Generalized Hamiltonian Formalism in Nonlinear Optics

Vladimir E. Zakharov[1]

[1]*L.D.Landau Institute for Theoretical Physics*
Russian Academy of Sciences
Kosygina Str. 2, Moscow, Russia 117940, zakharov@itp.ac.ru;
Department of Mathematics, University of Arizona
Tucson, AZ 85721, USA, zakharov@math.arizona.edu

1 Introduction

The basic mathematical apparatus of nonlinear optics consists of an array of nonlinear PDEs for the complex amplitudes of an envelope of interacting wave trains. In the general case, these equations include linear and nonlinear dissipative terms. However, in many important cases, they are small and can be neglected: therefore the equations are conservative, and the medium is transparent. According to the Kramers–Kronig relations, stemming from the principle of causality, the transparency can be realized at most in a limited spectral band, and even in this case some dissipation inevitably exists. Nevertheless, such fundamental nonlinear effects as the generation of high harmonics, induced Raman scattering, and self-focusing can be described by the conservative equations, preserving energy.

It is remarkable that these equations are not just conservative. It is an experimental fact that in all known cases, the conservative equations of nonlinear optics are also Hamiltonian systems. The nonlinear Schrödinger equation is a perfect example of that sort.

Actually, it is not astonishing. All macroscopic equations describing real media can be derived, at least in principle, from the microscopic quantum equations, which are Hamiltonian by definition. The original Hamiltonian structure of the interlying microscopic equations should be somehow inherited by the macroscopic equations. In the simplest cases, this heredity means that a Hamiltonian structure exists in the macroscopic equations. The situation is more complicated in the general case.

In nonlinear optics, a medium is described by the Maxwell equations, where the electric induction is a nonlinear operator on the electric field. In reality, the fields are sufficiently weak to expand in powers of the electric field. Usually, only the linear, quadratic, and cubic terms are considered. They are characterized by tensors of linear, quadratic, and cubic dielectric permittivity. Transparency of the media can be formulated in terms of symmetry properties of these tensors. This correspondence was first traced in 1962, in the classic paper by J. Amstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan [1]. It is discussed in detail in [2].

In this article, we claim the following : the symmetry conditions of the tensors of dielectric permittivity appearing in the transparent medium mean the existence of variotional principle for the nonlinear Maxwell equations. In other words, these equations can be obtained by minimizing the action, which is a functional on the vector potential. This functional is represented as power series starting from quadratic terms. In the general case, this functional is very nonlocal both in space and time.

This result is so simple and natural that it maybe is not new. However, we have not yet found it in the literature. The result seems important because the variational formulation of the nonlinear Maxwell equations is an ideal starting point for developing approximate models of any kind. If we properly approximate directly in the Lagrangian, we can be sure that the basic symmetries of the equation, including the Hamiltonian structure, are not violated. The conservation laws in such an approach are satisfied automatically.

This makes the variational approach an ideal tool for constructing an averaged equation for wave envelopes in all orders of nonlinearity and aspect ratio. In this article, we take just the first steps in this direction. We pay special attention to the isotropic medium with Kerr-type nonlinearity.

2 The Maxwell equations in the nonlinear transparent medium

We study the most general case of weakly nonlinear homogenous medium with spatial dispersion. For the Maxwell equations, in terms of Fourier transforms for electric field and induction we have:

$$(k^2\delta_{\alpha\beta} - k_\alpha k_\beta)E_\beta(k,\omega) = \frac{\omega^2}{c^2} D_\alpha(k,\omega), \tag{2.1}$$

where

$$\begin{aligned} D_\alpha(k,\omega) &= \epsilon^{(0)}_{\alpha\beta}(k,\omega)\, E_\beta(k,\omega) + \\ &+ \int \epsilon^{(1)}_{\alpha\beta\gamma}(k_1,\omega_1,k_2,\omega_2)\, E_\beta(k_1,\omega_1)\, E_\gamma(k_2,\omega_2)\delta_{k-k_1-k_2}\, \delta_{\omega-\omega_1-\omega_2}\, dk_1\, dk_2\, d\omega_1\, d\omega_2 + \\ &+ \int \epsilon^{(2)}_{\alpha\beta\gamma\delta}(k_1,\omega_1,k_2,\omega_2,k_3,\omega_3)\, E_\beta(k_1,\omega_1)\, E_\gamma(k_2,\omega_2)\, E_\delta(k_3,\omega_3) \times \\ &\times \delta_{k-k_1-k_2-k_3}\, \delta_{\omega-\omega_1-\omega_2-\omega_3}\, dk_1\, dk_2\, dk_3\, d\omega_1\, d\omega_2\, d\omega_3. \end{aligned} \tag{2.2}$$

It is clear that $\epsilon^{(1)}_{\alpha\beta\gamma}(k_1,\omega_1,k_2,\omega_2)$ is symmetric with respect to permutations

$$\beta \iff \gamma, \quad 1 \iff 2, \tag{2.3}$$

while $\epsilon^{(2)}_{\alpha\beta\gamma\delta}(k_1,\omega_1,k_2,\omega_2,k_3,\omega_3)$ is symmetric with respect to permutations

$$\begin{aligned} &\beta \iff \gamma, \quad 1 \iff 2; \\ &\beta \iff \delta, \quad 1 \iff 3; \\ &\gamma \iff \delta, \quad 2 \iff 3. \end{aligned} \tag{2.4}$$

