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Generalized Hardy Operators and Normalizing Measures

TIELING CHEN and GORD SINNAMON

Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7, Canada

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Necessary and sufficient conditions on the weight v and the measure σ for the operator

$$Kf(s) = \int_{a(s)}^{b(s)} k(s,y)f(y)dy$$

to be bounded from $L_v^p[0,\infty)$ to $L_\sigma^q(S)$ are given. Here $a(s)$ and $b(s)$ are similarly ordered functions and $k(s,y)$ satisfies a modified GHO condition. Nearly block diagonal decompositions of positive operators are introduced as is the concept of a normalizing measure. An application is made to estimates for the remainder in a Taylor approximation.

Keywords: Hardy inequality; Weight; Normalizing measure

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1 INTRODUCTION: MONOTONICITY

Generalized Hardy Operators are instances of integral operators having non-negative kernels:

$$Tf(s) = \int_0^\infty k(s,y)f(y)dy.$$

*Corresponding author. E-mail: sinnamon@uwo.ca

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Since the early 1970s there has been continual progess on the following question:

Between which weighted Lebesgue spaces is T a bounded operator (1.1)

Underlying the successes of the last 25 y has been the exploitation of the monotonicity of the kernel k. The kernel in Hardy's integral operator is $k(s,y) = \mathcal{X}_{(0,s)}(y)$ which is non-decreasing in s and non-increasing in y. The generalized Stieljtes kernel $k(s,y) = (s+y)^{-\lambda}$ and the Riemann–Liouville kernel $k(s,y) = \mathcal{X}_{(0,s)}(s-y)^{\lambda}$, $\lambda > 0$, are also monotone in each variable. See [9] for references. The above question has been answered for Generalized Hardy Operators, those whose kernel $k(s,y)$ is supported in $\{(s,y) : 0 \le y \le s\}$ and satisfies the GHO condition:

$$D^{-1}k(s,y) \le k(s,t) + k(t,y) \le Dk(s,y) \quad \text{for} \quad y \le t \le s.$$

Here D is some fixed positive constant. This condition, imposed in [1] and [7] and later in [2, 3, 5, 8, 12] was sometimes accompanied by (superfluous) monotonicity conditions. However, it is largely a monotonicity condition itself as we will see in Lemma 2.2 below.

Recently, Question (1.1) has been answered for some operators whose kernels are not monotone. This is a important step, especially since the necessary and sufficient conditions given have retained the simple character of those given for previously studied operators. The new operators include variable limits on the defining integral, essentially restricting the support of the kernel to the region between two curves. In [4], Question (1.1) was resolved for the operator

$$\int_{a(s)}^{b(s)} f(y)dy$$

with a and b smooth functions on $[0,\infty)$ which increase from 0 to ∞ with s. The paper [3] looks at the more general operator

$$Kf(s) = \int_{a(s)}^{b(s)} k(s,y)f(y)dy$$

with a and b non-decreasing but not necessarily smooth and k satisfying a modified GHO condition. The boundedness of K is established between certain Banach function spaces including the weighted Lebesgue spaces $K : L^p_v[0, \infty) \to L^q_u[0, \infty)$ for $p \leq q$ but not for $q < p$. The case $q < p$ was the difficult case in [4] and necessitated the introduction there of the concept of a normalizing function.

In this paper we answer (1.1) for the operator K in the case $q < p$. We also drop the monotonicity assumptions on a and b and as a result we are able to take the variable s off the half line and allow it to be in a general measure space. We explore the normalizing function concept further, placing it in the more general and more natural context of normalizing measures. We examine the GHO condition in some depth, showing its connection with monotonicity assumptions and formulating it for use when s is in a general measure space.

An orderly presentation of this investigation requires that we begin with our look at the GHO condition and prove some needed results over general measure spaces. This is done in Section 2. Section 3 contains technical results on nearly block diagonal decomposition of operators with positive kernels. These results are quite generally applicable and may be of independent interest. In Section 4 we define normalizing measures and use a block diagonal decomposition to prove our main result—giving necessary and sufficient conditions for K to be bounded from $L^p_v[0, \infty)$ to $L^q_\sigma(S)$ for an arbitrary measure space (S, σ). The existence of normalizing measures for a large class of pairs (a, b) is established in Section 5 where we also see the interesting form taken by what remains of our monotonicity assumptions. The final section is a brief presentation of the application of these results to approximation by Taylor polynomials. The integral form of the Taylor remainder is readily recognized as one of the operators we have been studying.

