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Generalized functional-differential equations of neutral type

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Dedicated to the memory of Jacek Szarski

Abstract. This paper contains an existence theorem for functional-differential generalized equations of the form

$$\dot{x}(t) \in F(t, x_t, \dot{x}_t),$$

where F is a multivalued mapping taking as values non-empty closed convex subsets of R^n, which satisfies the Carathéodory conditions and is Lipschitz continuous with respect to the third variable.

1. Introduction. The existence of solutions of generalized functional-differential equations of the form

$$\dot{x}(t) \in F(t, x_t, \dot{x}_t) \tag{1}$$

has been investigated in the author's paper [4], with F taking non-empty compact values and satisfying, in particular, a strong continuous dependence condition with respect to the third variable. It is natural to expect that if F satisfies the Carathéodory conditions and is Lipschitz continuous with respect to the third variable, then an appriopriate initial value problem for (1) has at leat one solution. It is the aim of this paper to present an existence theorem of this type for (1) under the assumption that the values of $F(t, x, y)$ are non-empty closed convex subsets of the n-dimensional Euclidean space R^n. The proof is based on a fixed point theorem for multivalued mappings which extends the well-known fixed point theorem of Krasnoselskii ([7]). This theorem can be proved in Section 2. The main result of this paper is given in Section 3.

2. Extension of the Krasnoselskii fixed point theorem. Let $(X, |\cdot|)$ be a Banach space and let $(Y, \|\cdot\|)$ be a Hilbert space. Denote by $\Omega(Y)$

the space of all non-empty closed convex subsets of Y with the generalized Hausdorff metric d defined for $A, B \in \Omega(Y)$ by

$$d(A, B) = \begin{cases} \inf\{\varepsilon > 0: A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\} \\ \qquad\qquad\qquad \text{if the infimum exists,} \\ \infty \quad \text{otherwise,} \end{cases}$$

where $N_\varepsilon(C) = \{y \in Y: \|y - c\| < \varepsilon \text{ for some } c \in C\}$ for $\varepsilon > 0$ and $C \in \Omega(Y)$.

Let $A \in \Omega(Y)$ be bounded and suppose that $\Gamma: A \to X$ is a given mapping. We will consider a multivalued mapping $G: A \times \Gamma(A) \to \Omega(A)$. It will be proved that if $G(\cdot, y)$ is a contraction, uniformly with respect to $y \in \Gamma(A)$, if $G(x, \cdot)$ is continuous on $\Gamma(A)$ (in the relative topology) and if Γ is completely continuous, then there exists $x \in A$ such that $x \in G(x, \Gamma(x))$.

This type theorem for single-valued mappings has been proved by Melvin ([8]) in the case where Y is a Banach space.

LEMMA 2.1. *Let A be a non-empty bounded subset of Y and suppose that K is a non-empty compact subset of X. If $H: K \to \Omega(A)$ is continuous, then for every $\varepsilon > 0$ there exists a continuous function $g_\varepsilon: K \to A$ such that $a(g_\varepsilon(x), H(x)) < \varepsilon$ and $|\|g_\varepsilon(x)\| - a(0, H(x))| \leqslant \varepsilon$ for $x \in K$, where $a(c, H(x))$ denotes the distance of $c \in A$ from $H(x) \subset A$.*

Proof. Let us first observe that the generalized Hausdorff metric d restricted to $\Omega(A) \times \Omega(A)$ is a metric on $\Omega(A)$. It can be defined by setting

$$d(A_1, A_2) = \max\big(\sup_{x \in A_1} a(x, A_2), \sup_{x \in A_2} a(x, A_1)\big).$$

Let $\varepsilon > 0$ be fixed and let $\delta > 0$ be such that $d(H(x), H(y)) < \varepsilon$ for $x, y \in K$ satisfying $|x - y| < \delta$. Let $(U_i)_{1 \leqslant i \leqslant N}$ be an open cover of K with $\operatorname{diam}(U_i) < \delta$, let $(P_i)_{1 \leqslant i \leqslant N}$ be a continuous partition of unity relative to $(U_i)_{1 \leqslant i \leqslant N}$ and let $x_i \in U_i$ for $i = 1, 2, \ldots, N$. Select $v_i \in H(x_i)$ such that $\|v_i\| = a(0, H(x_i))$. This is possible, because each $H(x_i)$ is a closed convex bounded subset of the Hilbert space Y ([2], Lemma IV. 4.2).