The elements of dielectic permittivity,

$$\epsilon^{(0)}_{\alpha\beta}(k,\omega),\ \epsilon^{(1)}_{\alpha\beta\gamma}(k_1,\omega_1,k_2,\omega_2), \epsilon^{(2)}_{\alpha\beta\gamma\delta}(k_1,\omega_1,k_2,\omega_2,k_3,\omega_3),$$

are Fourier transforms of real functions. Hence, they obey the following symmetry conditions,

$$\begin{aligned} &\epsilon^{(0)}_{\alpha\beta}(-k,-\omega) = \epsilon^{(0)*}_{\alpha\beta}(k,\omega), \\ &\epsilon^{(1)}_{\alpha\beta\gamma}(-k_1,-\omega_1,-k_2,-\omega_2) = \epsilon^{(1)*}_{\alpha\beta\gamma}(k_1,\omega_1,k_2,\omega_2), \\ &\epsilon^{(2)}_{\alpha\beta\gamma\delta}(-k_1,-\omega_1,-k_2,-\omega_2,-k_3,-\omega_3) = \epsilon^{(2)*}_{\alpha\beta\gamma\delta}(k_1,\omega_1,k_2,\omega_2,k_3,\omega_3). \end{aligned} \tag{2.5}$$

The symmetry conditions (2.3)-(2.5) are universal; they hold in any homogenous media. In the transparent media, the tensors ϵ^i obey the additional conditions (see, for instance, [3]),

$$\epsilon^{(0)*}_{\alpha\beta}(k,\omega) = \epsilon^{(0)}_{\alpha\beta}(k,\omega), \tag{2.6}$$

$$\epsilon^{(1)}_{\alpha\beta\gamma}(k_1,\omega_1,k_2,\omega_2) = \epsilon^{(1)}_{\beta\alpha\gamma}(-k,-\omega,k_2,\omega_2) = \epsilon^{(1)}_{\gamma\beta\alpha}(k_1,\omega_1,-k,-\omega),$$
$$(k = k_1 + k_2,\ \omega = \omega_1 + \omega_2) \tag{2.7}$$

$$\epsilon^{(2)}_{\alpha\beta\gamma\delta}(k_1,\omega_1,k_2,\omega_2,k_3,\omega_3) = e^{(2)}_{\beta\alpha\gamma\delta}(-k,-\omega,k_2,\omega_2,k_3,\omega_3) =$$
$$= \epsilon^{(2)}_{\gamma\beta\alpha\delta}(k_1,\omega_1,-k,-\omega,k_3,\omega_3) = \epsilon^{(2)}_{\delta\beta\gamma\alpha}(k_1,\omega_1,k_2,\omega_2,-k,-\omega).$$
$$(k = k_1 + k_2 + k_3,\ \omega = \omega_1 + \omega_2 + \omega_3) \tag{2.8}$$

The symmetry conditions (2.6)-(2.8) allow us to rewrite the relation (2.2) in the following form:

$$D_\alpha(k,\omega) = \epsilon^0_{\alpha\beta}(k,\omega)\, E_\beta(k,\omega) +$$
$$+ \int f_{\alpha\beta\gamma}(-k,-\omega,k_1,\omega_1,k_2,\omega_2)\, E_\beta(k_1,\omega_1)\, E_\gamma(k_2,\omega_2) \times$$
$$\times \delta_{k-k_1-k_2}\, \delta_{\omega-\omega_1-\omega_2}\, dk_1\, dk_2\, d\omega_1\, d\omega_2 +$$
$$+ \int g_{\alpha\beta\gamma\delta}(-k,-\omega,k_1,\omega_1,k_2,\omega_2,k_3,\omega_3)\, E_\beta(k_1,\omega_1)\, E_\gamma(k_2,\omega_2)\, E_\delta(k_3,\omega_3) \times$$
$$\times \delta_{k-k_1-k_2-k_3}\, \delta_{\omega-\omega_1-\omega_2-\omega_3}\, dk_1\, dk_2\, dk_3\, d\omega_1\, d\omega_2\, d\omega_3. \tag{2.9}$$

Here

$$f_{\alpha\beta\gamma}(\omega,\omega_1,\omega_2) = \frac{1}{3}\left[\epsilon^{(1)}_{\alpha\beta\gamma}(\omega_1\omega_2) + \epsilon^{(1)}_{\beta\alpha\gamma}(\omega_1\omega_2) + \epsilon^{(1)}_{\gamma\beta\alpha}(\omega_1\omega)\right], \tag{2.10}$$

$$g_{\alpha\beta\gamma\delta}(\omega,\omega_1,\omega_2,\omega_3) = \frac{1}{4}\left[\epsilon^{(2)}_{\alpha\beta\gamma\delta}(\omega_1,\omega_2,\omega_3) + \epsilon^{(2)}_{\beta\alpha\gamma\delta}(\omega,\omega_2,\omega_3) + \right.$$
$$\left. + \epsilon^{(2)}_{\gamma\beta\alpha\delta}(\omega_1,\omega,\omega_3) + \epsilon^{(2)}_{\delta\beta\gamma\alpha}(\omega_1,\omega_2,\omega)\right]. \tag{2.11}$$

In (2.10), (2.11) we omitted the vector arguments k_i for simplicity.

