2$. Indeed, for different values $\nu$, $\nu'$, we have by (4.10) and Hankel's formula (4.9)

$$\phi(z,x) = \int_0^\infty z'dz'\ K_{\nu\nu'}(z,z')\phi'(z',x) \tag{5.11}$$

with the kernel $K_{\nu\nu'}(z,z') = (z/z')^{\frac{d}{2}} \int_0^\infty m\,dm\ J_\nu(zm)J_{\nu'}(z'm)$. Since this kernel acts on the $z$ coordinate only, it takes test functions supported in $V(K)$ onto test functions supported in $V(K)$, and hence $A_{\rm KG}(V(K)) = A'_{\rm KG}(V(K))$.

Let us now turn to (5.7). We invoke a general theorem [1, 6] of Wightman QFT. Let $O$ be a double-cone and $T$ a timelike hypersurface passing through the apices of $O$. Then, in order to generate $A(O)$ as in (5.2), rather than smear the field in $O$ it suffices to smear the field and all its normal derivatives along $O \cap T$. In the present case, $O$ is an AdS wedge $W$, $T$ is the boundary, and $O \cap T$ is the corresponding boundary double-cone $K$. The normal derivatives of $\phi(z,x)$ are of the form $\lim_{z\to 0} \partial_z^N z^{-\Delta}\phi(z,x)$. By (4.17), and because $j_\nu$ is a power series in $-z^2\Box$, these derivatives vanish if $N$ is odd and are proportional to $\Box^n\varphi^{(\Delta)}(x)$ if $N = 2n$. With $(\Box\varphi)(f) = \varphi(\Box f)$ we conclude that $A_{\rm KG}(W)$ is in fact generated by $\varphi^{(\Delta)}(f)$, $\mathrm{supp}\, f \subset K$. This justifies our claim (5.7).

Now we turn to (5.8). Because the fields generating $A_{\rm tot}(K)$ are relatively local with respect to $\varphi^{(\Delta)}$, we know that $A_{\rm KG}(V(K)) \subset A_{\rm tot}(K) \subset A_\Delta(K^c)' \equiv A_\Delta^{\rm dual}(K)$. The claim is that equality holds.

$A_\Delta(K^c)$ is generated by all $A_\Delta(J)$ where $J$ are double-cones in $\mathbb{M}^d$ spacelike separated from $K$. Hence its commutant is the intersection, running over the same set of $J$, of algebras $A_\Delta(J)' = A_\Delta(J')$ (by (5.10)) $= A_{\rm KG}(W(J'))$ (by (5.7)). Now, $J$ is spacelike separated from $K$ and belongs to $\mathbb{M}^d$ iff $J'$ contains $K$ and the point $\omega = \mathbb{R}\cdot e_+$ of $\mathbb{CM}^d$ (spacelike infinity of $\mathbb{M}^d$, cf. Sect. 4), and iff the wedge $W(J')$

contains $W(K)$ and has $\omega$ as a boundary point ($\omega \in \partial W$). Thus,

$$A_\Delta(K^c)' = \bigcap_{\substack{W \supset W(K) \\ \partial W \ni \omega}} A_{\rm KG}(W). \tag{5.12}$$

We may choose $K = K_0$. The $x^0 = 0$ Cauchy surface of $V(K_0)$ is $C_0 = \{(z, 0, \vec{x}) : z > 0, \vec{x}^2 < 1\}$. Let $W_{\vec{e}}$ be the wedges with Cauchy surface $C_{\vec{e}} = \{(z, 0, \vec{x}) : z > 0, (\vec{e} \cdot \vec{x}) > -1\}$, $\vec{e} \in \mathbb{R}^{d-1}$, $\vec{e}^2 = 1$. Each $W_{\vec{e}}$ contains $W(K_0)$ and has $\omega$ as a boundary point, and every wedge which contains $W(K_0)$ and has $\omega$ as a boundary point, contains some $W_{\vec{e}}$. Thus, by isotony, the intersection of algebras in (5.12) may be taken over $W_{\vec{e}}$,

$$A_\Delta(K^c)' = \bigcap_{\vec{e}} A_{\rm KG}(W_{\vec{e}}). \tag{5.13}$$

Now, because of the Klein-Gordon equation, the field $\phi(z, x)$ is expressible in terms of its Cauchy data at $x^0 = 0$, hence $A_{\rm KG}(W_{\vec{e}}) = A_{\rm KG}(C_{\vec{e}})$ and $A_{\rm KG}(V(K_0)) = A_{\rm KG}(C_0)$ (time-slice property [24], see Sect. 5.2 below). The latter algebras are generated by the canonical $x^0 = 0$ Klein-Gordon fields $\phi$ and $\pi$ (cf. Sect. 4.1) smeared over the respective regions of the Cauchy surface. Thus (5.8) is reduced to the claim

$$\bigcap_{\vec{e}} A_{\rm KG}(C_{\vec{e}}) = A_{\rm KG}(C_0). \tag{5.14}$$

The independence of Cauchy data associated with disjoint regions entails [2]

$$\bigcap_{\vec{e}} A_{\rm KG}(C_{\vec{e}}) = A_{\rm KG}\big(\bigcap_{\vec{e}} C_{\vec{e}}\big). \tag{5.15}$$