Now we define the desired function g_ε by setting

$$g_\varepsilon(x) = \sum_{i=1}^{N} P_i(x) v_i \quad \text{for } x \in K.$$

It is clear that $g_\varepsilon: K \to A$ is continuous. Furthermore, for $x \in K$ we have

$$a(g_\varepsilon(x), H(x)) = a\Big(\sum_{i=1}^{N} P_i(x) v_i, \sum_{i=1}^{N} P_i(x) H(x_i)\Big) + \\ + \sum_{i=1}^{N} P_i(x) d(H(x_i), H(x)) < \varepsilon.$$

Finally,

$$\|g_\varepsilon(x)\| = \sum_{i=1}^{N} P_i(x)\|v_i\| \leqslant \sum_{i=1}^{N} P_i(x)a(0, H(x)) + \sum_{i=1}^{N} P_i(x)d(H(x), H(x_i))$$
$$< \varepsilon + a(0, H(x))$$

for $x \in K$. Hence it follows that

$$\big|\|g_\varepsilon(x)\| - a(0, H(x))\big| < \varepsilon,$$

which completes the proof.

LEMMA 2.2. *Suppose that the assumptions of Lemma* 2.1 *are satisfied. Then there exists a continuous function* $g\colon K \to A$ *such that* $g(x) \in H(x)$ *and* $\|g(x)\| = a(0, H(x))$ *for* $x \in K$.

Proof. Let $\varepsilon_0 > 0$ be an arbitrary number and let $g_0\colon K \to A$ be the continuous function defined in Lemma 2.1, corresponding to $\varepsilon = \varepsilon_0$. For any $\varepsilon_1 > 0$ we can select $\delta_1 > 0$ such that $d(H(x), H(y)) < \varepsilon_1$ and $\|g_0(x) - g_0(y)\| < \varepsilon_1$ for $x, y \in K$ satisfying $|x-y| < \delta_1$. Let $(U_i^1)_{1\leqslant i\leqslant N_1}$ be an open cover of K with $\operatorname{diam}(U_i^1) < \delta_1$, let (P_i^1) be a continuous partition of unity relative to $(U_i^1)_{1\leqslant i\leqslant N_1}$ and let $x_i^1 \in U_i^1$, $i = 1, 2, \ldots, N_1$. Select $V_i^1 \in H(x_i^1)$ such that

$$\|V_i^1 - g_0(x_i^1)\| = a(g_0(x_i^1), H(x_i^1))$$

for each $i = 1, 2, \ldots, N_1$.

Define a function $g_1\colon K \to A$ by setting

$$g_1(x) = \sum_{i=1}^{N_1} P_i^1(x) V_i^1$$

for $x \in K$. It is evident that g_1 is continuous and

$$a(g_1(x), H(x)) < \varepsilon_1.$$

Furthermore, for $x \in K$ we have

$$\|g_1(x) - g_0(x)\| \leqslant \Big\| \sum_{i=1}^{N_1} P_i^1(x) V_i^1 - \sum_{i=1}^{N_1} P_i^1(x) g_0(x_i^1)\Big\| +$$
$$+ \sum_{i=1}^{N_1} P_i^1(x)\|g_0(x_i^1) - g_0(x)\|$$
$$< \sum_{i=1}^{N_1} P_i^1(x)\|V_i^1 - g_0(x_i^1)\| + \varepsilon_1 < \varepsilon_0 + \varepsilon_1.$$

Moreover,

$$\big|\|g_1(x) - a(0, H(x))\big| \leqslant \big|\|g_1(x)\| - \|g_0(x)\|\big| + \big|\|g_0(x)\| - a(0, H(x))\big|$$
$$< 2\varepsilon_0 + \varepsilon_1$$

for each $x \in K$.

