Generalized Goal Programming: Polynomial Methods and Applications

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January 8, 2001

Abstract

In this paper we address a general Goal Programming problem with linear objectives, convex constraints, and an arbitrary componentwise nondecreasing norm to aggregate deviations with respect to targets. In particular, classical Linear Goal Programming problems, as well as several models in Location and Regression Analysis are modeled within this framework.

In spite of its generality, this problem can be analyzed from a geometrical and a computational viewpoint, and a unified solution methodology can be given. Indeed, a dual is derived, enabling us to describe the set of optimal solutions geometrically. Moreover, Interior-Point methods are described which yield an ε-optimal solution in polynomial time.

Keywords: Goal Programming, Closest points, Interior point methods, Location, Regression.

1 Introduction

1.1 Goal Programming

The origins of Goal Programming date back to the work of Charnes, Cooper and Ferguson [7], where an l_1-estimation regression model was proposed to estimate executive compensation. Since then, and thanks to its versatility and ease of use, it has become the by far most popular technique for tackling (linear) multiple-objective problems, as evidenced by the bulk of literature on theory and applications of the field. See, e. g., [40, 41, 44, 45] and the categorized bibliography of applications therein.

By a *Non-Preemptive Goal Programming problem* one usually means some particular instance of the following model: a polyhedron $K \subseteq \mathbb{R}^n$ is given as the set of decisions; there exist m *criteria* matrices, $C_1, \ldots, C_m$, with C_j in $\mathbb{R}^{n \times n_j}$; each decision $x \in K$ is valued according to criterion C_j by the vector $C_j^\top x$, to be compared with a given *target set* $T_j \subseteq \mathbb{R}^{n_j}$. With this, the *deviation* $d_j(x)$ of decision x with respect to the target set T_j is defined as

$$d_j(x) = \inf_{z_j \in T_j} \gamma_j(C_j^\top x - z_j)$$

for some given norm γ_j, while the overall deviation at x is measured by

$$\gamma(d_1(x), \ldots, d_m(x)),$$

where γ is a norm in $\mathbb{R}^m$ assumed to be *monotonic* in the nonnegative orthant $\mathbb{R}^m_+$ (see [4, 25]) i. e.

$$\gamma(u) \leq \gamma(v) \qquad \text{for all} \quad u, v \in \mathbb{R}^m_+ \text{ with } 0 \leq u_i \leq v_i \text{ for all } i = 1, \ldots, m.$$

Then, the solution(s) minimizing the overall deviation are sought. In other words, one solves the convex program

$$\inf_{x \in K} \gamma(d_1(x), \ldots, d_m(x)). \tag{1}$$

As pointed out e. g. in [8, 39, 40], Non-Preemptive Goal Programming and related models can be rephrased as minimum-distance problems. This follows from the previous formulation, since (1) is equivalent to

$$\begin{aligned} \min \quad & \gamma(\gamma_1(C_1^\top x - z_1), \ldots, \gamma_m(C_m^\top x - z_m)) \\ \text{s.t.} \quad & x \in K, \\ & z_j \in T_j \qquad \forall\, j = 1, \ldots, m. \end{aligned} \tag{2}$$

Denoting by $\widetilde{\gamma}$ the norm in $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}$ defined as

$$\widetilde{\gamma}(u_1, \ldots, u_m) = \gamma(\gamma_1(u_1), \ldots, \gamma_m(u_m)),$$

problem (2) can be written as the minimum $\widetilde{\gamma}$-norm problem

$$\begin{aligned} \min \quad & \widetilde{\gamma}(u_1, \ldots, u_m) \\ \text{s.t.} \quad & u_j = C_j^\top x - z_j \qquad \forall j = 1, \ldots, m \\ & (x, z) \in K \times \textstyle\prod_{1 \le j \le m} T_j \end{aligned} \tag{3}$$

In many applications, each criterion C_j is assumed to be a vector $c_j \in \mathbb{R}^n$, so it values x through the scalar $c_j^\top x$; each target set T_j is then a subset of $\mathbb{R}$ of one the forms

$$T_j = [t_j, +\infty), \tag{4}$$

$$T_j = (-\infty, t_j], \tag{5}$$

$$T_j = \{t_j\}, \tag{6}$$

or, in *Goal Range Programming* [20], of the form

$$T_j = [\underline{t}_j, \overline{t}_j]. \tag{7}$$

This corresponds to a *goal constraint* of type $c_j^\top x \ge t_j$, $c_j^\top x \le t_j$, $c_j^\top x = t_j$, or $c_j^\top x \in [\underline{t}_j, \overline{t}_j]$, respectively. In other words, one desires to have $c_j^\top x$ above t_j, below t_j, exactly at t_j, or between $\underline{t}_j$ and $\overline{t}_j$, respectively.

Whereas the choice of the aggregating norm γ is crucial, (although, in applications, mostly reduced to the cases l_1 or l_∞) the choice of the norms γ_j to measure deviations in the case $n_j = 1\ \forall\, j$ is irrelevant, and we can consider each γ_j to be equal to the absolute value function. Then, the deviations take on the more familiar form

$$d_j(x) = \begin{cases} \max\left\{t_j - c_j^\top x, 0\right\} & \text{if } T_j = [t_j, +\infty), \\ \max\left\{c_j^\top x - t_j, 0\right\} & \text{if } T_j = (-\infty, t_j], \\ |c_j^\top x - t_j| & \text{if } T_j = \{t_j\}, \\ \max\left\{\underline{t}_j - c_j^\top x, 0\right\} + \max\left\{c_j^\top x - \overline{t}_j, 0\right\} & \text{if } T_j = [\underline{t}_j, \overline{t}_j]. \end{cases}$$

From these expressions, it should become clear that target sets of type (7), (thus also of type (6)) are used only for modeling convenience, since they can be derived from sets of types (4) and (5): splitting criterion j into criteria j_1, j_2, and defining $T_j^1 = [\underline{t}_j, +\infty)$ and $T_j^2 = (-\infty, \overline{t}_j]$, the deviation $d_j(x)$ is simply the sum of the deviations with respect to T_j^1 and T_j^2.

1.2 Examples

Applications of Goal Programming abound in the literature; see e. g. the list of 351 applications papers cited in [40]. However, the range of applicability of (1) is by no means reduced to what is usually classified as Goal Programming: a vast series of important models in different fields of Optimization can also be seen as particular instances of (1), mainly from the perspective of minimum-distance problems. Some of them are briefly discussed below.

Overdetermined systems of (in)equalities

If a system of linear equalities and inequalities

$$\begin{array}{rcl} a_1^\top x & \geq & b_1 \\ a_2^\top x & \geq & b_2 \\ \vdots & \geq & \vdots \\ a_p^\top x & \geq & b_p \\ a_{p+1}^\top x & = & b_{p+1} \\ \vdots & = & \vdots \\ a_{p+q}^\top x & = & b_{p+q} \end{array} \tag{8}$$

is infeasible, one can look for a so-called *least infeasible* solution, i. e. a point x^* solving

$$\min_x \gamma(\max(0, b_1 - a_1^\top x), \ldots, \max(0, b_p - a_p^\top x), |b_{p+1} - a_{p+1}^\top x|, \ldots, |b_{p+q} - a_{p+q}^\top x|)$$

for some norm γ monotonic in $\mathbb{R}_+^{p+q}$. This is simply a Goal Programming problem in which the vectors a_i $(i = 1, \ldots, p+q)$ play the role of the criteria and the components b_i $(i = 1, \ldots, p+q)$ of the right hand side vector represent the targets, see Example 4 in Section 3.

When only equalities appear in (8), one obtains the problem of solving an overdetermined system of linear equations, classical in Approximation Theory [33, 43], or, equivalently, the Linear Regression problem [42]. Usually, γ is assumed to be an l_p norm, mainly $p = 2$, (yielding the well-known Least Squares problem [3]) $p = 1$, or $p = \infty$ [1].

Multifacility location

In Continuous Location [29, 36], distances are usually measured by *gauges*. For simplicity, we will consider throughout this paper only gauges γ of the form

$$\gamma(x) = \inf\{t \geq 0 : \, x \in tB\}$$

for some nonempty convex compact $B \subset \mathbb{R}^m$ (its *unit ball*) containing the origin in its interior. In applications, this additional assumption is usually fulfilled, see, e. g. [10, 29]. Observe that norms correspond to symmetric gauges. Moreover, since the origin is assumed to be an interior point, the gauge takes always finite values. See e. g. [17] for the case of gauges with values on $\mathbb{R}^+ \cup \{+\infty\}$.

