GENERALIZED GREEN'S EQUIVALENCES ON THE SUBSEMIGROUPS OF THE BICYCLIC MONOID

L. Descalço and P.M. Higgins

Department of Mathematical Sciences, University of Essex,
Wivenhoe Park, Colchester CO4 3SQ, United Kingdom

ABSTRACT

We study generalized Green's equivalences on all subsemigroups of the bicyclic monoid **B** and determine the abundant (and adequate) subsemigroups of **B**.

1 INTRODUCTION

The bicyclic monoid **B**, is one of the most fundamental semigroups, with many remarkable properties and generalizations; see [1, 2, 6, 7, 8, 9, 10, 11, 12].

A description of the subsemigroups of the bicyclic monoid was obtained in [3], and by using this description several properties about all subsemigroups of **B** have been proved in [4] . In this paper we use this description to study the generalized Green's relations $\mathcal{L}^*$ and $\mathcal{R}^*$ of the subsemigroups of **B**. This study is motivated by a J. Fountain's question, who asked if the description can be used to say which are the abundant and adequate subsemigroups of the bicyclic monoid.

Let S be a semigroup and $a, b \in S$. We say that $a\,\mathcal{L}^*\,b$, if there is an oversemigroup of S (a semigroup having S as a subsemigroup) where $a\,\mathcal{L}\,b$. It is known and it is easy to check that (see [5]) $a\,\mathcal{L}^*\,b$ if and only if,

$$\text{for all } x, y \in S^1 \text{ we have } ax = ay \;\Leftrightarrow\; bx = by. \tag{1}$$

The relation $\mathcal{R}^*$ is defined analogously as is the corresponding property. We say that a semigroup is *abundant* if every $\mathcal{L}^*$-class has an idempotent and every $\mathcal{R}^*$-class has an idempotent. An abundant semigroup is *adequate* if the set of its idempotents forms a semilattice.

The *bicyclic monoid* $\mathbf{B}$ is defined by the monoid presentation $\langle b, c \mid bc = 1\rangle$; a natural set of unique normal forms for $\mathbf{B}$ is $\{c^ib^j : i, j \geq 0\}$ and we shall identify $\mathbf{B}$ with this set. The normal forms multiply according to the following rule:

$$c^ib^jc^kb^l = \begin{cases} c^{i-j+k}b^l \text{ if } j \leq k \\ c^ib^{j-k+l} \text{ if } j > k. \end{cases}$$

We are going to study the $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes of all subsemigroups of the bicyclic monoid in order to determine the abundant subsemigroups. We note that every set of idempotents from the bicyclic monoid is a semilattice (indeed a chain) and so a subsemigroup of the bicyclic monoid is adequate if and only if it is abundant.

We start by noting that two idempotents in the bicyclic monoid are always in separated $\mathcal{L}^*$-classes ($\mathcal{R}^*$-classes). In fact, given two idempotents say, c^ib^i, c^jb^j with $i < j$ we can use (1) choosing $x = c^ib^i$ and $y = c^jb^j$. We have $c^jb^jx = c^jb^jy = c^jb^j$ but $c^ib^ix = c^ib^i$ which is not equal to $c^ib^iy = c^jb^j$.

We will consider the different types of semigroups of $\mathbf{B}$ separately. Diagonal subsemigroups, one of the types, are formed by idempotents and so trivially are abundant. We begin by presenting some previous results giving the description of the subsemigroups of $\mathbf{B}$ in Section 2, then in Sections 3 and 4 we make some remarks that will be useful to study their $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes. Finally, in Sections 5 and 6 we consider the two relevant types of subsemigroups, the Upper and Two-sided subsemigroups, respectively.

2 PREVIOUS RESULTS

In this section we introduce the necessary notation and present the main result from [3] with the description of the subsemigroups of $\mathbf{B}$.

In order to define subsets of the bicyclic monoid it is convenient to see $\mathbf{B}$ as an infinite square grid, as shown in Figure 1. We start by introducing some

	0	1	2	3	
0	1	b	b^2	b^3	$\cdots$
1	c	cb	cb^2	cb^3	$\cdots$
2	c^2	c^2b	c^2b^2	c^2b^3	$\cdots$
3	c^3	c^3b	c^3b^2	c^3b^3	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 1: The bicyclic monoid

basic subsets of $\mathbf{B}$:

$$\begin{aligned} &D = \{c^i b^i : i \geq 0\} - \text{ the } \textit{diagonal}, \\ &L_p = \{c^i b^j : 0 \leq j < p, i \geq 0\} - \text{ the } \textit{left strip (determined by p)}, \end{aligned}$$

for $p \geq 0$. For $0 \leq q \leq p \leq m$ we define the *triangle*

$$T_{q,p} = \{c^i b^j : q \leq i \leq j < p\}.$$

Note that for $q = p$ this set is empty. For $i, m \geq 0$ and $d > 0$ we define the *rows*

$$\Lambda_i = \{c^i b^j : j \geq 0\},\ \Lambda_{i,m,d} = \{c^i b^j : d \mid j - i,\ j \geq m\}$$

and in general for $I \subseteq \{0, \ldots, m-1\}$,

$$\Lambda_{I,m,d} = \bigcup_{i \in I} \Lambda_{i,m,d} = \{c^i b^j : i \in I, d \mid j - i, j \geq m\}.$$

For $p \geq 0$, $d > 0$, $r \in [d] = \{0, \ldots, d-1\}$ and $P \subseteq [d]$ we define the *squares*

$$\begin{aligned} &\Sigma_p = \{c^i b^j : i, j \geq p\},\ \Sigma_{p,d,r} = \{c^{p+r+ud} b^{p+r+vd} : u, v \geq 0\}, \\ &\Sigma_{p,d,P} = \bigcup_{r \in P} \Sigma_{p,d,r} = \{c^{p+r+ud} b^{p+r+vd} : r \in P; u, v \geq 0\}. \end{aligned}$$

Pictures illustrating some of these sets can be found in [3].

The function $\rho : \mathbf{B} \to \mathbf{B}$ defined by $c^i b^j \mapsto (c^i b^j)\rho = c^j b^i$ is an anti-isomorphism. Geometrically ρ is the reflection with respect to the main diagonal.

We can now present the main result from [3]:

Proposition 2.1 *Let S be a subsemigroup of the bicyclic monoid. Then one of the following conditions holds:*

1. *S is a subset of the diagonal;* $S \subseteq D$.