In a similar way we define for any $\varepsilon_2 > 0$, a continuous function $g_2\colon K \to A$ such that

$$a(g_2(x), H(x)) < \varepsilon_2, \quad \|g_2(x) - g_1(x)\| < \varepsilon_1 + \varepsilon_2$$

and

$$\big|\|g_2(x)\| - a(0, H(x))\big| < 2(\varepsilon_0 + \varepsilon_1) + \varepsilon_2$$

for $x \in K$.

We now define, by induction, for each natural $n \geqslant 1$ and $\varepsilon_n > 0$, a continuous function $g_n\colon K \to A$ such that

$$a(g_n(x), H(x)) < \varepsilon_n, \quad \|g_n(x) - g_{n-1}(x)\| < \varepsilon_n + \varepsilon_{n-1}$$

and

$$\big|\|g_n(x)\| - a(0, H(x))\big| < 2(\varepsilon_0 + \varepsilon_1 + \ldots + \varepsilon_{n-1}) + \varepsilon_n.$$

Let $\varepsilon_n = \eta/4\cdot 2^n$ for $n = 0, 1, \ldots$ and fixed $\eta > 0$. Denote by g_n^η the function g_n defined for this ε_n. It is easy to see that for every fixed $\eta > 0$ the sequence (g_n^η) converges uniformly on K. Then there exists a continuous function $g^\eta\colon K \to A$ such that $g^\eta(x) \in H(x)$ and $\big|\|g^\eta(x)\| - a(0, H(x))\big| < \eta$ for $x \in K$. Taking $\eta = 1/n$, we can easily see that the sequence $(y_n(x))$ given by $y_n(x) = \|g^{1/n}(x)\|$ converges uniformly on K to $a(0, H(x))$. Then

$$\lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} \|g^{1/n}(x)\| = a(0, H(x))$$

uniformly with respect to $x \in K$. We have, of course, $g^{1/n}(x) \in H(x)$ for $x \in K$ and $n \geqslant 1$. Therefore ([2], Lemma IV.4.2) the sequence $(g^{1/n}(x))$ is uniformly convergent on K. Let $g(x) = \lim_{n\to\infty} g^{1/n}(x)$ for $x \in K$. It is clear that g is a continuous mapping of K into A satisfying $g(x) \in H(x)$ and $\|g(x)\| = a(0, H(x))$ for $x \in K$. The proof is complete.

Now, as a corollary to Lemma 2.2, we get the following lemma.

LEMMA 2.3. *Suppose that the assumptions of Lemma 2.1 are satisfied and let $f\colon K \to A$ be continuous. Then there exists a continuous mapping $g\colon K \to A$ such that $g(x) \in H(x)$ and $\|f(x) - g(x)\| = a(f(x), H(x))$ for $x \in K$.*

Proof. Let $S\colon K \to \Omega(B)$ be a mapping defined by $S(x) = H(x) - f(x)$ for $x \in K$, where $B = \{a - f(x)\colon a \in A,\ x \in K\}$.

It is clear that S is continuous. Then, in virtue of Lemma 2.2, there exists a continuous mapping $h\colon K \to B$ such that $h(x) \in S(x)$ and $\|h(x)\| = a(0, S(x))$ for each $x \in K$.

By the definition of $S(x)$, for every $x \in K$ there exists $g(x) \in H(x)$ such that $h(x) = g(x) - f(x)$. It is evident that the mapping $g\colon K \ni x$

$\to g(x) \in A$ is continuous on K. Furthermore, we have

$$a(0, S(x)) = a(0, H(x)-f(x)) = \inf\{\|a-f(x)\|:\ a \in H(x)\} = a(f(x), H(x)).$$

Then $\|g(x)-f(x)\| = \|h(x)\| = a(f(x), H(x))$ for each $x \in K$, which completes the proof.

Now we can prove our fixed point theorem.