Let F be a nonempty finite set and let $\emptyset \neq E \subseteq F \times F$. Then (F, E) is a directed graph. Following e. g. [13, 27], F represents the set of *facilities* (some of which may have fixed locations in $\mathbb{R}^n$), whereas E represents the interactions between these facilities.

For each edge $e := (f, g) \in E$, let γ_e be a given gauge in $\mathbb{R}^n$, which measures the cost of the interaction between facility f and facility g. Let γ be a gauge in $\mathbb{R}^E$ monotonic in the non-negative orthant.

For a nonempty closed convex set $K \subseteq (\mathbb{R}^n)^F$, consider the optimization problem

$$\inf_{(x_f)_{f\in F} \in K} \gamma((\gamma_{(f,g)}(x_f - x_g))_{(f,g)\in E}). \tag{9}$$

The most popular instance of (9) is the *continuous minisum multifacility location* problem, see [36, 46, 47] and the references therein. There, the node set F is partitioned into two sets A and V, representing respectively the fixed and the free locations, and a family $(a_f)_{f\in A} \in (\mathbb{R}^n)^A$ of fixed locations is given. The feasible region K is then defined by

$$K = \left\{x = (x_f)_{f\in F} \in (\mathbb{R}^n)^F \mid x_f = a_f \text{ for all } f \in A\right\}, \tag{10}$$

while the gauge γ is taken as the l_1 norm, so that one minimizes the sum of all interactions between the facilities,

$$\inf_{x_f = a_f \, \forall f \in A} \sum_{(f,g) \in E} \gamma_{(f,g)}(x_f - x_g). \tag{11}$$

Let $J_{(F,E)}$ be the incidence matrix of the graph (F, E), i. e. $J_{(F,E)} \in \mathrm{IR}^{E \times F}$ is the matrix in which the row $e := (f, g) \in E$ has zeroes in all its positions except in the position f, where the entry is 1, and in position g, where the entry is -1. Moreover, define the matrix C by $C := J_{(F,E)} \otimes I_n$, the Kronecker product of $J_{(F,E)}$ with the unit matrix $I_n \in \mathrm{IR}^{n \times n}$.

Let γ be the gauge in $(\mathrm{IR}^n)^E$ defined by

$$\begin{aligned} \gamma : u := (u_e)_{e \in E} \longmapsto \gamma(u) \;&:=\; \|(\gamma_e(u_e))_{e \in E}\|_1 \\ &=\; \sum_{e \in E} \gamma_e(u_e). \end{aligned}$$

Then, problem (11) can also be written as

$$\begin{array}{ll} \min & \gamma(Cx) \;=\; \gamma(((Cx)_e)_{e \in E}) \\ \text{s.t} & x \in K \subseteq (\mathrm{IR}^n)^F, \end{array} \tag{12}$$

which is a particular instance of (1).

A similar representation can be obtained for the *continuous minimax multifacility location problem* [24], in which expression (12) holds for γ defined by

$$\begin{aligned} \gamma : u := (u_e)_{e \in E} \longmapsto \gamma(u) \;&:=\; \|(\gamma_e(u_e)_{e \in E})\|_\infty \\ &=\; \max_{e \in E} \gamma_e(u_e) \end{aligned}$$

General monotone gauges γ have been suggested by Durier [9, 10]. In the latter paper, he introduced problems with fixed costs, which can also be accommodated within this framework. Indeed, for

$$\inf_{(x_f)_{f \in F} \in K} \gamma\left((\omega_{(f,g)} + \gamma_{(f,g)}(x_f - x_g))_{(f,g) \in E} \right)$$

with a given vector $(\omega_e)_{e \in E} \in \mathrm{IR}^E$ with non-negative components, one may write

$$\inf_{(x_f)_{f \in F} \in K} \gamma\left((\hat{\gamma}_{(f,g)}(\omega_{(f,g)}, x_f - x_g))_{(f,g) \in E} \right),$$

where each $\hat{\gamma}_e$ is a gauge in $\mathbb{R} \times \mathbb{R}^n$ defined by

$$\hat{\gamma}_e(\omega, z) = |\omega| + \gamma_e(z).$$

With this, again an expression of type (12) is obtained.

Our aim is to study a generalized version of Problem (1) under some mild assumptions on the feasible set K, namely, K will be assumed to be an *asymptotically conical set.* To do this, we have structured the remaining of the paper as follows: In Section 2 the concept of asymptotically conical set is introduced, and some elementary properties are discussed. Then, in Section 3, the problem under study, (P), is formally defined, and its dual is derived. In Section 4, the existence of primal and dual optimal solutions is studied in detail, giving, in particular, sufficient conditions for the attainment of the optimal value. Then, an Interior-Point method is described in Section 5, yielding a unified methodology for solving problems which, until now, were solved by different (some not polynomial) techniques.

2 Asymptotically Conical Sets and their Properties

In what follows, for given nonempty subsets S_1, S_2 of $\mathbb{R}^n$, we mean by $S_1 + S_2$ the algebraic sum of S_1 and S_2,

$$S_1 + S_2 = \{s \in \mathbb{R}^n \mid s = s_1 + s_2 \text{ for some } s_1 \in S_1, s_2 \in S_2\}$$

When S_1 is a singleton, $S_1 = \{s_1\}$, we will write $s_1 + S_2$ to represent $\{s_1\} + S_2$.

Definition 1 *A nonempty set $S \subseteq \mathbb{R}^n$ is said to be* asymptotically conical *if it admits a representation of the form*

$$S = M + E, \tag{13}$$

*for some compact convex set M and some closed convex cone E. In such a case, the pair (M, E) is an asymptotically conical representation (*a. c. r.*) of S.*

The optimization problem addressed in this paper will have an asymptotically conical set as feasible set, (see Section 3). Here we take a quick look at the basic properties of such class of sets.

Denote by K_∞ the recession cone of K,

$$K_\infty = \{y \in \mathrm{IR}^n : K + y \subseteq K\},$$

see Theorem 8.1 in [37]. We now have the following properties.

Property 1 *If (M, E) is an a. c. r. of K then*

$$K_\infty = E. \tag{14}$$

Property 2 *Let $(M_1, E_1), (M_2, E_2)$ be a. c. r. of the asymptotically conical sets $K_1, K_2 \subseteq \mathrm{IR}^n$. Then,*

1. *$(M_1 \times M_2, E_1 \times E_2)$ is an a. c. r. of the asymptotically conical set $K_1 \times K_2$.*

2. *$(M_1 + M_2, E_1 + E_2)$ is an a. c. r. of the asymptotically conical set $K_1 + K_2$.*

Property 3 *Let $\mathcal{A}$ be an affine transformation of the form $\mathcal{A}(x) = Ax + b$ with a matrix A and a vector b. Then, $(AM + b, AE)$ is an a.c.r. of $\mathcal{A}(K)$.*

Remark 4 Compact sets, polyhedra, affine spaces, and cones are asymptotically conical. Although each of these classes is closed under intersections, this is not the case of the whole class of asymptotically conical sets. Indeed, take e. g. the following asymptotically conical sets in IR^3: $S_1 = \left\{(x_1, x_2, x_3)^\top \mid x_3^2 \geq x_1^2 + x_2^2, x_3 \geq 0\right\}$ and $S_2 = \left\{(x_1, x_2, x_3)^\top \mid x_1 = 1\right\}$, whereas, by Property 1, no a. c. r. for the hyperbola $S_1 \cap S_2$ exists. Moreover, this example shows that the inverse image of an asymptotically conical set under an affine mapping is not necessarily asymptotically conical. □

Denoting for each cone $S \subseteq \mathrm{IR}^n$ by S^* its *dual* cone,

$$S^* = \{x \in \mathrm{IR}^n \mid x^\top s \geq 0 \text{ for all } s \in S\},$$

one then has

Property 5 *Let (M, E) be an a. c. r. of K. Then, for any $u \in \mathrm{IR}^n$,*

$$\inf_{x \in K} u^\top x = \begin{cases} \min_{x \in M} u^\top x & \text{if } u \in E^*, \\ -\infty & \text{else.} \end{cases} \tag{15}$$

3 The Problem Addressed and its Dual

The problem addressed in this paper has the form

$$\begin{array}{ll} \inf & g(x) := \gamma(Cx+c) + d^\top x, \\ \text{s. t.} & x \in K, \end{array} \tag{P}$$

where γ is a gauge in the $\mathbb{R}^m$, C is a matrix in $\mathbb{R}^{m\times n}$, $c \in \mathbb{R}^m$ and $d \in \mathbb{R}^n$ are vectors, and $K = M + E \subseteq \mathbb{R}^n$ is a nonempty asymptotically conical set with a. c. r. (M, E).