We see that $f_{\alpha\beta\gamma}$ and $g_{\alpha\beta\gamma\delta}$ are completely symmetric functions. The permutation of their tensors should be done simultaneously with the permutation of corresponding frequencies and wave vectors.

Further, let us introduce a vector-potential,

$$E_\alpha(k,\omega) = i\omega\, A_\alpha(k,\omega), \tag{2.12}$$

that satisfies the following equations,

$$\left(k^2\,\delta\alpha\beta - k_\alpha k_\beta \frac{\omega^2}{c^2}\,\epsilon^{(0)}_{\alpha\beta}\right)\,A_\beta(k,\omega) =$$
$$+i\int \omega\omega_1\omega_2\, f_{\alpha\beta\gamma}(-k,-\omega,k_1,\omega_1,k_2,\omega_2)\,A_\beta(k_1,\omega_1)\,A_\gamma(k_2,\omega_2)\,\times$$
$$\times\delta_{k-k_1-k_2}\,\delta_{\omega-\omega_1-\omega_2}\,dk_1\,dk_2\,d\omega_1\,d\omega_2\,-$$
$$-\int \omega\omega_1\omega_2\omega_3\, g_{\alpha\beta\gamma\delta}\,(-k,-\omega,k_1,\omega_1,k_2,\omega_2,k_3,\omega_3)\,A_\beta(k_1,\omega_1)\,A_\gamma(k_2,\omega_2)\,A_\delta(k_3,\omega_3)\,\times$$
$$\times\delta_{k-k_1-k_2-k_3}\,\delta_{\omega-\omega_1-\omega_2-\omega_3}\,dk_1dk_2dk_3d\omega_1d\omega_2d\omega_3, \qquad (2.13)$$

and mention that

$$E_\alpha(-k,-\omega) = E^*_\alpha(k,\omega),$$
$$A_\alpha(-k,-\omega) = A^*_\alpha(k,\omega). \qquad (2.14)$$

3 Variational principle and normal variables

We can easily check that the symmetry properties of $\epsilon^{(0)}_{\alpha\beta}$, $f_{\alpha\beta\gamma}$ and $g_{\alpha\beta\gamma\delta}$ in the transparent media allow us to rewrite the equation (2.13) in a variational form,

$$\frac{\delta S}{\delta A^*_\alpha} = 0, \qquad (3.1)$$

where the action S can be given by a non-local functional on $A_\alpha(k,\omega)$,

$$S = \frac{1}{2}\int \left(k^2\,\delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2}\,\epsilon^0_{\alpha\beta}(k,\omega)\right)\,A^*_\alpha(k,\omega)\,A_\beta(k,\omega)\,dk\,d\omega\,+$$
$$+\ \frac{i}{3}\int \omega\omega_1\omega_2\, f_{\alpha\beta\gamma}(k,\omega,k_1,\omega_1,k_2,\omega_2)\,A_\alpha(k,\omega)\,A_\beta(k_1,\omega_1)\,A_\gamma(k_2,\omega_2)\,\times$$
$$\times\delta_{k+k_1+k_2}\,\delta_{\omega+\omega_1+\omega_2}\,dk\,dk_1\,dk_2\,d\omega\,d\omega_1\,d\omega_2\,-$$
$$-\ \frac{1}{4}\int \omega\omega_1\omega_2\omega_3\, g_{\alpha\beta\gamma\delta}(k,\omega,k_1,\omega_1,k_2,\omega_2,k_3,\omega_3)\,A_\alpha(k,\omega)\,A_\beta(k_1,\omega_1)\,A_\gamma(k_2,\omega_2)\,A_\delta(k_3,\omega_3)\,\times$$
$$\times\delta_{k+k_1+k_2+k_3}\,\delta_{\omega+\omega_1+\omega_2+\omega_3}\,dk\,dk_1\,dk_2\,dk_3\,d\omega\,d\omega_1\,d\omega_2\,d\omega_3. \qquad (3.2)$$

In the general case, it is not possible to develop a regular Hamiltonian dynamics for a non-local functional. However, in some cases the nonlocality is even convinient. Operating with the non-local action functional, we can broadly extend the class of admittable transformations. In Hamiltonian dynamics, it is admittable only the canonical transformation that preserves a simplest form of a simplectic structure. In our case, we can study absolutely general transformations from initial variables $A_\alpha(k,\omega)$ to any arbitrary new variables $B_\alpha(k,\omega)$. We choose these new variables depending on the physical problem, which is solved. In a sense, this is a question of ability and skill of the analyst. Still, some simplification principles for the action S and the Maxwell equations can be formulated in the general form.

Let us consider the linearized equation

$$\left(k^2\,\delta\alpha\beta - k_\alpha k_\beta - \frac{\omega^2}{c^2}\,\epsilon^{(0)}_{\alpha\beta}(k,\omega)\right) A_\beta(k,\omega) = 0. \tag{3.3}$$

The general solution of this equation can be found in the following form:

$$A_\alpha(k\omega) = \frac{A^0_\alpha(k)}{\omega}\delta(\omega) + \sum_{n=-N}^{N} A^{(n)}_\alpha(k)\,\delta\left(\omega - \omega^{(n)}(k)\right). \tag{3.4}$$

Here the first term is a constant potential electric field, which can exist in a dielectric when $\epsilon^{(0)}_{ij}(0) = const$. In plasmas, metals, and superconductors $A^{(0)}_\alpha(k) = 0$. The second term in (3.4) corresponds to electromagnetic waves.