THEOREM 2.4. *Suppose that A is a non-empty closed convex bounded subset of the Hilbert space $(Y, \|\cdot\|)$ and let Γ be an operator with domain A and range in the Banach space $(X, |\cdot|)$. Suppose further that G: $A \times \Gamma(A) \to \Omega(A)$ is such that*

(i) *$G(\cdot, y)$ is a contraction, uniformly with respect to $y \in \Gamma(A)$*

and

(ii) *$G(x, \cdot)$ is continuous on $\Gamma(A)$ in the relative topology.*

If Γ is completely continuous, then there exists $x \in A$ such that $x \in G(x, \Gamma(x))$.

Proof. Let $K = \Gamma(A)$. By the properties of Γ, K is a compact subset of $(X, |\cdot|)$. Suppose that $L \in [0, 1)$ is such that

$$d(G(x, z), G(\bar{x}, z)) \leqslant L\|x-\bar{x}\| \quad \text{for } x,\ \bar{x} \in A$$

and every $z \in K$. Fix $g_0 \in A$ and let g_1 be a continuous mapping of K into A such that $g_1(z) \in G(x_0, z)$ and $\|g_1(z)-g_0\| = a(g_0, G(g_0, z))$. Such a mapping exists in virtue of Lemma 2.3.

Now, let g_2: $K \to A$ be a continuous mapping such that $g_2(z) \in G(g_1(z), z)$ and $\|g_2(z)-g_1(z)\| = a(g_1(z), G(g_1(z), z))$. Using induction we can select for each $n \geqslant 1$ a continuous function g_n: $K \to A$ satisfying $g_n(z) \in G(g_{n-1}(z), z)$ and $\|g_n(z)-g_{n-1}(z)\| = a(g_{n-1}(z),\ G(g_{n-1}(z), z))$.

Since $\|g_n(z)-g_{n-1}(z)\| \leqslant L\|g_{n-1}(z)-g_{n-2}(z)\|$ for $n \geqslant 2$ and each $z \in K$, a sequence (g_n) of mappings from K into A is uniformly convergent on K. Thus there exists a continuous mapping g: $K \to A$ such that

$$\sup\{\|g_n(z)-g(z)\|:\ z \in K\} \to 0 \quad \text{as } n \to \infty.$$

Since

$$a(g(z), G(g(z), z)) \leqslant \|g(z)-g_n(z)\| + a(g_n(z), G(g_{n-1}(z), z)) + d(G(g_{n-1}(z), z), G(g(z), z))$$

for $z \in K$ and $n \geqslant 1$, therefore $g(z) \in G(g(z), z)$ for each $z \in K$.

Let us now consider the operator T: $A \to A$ given by $T(x) = g(\Gamma(x))$ for $x \in A$. Since Γ is completely continuous and g is continuous on $K = \Gamma(A)$, T is completely continuous on A. Hence, in virtue of the Schauder fixed point theorem, follows the existence of $x \in A$ such that $x = g(\Gamma(x))$.

But $x = g(\Gamma(x)) \in G(g(\Gamma(x)), \Gamma(x)) = G(x, \Gamma(x))$, and so x is a fixed point of $G(x, \Gamma(x))$ and the proof is complete.

3. Existence of solutions of a generalized functional-differential equations of neutral type. Let $r \geqslant 0$ and $\beta \geqslant 0$ be given and let us denote by C_β and L^2_β, respectively, the space of all continuous and the space of all square (Lebesgue) integrable functions of $[-r, \beta]$ into $\boldsymbol{R}^n$ with the usual norms. By $\Omega(\boldsymbol{R}^n)$ we will denote, as in Section 2, the space of all non-empty closed convex subsets of $\boldsymbol{R}^n$ with the generalized Hausdorff metric h.

Let f: $[0, T] \times C_T \times L^2_T \to \Omega(\boldsymbol{R}^n)$ be a multivalued mapping satisfying the following Carathéodory type conditions:

(i) $f(\cdot, x, y)$ is measurable for fixed $(x, u) \in C_T \times L^2_T$,

(ii) $f(t, \cdot, \cdot)$ is continuous for fixed $t \in [0, T]$

and

(iii) there exists a square (Lebesgue) integrable function m_f: $[0, T] \to R$ such that

$$h(f(t, x, u), \{0\}) \leqslant m_f(t) \quad \text{for } (x, u) \in C_T \times L^2_T$$

and almost all $t \in [0, T]$.