2. *S is a union of a subset of a triangle, a subset of the diagonal above the triangle, a square below the triangle and some rows belonging to a strip determined by the square and the triangle, or the reflection of such a union with respect to the diagonal. Formally there exist* $q, p \in \mathbb{N}_0$ *with* $q \leq p$, $d \in \mathbb{N}$, $I \subseteq \{q, \dots, p-1\}$ *with* $q \in I$, $P \subseteq \{0, \dots, d-1\}$ *with* $0 \in P$, $F_D \subseteq D \cap L_q$, $F \subseteq T_{q,p}$ *such that S is of one of the following forms:*

   (i) $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$; *or*

   (ii) $S = F_D \cup (F)\rho \cup (\Lambda_{I,p,d})\rho \cup \Sigma_{p,d,P}$.

3. *There exist* $d \in \mathbb{N}$, $I \subseteq \mathbb{N}_0$, $F_D \subseteq D \cap L_{\min(I)}$ *and sets* $S_i \subseteq \Lambda_{i,i,d}$ $(i \in I)$ *such that S is of one of the following forms:*

   (i) $S = F_D \cup \bigcup_{i \in I} S_i$; *or*

   (ii) $S = F_D \cup \bigcup_{i \in I} (S_i)\rho$;

   where each S_i *has the form*

   $$S_i = F_i \cup \Lambda_{i,m_i,d}$$

   for some $m_i \in \mathbb{N}_0$ *and some finite set* F_i, *and*

   $$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}_0, r + ud \geq N\}$$

   for some (possibly empty) $R \subseteq \{0, \dots, d-1\}$, *some* $N \in \mathbb{N}_0$ *and some finite set* $I_0 \subseteq \{0, \dots, N-1\}$.

We call *diagonal subsemigroups* those defined by *1.*, *two-sided subsemigroups* those defined by *2.*, *upper subsemigroups* those defined by *3.(i)* and *lower subsemigroups* those defined by *3.(ii)*. Pictures illustrating the several types of semigroups can be found in [3].

3 $\mathcal{L}^*$-CLASSES

In general, to study the $\mathcal{L}^*$-classes of a subsemigroup S of $\mathbf{B}$ we have to consider the equation $ax = ay$ appearing in (1), in our introductory section, and the following fact will be useful:

Lemma 3.1 *We have $ax = ay$ with $a = c^ib^j, x = c^rb^s, y = c^ub^v \in S$, $x \neq y$, i.e,*

$$c^ib^j\ c^rb^s = c^ib^j\ c^ub^v$$

if and only if

$$j \geq r, j \geq u \text{ and } s - r = v - u.$$

PROOF. If $j \geq r, j \geq u$ and $s - r = v - u$ then $c^ib^jc^rb^s = c^ib^{j+s-r} = c^ib^{j+v-u} = c^ib^jc^ub^v$. For the converse let's consider the four cases in the equation $c^ib^j\ c^rb^s = c^ib^j\ c^ub^v$. (i) $j \geq r, j \geq u$. In this case the equation becomes $c^ib^{j-r+s} = c^ib^{j-u+v}$ and so $s - r = v - u$ as stated. (ii) $j \geq r, j < u$. In this case we obtain $c^ib^{j-r+s} = c^{i-j+u}b^v$ and so we have $i = i - j + u$ (and $j - r + s = v$) which implies $j = u$, a contradiction. Analogously we cannot have (iii) $j < r, j \geq u$. (iv) Finally we show that is also not possible to have $j < r, j < u$. In this case the equation becomes $c^{i-j+r}b^s = c^{i-j+u}b^v$ which implies $r = u$, $s = v$ and so $x = y$, which contradicts the hypothesis. ■

Lemma 3.2 *Let $c^ib^j, c^kb^l \in \mathbf{B}$, with $j \leq l$. If $c^ib^jx = c^ib^jy$ for some $x, y \in \mathbf{B}$ then $c^kb^lx = c^kb^ly$.*

PROOF. The statement holds trivially if $x = y$, so assume that $x \neq y$. Let $x = c^rb^s$ and $y = c^ub^v$. Since $c^ib^j\ c^rb^s = c^ib^j\ c^ub^v$ with $c^rb^s \neq c^ub^v$, using Lemma 3.1, we have $j \geq r, j \geq u$ and $s - r = v - u$. So, since $l \geq j \geq r$ and $l \geq j \geq u$, we have $c^kb^l\ c^rb^s = c^kb^{l-r+s} = c^kb^{l+v-u} = c^kb^l\ c^ub^v$. ■

As an immediate consequence of this fact, we just have to check one of the equivalences in (1):

Corollary 3.3 *Two elements c^ib^j, c^kb^l $(j \leq l)$ in a subsemigroup S of $\mathbf{B}$ are $\mathcal{L}^*$-related if and only if*

$$c^kb^lx = c^kb^ly \implies c^ib^jx = c^ib^jy, \forall x, y \in S^1.$$

Using this we can state a necessary and sufficient condition for two elements A and B in a subsemigroup of $\mathbf{B}$ to be $\mathcal{L}^*$-related, illustrated in Figure 2 (x is in the horizontal shaded strip determined by the columns of A and B and y in the shaded diagonal):

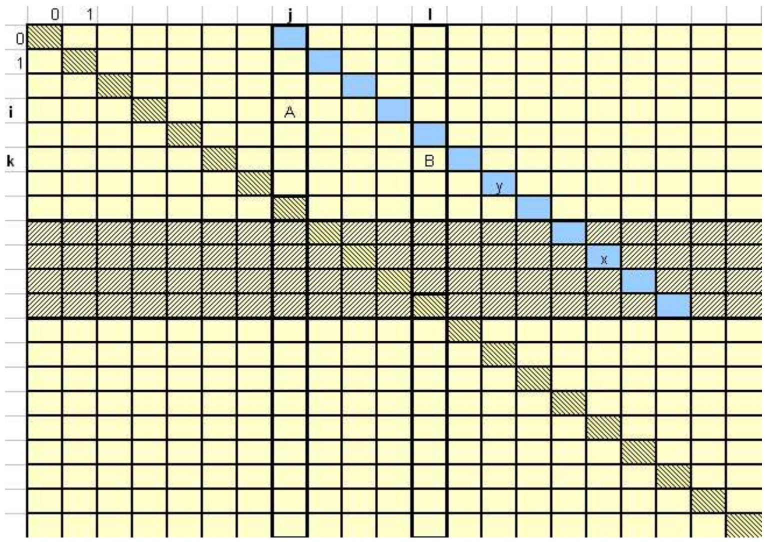

Figure 2: $\mathcal{L}^*$-relation in the subsemigroups of the bicyclic monoid

Lemma 3.4 *Two elements c^ib^j, c^kb^l $(j \leq l)$ in a subsemigroup S of $\mathbf{B}$ are not $\mathcal{L}^*$-related if and only if there exist two different elements $x = c^rb^s, y = c^ub^v \in S$ such that $j < r \leq l$, $u \leq l$ and $s - r = v - u$.*