Observe that, in particular, problem (P) contains as instances all the examples discussed in Section 1.2. Moreover, the case $K = \mathbb{R}^n$ has been addressed in [26], whereas the case $d = 0$ leads to the so-called *gauge-* or *homogeneous program*, addressed, among others in [17, 12, 19, 21].

In these references, duals are derived and Slater-type assumptions are made to link primal and dual optimality. We show below and illustrate by examples how the knowledge of an a. c. r. can be successfully used to address duality questions and to design efficient algorithms as well.

A dual for (P) can easily be derived using minmax theorems as basic tool. Indeed, one has

$$\inf_{x\in K} \gamma(Cx+c) + d^\top x = \inf_{x\in K} \sup_{\gamma^\circ(u)\le 1} \left(u^\top (Cx+c) + d^\top x\right) \tag{16}$$

$$= \sup_{\gamma^\circ(u)\le 1} u^\top c + \inf_{x\in K} \left(u^\top C + d^\top\right) x, \tag{17}$$

$$= \sup_{\substack{\gamma^\circ(u)\le 1,\\ C^\top u + d \in E^*}} u^\top c + \min_{x\in M} \left(u^\top C + d^\top\right) x \tag{18}$$

where (16) follows from the representation of a gauge as the support of its polar unit ball, see Theorem 14.5 of [37], and (17) follows from the Minimax Theorem stated as Corollary 37.3.2 in [37] and the fact that γ°, the dual gauge of γ, has compact level sets (recall that we are assuming that gauges γ has the origin in its interior, which guarantees the compactness of its dual ball). Finally, (18) follows from Property 5.

Denoting by δ^*_S the support of the set $S \subseteq \mathbb{R}^n$,

$$\delta^*_S(x) = \sup\left\{x^\top y \mid y \in S\right\},$$

the chain of equalities above yields

$$\begin{array}{llcll} \inf & \gamma(Cx+c)+d^\top x & = & \max & u^\top c - \delta^*_M(-C^\top u - d) \\ \text{s.t.} & x \in M+E & & \text{s.t.} & C^\top u + d \in E^* \\ & & & & \gamma^\circ(u) \le 1 \end{array} \tag{19}$$

From this equivalence, we will call the optimization problem in the right-hand side of (19) the *dual* (D) of problem (P), and we have already shown that (P) and (D) have identical optimal value.

Before exploring further the relations between (P) and (D) we now present some particular instances of (P), whose corresponding dual (D) has a simple (explicit) form.

Example 1 Let $\tilde\gamma$ be a gauge in $\mathbb{R}^n$, let $x_0 \in \mathbb{R}^n$ and let $M = \{x \in \mathbb{R}^n \mid \tilde\gamma(x-x_0) \le r\}$ for some constant $r \ge 0$. Since, by definition of dual gauges,

$$\min\left\{\left(u^\top C + d^\top\right) x \mid \tilde\gamma(x-x_0) \le r\right\} = \left(u^\top C + d^\top\right) x_0 - r\tilde\gamma^\circ(-C^\top u - d),$$

we get the dual

$$\begin{array}{ll} \max & u^\top c + \left(u^\top C + d^\top\right) x_0 - r\tilde\gamma^\circ(-C^\top u - d) \\ \text{s.t.} & C^\top u + d \in E^* \\ & \gamma^\circ(u) \le 1. \end{array} \tag{20}$$

Example 2 Setting $K = \mathbb{R}^n$ and $M = \{0\}$, we have $E = \mathbb{R}^n$ and $E^* = \{0\}$. Hence, the dual (D) takes the form

$$\begin{array}{llcl} \max & u^\top c & & \\ \text{s.t.} & C^\top u + d & = & 0, \\ & \gamma^\circ(u) & \le & 1. \end{array}$$

This dual has been derived in [26] using the same idea but lengthier arguments, see their Theorem 1, Remark 4 and Remark 5. □

Example 3 Given two asymptotically conical sets K_1, K_2 in $\mathbb{R}^n$, with respective a. c. r. $K_1 = M_1 + E_1, K_2 = M_2 + E_2$ and a gauge γ in $\mathbb{R}^n$, formula (19) provides an alternative expression for the distance $\delta_\gamma(K_1, K_2)$ between K_1 and K_2. Indeed, since

$$\begin{aligned}\delta_\gamma(K_1, K_2) &= \inf\{\gamma(x_1 - x_2) \mid x_1 \in K_1, x_2 \in K_2\} \\ &= \inf\{\gamma(Cx) \mid x \in K\},\end{aligned}$$

for $K := K_1 \times K_2$ and $C = (I_n, -I_n)$ and I_n the $n \times n$ identity matrix, one gets

$$\begin{aligned}\delta_\gamma(K_1, K_2) &= \max_{u \in E_1^* \cap (-E_2^*) \cap B^\circ} \left(\min_{x_1 \in M_1, x_2 \in M_2} u^\top (x_1 - x_2) \right) \\ &= \max_{u \in (E_1^* \cap (-E_2^*) \cap B^\circ)} \min_{x \in M_1 - M_2} u^\top x\end{aligned}$$

where B° is the unit ball of the gauge γ°. When K_1 is an affine manifold, $K_1 = \{p_0\} + E_1$ for some vector space $E_1 \subseteq \mathbb{R}^n$, and K_2 is a cone, thus having $(\{0\}, K_2)$ as a. c. r. decomposition, we get

$$\delta_\gamma(K_1, K_2) = \max\left\{u^\top p_0 \mid u \in E_1^* \cap (-E_2^*) \cap B^\circ\right\}$$

This expression yields a simple characterization for $K_1 \cap K_2 \neq \emptyset$. Indeed, for any γ one has that $K_1 \cap K_2 \neq \emptyset$ iff $\delta_\gamma(K_1, K_2) \leq 0$ (see Corollary 17), thus

$$\begin{aligned}K_1 \cap K_2 \neq \emptyset \quad &\text{iff} \quad u^\top p_0 \leq 0 \qquad \forall u \in E_1^* \cap (-E_2^*) \cap B^\circ \\ &\text{iff} \quad u^\top p_0 \leq 0 \qquad \forall u \in E_1^* \cap (-E_2^*) \\ &\text{iff} \quad p_0 \in -\left(E_1^* \cap (-E_2^*)\right)^*\end{aligned}$$

□

The following result will be useful to rephrase the dual (D) if the gauge γ in use is a composite gauge:

Lemma 6 *Let $\gamma_1, \ldots, \gamma_k$ be gauges in $\mathbb{R}^{m_1}, \ldots, \mathbb{R}^{m_k}$, and let $\tilde{\gamma}$ be a gauge in $\mathbb{R}^k$, monotone in $\mathbb{R}^k_+$. The gauge γ in $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}$ defined by*

$$\gamma(u_1, \ldots, u_k) = \tilde{\gamma}(\gamma_1(u_1), \ldots, \gamma_k(u_k)) \tag{21}$$

has as dual the gauge γ° with

$$\gamma^\circ(u_1, \ldots, u_k) = \tilde{\gamma}^\circ(\gamma_1^\circ(u_1), \ldots, \gamma_k^\circ(u_k)).$$