Furthermore, it will be assumed that $f(t, x, \cdot)$ is Lipschitz continuous uniformly with respect to $x \in C_T$, i.e. that

(iv) there exists a square (Lebesgue) integrable function k: $[0, T] \to R$ such that

$$h(f(t, x, u), f(t, x, \bar{u})) \leqslant k(t) \int_{-r}^{t} |u(s) - \bar{u}(s)| ds$$

for $x \in C_T$, $u, \bar{u} \in L^2_T$ and almost all $t \in [0, T]$.

We will consider a generalized functional-differential equation of the form

(2) $$\dot{x}(t) \in f(t, x, \dot{x}) \quad \text{for almost all } t \in [0, T]$$

together with the initial condition

(3) $$x(t) = \varphi(t) \quad \text{for } t \in [-r, 0],$$

where φ: $[-r, 0] \to \boldsymbol{R}^n$ is an absolutely continuous function such that $\dot{\varphi} \in L^2_0$.

It will be shown that conditions (i)–(iv) imply the existence of an absolutely continuous function x: $[-r, T] \to R$ satisfying (2) and (3). Hence, in particular, follows the existence of solutions of (1) with the initial condition (3).

THEOREM 3.1. *Suppose that* f: $[0, T] \times C_T \times L^2_T \to \Omega(\boldsymbol{R}^n)$ *satisfies the Carathéodory conditions* (i)–(iii) *and the Lipschitz condition* (iv). *Let* φ:

$[-r, 0] \to \boldsymbol{R}^n$ *be an absolutely continuous function such that* $\dot{\varphi} \in L_0^2$. *Then there exists at least one absolutely continuous function* $x \in C_T$ *such that*

$$(4) \qquad \begin{aligned} &\dot{x}(t) \in f(t, x, \dot{x}) && \text{for almost all } t \in [0, T], \\ &x(t) = \varphi(t) && \text{for } t \in [-r, 0]. \end{aligned}$$

Proof. Let A be a subset of L_T^2 containing all functions $u \in L_T^2$ such that $u(t) = \dot{\varphi}(t)$ for almost all $t \in [-r, 0]$ and $|u(t)| \leqslant m_f(t)$ for almost every $t \in [0, T]$. It is not difficult to see that A is a closed convex bounded subset of L_T^2.

Let us introduce in L_T^2 the inner product $\langle \cdot, \cdot \rangle$ by setting

$$\langle u, v \rangle = \int_{-r}^{T} |u(t)|\,|v(t)|\, e^{-L \cdot K(t)}\, dt,$$

where $L > T$, $K(t) = 0$ for $t \in [-r, 0]$ and $K(t) = \int_0^t k^2(s)\,ds$ for $t \in [0, T]$. Let $\|\cdot\|_T$ be a norm in L_T^2 defined by $\|u\|_T = \sqrt{\langle u, u \rangle}$ for $u \in L_T^2$.

As in Section 2, let us denote by $\Omega(L_T^2)$ the space of all non-empty closed convex subsets of L_T^2 with the generalized Hausdorff metric d generated by $\|\cdot\|_T$.

Define an operator Γ on A by putting

$$(\Gamma u)(t) = \begin{cases} \varphi(t) & \text{for } t \in [-r, 0], \\ \varphi(0) + \int_0^t u(s)\,ds & \text{for } t \in [0, T]. \end{cases}$$

It is easy to see that Γ is a completely continuous mapping of A into C_T.