PROOF. Using Corollary 3.3, c^ib^j and c^kb^l are not $\mathcal{L}^*$-related if and only if there exist two elements $x, y \in S$ such that $c^kb^lx = c^kb^ly$ and $c^ib^jx \neq c^ib^jy$. Let $x = c^rb^s$ and $y = c^ub^v$. Using Lemma 3.1, $c^kb^lx = c^kb^ly$ is equivalent to $l \geq r, l \geq u$ and $s - r = v - u$, and $c^ib^jx \neq c^ib^jy$ is equivalent to $j < r \vee j < u \vee s - r \neq v - u$. Since $s - r = v - u$ it must be that $j < r \vee j < u$. We can assume, without loss of generality, that $j < r$, whence we have $j < r \leq l, u \leq l, s - r = v - u$. ■

As a trivial consequence we have the following useful sufficient condition for two elements to be $\mathcal{L}^*$-related:

Lemma 3.5 *Let S be a subsemigroup of* $\mathbf{B}$ *and let $c^ib^j, c^kb^l \in S$ $(j \leq l)$. If S has no elements in rows $j+1, \ldots, l$ then $c^ib^j \, \mathcal{L}^* \, c^kb^l$.*

And we have the following corollary:

Corollary 3.6 *Two elements of a subsemigroup S of* $\mathbf{B}$ *in the same column are $\mathcal{L}^*$-related.*

This we knew already because two elements in the same column are $\mathcal{L}$-related in the bicyclic monoid.

Another consequence of Lemma 3.4 is the following:

Corollary 3.7 *An $\mathcal{L}^*$-class of S consists of a union of adjacent columns, i.e., there cannot exist two $\mathcal{L}^*$-related elements A and B and another element C not $\mathcal{L}^*$-related to A and B in a column between them.*

Proof. Let $A = c^ib^j$, $B = c^kb^l$ and $C = c^mb^n$ $(j \leq n \leq l)$. If A and B are $\mathcal{L}^*$-related then, by Lemma 3.4, elements $x = c^rb^s, y = c^ub^v$ with $j < r \leq l, u \leq l$ and $s - r = v - u$ cannot exist. Hence such elements cannot exist with $j < r \leq n \leq l$ and $u \leq n$, so $C \, \mathcal{L}^* \, A \, \mathcal{L}^* B$ and $C \, \mathcal{L}^* \, B$ by transitivity. ■

4 $\mathcal{R}^*$-CLASSES

To obtain the corresponding facts for $\mathcal{R}^*$-classes we will use the standard anti-isomorphism of an inverse semigroup T to itself, $\rho : T \to T;\ x \mapsto x^{-1}$. We note that $(xy)\rho = (xy)^{-1} = y^{-1}x^{-1}$. If S is a subsemigroup of T, we denote by S^{-1} the subsemigroup $S\rho$. If T is the bicyclic monoid $\mathbf{B}$ then $\rho : \mathrm{B} \to \mathrm{B};\ c^ib^j \mapsto c^jb^i$ and

$$(c^ib^jc^kb^l)^{-1} = c^lb^kc^jb^i. \tag{2}$$

The following fact will be useful:

Lemma 4.1 *If S is a subsemigroup of an inverse semigroup T and $a, b \in S$ then $(a, b) \in \mathcal{L}^*_S$ if and only if $(a^{-1}, b^{-1}) \in \mathcal{R}^*_{S^{-1}}$.*

Proof. Let $a, b \in S$. We have $(a, b) \in \mathcal{L}^*_S$ if and only if $ax = ay \Leftrightarrow bx = by$ $(\forall x, y \in S^1)$. This happens if and only if $(ax)^{-1} = (ay)^{-1} \Leftrightarrow (bx)^{-1} = (by)^{-1}$ $(\forall x, y \in S^1)$. This is equivalent to $x^{-1}a^{-1} = y^{-1}a^{-1} \Leftrightarrow x^{-1}b^{-1} = y^{-1}b^{-1}$ $(\forall x, y \in S^1)$ what is the same as $ua^{-1} = va^{-1} \Leftrightarrow ub^{-1} = vb^{-1}$ $(\forall u, v \in (S^{-1})^1)$ and so $(a^{-1}, b^{-1}) \in \mathcal{R}^*_{S^{-1}}$. ■

In the case where $T = S$ we have $S = S^{-1}$ and so we can say:

Lemma 4.2 *If S is an inverse semigroup and $a, b \in S$ then $(a, b) \in \mathcal{L}^*$ if and only if $(a^{-1}, b^{-1}) \in \mathcal{R}^*$.*

Lemma 4.3 *If $xa = ya$ with $a = c^i b^j, x = c^r b^s, y = c^u b^v \in S$, $x \neq y$, i.e,*

$$c^r b^s \; c^i b^j = c^u b^v \; c^i b^j$$

then

$$i \geq s, i \geq v \text{ and } r - s = u - v.$$

Proof.

We have $xa = ya$ if and only if $(xa)^{-1} = (ya)^{-1}$. By (2), we have $(xa)^{-1} = c^j b^i c^s b^r$ and $(ya)^{-1} = c^j b^i c^v b^u$ and so, by Lemma 3.1, we have $i \geq s, i \geq v$ and $r - s = u - v$.

■

Lemma 4.4 *Let $c^i b^j, c^k b^l \in \mathbf{B}$, $i \leq k$. If $x \; c^i b^j = y \; c^i b^j$ then $x \; c^k b^l = y \; c^k b^l$, for any $x, y \in \mathbf{B}$, $x \neq y$.*

Proof. If $xc^i b^j = yc^i b^j$ then $(xc^i b^j)^{-1} = (yc^i b^j)^{-1}$. So $c^j b^i x^{-1} = c^j b^i y^{-1}$ and, by Lemma 3.2, $c^l b^k x^{-1} = c^l b^k y^{-1}$. Hence, $(xc^k b^l)^{-1} = (yc^k b^l)^{-1}$ and so $xc^k b^l = yc^k b^l$. ■

As an immediate consequence of this fact we have

Corollary 4.5 *Two elements $c^i b^j, c^k b^l$ $(i \leq k)$ in a subsemigroup S of $\mathbf{B}$ are $\mathcal{R}^*$-related if and only if*

$$x \; c^k b^l = y \; c^k b^l \implies x \; c^i b^j = y \; c^i b^j, \forall x, y \in S^1.$$

The following lemma gives a necessary and sufficient condition for two elements to be $\mathcal{R}^*$-related and it is illustrated by Figure 3.