Proof: Let $x := (x_1, \ldots, x_k) \in \mathrm{IR}^{m_1} \times \ldots \times \mathrm{IR}^{m_k}$. Since $\tilde{\gamma}$ is monotone in IR^k_+, by definition of a dual gauge one has

$$\begin{aligned}
&\tilde{\gamma}^\circ(\gamma_1^\circ(x_1), \ldots, \gamma_k^\circ(x_k)) \\
&= \max \left\{ \sum_{j=1}^k \alpha_j \gamma_j^\circ(x_j) : \alpha \in \mathrm{IR}^k_+, \gamma(\alpha) = 1 \right\} \\
&= \max \left\{ \sum_{j=1}^k \alpha_j u_j^\top x_j : \alpha \in \mathrm{IR}^k_+, \gamma(\alpha) = 1, \gamma_j(u_j) = 1 \, \forall j \right\} \\
&= \max \left\{ \sum_{j=1}^k (\alpha_j u_j)^\top x_j : \alpha \in \mathrm{IR}^k_+, \gamma(\alpha_1\gamma_1(u_1), \ldots, \alpha_k\gamma_k(u_k)) = 1 \right\} \\
&= \max \left\{ \sum_{j=1}^k (\alpha_j u_j)^\top x_j : \alpha \in \mathrm{IR}^k_+, \gamma(\gamma_1(\alpha_1 u_1), \ldots, \gamma_k(\alpha_k u_k)) = 1 \right\}.
\end{aligned}$$

With the change of variables $\alpha_j u_j = \omega_j$ $(j = 1, \ldots, k)$, we get

$$\begin{aligned}
\tilde{\gamma}^\circ(\gamma_1^\circ(x_1), \ldots, \gamma_k^\circ(x_k)) &= \max \left\{ \sum_{j=1}^k (\omega_j)^\top x_j : \gamma(\gamma_1(\omega_1), \ldots, \gamma_k(\omega_k)) = 1 \right\} \\
&= (\tilde{\gamma}(\gamma_1(x_1), \ldots, \gamma_k(x_k)))^\circ
\end{aligned}$$

□

This lemma yields a very simple dual for gauges of the form (21):

Corollary 7 *Let $C_i \in \mathrm{IR}^{m_i \times n}$ $(i = 1, \ldots, k)$ be matrices and define the matrix C by $C^\top := \left(C_1^\top, C_2^\top, \ldots, C_k^\top\right)$. Let $c_i \in \mathrm{IR}^{m_i}$ $(i = 1, \ldots, k)$ be vectors and set $c^\top := (c_1^\top, \ldots, c_k^\top)$. Moreover, let γ be defined as in (21) and let K be asymptotically conical with a. c. r. (M, E). Then (D) admits the form*

$$\begin{aligned}
\max \quad & \sum_{j=1}^k u_j^\top c_j + \min_{x \in M} x^\top \left(\sum_{j=1}^k C_j^\top u_j + d \right) \\
s.t. \quad & \sum_{j=1}^k C_j^\top u_j + d \in E^* \\
& \tilde{\gamma}^\circ(\gamma_1^\circ(u_1), \ldots, \gamma_k^\circ(u_k)) \leq 1
\end{aligned}$$

We illustrate the power of our strategy for deriving the dual by applying it to two problems previously addressed in the literature, as discussed in the following examples.

Example 4 We consider the flow problem of [28]. Let (F, E) be a directed graph. Associate with each arc $e \in E$ a lower bound l_e and an upper bound u_e on its capacity, $l_e \in [-\infty, +\infty), u_e \in (-\infty, +\infty]$. Associate with each node $f \in F$ its demand $d_f \in \mathrm{IR}$. Flows on (F, E) are vectors x in IR^E; a flow x is said feasible if it satisfies both flow conservation,

$$\sum_{f:\,(f,g)\in E} x_{(f,g)} - \sum_{f:\,(g,f)\in E} x_{(g,f)} = d_g \qquad \forall g \in F \tag{22}$$

and boundedness,

$$\begin{array}{rcl} x_e & \geq & l_e \\ x_e & \leq & u_e \end{array} \qquad \forall e \in E. \tag{23}$$

When no feasible flow exists, McCormick proposed in [28] to consider (22)-(23) as goal constraints and to solve the corresponding problem (1) for γ equal to the (weighted) l_1, l_∞, and l_2 norm.

We first reformulate (23) as distance constraints following (2):

$$\begin{array}{rcl} x_e & \in & [l_e, +\infty) \\ x_e & \in & (-\infty, u_e] \end{array} \qquad \forall e \in E. \tag{24}$$

Then, the problem can be written as

$$\begin{array}{ll} \min & \gamma\left(\left(\sum_{f:\,(f,g)\in E} x_{(f,g)} - \sum_{f:\,(g,f)\in E} x_{(g,f)} - d_g\right)_{g\in F}, (x_e - \underline{u}_e)_{e\in E}, \left(x_e - \overline{l}_e\right)_{e\in E}\right) \\ \text{s.t.} & \underline{u}_e \in (-\infty, u_e] \qquad \forall e \in E, \\ & \overline{l}_e \in [l_e, +\infty) \qquad \forall e \in E, \end{array}$$

or

$$\begin{array}{ll} \min & \gamma\left(\left(\sum_{f:\,(f,g)\in E} x_{(f,g)} - \sum_{f:\,(g,f)\in E} x_{(g,f)} - d_g\right)_{g\in F}, (x_e - \underline{u}_e)_{e\in E}, \left(x_e - \overline{l}_e\right)_{e\in E}\right) \\ \text{s.t.} & \left(x, \underline{u}, \overline{l}\right) \in \{((0)_{e\in E}, (u_e)_{e\in E}, (l_e)_{e\in E})\} + \left(\mathrm{IR}^E \times \mathrm{IR}^E_- \times \mathrm{IR}^E_+\right). \end{array} \tag{25}$$

Associate dual variables $\pi := (\pi_g)_{g\in F}$, $\pi^+ := (\pi^+_e)_{e\in E}$, and $\pi^- := (\pi^-_e)_{e\in E}$ with the three blocks of components in (25). Since, in the a. c. r. of (25), one has

$$\left(\mathrm{IR}^E \times \mathrm{IR}^E_- \times \mathrm{IR}^E_+\right)^* = \{0\}^E \times \mathrm{IR}^E_- \times \mathrm{IR}^E_+,$$

one obtains the dual

$$\begin{array}{ll} \max & \sum_{g \in F} \pi_g(-d_g) + \min_{x=0, \underline{u}=u, \bar{l}=l} \sum_{e \in E} (-\pi_e^+)\underline{u}_e + \sum_{e \in E} (-\pi_e^-)\bar{l}_e \\ \text{s.t.} & \pi_e^+ \geq 0 \qquad \forall e \in E, \\ & \pi_e^- \leq 0 \qquad \forall e \in E, \\ & \pi_e^+ + \pi_e^- = \pi_f - \pi_g \qquad \forall e := (f,g) \in E, \\ & \gamma^\circ(\pi, \pi^+, \pi^-) \leq 1, \end{array}$$

i. e.

$$\begin{array}{ll} \max & -\sum_{g \in F} \pi_g d_g - \sum_{e \in E} \pi_e^+ u_e - \sum_{e \in E} \pi_e^- l_e \\ \text{s.t.} & \pi_e^+ \geq 0 \qquad \forall\, e \in E, \\ & \pi_e^- \leq 0 \qquad \forall\, e \in E, \\ & \pi_e^+ + \pi_e^- = \pi_f - \pi_g, \qquad \forall\, e := (f,g) \in E, \\ & \gamma^\circ(\pi, \pi^+, \pi^-) \leq 1, \end{array}$$

an expression which includes the particular cases derived in [28]. □

As another application, we derive the dual of the quite general unconstrained multifacility location problem (9) introduced in Section 1.2.

Example 5 With the notation as used in (9), for K defined in (10), one has that

$$K_\infty = \prod_{f \in F} K^f,$$

where $K^f = \{0\} \subset \mathbb{R}^n$ if $f \in A$ and $K^f = \mathbb{R}^n$ if $f \notin A$. Then, one obtains the dual

$$\begin{array}{ll} \max & \sum_{f \in A} a_f^\top \left(\sum_{h:\,(f,h) \in E} u_{(f,h)} - \sum_{h:\,(h,f) \in E} u_{(h,f)} \right) \\ \text{s.t.} & \sum_{g:\,(f,g) \in E} u_{(f,g)} - \sum_{g:\,(g,f) \in E} u_{(g,f)} = 0 \qquad \forall f \notin A, \\ & \gamma^\circ((\gamma_e^\circ(u_e))_{e \in E}) \leq 1, \end{array}$$

which covers most of the instances previously addressed in the literature, e. g. [16, 19]. □

4 Existence of Primal and Dual Solutions

In this section we study the finiteness and attainment of the optimal value v of problems (P) and (D).