Further, from the condition $A^*_\alpha(-k,-\omega) = A_\alpha(k,\omega)$, we obtain

$$\begin{aligned} A^{0*}_\alpha(-k) &= A^0_\alpha(k), \\ \omega^{-n}(k) &= -\omega^n(-k), \\ A^{-n}_\alpha(k) &= A^{n*}_\alpha(k). \end{aligned} \tag{3.5}$$

Thereafter, we assume that the medium is invariant with respect to the reflection of coordinates. This implies that

$$\omega^{+n}(-k) = \omega^n(k), \quad \omega^{-n}(k) = -\omega^n(k). \tag{3.6}$$

We can consider that $\omega^{\pm(n)}(k)$ belongs to the same branch of oscillations. For the total number of branches, N, the minimal value is $N = 2$. This case corresponds to electromagnetic waves of different polarization in dielectric, in absence of non-decaying optical oscillations. In plasmas, metals, and semiconductors as well as in dielectrics, N can be arbitrary large, depending on complexity of situation. For instance, in the magnetized plasma $N = 7$, in the isotropic plasma $N = 3$.

In the general case, we can perform a decomposition,

$$A_\alpha(k\omega) = C_\alpha(k\omega) + C_\alpha^*(-k,-\omega), \tag{3.7}$$

where

$$\begin{aligned} C_\alpha(k,\omega) &= A_\alpha(k,\omega)\,\Theta(\omega-\omega_k), \\ \Theta &= \begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0. \end{cases} \end{aligned} \tag{3.8}$$

The action S can now be represented in the following form:

$$\begin{aligned} S &= S_2 + S_3 + S_4, \\ S_2 &= \int_{\omega>0} \left[k^2\delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2}\,\epsilon^{(0)}_{\alpha\beta}(k,\omega)\right] C_\alpha^*(k,\omega)\,C_\beta(k,\omega)\,dk\,d\omega, \quad (3.9) \\ S_3 &= S_3^{(1)} + S_3^{(2)}, \\ S_4 &= S_4^{(1)} + S_4^{(2)} + S_4^{(3)}. \end{aligned} \tag{3.10}$$

Here

$$\begin{aligned} S_3^{(1)} &= \frac{1}{3}\int_{\omega_i>0} F^{(1)}_{\alpha\beta\gamma}(\omega\omega_1\omega_2, kk_1k_2)\,[C_\alpha(k,\omega)\,C_\beta(k_1,\omega_1)\,C_\gamma(k_2,\omega_2) \\ &+ C_\alpha^*(k,\omega)\,C_\beta^*(k_1,\omega_1)\,C_\gamma^*(k_2,\omega_2)\big]\,\delta_{k+k_1+k_2}\delta_{\omega+\omega_1+\omega_2}dk\,dk_1\,dk_2\,d\omega\,d\omega_1\,d\omega_2, \\ F^{(1)}_{\alpha\beta\gamma} &= i\omega\,\omega_1\,\omega_2\,f_{\alpha\beta\gamma}(k,\omega,k_1,\omega_1,k_2,\omega_2), \end{aligned} \tag{3.11}$$

$$\begin{aligned} S_3^{(2)} &= \int_{\omega_i>0} F^{(2)}_{\alpha\beta\gamma}(\omega\omega_1\omega_2, kk_1k_2)\,[C_\alpha^*(k,\omega)\,C_\beta(k_1,\omega_1)\,C_\gamma(k_2,\omega_2) + \\ &+ C_\alpha(k,\omega)\,C_\beta^*(k_1,\omega_1)\,C_\gamma^*(k_2,\omega_2)\big]\,\delta_{k-k_1-k_2}\delta_{\omega-\omega_1-\omega_2}dk\,dk_1\,dk_2\,d\omega\,d\omega_1\,d\omega_2, \\ F^{(2)}_{\alpha\beta\gamma} &= -i\omega\,\omega_1\,\omega_2\,f_{\alpha\beta\gamma}(-k,-\omega,k_1,\omega_1,k_2,\omega_2), \end{aligned} \tag{3.12}$$

$$\begin{aligned} S_4^{(1)} &= \frac{1}{4}\int_{\omega_i>0} F^{(1)}_{\alpha\beta\gamma\delta}(\omega\omega_1\omega_2\omega_3, kk_1k_2k_3)\,[C_\alpha(k,\omega)\,C_\beta(k_1,\omega_1)\,C_\gamma(k_2,\omega_2)\,C_\delta(k_3,\omega_3) + \\ &\quad + C_\alpha^*(k,\omega)\,C_\beta^*(k_1,\omega_1)\,C_\gamma^*(k_2,\omega_2)\,C_\delta^*(k_3,\omega_3)\big] \times \\ &\quad \times \delta_{\omega+\omega_1+\omega_2+\omega_3}\delta_{k+k_1+k_2+k_3}d\omega\,d\omega_1\,d\omega_2\,d\omega_3\,dk\,dk_1\,dk_2\,dk_3, \qquad (3.13) \\ F^{(1)}_{\alpha\beta\gamma\delta}&(\omega\omega_1\omega_2\omega_3, kk_1k_2k_3) = -\omega\omega_1\omega_2\omega_3\,g_{\alpha\beta\gamma\delta}(k,\omega,k_1,\omega_1,k_2,\omega_2,k_3,\omega_3), \end{aligned}$$