Lemma 4.6 *Two elements* c^ib^j, c^kb^l $(i \leq k)$ *in a subsemigroup* S *of* $\mathbf{B}$ *are not* $\mathcal{R}^*$*-related if and only if there exist two different elements* $x = c^rb^s, y = c^ub^v \in S$ *such that* $i < s \leq k$, $v \leq k$ *and* $r - s = u - v$.

PROOF. The elements c^ib^j, c^kb^l are not $\mathcal{R}^*_S$-related if and only if the elements c^jb^i, c^lb^k are not $\mathcal{L}^*_{S^{-1}}$-related. By Lemma 3.4 this happens if and only if there exists $x^{-1} = c^sb^r \neq c^vb^u = y^{-1}$ in S^{-1} such that $i < s \leq k$, $v \leq k$ and $r - s = u - v$. And so, if and only if, there exists $x = c^rb^s \neq c^ub^v = y$ in S such that $i < s \leq k$, $v \leq k$ and $r - s = u - v$. ■

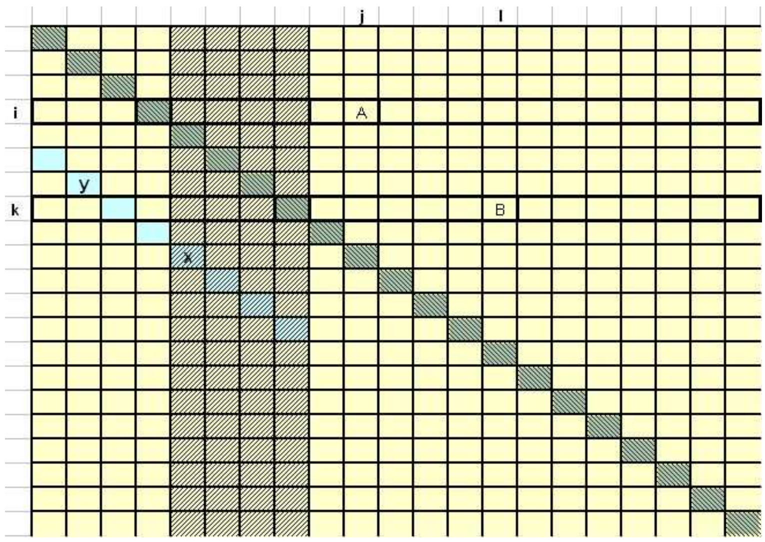

Figure 3: $\mathcal{R}^*$-relation in subsemigroups of the bicyclic monoid

Lemma 4.7 *Let* S *be a subsemigroup of* $\mathbf{B}$ *and let* $c^ib^j, c^kb^l \in S$ $(i \leq k)$. *If* S *has no elements in columns* $i+1, \ldots, k$ *then* $c^ib^j\, \mathcal{R}^*\, c^kb^l$.

Corollary 4.8 *Two elements of a subsemigroup* S *of* $\mathbf{B}$ *in the same row are* $\mathcal{R}^*$*-related.*

Corollary 4.9 *An* $\mathcal{R}^*$*-class of* S *consists of adjacent rows, i.e., there cannot exist two* $\mathcal{R}^*$*-related elements* A *and* B *in* S *and another element* C *in* S *not related with* A *and* B *in a row between them.*

Proof. If $(c^ib^j, c^kb^l) \in \mathcal{R}^*_S$ then, by Lemma 4.1, $(c^jb^i, c^lb^k) \in \mathcal{L}^*_{S^{-1}}$. By Corollary 3.7, c^jb^i and c^lb^k are in union of adjacent rows in S^{-1}, which means that c^ib^j and c^kb^l are in a union of adjacent columns in S. ∎

5 UPPER SUBSEMIGROUPS

Upper semigroups may be abundant or not. A simple example is the free monogenic semigroup, generated by b, which is a non abundant upper semigroup, since it has no idempotents. We note that, since this semigroup is cancellative, it has a unique $\mathcal{L}^*$-class and a unique $\mathcal{R}^*$-class. If we adjoin the identity to it, we obtain the free monogenic monoid, which is an abundant upper subsemigroup of the bicyclic monoid, having one $\mathcal{L}^*$-class and one $\mathcal{R}^*$-class; both contain an idempotent, the identity of the monoid.

We start by considering finitely generated upper subsemigroups. They have the form $S = F_D \cup F \cup \Lambda_{I,m,d}$ where $I \subseteq \mathbb{N}_0$, $q = \min(I) \leq p = \max(I) \leq m$, $d \in \mathbb{N}$, $F_D \subseteq \{c^ib^i : i < q\}$, $F \subseteq \{c^ib^j : q \leq i \leq p, i \leq j < m\}$ are finite sets, and $\Lambda_{I,m,d} = \{c^ib^j : i \in I, d \mid j-i, j \geq m\}$ (see [4]). This semigroup is illustrated by Figure 4. In this section we assume that S is a semigroup of this kind.

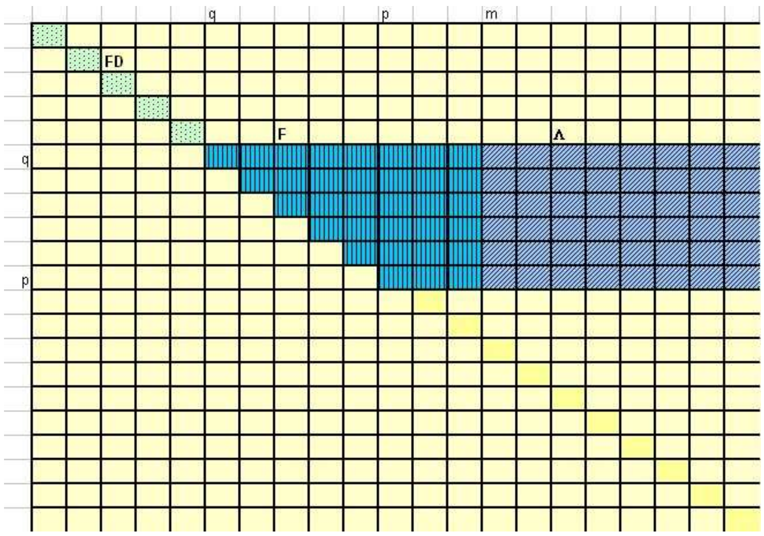

Figure 4: The region containing a semigroup $S = F_D \cup F \cup \Lambda_{I,m,d}$