Theorem 8 *Let B° be the polar of the unit ball B of γ. The following statements are equivalent.*

1. *(P) (and (D)) have finite optimal value.*
2. *For all $y \in K_\infty$ one has $\gamma(Cy) + d^\top y \geq 0$.*
3. *$d \in (K_\infty)^* - C^\top B^\circ$.*

Proof: Denote by g_∞ the recession function of g and let $y \in \mathrm{I\!R}^n$. For arbitrary $x \in \mathrm{I\!R}^n$ we have that

$$\begin{aligned} g_\infty(y) &= \sup_{\lambda>0} \frac{g(x+\lambda y) - g(x)}{\lambda} \\ &= \lim_{\lambda\to+\infty} \frac{g(x+\lambda y) - g(x)}{\lambda} \\ &= \lim_{\lambda\to+\infty} \frac{\gamma(C(x+\lambda y)+c) + d^\top(x+\lambda y) - \gamma(Cx+c) - d^\top x}{\lambda} \\ &= \gamma(Cy) + d^\top y, \end{aligned} \tag{26}$$

where the second equation is due to Theorem 8.5 of [37] and the last equation follows because of the homogeneity of γ. If (P) has a finite optimal value, then Part 2 is a consequence of (26) and Theorem 27.1 (parts (a) and (i)) of [37]. Conversely, if Condition 2 holds, we have for any a. c. r. (M, E) of K and for any $x \in K$, $x = x_M + x_E$ with $x_M \in M$, $x_E \in E = K_\infty$, that

$$\begin{aligned} \gamma(Cx+c) + d^\top x &= \gamma(Cx_M + Cx_E + c) + d^\top x_M + d^\top x_E \\ &\geq \gamma(Cx_E) + d^\top x_E - \gamma(-Cx_M - c) + d^\top x_M \\ &\geq \min_{x_M \in M} \left(-\gamma(-Cx_M - c) + d^\top x_M\right) > -\infty, \end{aligned}$$

showing that Condition 1 holds.

The equivalence between Condition 1 and Condition 3 follows from (19) and (14).

□

Remark 9 Conditions 2 and 3 do not imply Condition 1 for sets K which are not asymptotically conical. As a simple counterexample, take $K = \{x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid x_2 \geq x_1^2\}$, let C be the 1×2 matrix $C = (1, 0)$, $d := (2, 0)^\top$, $c := (0)$, and let $\gamma(s) = |s|$ for all $s \in \mathbb{R}$. Then K_∞ is the ray expanded by the vector $(0, 1)^\top$, and thus $\gamma(Cx) + d^\top x = 0$ for all $x \in K_\infty$. Hence, condition 2 holds. Moreover,

$$(K_\infty)^* = \{(x_1, x_2)^\top \mid x_2 \geq 0\},$$

thus, taking $u = 0 \in B^\circ$, one obtains $d \in (K_\infty)^* - C^\top B^\circ$, and Condition 3 holds also. However, $\gamma(C(-n, n^2)^\top + c) + d^\top(-n, n^2)^\top = -n$ for every natural n, thus the optimal value of (P) is $-\infty$. □

The duality scheme previously described enables us to easily characterize the (possibly empty) set of optimal solutions of (P) in terms of any optimal solution $\bar{u}$ of (D). See also Theorem 1 of [17] for the case of polyhedral feasible set K, or Theorem 1. 1 of [19] for related constraint qualification assumptions. One has

Theorem 10 *Let one of the equivalent conditions of Theorem 8 hold. Then,*

1. *The set of optimal solutions of (D) is not empty.*
2. *Let $\bar{x}$ be feasible for (P) and $\bar{u}$ feasible for (D). Then $\bar{x}$ is optimal for (P) and $\bar{u}$ is optimal for (D) iff the pair $(\bar{x}, \bar{u})$ is a saddle-point for the problem*

$$\inf_{x \in K} \sup_{\gamma^\circ(u) \leq 1} u^\top (Cx + c) + d^\top x. \tag{27}$$

Proof: By (19), under the assumptions of Theorem 8, the dual (D) consists of the minimization of the continuous function $u \mapsto u^\top c + \min_{x \in M} x^\top (C^\top u + d)$ over the nonempty compact set $\{u \mid C^\top u + d \in E^*, \gamma^\circ(u) \leq 1\}$. Thus an optimal solution $\bar{u}$ for (D) always exists.

For Part 2, observe that, under the assumptions of Theorem 8 and (19), the saddle value exists and is finite. Hence, saddle points exist, cmp. e. g. Theorem 4.2.5. in [23]. Moreover, the set of saddle points coincides with the cartesian product of the set of optimal solutions for (P) and (D), as asserted. □

For a nonempty set S and $\bar{x} \in S \subseteq \mathbb{R}^m$, let $N_S(\bar{x})$ denote the normal cone of S at $\bar{x}$,

$$N_S(\bar{x}) = \left\{ y \in \mathbb{R}^m \mid y^\top (x - \bar{x}) \leq 0 \text{ for all } x \in S \right\}.$$

The characterization of optimal solutions of (P) as part of saddle points yields the following.

Theorem 11 *Let (M, E) be an a. c. r. of K. The feasible point $\bar{x} := \bar{x}_M + \bar{x}_E$, $\bar{x}_M \in M$, $\bar{x}_E \in E$, is optimal for (P) iff there exists a point $\bar{u} \in \mathbb{R}^m$ satisfying*

$$\begin{aligned} \gamma^\circ(\bar{u}) &\leq 1, \\ C^\top \bar{u} + d &\in E^* \cap -N_M(\bar{x}_M), \\ \bar{x}_E^\top \left(C^\top \bar{u} + d\right) &= 0, \\ C\bar{x} + c &\in N_{B^\circ}(\bar{u}). \end{aligned}$$

In that case, such a $\bar{u}$ is an optimal solution for (D).

Proof: By Theorem 10, $\bar{x}$ is optimal for (P) iff there exists $\bar{u}$ optimal for (D) such that the pair $(\bar{x}, \bar{u})$ is a saddle point. In other words, $\bar{x}$ is optimal for (P) iff there exists some $\bar{u} \in \mathbb{R}^m$ satisfying

$$\begin{aligned} C^\top \bar{u} + d &\in E^*, && (28) \\ \gamma^\circ(\bar{u}) &\leq 1, && (29) \\ \gamma(C\bar{x} + c) + d^\top \bar{x} &= \bar{u}^\top (C\bar{x} + c) + d^\top \bar{x}, && (30) \\ &= \bar{u}^\top c + \inf_{x_M \in M, x_E \in E} \bar{u}^\top C(x_M + x_E) + d^\top (x_M + x_E). && (31) \end{aligned}$$

But (30) holds iff

$$C\bar{x} + c \in N_{B^\circ}(\bar{u}).$$

Moreover, for vectors u satisfying (28), it follows from Property 5 that

$$\inf_{x \in K} u^\top Cx + d^\top x = \min_{x_M \in M} u^\top Cx_M + d^\top x_M,$$

and thus condition (31) is equivalent (for vectors $\bar{u}$ satisfying (28)) to

$$\begin{aligned} \bar{u}^\top C\bar{x}_M + d^\top \bar{x}_M &= \min_{x_M \in M} \bar{u}^\top Cx_M + d^\top x_M, && (32) \\ \bar{u}^\top C\bar{x}_E + d^\top \bar{x}_E &= 0. \end{aligned}$$

Since (32) is equivalent to

$$C^\top \bar{u} + d \in -N_M(\bar{x}_M),$$

the result follows. $\square$

Note that the conditions derived in [10] and [11] for the single-facility location model (see Subsection 1.2) are special cases of the ones derived in the last theorem.

Remark 12 Let $x = x_M + x_E$ ($x_M \in M$, $x_E \in E$) be primal feasible and let u be dual feasible. Additionaly, let $\gamma(Cx + c) = u^\top(Cx + c)$ and let $\delta^*_M(-C^\top u - d) = -x_M(C^\top u + d)$. (Note that these two additional conditions are equivalent to $u \in \partial\gamma(Cx + c)$ and $x_M \in \partial\delta^*_M(-C^\top u - d)$.) A simple calculation then shows that $x_E^\top \left(C^\top u + d\right)$ is the dual gap with respect to the feasible points x and u.