The notation of the paper is standard. The harmonic conjugate of the Lebesgue index p is denoted p' so that $1/p + 1/p' = 1$. Weight functions are non-negative and allowed to take the value ∞. As usual, $0.\infty = 0$. The supremum of the empty set is taken to be zero. Integrals with limits are assumed to include the endpoints when possible so that

$$\int_a^b = \int_{[a,b]} \quad \text{but} \quad \int_a^\infty = \int_{[a,\infty)},$$

The expression "A is comparable to B," written $A \approx B$, means that there are positive constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. If $X_0 \subset X$ then counting measure on X_0 is the measure defined on the σ-algebra of all subsets of X whose value on E is just $\sharp(E \cap X_0)$, the number of elements in $E \cap X_0$.

2 THE CASE $a(s) = 0$

The operators we consider in this section take the function $f(y), y \in [0, \infty)$ to the function $Kf(s), s \in S$, with the formula

$$Kf(s) = \int_0^{b(s)} k(s, y) f(y) dy.$$

Here (S, σ) is an arbitrary measure space, $b : S \to [0, \infty)$ is σ-measurable, and $k : S \times [0, \infty) \to [0, \infty)$ satisfies the GHO condition given in Definition 2.1 below. The main result of this section, Theorem 2.6, gives simple integral conditions on k, b, υ and σ which are necessary and sufficient for the operator K to be bounded as a map from $L^p_\upsilon[0, \infty)$ to $L^q_\sigma(S)$.

DEFINITION 2.1 *Suppose that (S, σ) is a measure space and $b : S \to [0, \infty)$ is σ-measurable. A kernel k satisfies the* GHO condition *on $\{(s, y) : 0 \leq y \leq b(s)\}$ provided there exists a $D \geq 1$ such that*

$$D^{-1} k(s, y) \leq k(s, b(t)) + k(t, y) \leq D k(s, y) \textit{ for } y \leq b(t) \leq b(s) \quad (2.1)$$

and

$$D^{-1} k(s, y) \leq k(s, w) \leq D k(s, y) \textit{ for } y \leq w \leq b(s), w \notin b(S). \quad (2.2)$$

If $S = [0, \infty)$ and $b(s) = s$ then the case (2.2) does not arise and we see that this definition agrees with the usual GHO condition.

LEMMA 2.2 *Suppose (S, σ) is a measure space, $b : S \to [0, \infty)$ is σ-measurable and k satisfies the GHO condition on $\{(s, y), 0 \leq y \leq b(s)\}$. Then there exists a kernel l satisfying the GHO condition on $\{(x, z) : 0 \leq z \leq x\}$ such that $l(x, z)$ is non-decreasing in x, $l(x, z)$ is non-increasing in z, and $k(s, y) \approx l(b(s), y)$ for $0 \leq y \leq b(s)$.*

Proof Define $l : \{(x,z) : 0 \le z \le x\} \to [0,\infty]$ by

$$l(x,z) = \sup\{k(t,y) : z \le y \le b(t) \le x\}. \tag{2.3}$$

It is clear that $l(x,z)$ is non-decreasing in x and non-increasing in z. It is also clear that $k(s,y) \le l(b(s),y)$ whenever $0 \le y \le b(s)$. Let D be the constant in the GHO condition satisfied by k. If we show that $l(b(s),z) \le D^2 k(s,z)$ whenever $0 \le z \le b(s)$ we will have shown that $k(s,y) \approx l(b(s),y)$. To this end, fix $z \ge 0$ and $s \in S$ such that $z \le b(s)$ and suppose that $y \ge 0$ and $t \in S$ satisfy $z \le y \le b(t) \le b(s)$. First observe that $k(t,y) \le Dk(s,y)$ by the second inequality in (2.1). If $y \notin b(S)$ we have $k(s,y) \le Dk(s,z)$ by the second inequality in (2.2) but if $y \in b(S)$, say $y = b(t_1)$, then $k(s,y) = k(s,b(t_1)) \le Dk(s,z)$ by the second inequality in (2.1). In either case we have $k(t,y) \le Dk(s,y) \le D^2 k(s,z)$ and, taking the supremum over all y and t we get $l(b(s),z) \le D^2 k(s,z)$ as required.