We will first consider the case where $F_D = F = \emptyset$. In this case, $S = \Lambda_{I,m,d}$ is a finite union of special subsemigroups of $\mathbb{N}_0$ (numerical semigroups of the form $\{kd : k \in \mathbb{N}_0,\ kd \geq N\}$ with $d, N \in \mathbb{N}_0, d > 0$). We will show that this subsemigroup has only one $\mathcal{L}^*$-class and only one $\mathcal{R}^*$-class. In fact, given two elements $c^ib^j, c^kb^l \in S$, with $j \leq l$, there are no elements of S in rows $j+1, \ldots, l$ because $j+1 > m$ and all elements of S are in rows $q, \ldots, p$ with $p \leq m$. So using Lemma 3.5 we see that c^ib^j, c^kb^l are $\mathcal{L}^*$-related. To see that there is also only one $\mathcal{R}^*$-class, we can take two arbitrary elements $c^ib^j, c^kb^l \in S$ with $i \leq k$ $(\leq p \leq m)$. Since S has no elements in columns $i+1, \ldots, k-1$, we cannot find two different elements x, y in the conditions of Lemma 4.6, not even in the case where $p = m$. Hence c^ib^j, c^kb^l are $\mathcal{R}^*$-related. Having only a $\mathcal{L}^*$-class and only a $\mathcal{R}^*$-class, for the subsemigroup to be abundant it just needs to contain one idempotent. This is only possible if $m = p$ and the idempotent c^pb^p belongs to S. We summarize this in the following

Proposition 5.1 *An upper subsemigroup S of* $\mathbf{B}$ *of the form $S = \Lambda_{I,m,d}$ has a unique $\mathcal{R}^*$-class and a unique $\mathcal{L}^*$-class. It is abundant if and only if $m = p$ and $c^pb^p \in S$.*

We consider now subsemigroups of the form $S = F_D \cup \Lambda_{I,m,d}$ where $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 1)$. If I has only one element, say $I = \{p\}$, then S is in fact obtained starting from the numerical semigroup $\Lambda_{I,m,d}$ and adding successively the identities $e_n, e_{n-1}, \ldots e_1$. The elements of $\Lambda_{p,m,d}$ together with the idempotent of S that is lower in the diagonal (which may be c^pb^p) form a cancellative monoid, hence having a unique $\mathcal{L}^*$-class and $\mathcal{R}^*$-class. Each other idempotent in F_D is by itself an $\mathcal{L}^*$-class and an $\mathcal{R}^*$-class. So, if $c^pb^p \in S$ the classes are $\{e_1\}, \{e_2\}, \ldots, \{e_n\}, \Lambda_{p,m,d}$. Otherwise the classes are $\{e_1\}, \{e_2\}, \ldots, \{e_n\} \cup \Lambda_{p,m,d}$. In any case the subsemigroup is abundant.

Proposition 5.2 *An upper semigroup of the form $S = F_D \cup \Lambda_{p,m,d}$ with $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 1)$ is abundant. If $\Lambda_{p,m,d}$ has an idempotent the $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes are $\{e_1\}, \{e_2\}, \ldots, \{e_n\}, \Lambda_{p,m,d}$. Otherwise they are $\{e_1\}, \{e_2\}, \ldots, \{e_n\} \cup \Lambda_{p,m,d}$.*

We continue with $S = F_D \cup \Lambda_{I,m,d}$ where $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 1)$ but assuming now that I has more than one element. Let's first consider the case where $c^pb^p \in \Lambda_{I,m,d}$ $(m = p)$. In this case, two elements $c^ib^j, c^pb^k \in \Lambda_{I,m,d}$ with $i < p$ are not $\mathcal{R}^*$-related since, letting x be the lower idempotent in F_D and $y = c^pb^p$, the elements x and y are in the conditions of Lemma 4.6. But any two elements $c^ib^j, c^kb^l \in \Lambda_{I,m,d}$ with $i, j < p$ are $\mathcal{R}^*$-related since S has no elements in columns $q, \ldots, p-1$ and we can use Lemma 4.7. The lower idempotent in F_D, say $e = c^ib^i$, is $\mathcal{R}^*$-related to any element c^kb^l in rows $I\backslash\{p\}$ because $k < p$ and S has no elements in columns $i+1, \ldots, p-1$. The other idempotents in F_D are $\mathcal{R}^*$-classes by themselves. So every $\mathcal{R}^*$-class has an idempotent. The $\mathcal{R}^*$-classes of these subsemigroups are illustrated by the example in Figure 5.

Two elements $c^ib^j, c^kb^l \in \Lambda_{I,m,d}$ are $\mathcal{L}^*$-related because $j, l \geq p$, there are no elements in S below row p and so we can use Lemma 3.5. These $\mathcal{L}^*$-class contains already one idempotent, c^pb^p, and so the other idempotents in F_D are $\mathcal{L}^*$-classes by themselves. Hence, also every $\mathcal{L}^*$-class has an idempotent and the semigroup is abundant.

The case where $c^pb^p \notin S$ can also be illustrated by Figure 5, removing the last row. We show that, in this case, the set $\Lambda_{I,m,d}$ is an $\mathcal{L}^*$-class of S without an idempotent, and so the semigroup is not abundant. The elements in $\Lambda_{I,m,d}$ are still $\mathcal{L}^*$-related. But the lower idempotent $c^ib^i \in F_D$ is not related to them, since, given $c^kb^l \in \Lambda_{I,m,d}$, we can find two elements $x = c^jb^{j+ud}, y = c^kb^{k+vd} \in \Lambda_{I,m,d}$ in the same diagonal and in different rows, in the conditions of Lemma 3.4.

Proposition 5.3 *An upper semigroup of the form* $S = F_D \cup \Lambda_{I,m,d}$ *with* $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 1)$ *and* $|I| > 1$ *is abundant if and only if* $m = p$ *and* $c^pb^p \in \Lambda_{I,m,d}$. *In this case the* $\mathcal{L}^*$*-classes and* $\mathcal{R}^*$*-classes are* $\{e_1\}, \{e_2\}, \ldots, \{e_n\}, \Lambda_{I,m,d}$.