Remark 13 If K_∞ is a linear space then $K^*_\infty = K^\perp_\infty$, thus the complementary condition $(\overline{u}^\top C + d^\top)\overline{x}_E = 0$ in Theorem 11 is redundant.

We have shown in Theorem 10 that primal and dual optimal solutions (when they exist!) are related with each other as saddle point solutions of (27). However, the existence of optimal solutions for (P) is not guaranteed when (P) has a finite optimal value. Since in applications $E = \{0\}$ usually does not hold (see Section 1), a deeper analysis is required. This is the purpose of the rest of the section.

For certain instances of (P), the non-emptiness and compactness of the set of optimal primal solutions can be derived by ad-hoc procedures, as done, e.g., in [5, 6, 34]. For the general situation we have the following.

Theorem 14 *If all nonzero $y \in K_\infty$ satisfy $\gamma(Cy) + d^\top y > 0$, then*

1. *The set of optimal solutions is nonempty, convex, and compact.*

2. *Let (M, E) be an a. c. r. of K and suppose $E \neq \{0\}$. Let*

$$\begin{aligned} L &\geq \max_{x \in M} \left(\gamma(-Cx - c) - d^\top x\right), \\ \overline{v} &\geq \min_{x \in K} \left(\gamma(Cx + c) + d^\top x\right), \\ r &\in \left]0, \min_{e \in E, \|e\|_2 = 1} \gamma(Ce) + d^\top e\right], \end{aligned}$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Then, any optimal solution $x_M + x_E$ for (P) with $x_M \in M$ and $x_E \in E$ satisfies

$$\|x_E\| \leq \frac{\overline{v} + L}{r}$$

Proof: Part 1 follows from the fact that, under these assumptions, $(g + \delta_K)$ is coercive and thus level bounded. See, e. g., [2].

To show Part 2, let $x_E \in E$ be given with $\|x_E\| > (\overline{v} + L)/r$ and let $x_M \in M$. The triangle inequality now shows that

$$\begin{aligned}
&\gamma(Cx_M + Cx_E + c) + d^\top x_M + d^\top x_E \geq \\
&\geq \gamma(Cx_E) + d^\top x_E - \gamma(-Cx_M - c) + d^\top x_M \\
&= \|x_E\| \left(\gamma \left(C \frac{1}{\|x_E\|} x_E \right) + d^\top \frac{x_E}{\|x_E\|} \right) + d^\top x_M - \gamma(-Cx_M - c) \\
&> \frac{\overline{v} + L}{r} \left(\gamma \left(Cx_E / \|x_E\| \right) + d^\top \frac{x_E}{\|x_E\|} \right) - L \\
&\geq \overline{v},
\end{aligned}$$

contradicting its optimality. □

Remark 15 In Theorem 1 of [26], it is assumed that C is a $p \times q$ matrix ($q < p$) with rank q, $K = \mathbb{R}^q$, and $0 \in C^\top \mathrm{int}(B^\circ) + d$, where int denotes the interior. This is clearly stronger than the assumption in Theorem 14. Indeed, for any $e \in K_\infty$, one of the two following conditions hold:

$$\max_{u \in B^\circ} e^\top (C^\top u + d) \ > \ 0, \tag{33}$$

$$e^\top (C^\top u + d) \ = \ 0 \qquad \forall\, u \in B^\circ. \tag{34}$$

If (33) holds, then

$$\gamma(Ce) + d^\top e > 0,$$

whereas if (34) holds, then

$$d + C^\top B^\circ \in \{e\}^\perp,$$

which, since $C^\top B^\circ$ has full dimension, implies that $e = 0$. Hence, $\gamma(Ce) + d^\top e > 0$ for all nonzero $e \in K_\infty$. □

When the condition in Theorem 14 is dropped, non-emptiness of the set of optimal solutions cannot be guaranteed in general, even for the case of polyhedral recession cone K_∞. This is shown in the following example.

Example 6 Let $C = (1, 0)^\top$, $d = (-1)$, $c = (0, 1)^\top$, the feasible set $K = [0, +\infty) = K_\infty$, and let γ be the Euclidean norm in the plane. Then,

$$\gamma(Cx + c) + d^\top x = \sqrt{x^2 + 1} - x,$$

which is always non-negative, but tends to zero when x grows to infinity. Hence, no optimal solution exists. □

The case $d = 0$ (in fact the common one in applications) simplifies the analysis since then the objective function of (P) is bounded below. However, this does not guarantee the attainment of the optimal value, as shown in the following example.

Example 7 Let $K = \{(x_1, x_2, x_3) : x_3^2 \geq x_1^2 + x_2^2, x_i \geq 0, i = 1, 2, 3\}$ and let C be the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let $c = (0, 1)^\top$, $d = 0$, and let γ be the Euclidean norm. Then,

$$\gamma(Cx + c) + d^\top x = \gamma(x_1 - x_3, 1 - x_2).$$

Since the system

$$\begin{cases} x_1 & = & x_3 \\ x_2 & = & 1 \end{cases}$$

has no solution on K, the objective function value is strictly positive on K. However, for the feasible sequence $\{(n, 1, \sqrt{1 + n^2})\}_n$ the objective value tends to zero, showing that the infimum (zero) is not attained. □

In spite of this negative result, a geometrical condition can be given to guarantee the attainment of the optimal value for $d = 0$:

Theorem 16 *Let $d = 0$. The following conditions are equivalent.*

1. *(P) attains its optimal value for each $c \in \mathbb{R}^m$.*

2. *The set CK is closed.*

Proof: Assume Condition 1 and suppose CK is not closed. Then there exist a vector v^* and a sequence $\{x_j\}_j \subseteq K$ such that

$$\lim_j Cx_j = v^* \notin CK. \tag{35}$$

However, this would imply that, for $c = -v^*$, the objective function value is always strictly positive but converging to zero for the feasible sequence $\{x_j\}_j$. This contradicts the assumption.

Conversely, if Condition 2 holds, then, formulating (P) as the problem

$$\begin{array}{ll} \min & \gamma(y) \\ \text{s.t.} & y \in CK + c \end{array}$$

one immediately obtains that (P) amounts to finding the point in the *closed* set $CK + c$ closest to the origin, which always admits an optimal solution. □

Corollary 17 *Let K_1, K_2 be asymptotically conical sets such that* $\inf\{\gamma(x-y) \mid x \in K_1, y \in K_2\} = 0$. *Then, $K_1 \cap K_2 \neq \emptyset$.*

Proof: The set $K := K_1 \times K_2$ is asymptotically conical, see Property 2. Let $C = (I_n, -I_n)$, where I_n is the $n \times n$ identity matrix. Then $CK = K_1 - K_2$, which is asymptotically conical as well, see Property 2, thus it is closed. By Theorem 16, the problem

$$\inf_{x \in K} \gamma(Cx)$$

attains its infimal value (zero), which means that some x exists in $K_1 \cap K_2$. □

5 Solving the Problem Efficiently

The aim of this section is to show how the structure of (P) can be exploited to derive polynomial time interior-point schemes for solving the problem at hand. While the actual development of new methods for the general case falls outside the scope of this paper, the discussion below shows that particular instances of the general problem can be solved by interior-point methods of various types (e. g. primal, primal-dual, short-step, long-step, etc.), provided that self-concordant barriers for the unit ball of the gauge γ and M as well as for the cone E are given.

For instance, consider the primal problem (P) restated in the form

$$\begin{array}{ll} \min & s + t, \\ \text{s. t.} & (Cx + c, s) \in \text{epi}(\gamma), \\ & d^\top x \leq t, \\ & x \in K \end{array}$$

and let B be the unit ball of the gauge γ. Then epi(γ) is the conic hull of this unit ball, i. e.

$$\text{epi}(\gamma) = \{(y, \lambda) \in \mathbb{R}^{m+1} \mid y \in \lambda B;\ \lambda \geq 0\}.$$

The reformulated problem has therefore a linear objective function, a conic constraint, and a convex constraint of a rather special structure, which makes it easily exploitable for interior-point methods.