$$S_4^{(2)} = \frac{1}{3}\int_{\omega_i>0} F^{(2)}_{\alpha\beta\gamma\delta}(\omega\omega_1\omega_2\omega_3, kk_1k_2k_3)\,\big[C^*_\alpha(k,\omega)\,C_\beta(k_1,\omega_1)\,C_\gamma(k_2,\omega_2)\,C_\delta(k_3,\omega_3)+$$
$$+C_\alpha(k,\omega)\,C^*_\beta(k_1,\omega_1)\,C^*_\gamma(k_2,\omega_2)\,C^*_\delta(k_3,\omega_3)\big]\times$$
$$\times\delta_{\omega-\omega_1-\omega_2-\omega_3}\delta_{k-k_1-k_2-k_3}d\omega\,d\omega_1\,d\omega_2\,d\omega_3\,dk\,dk_1\,dk_2\,dk_3, \tag{3.14}$$
$$F^{(2)}_{\alpha\beta\gamma\delta}(\omega\omega_1\omega_2\omega_3, kk_1k_2k_3) = 3\omega\omega_1\omega_2\omega_3\,g_{\alpha\beta\gamma\delta}(-k,-\omega,k_1,\omega_1,k_2,\omega_2,k_3,\omega_3),$$

$$S_4^{(3)} = \frac{1}{2}\int_{\omega_i>0} F^{(3)}_{\alpha\beta\gamma\delta}(\omega\omega_1\omega_2\omega_3, kk_1k_2k_3)\,C^*_\alpha(k,\omega)\,C^*_\beta(k_1,\omega_1)\,C_\gamma(k_2,\omega_2)\,C_\delta(k_3,\omega_3)\times$$
$$\times\delta_{\omega+\omega_1-\omega_2-\omega_3}\delta_{k+k_1-k_2-k_3}d\omega\,d\omega_1\,d\omega_2\,d\omega_3\,dk\,dk_1\,dk_2\,dk_3, \tag{3.15}$$
$$F^{(3)}_{\alpha\beta\gamma\delta}(\omega\omega_1\omega_2\omega_3, kk_1k_2k_3) = -6\omega\omega_1\omega_2\omega_3\,g_{\alpha\beta\gamma\delta}(-k,-\omega,-k_1,-\omega_1,k_2,\omega_2,k_3,\omega_3).$$

In (3.9)-(3.15), we integrate along the positive frequencies only. We should stress once more that the transparency takes place in some limited band of frequencies,

$$\omega_{min} < \omega < \omega_{max}.$$

For the further simplification of the action, we should mention that in the transparent medium the matrix

$$L_{\alpha\beta} = k^2\delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2}\,\epsilon^{(0)}_{\alpha\beta}(k,\omega), \tag{3.16}$$

is Hermitian for all k,ω,

$$L_{\alpha\beta} = L^*_{\beta\alpha},$$

and can be transformed to a diagonal form by some unitary transformation,

$$B = UA, \quad UU^+ = I.$$

The action S can be represented in new variables as

$$S = \sum_{i=1}^{3}\int_{\omega>0}\lambda_i(k,\omega)\,|B_i(k,\omega)|^2\,dk\,d\omega, \tag{3.17}$$

where $\lambda_i(k,\omega)$ are eigenvalues of $L_{\alpha\beta}$. In the linear approximation, the motion equations are

$$\lambda_i(k,\omega)\,B_i(k,\omega) = 0, \tag{3.18}$$

therefore each eigenvalue λ_i can be written in the following way:

$$\begin{aligned}
\lambda_1(k,\omega) &= l_1(k,\omega)\,(\omega-\omega_1(k))\cdots(\omega-\omega_p(k))\,,\\
\lambda_2(k,\omega) &= l_2(k,\omega)\,(\omega-\omega_{p+1}(k))\cdots(\omega-\omega_{p+q}(k))\,,\\
\lambda_3(k,\omega) &= l_3(k,\omega)\,(\omega-\omega_{p+q+1}(k))\cdots(\omega-\omega_N(k))\,. \qquad (3.19)
\end{aligned}$$

Here $\omega_i(k)$ are normal modes. In general, the positions of all normal modes are different. If degeneration takes place and $\omega_l(k)=\omega_m(k)$, $i\neq j$, then factors $\omega-\omega_l(k)$ and $\omega-\omega_m(k)$ must be divisors of different eugenvalues.

Let us choose an eigenvalue λ_i and divide the half-axis, $0<\omega<\infty$, into intervals such that each interval contains one normal mode $\omega_l(k)$ only. Each of these intervals makes some contribution to the action S_2. In other words,

$$S_2=\sum_{n=0}^{N}\int dk\int_{\omega_n^-}^{\omega_n^+} d\omega\, r_n(k,\omega)\,(\omega-\omega_n(k))\,|B_{in}(k,\omega)|^2, \quad r_n\neq 0. \qquad (3.20)$$

The total sum of all intervals $(\omega_n^-,\,\omega_n^+)$ covers the half-axis three times. Inside of each interval, the function $r_n(k,\omega)$ has a definite sign.