Finally we consider arbitrary finitely generated upper semigroups $S = F_D \cup F \cup \Lambda_{I,m,d}$ where $F \neq \emptyset$. We note that, if I has only one element, then S is again obtained from a numerical semigroup adding finitely many idempotents and so we have:

Proposition 5.4 *A subsemigroup of the form* $F_D \cup F \cup \Lambda_{p,m,d}$, $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 0)$ *is abundant if and only if it contains at least one idempotent. If* $c^pb^p \in S$ *the* $\mathcal{L}^*$*-classes and* $\mathcal{R}^*$*-classes are* $\{e_1\}, \{e_2\}, \ldots, \{e_n\}, \Lambda_{I,m,d}$. *Otherwise the classes are* $\{e_1\}, \{e_2\}, \ldots, \{e_n\} \cup \Lambda_{I,m,d}$.

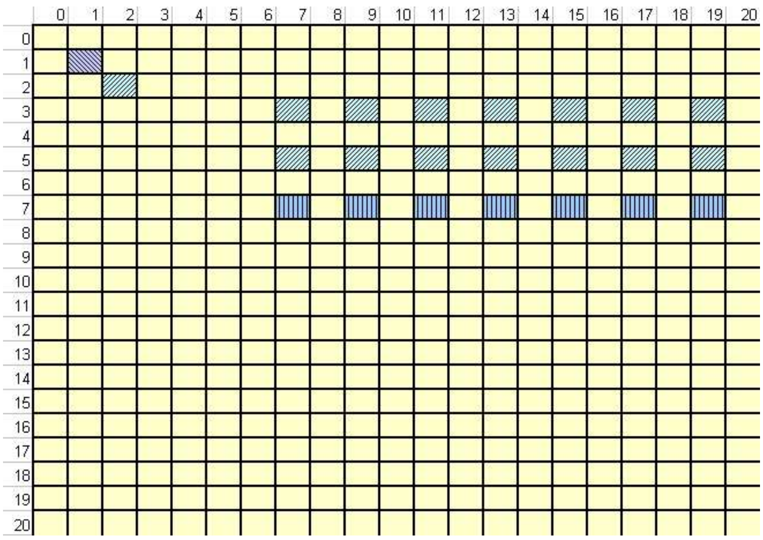

Figure 5: $\mathcal{R}$-classes of $S = F_D \cup \Lambda_{I,m,d}$

So we assume now that I has a at least two elements. To identify the $\mathcal{L}^*$-classes of S, it is convenient to write $S = F_D \cup F' \cup S'$, where $F' = S \cap \{c^i b^j : q \leq i \leq j < p\}$ and $S' = S \cap \{c^i b^j : j \geq p\}$, as illustrated in Figure 6.

The elements in S' are $\mathcal{L}^*$-related because they are on columns $p, p+1, \ldots$ and, since there are no elements in S below row p, we can use Lemma 3.5. An element $c^i b^j \in F'$ is not $\mathcal{L}^*$-related to an element $c^k b^l \in S'$. In fact, since S has elements in row p and $i, j < p$, we can choose two different elements $x = c^p b^{p+ud}$, $y = c^i b^{i+vd}$ in the same diagonal, in the conditions of Lemma 3.4. Hence, if $F' \neq \emptyset$ then S' is an $\mathcal{L}^*$-classe of S. Also in the case where $F' = \emptyset$ the set S' is an $\mathcal{L}^*$-class of S. This is shown if $F_D = \emptyset$. And if $F_D \neq \emptyset$, we can see that the lower idempotent in F_D is not $\mathcal{L}^*$-related say, with an element $c^p b^k \in S'$ because we can choose two different elements $x = c^p b^{p+ud}$ and $y = c^i b^{i+vd}$ in S with $i < p$ in the same diagonal, and use Lemma 3.4. So, in any case, S' is an $\mathcal{L}^*$-class of S and for S to be abundant it must contain the idempotent $c^p b^p$.

We continue the study of $\mathcal{L}^*$-classes considering now the elements in F', which are in finitely many columns. For the semigroup to be abundant, each $\mathcal{L}^*$-classe, that may be formed by the elements in one or more columns, must contain an idempotent. In Figure 7, we find an example of an abundant

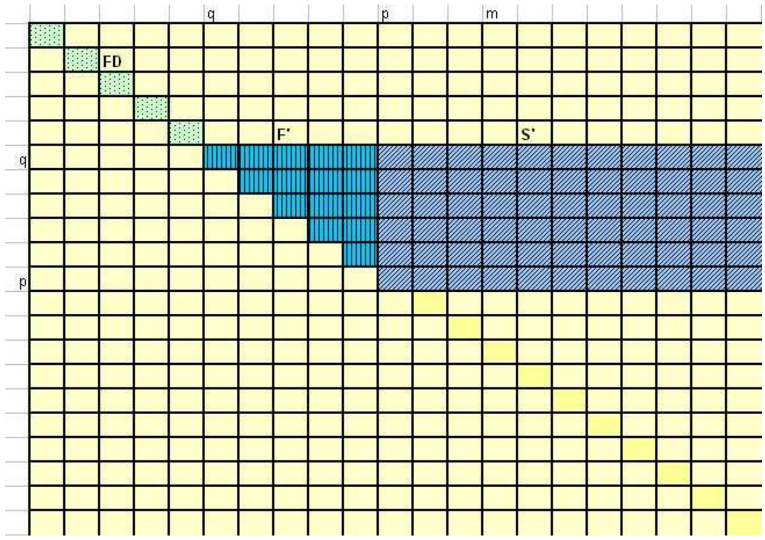

Figure 6: Determining the $\mathcal{L}^*$-classes of $S = F_D \cup F \cup \Lambda_{I,m,d}$

subsemigroup where some columns with elements do not have idempotents.