As a simple example, we might use the standard primal path-following algorithm from Nesterov and Nemirovskii [31]. For this, we need not only a starting point in the interior of a compact set of feasible points, but also a lower bound on the *asymmetry coefficient* of this starting point. The asymmetry coefficient $a(x, G)$ of a point x lying in the strict interior of a convex compact set G is defined as

$$a(x, G) := \sup\{\alpha \geq 0 \mid x + \alpha(x - G) \subseteq G\}.$$

Denoting by γ_{G-x} the gauge with unit ball $G - x$, one immediately obtains from the definition that

$$a(x, G) = \Big(\max_{y \in G-x} \gamma_{G-x}(-y)\Big)^{-1}. \qquad (36)$$

Let B_1 and B_2 be the unit balls of the ℓ_1 and ℓ_2 norm, respectively, and suppose that we are given constants $r_1, r_2 > 0$ such that $x + r_1 B_1 \subseteq G$ and $G \subseteq x + r_2 B_2$ holds. Then,

$$a(x, G) \geq \Big(\max_{y \in r_2 B_2} \gamma_{r_1 B_1}(-y)\Big)^{-1} = \frac{r_1}{r_2}\Big(\max_{y \in B_2} \gamma_{B_1}(-y)\Big)^{-1} = \frac{r_1}{r_2\sqrt{n}} \qquad (37)$$

follows with some easy calculations.

Suppose now that we are given a self-concordant barrier b_B for the unit ball B with self-concordancy parameter $\vartheta_B \geq 1$ and a self-concordant barrier b_K for the set K with self-concordancy parameter $\vartheta_K \geq 1$.

If an a.c.r $K = M + E$ for K is known, the latter barrier will usually be written as $b_K = b_M + b_E$, where b_M is a barrier for the compact set M, while b_E is the corresponding barrier for the cone E.

Theorem 4 from [18] tells us that it is relatively easy to construct a self-concordant barrier for the epigraph of γ explicitly. Indeed, such a barrier takes the form

$$b^+(x, t) = \beta b_B(x/t) - \alpha\vartheta_B \ln t,$$

where $\alpha, \beta > 0$ are explicitly given constants, depending only on ϑ_B. Both constants are of magnitude $O(1)$. Moreover, b^+ has a self-concordancy parameter of order $O(\vartheta_B)$.

Let there be given a point $y \in \operatorname{int}(K)$. In the next step, we have to assume that we know a bound $\tilde{r} > 0$ such that for every solution $x \in K$ of our primal problem the relation $\|x\|_2 \le \tilde{r}$ holds. See Theorem 14 for methods for constructing such $\tilde{r}$ in particular cases. Moreover, we assume that $\tilde{r}$ is chosen in such a way that $\|y\|_2 + 1 \le \tilde{r}$. Define now

$$\varrho_P := \gamma(Cy + c) + d^\top y + 3$$

and

$$G_P := \{(x, s, t) \in \mathrm{IR}^{n+2} \,|\, x \in K,\ \|x\|_2 \le \tilde{r},\ (Cx+c, s) \in \operatorname{epi}(\gamma),\ d^\top x \le t,\ s+t \le \varrho_P\}.$$

Obviously, G_P is a convex compact set. With $u := \gamma(Cy + c) + 1$ and $v := d^\top y + 1$ we have that the point $\hat{y} := (y, u, v)$ is in the strict interior of G_P. Moreover,

$$b_P(x, t) := b^+(Cx + c, t) + b_K(x) - \ln(\tilde{r}^2 - \|x\|_2^2) - \ln(t - d^\top x) - \ln(\varrho_P - s - t)$$

is a self-concordant barrier for G_P with self-concordancy parameter

$$\vartheta_P := O(1)\vartheta_B + \vartheta_K + 3 = O(\vartheta_B + \vartheta_K)$$

(see [31, Proposition 5.1.1 and 5.1.2]). This means that we can opt to solve the problem

$$\begin{array}{ll} \min & s + t \\ \text{s. t.} & (x, s, t) \in G_P \end{array}$$

with an interior-point method, using $\hat{y}$ as a starting point.

Lemma 18 *Let e_i be the ith euclidean unit vector ($i = 1, \ldots, n$) in the IR^n. Let $\delta_K > 0$ be such that $y + \delta_K e_i, y - \delta_K e_i \in K$ for all $i = 1, \ldots, n$ and denote by $c_i \in \mathrm{IR}^m$, $i = 1, \ldots, n$, the columns of the matrix C. Define*

$$r_P := \min\{1, \delta_K, 1/\gamma(c_i), 1/\gamma(-c_i), 1/|d_i| : i = 1, \ldots, n\}$$

and

$$R := \left(\tilde{r}^2 + (\varrho_P + \tilde{r}\|d\|_2)^2 + (\max\{|\varrho_P|, \tilde{r}\|d\|_2\})^2\right)^{1/2}.$$

It then follows that

$$a(\hat{y}, G_P) \ge \frac{r_P}{2\sqrt{n + 2}R}.$$

Proof: First, let $(x, s, t) \in G_P$. It then follows that $\|x\|_2 \le \tilde{r}$, $0 \le s$, and $-\tilde{r}\|d\|_2 \le t \le \varrho_P$. As a consequence, we have $s \le \varrho_P + \tilde{r}\|d\|_2$. This means that

$$\|(x, s, t)\|_2^2 = \|x\|_2^2 + s^2 + t^2 \le \tilde{r}^2 + (\varrho_P + \tilde{r}\|d\|_2)^2 + (\max\{|\varrho_P|, \tilde{r}\|d\|_2\})^2$$

Therefore,

$$G_P \subseteq \hat{y} + 2RB_2,$$

where B_2 is the unit ball of the 2-norm in $\mathbb{R}^{n+2}$.

Second, with the ith euclidean unit vector $e_i \in \mathbb{R}^n$ we have for all $z \in \mathbb{R}^n$ that $\gamma(C(z + r_P e_i) + c) \le \gamma(Cz + c) + 1$ and $\gamma(C(z - r_P e_i) + c) \le \gamma(Cz + c) + 1$ for $i = 1, \ldots, n$. Moreover, $d^\top(y + r_P e_i) \le v$ as well as $d^\top(y - r_P e_i) \le v$ $(i = 1, \ldots, n)$. This means that $\hat{y} + r_P B_1 \subseteq G_P$, where B_1 is the unit ball of the ℓ_1 norm in $\mathbb{R}^{n+2}$. The result follows with (37). □

With the last lemma, it is easy to see that Stage 1 of the standard primal path-following algorithm from [31] takes

$$O(\sqrt{\vartheta_P}(\ln \vartheta_P + \ln n + \ln R - \ln r_P))$$

iterations, while Stage 2 of this method takes

$$O(\sqrt{\vartheta_P}(\ln \vartheta_P + \ln(1/\varepsilon) + \ln(\varrho_P + 2\tilde{r}\|d\|_2)))$$

iterations to achieve ε-accuracy.

Bounding r_P, ϑ_B, and ϑ_K depends on the nature of the actual data at hand. We will consider typical examples of goal programming problems in Section 5.1.

Note that the dual problem in the formulation (20) can be treated in the same way as the primal one, provided that there are known self-concordant barriers for M°, B°, and E^*. A latter one can, at least in principle, be constructed from a self-concordant barrier for E, see Section 2.4.1 in [31]. One just has to change the maximization problem into an equivalent minimization problem and add one slack variable $s \in \mathbb{R}$ for the constraint $(-C^\top u - d, s) \in \operatorname{epi}(\tilde{\gamma}^\circ)$.

Another possibility is to consider a primal-dual reformulation of the problem. Assume M has non-empty interior, and let $x_0 \in \operatorname{int}(M)$ be given. Define the gauge $\tilde{\gamma}$ by its unit ball: $\tilde{\gamma} := \gamma_{M - x_0}$. Example 1 now shows that

$$\min \quad \gamma(Cx + c) + \tilde{\gamma}^\circ(-C^\top u - d) + d^\top x - (Cx_0 + c)^\top u - d^\top x_0$$

$$\begin{aligned} \text{s.t.} \quad & x \in M + E, \\ & C^\top u + d \in E^* \\ & \gamma^\circ(u) \leq 1. \end{aligned}$$

is the primal-dual reformulation of problem (P) with objective function value 0. Note that this again is a problem of the same type as (P).