Now, we can introduce new variables,

$$a_n(k,\omega)=\frac{B_{in}(k,\omega)}{\sqrt{|r_n(k,\omega)|}}\,,$$

and obtain finally

$$S_2=\sum_{n=0}^{N}\int dk\int_{\omega_n^-}^{\omega_n^+}(-1)^{\lambda_n}\,(\omega-\omega_n(k))\,|a_n(k)|^2 d\omega. \qquad (3.21)$$

Formula (3.21) is a canonic form for the quadratic part of the action S in a nonlinear medium. In the dielectric $\omega_0(k)=0$, as well as $\omega_0^-=0$. In the isotropic plasma $\omega_0^-=\omega_{pl}$, where ω_{pl} is a Langmuir plasma frequency.

We call the variables $a_n(k)$ normal variables. If we introduce new variables, $a_n^s(k)$, $s=\pm1$, such that $a_n^1(k)=a_n(k)$, $a_n^{-1}(k)=a_n^*(k)$, then the cubic and the quartic parts of the action S take more compact form:

$$\begin{aligned}
S_3 = \frac{1}{3}\int & V_{nmp}^{ss_1s_2}(\omega\omega_1\omega_2,kk_1k_2)\delta_{sk+s_1k_1+s_2k_2}\delta_{s\omega+s_1\omega_1+s_2\omega_2}\times\\
&\times a_n^s(k,\omega)\,a_m^{s_1}(k_1,\omega_1)\,a_p^{s_2}(k_2,\omega_2)\,dk\,dk_1\,dk_2\,d\omega\,d\omega_1\,d\omega_2, \qquad (3.22)
\end{aligned}$$

$$\begin{aligned}
S_4 = \frac{1}{4}\int & W_{nmpl}^{ss_1s_2s_3}(\omega\omega_1\omega_2\omega_3,kk_1k_2k_3)\delta_{sk+s_1k_1+s_2k_2+s_3k_3}\delta_{s\omega+s_1\omega_1+s_2\omega_2+s_3\omega_3}\times\\
\times & a_n^s(k,\omega)a_m^{s_1}(k_1,\omega_1)a_p^{s_2}(k_2,\omega_2)a_l^{s_3}(k_3,\omega_3)\,dk\,dk_1\,dk_2\,dk_3\,d\omega\,d\omega_1\,d\omega_2\,d\omega_3 \qquad (3.23)
\end{aligned}$$

The coefficients $V^{ss_1s_2}_{nmp}$ and $W^{ss_1s_2s_3}_{nmpl}$ can be easily expressed through $F^a_{\alpha\beta\gamma}$ and $F^b_{\alpha\beta\gamma\delta}$.

In the normal variables the Maxwell equations take the form

$$(\omega-\omega_k)\,a_{k\omega}=\frac{\delta H}{\delta a^*_{k\omega}},$$
$$H=(-1)^{a_l}(S_3+S_4), \tag{3.24}$$

which can be called the generalized Hamiltonian equations. These equations are not rigorously Hamiltonian. The functional H after the Fourier transformation, $\omega\to i\frac{\partial}{\partial t}$, becomes non-local in time. Nevertheless, the equations (3.24) do inherit many important properties of the Hamiltonian systems.

For example, we can perform transformations, which preserve the linear part of equation (3.24):

$$\begin{aligned}
a^{(n)}_{k\omega}=b^{(n)}_{k\omega} \;&+\; \int A_{nmp}(\omega,\omega_1,\omega_2,k,k_1,k_2)\,b^m(k_1,\omega_1)\,b^p(k_2,\omega_2)\times\\
&\qquad\qquad\times\delta_{\omega-\omega_1-\omega_2}\delta_{k-k_1-k_2}d\omega_1\,d\omega_2\,k_1\,k_2+\\
&+\; \int B_{nmp}(\omega,\omega_1,\omega_2,k,k_1,k_2)\,b^{*m}(k_1,\omega_1)\,b^p(k_2,\omega_2)\times\\
&\qquad\qquad\delta_{\omega+\omega_1-\omega_2}\delta_{k+k_1-k_2}dk_1\,dk_2\,d\omega_1\,d\omega_2+\\
&+\; \int C_{nmp}(\omega,\omega_1,\omega_2,k,k_1,k_2)\,b^{*m}(k_1,\omega_1)\,b^{*p}(k_2,\omega_2)\times\\
&\qquad\qquad\times\delta_{\omega+\omega_1+\omega_2}\delta_{k+k_1+k_2}dk_1\,dk_2\,d\omega_1\,d\omega_2.
\end{aligned} \tag{3.25}$$

Here A, B and C are arbitrary coefficients obeying trivial symmetry relations.