To check if all $\mathcal{L}^*$-classes of elements in $F_D \cup F'$ have idempotents we just have to form unions with the $p-q$ columns. To do that, we start by observing that if two columns i, j with elements with $q \leq i < j < p$, are in the same $\mathcal{L}^*$-class, then $c^j b^j$ cannot be in the class. In fact, if $c^j b^j$ were in the class then, given two elements $c^k b^i, c^l b^j$ we could obtain $x = c^j b^{j+ud}, y = c^k b^{k+vd}$ in the conditions of Lemma 3.4 and $c^k b^i, c^l b^j$ would not be related. So, the idempotent in an $\mathcal{L}^*$-class with elements from F' is either in the leftmost column or in F_D. Hence, to check if all classes have idempotents we can proceed the following way. We begin by forming a union of rows with elements, L, starting from first column $i \leq p-1$ with elements and going left. If we have already an idempotent we start forming next class. If not we add next column $j < i$ with elements to L, if there are no elements in rows $j+1, \ldots, i$. We proceed adding columns until no more columns can be added. After that, if the last column does not have an idempotent in S (and there are still other columns with elements in F' on the left) we have found an $\mathcal{L}^*$-class without an idempotent. Otherwise L is an $\mathcal{L}^*$-class with idempotent and we start forming the next class. After going through all columns of F' two things may happen. If the final union of rows L has an idempotent then all $\mathcal{L}^*$-classes have idempotents.

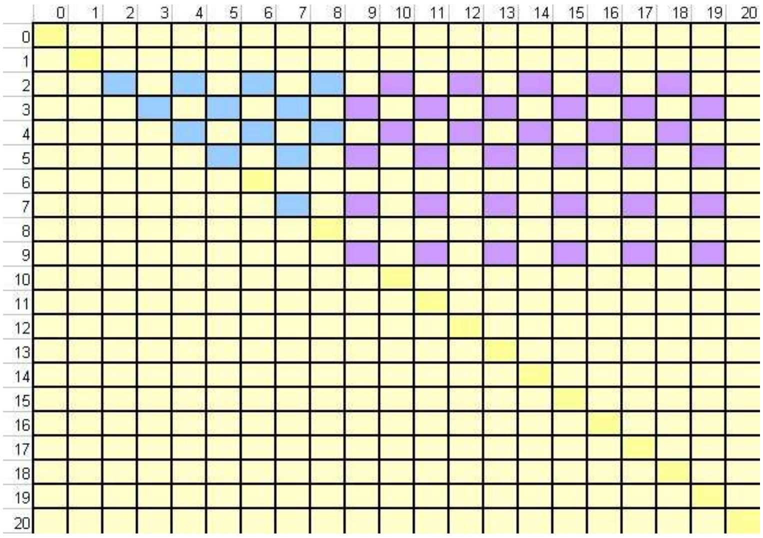

Figure 7: An abundant upper subsemigroup

If not, it may be possible that all $\mathcal{L}^*$-classes have idempotents if $F_D \neq \emptyset$. This can only occur if there is only one row in S with elements in rows $k+1, \ldots, l$, where k is the minimum of column indices in L and l is the maximum. In fact, we see using Lemma 3.4 that only in this case the lower idempotent in F_D is $\mathcal{L}^*$-related to the elements in L.

To check if the $\mathcal{R}^*$-classes have idempotents we can proceed in a similar way. There are finitely many rows with elements and $\mathcal{R}^*$-classes are unions of adjacent rows. We start from row $i = p$ and go up adding rows and forming $\mathcal{R}^*$-classes. We consider the next row j $(= \max(I \backslash \{i\})$ in I. Using Lemma 4.6 we see that, elements in rows i and j are $\mathcal{R}^*$-related if and only if S has no elements in columns $i+1, \ldots, j$ or, for each element A in columns $i+1, \ldots, j$ we cannot find another element $\mathbf{B}$ in columns $0, \ldots, j$ in the same diagonal as A. Proceeding this way we can form the $\mathcal{R}^*$-classes, which are at most $|I| + |F_D|$, and check if they all have idempotents.

These algorithms allows us to check if a general upper subsemigroup of the form $F_D \cup F \cup \Lambda_{I,m,d}$ is abundant. Hence we can say the following

Proposition 5.5 *Let $S = F_D \cup F \cup \Lambda_{I,m,d}$ be an upper subsemigroup of $\mathbf{B}$. Writing $S = F_D \cup F' \cup S'$, where $F' = S \cap \{c^i b^j : q \leq i \leq j < p\}$ and $S' = S \cap \{c^i b^j : j \geq p\}$, the set S' is an $\mathcal{L}^*$-class of S. For S to be abundant* *it is necessary that $c^p b^p \in S$. There exist algorithms to construct the other finitely many $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes of S from the finitely many elements of S in columns $0, \ldots, p$ and the finite set I. The semigroup is abundant if and only if these finitely many $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes have idempotents.*

The algorithms to check if a upper subsemigroup is abundant follows. We have seen it is necessary that $c^p b^p \in S$ for S to be abundant. Then we can check if all $\mathcal{L}^*$-classes have idempotents with the algorithm in Figure 8, where C is the set of indices of columns having elements in $F' \cap S$.

```
isabundant ← true
L ← ∅
while C ≠ ∅ and isabundant
do
    i ← max(C); C ← C\{i}; L ← L ∪ {i}
    if c^i b^i ∈ S then L ← ∅
    else
        if C = ∅ then
            if F_D = ∅ or {min(C) + 1, ..., max(C)} ∩ I > 1
            then isabundant ← false
        else
            j ← max(C)
            if {j + 1, ..., i} ∩ I = ∅ then isabundant ← false fi
        fi
    fi
od
```

Figure 8: Algorithm to check if all $\mathcal{L}^*$-classes have indempotents

To check if all $\mathcal{R}^*$-classes have idempotents we can the use the algorithm in Figure 9.

Finitely generated lower subsemigroups are similar, just replacing rows by columns.

If S is a non finitely generated upper subsemigroup, so with elements in an infinite number of rows, then there is no algorithm to check if S is abun-

```
isabundant ← true
R ← ∅
while I ≠ ∅ and isabundant
do
    i ← max(I); I ← I\{i}; R ← R ∪ {i}
    if I = ∅ then
        if ⋃_{k∈R} {c^k b^k} = ∅ and F_D = ∅ then isabundant ← false fi
    else
        j ← max(I)
        if ¬R_related(j, i) then
            if ⋃_{k∈R} {c^k b^k} = ∅ then isabundant ← false fi
        fi
    fi
od
where
    R_related(j, i) = (∃u : −p ≤ u ≤ p : c^{j+u} b^j, c^{i+u} b^i ∈ S)
```

Figure 9: Algorithm to check if all $\mathcal{R}^*$-classes have indempotents

dant. In fact, we cannot decide if S is abundant, looking to finitely many rows, because we can always add a row without idempotent to an abundant semigroup obtaining a non abundant subsemigroup. For example, the semigroup $S = \{c^i b^j : 0 \leq i < p, j \geq i\}$ $(p > 0)$ is abundant and the semigroup $S \cup \{c^p b^j : j > p\}$ is not.