Let us now consider a generalized functional equation

$$(5) \qquad u(t) \in f\big(t, \Gamma(u), u\big) \quad \text{for almost all } t \in [0, T].$$

It is clear that for each $u \in A$, satisfying (5), $x = \Gamma(u)$ satisfies (4). For any fixed $(u, x) \in L_T^2 \times C_T$ let $G(u, x)$ denote the subset of L_T^2 containing all $y \in L_T^2$ of the form

$$y(t) = \begin{cases} \dot{\varphi}(t) & \text{for almost all } t \in [-r, 0], \\ v(t) & \text{for } t \in [0, T], \end{cases}$$

where $v\colon [0, T] \to \boldsymbol{R}^n$ is a measurable selector of $f(\cdot, x, u)$. We have, of course, $G(u, x) \neq \emptyset$ for each $(u, x) \in L_T^2 \times C_T$ ([6]). It is not difficult to verify that $G(u, x)$ is a closed convex subset of A for each $(u, x) \in L_T^2 \times C_T$. Then $G\colon A \times C_T \to \Omega(A)$.

We now show that $G(\cdot, x)$ is a contraction, uniformly with respect to $x \in C_T$. Let $x \in C_T$ be fixed and let $u_1, u_2 \in A$.

For every $y \in G(u_1, x)$ and almost all $t \in [0, T]$ there exists a $z_t \in f(t, x, u_2)$ such that

$$|y(t) - z_t| \leqslant k(t) \int_0^t |u_1(s) - u_2(s)| ds.$$

Thus we can define a multivalued mapping $\mathcal{R}$ by

$$\mathcal{R}(t) = f(t, x, y_2) \cap S(t),$$

where

$$S(t) = \{z \in \boldsymbol{R}^n\colon \ |y(t) - z| \leqslant k(t) \int_0^t |u_1(s) - u_2(s)| ds.$$

We have $\mathcal{R}(t) \neq \emptyset$ for almost all $t \in [0, T]$, because $z_t \in \mathcal{R}(t)$ for almost every $t \in [0, T]$.

It is easy to verify that S is a measurable multivalued mapping of $[0, T]$ into $\Omega(\boldsymbol{R}^n)$. Thus $\mathcal{R}$ is also a measurable mapping of $[0, T]$ into $\Omega(\boldsymbol{R}^n)$. Therefore there exists ([6]) a measurable selector w for $\mathcal{R}$. We have

$$|y(t) - \tilde{w}(t)| \leqslant k(t) \int_0^t |u_1(s) - u_2(s)| ds$$

for almost all $t \in [0, T]$, where all $\tilde{w}(t) = \dot{\varphi}(t)$ for almost all $t \in [-r, 0]$ and $\tilde{w}(t) = w(t)$ for $t \in [0, T]$. Then

$$\begin{aligned}
\int_{-r}^{T} |y(t) - \tilde{w}(t)|^2 e^{-L\cdot K(t)} dt &= \int_0^T |y(t) - \tilde{w}(t)|^2 e^{-L\cdot K(t)} dt \\
&\leqslant \int_0^T k^2(t) \Big(\int_0^t |u_1(s) - u_2(s)| ds\Big)^2 e^{-L\cdot K(t)} dt \\
&\leqslant T \int_0^T k^2(t) e^{-L\cdot K(t)} \int_0^t |u_1(s) - u_2(s)|^2 ds\, dt \\
&= T \int_0^T \int_s^T k^2(t) e^{-L\cdot K(t)} |u_1(s) - u_2(s)|^2 ds\, dt \\
&= T \int_0^T |u_1(s) - u_2(s)|^2 \int_s^T k^2(t) e^{-L\cdot K(t)} dt\, ds \\
&= (T/L) \int_0^T |u_1(s) - u_2(s)|^2 [e^{-L\cdot K(s)} - e^{-L\cdot K(T)}] ds \\
&\leqslant (T/L) \int_0^T |u_1(s) - u_2(s)|^2 e^{-L\cdot K(s)} ds.
\end{aligned}$$

Therefore

$$\|y-\tilde{w}\|_T \leqslant (T/L)\|u_1-u_2\|_T.$$

Hence and from the analogous inequality obtained by interchanging the roles of u_1 and u_2 we get

$$d\big(G(u_1, x), G(u_2, x)\big) \leqslant (T/L)\|u_1-u_2\|_T$$

for each $x \in C_T$ and $u_1, u_2 \in A$. Since $L > T$, the mapping $G(\cdot, x)$ is a contraction, uniformly with respect to $x \in C_T$.