Transformation (3.25) is a substitute of canonical transformations in Hamiltonian dynamics. However, this class of transformations is much broader than the class of canonical transformations defined by a single generating functional. In particular, the coefficients A_{nmp} and B_{nmp} for canonical transformations are connected. Plugging (3.25) to S we can try to simplify the cubic and the quartic parts of the action. Moreover, we can try to eliminate the cubic terms. This procedure leads to appearance of resonant denominators,

$$\frac{\delta_{k-k_1-k_2}\delta_{\omega-\omega_1-\omega_2}}{\omega^n_k-\omega^m_{k_1}-\omega^p_{k_2}}. \tag{3.26}$$

If the denominators are not zero, the elimination of cubic terms is possible; but we will not discuss this interesting question now. Another interesting question is more mathematical. Is it possible to transform the nonlocal generalized Hamiltonian H to a classical, local in time Hamiltonian by a proper choice of variables? In some important cases, as an example, for the plasma described by hydrodynamic equations, it is certainly possible. However, these questions are out of the scope of this article.

4 Isotropic medium with Kerr-type nonlinearity

In the most general case of a transparent isotropic medium, the linear parts of electrical induction and electric field are connected by the expession

$$\vec{D}_k = \epsilon^{tr}\,\vec{E}_k + (\epsilon^l - \epsilon^{tr})\frac{\vec{k}}{k^2}(\vec{k}\vec{E}) + \frac{i\gamma}{|k|}[\vec{k},\vec{E}], \tag{4.1}$$

where ϵ^{tr}, ϵ^l and γ are real functions. We suppose that they depend on frequency ω only.

According to (4.1), the quadratic part of the action takes the following form:

$$S_2 = \int_0^\infty d\omega \int dk \left\{ \left(k^2 - \frac{\omega^2}{c^2}\,\epsilon^{tr}(\omega)\right)|\vec{A}_k|^2 - \right.$$
$$\left. - \left(1 + \frac{\omega^2}{c^2}\frac{\epsilon^l(\omega) - \epsilon^{tr}(\omega)}{k^2}\right)\left(|(\vec{k}\vec{A}_k)|^2 + \frac{i\gamma_k}{k}(\vec{A}^*[\vec{k},\vec{A}])\right)\right\}. \tag{4.2}$$

Now, we assume that the cubic part of the action vanishes and consider a simple quartic action,

$$S_4 = -\frac{1}{2}\int_{\omega_i} \omega\omega_1\omega_2\omega_3 \left\{\alpha(\vec{A}_k\vec{A}^*_{k_1})(\vec{A}_{k_2}\vec{A}^*_{k_3}) + \beta(\vec{A}_{k_1}\vec{A}_{k_2})(\vec{A}^*_{k_1}\vec{A}^*_{k_3})\right\} \times$$
$$\times\delta_{k-k_1+k_2-k_3}\delta_{\omega-\omega_1+\omega_2-\omega_3}\, dk\, dk_1\, dk_2\, dk_3\, d\omega\, d\omega_1\, d\omega_2\, d\omega_3, \tag{4.3}$$

where α, β are real constants. This is the most simple action in an isotropic medium with instant nonlinearity and absence of spatial dispersion.

Let us introduce the vector field $\vec{S}(k)$ satisfying conditions

$$i[\vec{k},\vec{S}] = |k|\vec{S}, \;\; \vec{S}(-k) = \vec{S}(k), \;\; |\vec{S}(k)|^2 = 1. \tag{4.4}$$

We can represent the electric potential as

$$A = A^+(k,\omega)\vec{S}(k) + A^-(k,\omega)\vec{S^*}(k) + A^0(k,\omega)\frac{\vec{k}}{k} \tag{4.5}$$

and obtain for the quadratic part of the action the following expression:

$$S_2 = \int d\omega \int d\vec{k} \left\{ \left(k^2 - \frac{\omega^2}{c^2}\epsilon^+\right)|A^+|^2 + \left(k^2 - \frac{\omega^2}{c^2}\epsilon^-\right)|A^-|^2 + \right.$$
$$\left. + \frac{\omega^2}{c^2}\epsilon^l|A^0|^2 \right\}. \tag{4.6}$$

Here $\epsilon^\pm(\omega) = \epsilon^{tr}(\omega) \pm \gamma(\omega)$; if $\gamma \neq 0$, the medium is birefringerant. The components $A^\pm$ are amplitudes of circular polarized waves and A^0 is a longitudinal wave. If $\epsilon^l(\omega) \neq 0$, the wave is a "slave" wave. It appears as a result of interaction of transverse waves.

From (4.6) we obtain

$$(\vec{A}_{k_1\omega_1}, \vec{A}_{k_2\omega_2}) = \left(\vec{S}(k_1)\vec{S^*}(k_2)\right) A^+_{k_1\omega_1}A^-_{k_2\omega_2} + \vec{S^*}(k_1)\vec{S}(k_2)A^-_{k_1\omega_1}A^+_{k_2\omega_2} +$$
$$+ \left(\vec{S}(k_1)\vec{S}(k_2)\right) A^+_{k_1\omega_1}A^+_{k_2\omega_2} + \left(\vec{S^*}(k_1)\vec{S^*}(\vec{k_2})\right) A^-_{k_1\omega_1}A^-_{k_2\omega_2} +$$
$$+ \frac{A^0(k_1\omega_1)}{k_1}\left(\vec{k_1}, \vec{S}(k_2)A^+(k_2) + \vec{S^*}(k_2)A^-(k_2)\right) + \tag{4.7}$$
$$+ \frac{A^0(k_2,\omega_2)}{k_2}\left(\vec{k_2}, \vec{S}(k_1)A^+(k_1) + \vec{S^*}(k_1)A^-(k_1)\right) + \frac{(k_1k_2)}{k_1k_2}A^0(k_1\omega_1)A^0(k_2\omega_2),$$