Of course there is a procedure to check if S is not abundant. It suffices to construct the $\mathcal{L}^*$-classes ($\mathcal{R}^*$-classes), which are unions of columns (rows), until a class without idempotent is found, using Lemma 3.4 (Lemma 4.6).

6 TWO SIDED SUBSEMIGROUPS

In general, a two-sided semigroup has the form $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ (or the corresponding anti-isomorphic image) where $q, p \in \mathbb{N}_0$ with $q \leq p$, $d \in \mathbb{N}$, $I \subseteq \{q, \ldots, p-1\}$ with $q \in I$, $P \subseteq \{0, \ldots, d-1\}$ with $0 \in P$, $F_D \subseteq \{c^i b^i : i = 0, \ldots, q-1\}$, $F \subseteq \{c^i b^j : q \leq i < p, i \leq j < p\}$, $\Sigma_{p,d,P} = \{c^{p+r+ud} b^{p+r+vd} : r \in P; u, v \geq 0\}$ (see [3]). Figure 10 shows an example of one of this subsemigroups.

We note that a two-sided semigroup of the form $F_D \cup \Sigma_{p,d,P}$ is regular (see [3]) and so abundant. Each of its $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes is contained in single row or column and all have an idempotent. We start by showing the following:

Proposition 6.1 *Subsemigroups of the form $S = \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ (with $F_D = F = \emptyset$) are abundant.*

Proof. If $I = \emptyset$ we have seen that S is abundant, so we assume $I \neq \emptyset$. We begin by showing that two columns i, j $(i < j)$ such that, the set $\{c^k b^k : i \leq k \leq j\} \cap S$ is either empty or equal to $\{c^i b^i\}$, are $\mathcal{L}^*$-related. In fact, since $i, j \geq p$, the rows $i+1, \ldots, j$ are in Σ_p. A row k in Σ_p has elements from S if and only if $c^k b^k \in S$. Hence S has no elements in rows $i+1, \ldots, j$ and Lemma 3.5 can be applied.

Using this, and observing thet $0 \in P$ and so $c^p b^p \in S$, we see that every $\mathcal{L}^*$-class is a union of columns starting from a column with an idempotent together with all columns on its right hand side not having idempotents. Hence, every $\mathcal{L}^*$-class has an idempotent.

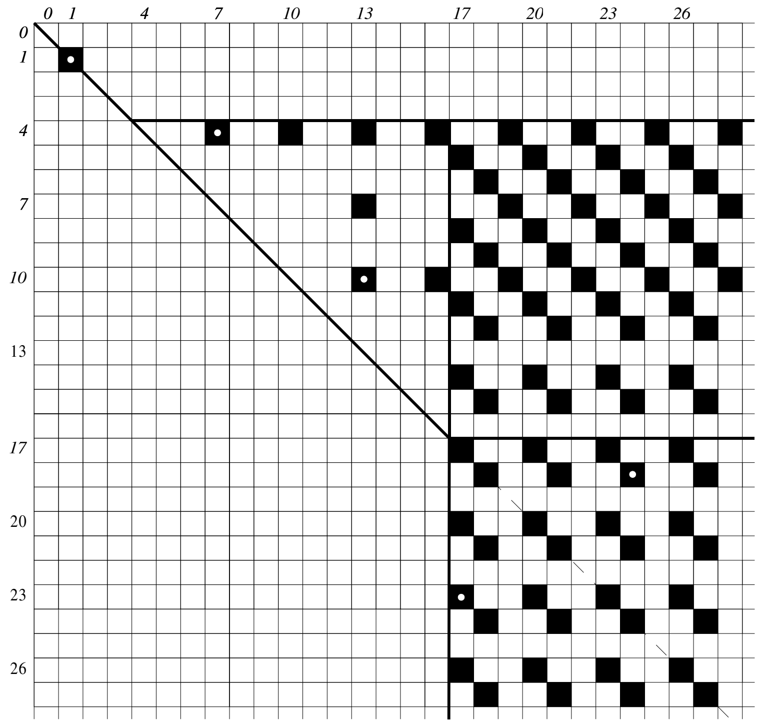

Figure 10: Two-sided subsemigroup generated by $\{cb, c^4b^7, c^{10}b^{13}, c^{18}b^{24}, c^{23}b^{17}\}$.

Each row in $p+1, p+2, \ldots$ with elements in S is an $\mathcal{R}^*$-class with idempotent. The elements in $S \cap \{c^ib^j : 0 \leq i \leq p, j \geq p\}$ are $\mathcal{R}^*$-related because S has no elements in columns $q, \ldots, p-1$ and Lemma 4.6 can be applied. Hence every $\mathcal{R}^*$-class has an idempotent. ■

Corollary 6.2 *Every simple subsemigroup of the bicyclic monoid is abundant.*

PROOF. As shown in [3] these are the simple subsemigroups of the bicyclic monoid. ■

We consider now a general two sided semigroup $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$. For columns in $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$, the argument in the proof of Proposition 6.1 can be applied and so the classes with these elements have idempotents. Elements in columns $p-1$ and p are note related because we can take $x = c^pb^{p+ud}, y = c^ib^{i+vd}$ $(i < p)$ in the conditions of Lemma 3.4. To form the $\mathcal{L}^*$-classes with the elements in columns $p-1, p-2, \ldots$ we just look to rows $i < p$ and so to $F_D \cup F \cup \Lambda_{I,p,d}$. Hence we just have to apply the algorithm in Figure 8, where C is the set of indices of columns $i < p$ with elements in S.

Each row in $\Sigma_{p,d,P}$ is in a separate $\mathcal{R}^*$-class with idempotent. We note that rows p and $p-1$ are not related if $F_D \cup F \neq \emptyset$. In fact, if $F_D \neq \emptyset$ then the elements $x = c^p b^p$, $y \in F_D$ are in the conditions of Lemma 4.6. And if $F \neq \emptyset$ we can also find two elements $x = c^{p-ud} b^p$, $y \in F$ in the conditions of Lemma 4.6. Hence, to check if all $\mathcal{R}^*$-classes have idempotents, we just have to look to rows in $F_D \cup F \cup \Sigma_{p,d,P}$ and columns $0, \ldots, p-1$ and we can use the algorithm in Figure 9.

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