In a similar way we can verify that $G(u, \cdot)$ is continuous for any fixed $u \in A$.

Therefore, in virtue of our fixed point theorem, there exists $u \in A$ such that $u \in G\big(u, \Gamma(u)\big)$, which means that $u(t) \in f\big(t, \Gamma(u), u\big)$ for almost all $t \in [0, T]$ and $u(t) = \dot{\varphi}(t)$ for almost every $t \in [-r, 0]$. Hence it follows that $x = \Gamma(u)$ satisfies (4) and the proof is complete.

Now, as a consequence of the above existence theorem we get

THEOREM 3.2. *Suppose that* $F\colon [0, T]\times C_0 \times L_0^2 \to \Omega(\boldsymbol{R}^n)$ *satisfies the Carathéodory conditions with* $m_F \in L_2([0, T], R)$ *and assume that there exists a square (Lebesgue) integrable function* $k\colon [0, T] \to R$ *such that*

$$h\big(F(t, x, y_1), F(t, x, y_2)\big) \leqslant k(t) \int_{-r}^{0} |y_1(s)-y_2(s)|\,ds$$

for $x \in C_0$, $y_1, y_2 \in L_0^2$ *and almost all* $t \in [0, T]$. *Let* $\varphi\colon [-r, 0] \to \boldsymbol{R}^n$ *be an absolutely continuous function such that* $\dot{\varphi} \in L_0^2$. *Then there exists an absolutely continuous function* $x\colon [-r, T] \to \boldsymbol{R}^n$ *such that*

$$\begin{aligned} &\dot{x}(t) \in F(t, x_t, \dot{x}_t) && \text{for almost every } t \in [0, T],\\ &x(t) = \varphi(t) && \text{for } t \in [-r, 0], \end{aligned}$$

where $x_t(s) = x(t+s)$ *and* $\dot{x}_t(s) = \dot{x}(t+s)$ *for* $t \in [0, T]$ *and* $s \in [-r, 0]$.

Proof. Let $f\colon [0, T]\times C_T \times L_T^2 \to \Omega(\boldsymbol{R}^n)$ be the mapping defined by $f(t, x, y) = F(t, x_t, y_t)$ for $t \in [0, T]$ and $(x, y) \in C_T \times L_T^2$.

It is not difficult to check that f satisfies the Carathéodory conditions. Furthermore, for fixed $x \in C_T$ and $y, \bar{y} \in L_T^2$ we have

$$\begin{aligned} h\big(f(t, x, y), f(t, x, \bar{y})\big) &= h\big(F(t, x_t, y_t), F(t, x_t, \bar{y}_t)\big)\\ &\leqslant k(t) \int_{-r}^{0} |y(t+s)-\bar{y}(t+s)|\,ds\\ &= k(t) \int_{t-r}^{t} |y(\tau)-\bar{y}(\tau)|\,d\tau \leqslant k(t) \int_{-r}^{t} |y(\tau)-\bar{y}(\tau)|\,d\tau \end{aligned}$$

for almost all $t \in [0, T]$. Therefore, Theorem 3.1 implies the existence of an absolutely continuous function $x\colon [-r, T] \to \boldsymbol{R}^n$ such that $x(t)$

$= \varphi(t)$ for $t \in [-r, 0]$ and $\dot{x}(t) \in f(t, x, \dot{x}) = F(t, x_t, \dot{x}_t)$ for almost all $t \in [0, T]$. This completes the proof.

Remark 3.1. The idea of renorming certain function spaces in order to change a locally contracting operator into a global contraction is due to Bielecki ([1]); we have used here this idea, just as Himmelberg and Van Vleck ([3]) have done with contracting multifunctions. Lasota and Opial ([7]) have also indicated its usefulness in connection with solving generalized differential equations via the Fan fixed point theorem.

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