Thus (5.8) is a consequence of the geometric fact $\bigcap_{\vec{e}} C_{\vec{e}} = C_0$ for $K = K_0$, and by Poincaré and dilation covariance for all $K \subset \mathbb{M}^d$.

## 5.2 Time-slice property

Finally, we consider the validity of the "time-slice property" (also called "primitive causality" [24], or "weak additivity" in [33])

$$A(O) = A(C) \tag{5.16}$$

if $O$ is the causal completion of its Cauchy surface $C$. $A(C)$ is generated by the fields and their time derivatives smeared over $C$. This property holds for canonical free fields [24], and we have just used it in the case of the Klein-Gordon field on AdS.

The time-slice property does *not* hold for generalized free fields (with $h$ fixed) [24]. E.g., for the above double-cone $K_0$ the Cauchy surface is the ball $B_0 = \{(x^0 = 0, \vec{x}) : \vec{x}^2 < 1\}$. Smearing the generalized free field $\varphi^{(\Delta)}$ together with its time derivatives over this surface, tests only the boundary limits of the corresponding canonical $x^0 = 0$ Klein-Gordon fields $\phi$ and $\pi$ and their $z$-derivatives at $\{(z = 0, x^0 = 0, \vec{x}) : \vec{x}^2 < 1\}$. Since this set does not constitute a Cauchy surface for the wedge $W(K_0)$, the generalized free field smeared over the Cauchy surface of $K_0$ generates only a proper subalgebra of $A_\Delta(K_0) = A_{\rm KG}(W(K_0))$. This violation of the time-slice property is a necessary feature of algebraic holography [33].

We want to show that the time-slice property is restored for the dual completion $A_\Delta^{\rm dual}(K) = A_{\rm tot}(K)$:

$$A_{\rm tot}(K) = A_{\rm tot}(C). \tag{5.17}$$

Again we may choose $K = K_0$. The algebra $A_{\rm tot}(K_0)$ equals $A_{\rm KG}(V(K_0))$ by (5.8) and hence is generated by the canonical $x^0 = 0$ Klein-Gordon fields $\phi$ and $\pi$ smeared over the Cauchy surface $C_0$ of $V(K_0)$. By (4.11), such smearing are smearing of $x^0 = 0$ generalized free fields $\varphi_h(0, \vec{x})$ and their time derivatives over the Cauchy surface $B_0$ of $K_0$. Hence the latter generate $A_{\rm tot}(K_0)$. This justifies (5.17).

It is natural to consider this restoration of the time-slice property as being related to the existence of the singular stress-energy tensor (3.6) which is in some weak technical sense associated with the algebras $A_{\rm tot}(K)$ but not with $A_\Delta(K)$, and whose integral over a Cauchy surface generates the causal time evolution. Even though the stress-energy tensor is not itself a Wightman field, it is not too singular to have this desirable dynamical consequence for the structure of the local algebras associated with the Wightman fields of the theory.

## 6 Conclusion

We have studied generalized free fields from a general perspective and established, in spite of the non-canonical nature of these fields, the existence of a stress-energy tensor which serves as a density for the generators of the Poincaré group as in the canonical framework. We have pointed out that this stress-energy tensor is a mathematical object which is more singular than a Wightman field, but can be obtained as a certain limit of generalized Wick products.

We have then studied the free field AdS-CFT-correspondence in the light of the previous results on generalized free fields. In particular, we have given an explicit "holographic" formula expressing the Klein-Gordon field on AdS in terms of generalized free fields on the boundary. We have identified the above stress-energy tensor for generalized free fields as an integral "along the $z$-axis of AdS" over the canonical Klein-Gordon stress-energy tensor $\Theta^{\rm KG}_{\mu\nu}(z, \cdot)$ on AdS.

These results should be useful as a starting point for a perturbation theory of the AdS-CFT-correspondence. If the AdS field is perturbed by a local interaction Lagrangean density $L_I(\phi)$ on AdS, it is naively expected that the effect on the conformal field (generalized free field) on the boundary is that of a Lagrangean perturbation by the integral "along the $z$-axis of AdS" of $L_I(\phi(z, \cdot))$. This integral, akin to (4.24), is again of the form of a (regular or singular) generalized

Wick product of the boundary field. Further analysis of this issue will be pursued elsewhere.

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