Other algorithms, especially long-step methods, can be derived when more knowledge is available about the problem structure. As a trivial example, if K as well as B is polyhedral, the problem reduces to a linear one, for which methods of higher efficiency then the one depicted above are readily available. (See also Examples 9, 10, and 11 in the next subsection.) Other possibilities include the cases in which the cones E and $\text{epi}(\gamma)$ appearing in the problem formulation are direct sums of cones of positive semidefinite symmetric real matrices and cones of the form $\text{epi}(\|\cdot\|_2)$. (See Example 8 in the next subsection.) The standard barriers for these cones are self-scaled, allowing for especially efficient algorithms, cmp. [32]. Note, however, that the use of a p-norm with $p \neq 2$ (an important case in applications [36]) does not allow for a self-scaled cone, and that a self-dual formulation for the corresponding problem is not readily at hand. Indeed, interior-point methods proposed up to now for this class of problems do not use a self-dual formulation, see [47] and Example 12 in the next subsection.

5.1 Particular Cases

In this section we take a quick look at how self-concordant barriers for the unit balls of typical gauges encountered in applications can be easily derived. Some of these cases have already been discussed in [15], in the context of interior point algorithms applied to specific location problems similar to the one discussed in Subsection 1.2.

Example 8 (cp. Proposition 5.4.2 and 5.4.3 in [31]) If γ is the euclidean norm, a self-concordant barrier with self-concordancy parameter $\vartheta_B = 1$ for the unit ball of γ is given by $b_B(x) := -\ln(1 - \|x\|_2^2)$. Likewise, if the unit ball of γ is an ellipsoid, $B = \{x \in \mathbb{R}^n \mid x^\top Q x \leq 1\}$, where Q is a positive definite $n \times n$-matrix, a self-concordant barrier with self-concordancy parameter $\vartheta_B = 1$ is given by $b_B(x) := -\ln(1 - x^\top Q x)$. In both cases, the tedious general construction of a self-concordant barrier for epi(γ) can be avoided by noting that $b^+(x,t) = -\ln(t^2 - x^\top Qx)$ is a self-concordant barrier for this cone with self-concordancy parameter $\vartheta_B^+ = 2$. Note that epi($\|\cdot\|_2$) is just the standard second-order cone, while b^+ is the corresponding self-scaled barrier for this cone, see [32]. □

Example 9 Let γ be a polyhedral gauge whose unit ball is given by a set of k linear inequalities: $B = \{x \in \mathbb{R}^n \mid Ax \leq g\}$, $A \in \mathbb{R}^{k\times n}$, $g \in \mathbb{R}^k$. Of course, the standard logarithmic barrier $b_B(x) = -\sum_{i=1}^k \ln(g_i - a_i^\top x)$ for the polytope B can be used to define $b_B^+(x,t) = -\sum_{i=1}^k \ln(g_i t - a_i^\top x)$, a self-concordant barrier for the epigraph of γ with self-concordancy parameter $\vartheta_B^+ = k$. □

Example 10 Let γ be a gauge, $A \in \mathbb{R}^{n\times n}$ be a regular matrix and $c \in \mathbb{R}^n$ be a vector with $\gamma^\circ(A^\top c) < 1$. Then $\tilde{\gamma}(x) := \gamma(Ax) + c^\top x$ defines a gauge [35]. Gauges defined like this have important applications in location science, see, e. g., [35, 14]. It is easy to see that the unit ball of $\tilde{\gamma}$ is given by $(c + A^\top(B^\circ))^\circ$. However, finding a barrier for this unit ball does not seem to be so easy. On the other hand, finding a self-concordant barrier for epi($\tilde{\gamma}$) is simple, as long as such a barrier for the unit ball B of γ is given. Let b be such a barrier with self-concordancy parameter ϑ_B and let b^+ be a barrier for epi(γ) with self-concordancy parameter ϑ_B^+ (note that we have $\vartheta_B^+ = O(\vartheta_B)$, according to [18]). Then,

$$\tilde{b}^+(x,t) := b^+(Ax, t - c^\top x) - \ln t$$

is a barrier for the epigraph of $\tilde{\gamma}$ with self-concordancy parameter $\tilde{\vartheta}^+ = \vartheta_B^+ + 1$. This is a simple application of Proposition 5.1.1 and 5.2.5 from [31]. □

Example 11 Suppose that we are given k gauges γ_i $(i = 1,\ldots,k)$ and we want to use the gauge γ defined by $\gamma(x) := \max_{i=1}^k \gamma_i(x)$. If B_i is the unit ball of γ_i $(i = 1,\ldots,k)$, it is easy to see that $B = \bigcap_{i=1}^k B_i$ is the unit ball of γ. Given self-concordant barriers b_i for the unit balls B_i with self-concordancy parameter ϑ_i, we have that $b_B := \sum_{i=1}^k b_i$ is a self-concordant barrier for B with self-concordancy parameter $\vartheta_B = \sum_{i=1}^k \vartheta_i$. □

Example 12 We are now considering the case that γ is a p-norm with $p \in]1,\infty[$. By introducing slack variables for the inequalities describing the unit ball B of γ, we can consider the set

$$\hat{B} := \{(x, y, z) \in \mathbb{R}^{3n} \mid -y_i \le x_i \le y_i,\, 0 \le y_i,\, y_i^p \le z_i\, (i = 1, \ldots, n),\, \sum_{i=1}^{n} z_i \le 1\}$$

Using Proposition 5.3.1 from [31], we see that

$$\hat{b}(x, y, z) := -\sum_{i=1}^{n} (\ln(y_i - x_i) + \ln(y_i + x_i) + \ln y_i + \ln z_i + \ln(z_i^{1/p} - y_i)) - \ln\left(1 - \sum_{i=1}^{n} z_i\right)$$

is self-concordant barrier for $\hat{B}$ with self-concordancy parameter $\hat{\vartheta} = 6n + 1$. Of course, covering B is achieved by $b(x) := \hat{b}(x, y, z)$. Constructing a barrier for the conic hull of $\hat{B}$ is now straightforward, see also [15].

The construction of a starting point lying in the strict interior of the set of feasible points and the estimate of the asymmetry of this point can be done as shown in Section 5 and is discussed in more detail in [15]. □

Example 13 Let γ be a gauge as in (21). Using the fact that $\tilde{\gamma}$ is monotonic, it is sufficient to consider the set

$$\hat{B} = \{(u_1, t_1, \ldots, u_k, t_k) \in \mathbb{R}^{m_1+1} \times \cdots \times \mathbb{R}^{m_k+1} \mid \gamma_i(u_i) \le t_i\, (i = 1, \ldots, k),\; \tilde{\gamma}(t_1, \ldots, t_k) \le 1\}.$$

Obviously, to construct a self-concordant barrier for the set $\tilde{B}$, one can use self-concordant barriers b_i^+ with self-concordancy parameter ϑ_i^+ for the cones $\mathrm{epi}(\gamma_i)$ and a self-concordant barrier $\tilde{b}$ with self-concordancy parameter $\tilde{\vartheta}$ for the unit ball of $\tilde{\gamma}$ to define

$$\hat{b}(u_1, t_1, \ldots, u_k, t_k) := \tilde{b}(t_1, \ldots, t_k) + \sum_{i=1}^{k} b_i^+(u_i, t_i),$$

a self-concordant barrier for $\hat{B}$ with self-concordancy parameter $\tilde{\vartheta} + \sum_{i=1}^{k} \vartheta_i^+$. □

6 Conclusions

In this paper we have addressed a generalization of the basic Non-Preemptive Linear Goal Programming model, which includes its variants (Range or Interval Programming) as well other important optimization problems in areas as diverse as Continuous Location or Regression Analysis.

Duality is used as a tool for describing the set of optimal solutions geometrically, and is derived here by using well-known minimax theorems.

A unified solution methodology is proposed, yielding ε-optimal solutions of the primal or the dual problem in polynomial time.

Acknowledgements The authors want to thank Dr. Carlos Romero for his helpful comments.

The research of the first author is partially supported by Grant PB96-1416-C02-02 of D.G.E.S., Spain.

The research of the second author was partially supported by a grant of the Deutsche Forschungsgesellschaft (DFG).

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