$$(\vec{A}_{k_1\omega_1}, \vec{A^*}_{k_2\omega_2}) = \left(\vec{S}(k_1)\vec{S^*}(k_2)\right) A^+_{k_1\omega_1}A^{+*}_{k_2\omega_2} + \vec{S^*}(k_1)\vec{S}(k_2)A^-_{k_1\omega_1}A^{-*}_{k_2\omega_2} +$$
$$+ \left(\vec{S}(k_1)\vec{S}(k_2)\right) A^+_{k_1\omega_1}A^{-*}_{k_2\omega_2} + \left(\vec{S^*}(k_1)\vec{S^*}(k_2)\right) A^{+*}_{k_1\omega_1}A^-_{k_2\omega_2} +$$
$$+ \frac{A^0(k_1\omega_1)}{k_1}\left(\vec{k_1}, \vec{S}(k_2)A^+(k_2) + \vec{S^*}(k_2)A^-(k_2)\right) + \tag{4.8}$$
$$+ \frac{A^{0*}(k_2,\omega_2)}{k_2}\left(\vec{k_2}, \vec{S}(k_2)A^+(k_2) + \vec{S^*}(k_2)A^-(k_2)\right) + \frac{(\vec{k_1}\vec{k_2})}{k_1k_2}A^0(k_1\omega_1)A^{0*}(k_2\omega_2).$$

By substituting (4.7),(4.8) to (4.3), we can express S_4 in terms of $A^\pm(k,\omega)$, $A^0(k,\omega)$. Let us suppose that the waves are almost monochromatic. This means that $\vec{A}(k,\omega)$ is supported at

$$\begin{aligned} \vec{k} &= k_0(\vec{n_3} + \epsilon\vec{\kappa}), \quad \epsilon \ll 1, \\ \omega &= \omega_0(1 + \epsilon\eta), \end{aligned} \tag{4.9}$$

only. Further, we can expand the vector field $\vec{S}(k)$ in powers of ϵ,

$$\vec{S}(k) = \vec{S_0} + \epsilon \vec{S_1} + \epsilon^2 \vec{S_2} + \cdots , \tag{4.10}$$

where

$$\vec{S_0} = \frac{1}{\sqrt{2}}(\vec{n_1} + i\vec{n_2}),$$

$$\vec{S_1} = -\frac{1}{\sqrt{2}}(\kappa^{(1)} + i\kappa^{(2)})\vec{n_3} + \frac{i\kappa_3}{2\sqrt{2}}(\vec{n_1} - i\vec{n_2}). \tag{4.11}$$

In zero order of ϵ we can put $A^0 = 0$. Then, in *this* approximation,

$$\begin{aligned} (\vec{A}_{k_1\omega_1}, \vec{A}_{k_2\omega_2}) &= A^+_{k_1\omega_1} A^-_{k_2\omega_2} + A^-_{k_1\omega_1} A^+_{k_2\omega_2}, \\ (\vec{A}_{k_1\omega_1}, \vec{A^*}_{k_2\omega_2}) &= A^+_{k_1\omega_1} A^{+*}_{k_2\omega_2} + A^-_{k_1\omega_1} A^{-*}_{k_2\omega_2}, \end{aligned} \tag{4.12}$$

and for the quartic part of the action S we obtain the following expression:

$$\begin{aligned} S_4 = -\frac{1}{2}\omega_0^4 \int \alpha \Big[&(A^+_{k_1\omega_1} A^{+*}_{k_2\omega_2} + A^-_{k_1\omega_1} A^{-*}_{k_2\omega_2})\,(A^{+*}_{k_3\omega_3} A^+_{k_4\omega_4} + A^{-*}_{k_3\omega_3} A^-_{k_4\omega_4}) \times \\ &\times \delta_{k_1-k_2-k_3+k_4} \delta_{\omega_1-\omega_2-\omega_3+\omega_4} + \\ &+ \beta (A^+_{k_1\omega_1} A^-_{k_2\omega_2} + A^-_{k_1\omega_1} A^+_{k_2\omega_2})(A^{+*}_{k_3\omega_3} A^{-*}_{k_4\omega_4} + A^{-*}_{k_3\omega_3} A^{+*}_{k_4\omega_4}) \times \\ &\times \delta_{k_1+k_2-k_3-k_4} \delta_{\omega_1+\omega_2-\omega_3-\omega_4} \Big]\, dk_1\, dk_2\, dk_3\, dk_4\, d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4. \end{aligned} \tag{4.13}$$

In this local approximation, the Maxwell equations describing the media can be reduced to the system of nonlinear Schrödinger equations, which was first derived by Zakharov and Schulman[4]. To obtain a specific form of the equation, we must choose one variable as the efficient time. In the original paper[5], the physical time plays this role.

The variational approach formulated in this article makes it possible to derive a correction to the NSLE model in a regular way. The detailed description of these calculations will be published in a separated paper.

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