To complete the proof it remains to show that l satisfies the GHO condition on $\{(x,z) : 0 \le z \le x\}$. To do this it is enough to show that

$$D^{-1} l(x,z) \le l(x,w) + l(w,z) \le 2l(x,z) \quad \text{for } 0 \le z \le w \le x. \tag{2.4}$$

The monotonicity of l, already established, proves the second inequality in (2.4). To prove the first we suppose that y and t satisfy $z \le y \le b(t) \le x$ and show that

$$k(t,y) \le D(l(x,w) + l(w,z)) \tag{2.5}$$

whenever $z \le w \le x$ by looking at four cases.

Case 1 $z \le y \le b(t) \le w \le x$. The definition of l yields $k(t,y) \le l(w,z)$ so (2.5) holds. (Recall that $D \ge 1$.)

Case 2 $z \le w \le y \le b(t) \le x$. The definition of l shows that $k(t,y) \le l(x,w)$ so again (2.5) holds.

Case 3 $z \leq y \leq w \leq b(t) \leq x$ and $\omega \notin b(S)$. By the first inequality in (2.2), $k(t,y) \leq Dk(t,w)$ and by the definition of l, $k(t,w) \leq l(x,w)$ so we have $k(t,y) \leq Dl(x,w)$ and (2.5) follows.

Case 4 $z \leq y \leq w \leq b(t) \leq x$ and $w = b(s)$ for some $s \in S$. The first inequality in (2.2), with s and t interchanged, shows that $k(t,y) \leq D(k(t,b(s)) + k(s,y))$. The definition of l, used twice, shows that $k(t,b(s)) \leq l(x,w)$ and $k(s,y) \leq l(w,z)$ so in this case too we have (2.5).

Taking the supremum over all t and y satisfying $z \leq y \leq b(t) \leq x$, (2.5) becomes $l(x,z) \leq D(l(x,w) + l(w,z))$ which completes the proof of (2.4) and the lemma.

Lemma 2.2 permits us to move from the kernel k depending on the variable $s \in S$ to a kernel l defined in the familiar triangle $\{(x,y) : 0 \leq y \leq x\}$. We must also be able to move from the measure σ on S to a measure on $[0,\infty)$ and, in order to apply Stepanov's results on Generalized Hardy Operators, from there to weight functions on $[0,\infty)$. Somewhat surprisingly, the latter move proves to be more problematic than the former.

LEMMA 2.3 *Suppose (S,σ) is a measure space and $b : S \to [0,\infty)$ is σ-measurable. Then there exists a measure μ defined on the Borel subsets of $[0,\infty)$ and satisfying*

$$\int_{[0,\infty)} F(x)d\mu(x) = \int_S F(b(s))d\sigma(s) \tag{2.6}$$

for every Borel measurable function $F : [0,\infty) \to [0,\infty)$.

Proof Since b is σ-measurable, $b^{-1}(E)$ is σ-measurable for every Borel set $E \subset [0,\infty)$. Define μ by

$$\mu(E) = \sigma(b^{-1}(E)). \tag{2.7}$$

It is routine to check that μ is a measure and that (2.6) holds.

THEOREM 2.4 *Suppose* (S, σ) *is a measure space and* $b : S \to [0, \infty)$ *is* σ*-measurable. Let* k *be a kernel satisfying the GHO condition on* $\{(s,y) : 0 \le y \le b(s)\}$ *and define* l *by* (2.3). *Define* μ *by* (2.7). *If* $q > 0$ *then*

$$\int_S \left(\int_0^{b(s)} k(s,y)f(y)dy\right)^q d\sigma(s) \approx \int_{(0,\infty)} \left(\int_0^x l(x,y)f(y)dy\right)^q d\mu(x)$$

for all $f \ge 0$.

Proof The work has been done. By Lemma 2.2, $k(s,y) \approx l(b(s),y)$ so we have

$$\int_S \left(\int_0^{b(s)} k(s,y)f(y)dy\right)^q d\sigma(s) \approx \int_S \left(\int_0^{b(s)} l(b(s),y)f(y)dy\right)^q d\sigma(s)$$

with constants independent of f. Now let $F(x) = (\int_0^x l(x,y)f(y)dy)^q$ and note that F is non-decreasing and hence Borel measurable. Lemma 2.3 provides

$$\int_S \left(\int_0^{b(s)} l(b(s),y)f(y)dy\right)^q d\sigma(s) = \int_{[0,\infty)} \left(\int_0^x l(x,y)f(y)dy\right)^q d\mu(x).$$

The point 0 may be omitted from the range of integration because the integrand is zero there. This completes the proof.

Theorem 2.4 takes us from the measure space (S, σ) back to the half line but the measure μ may not be a weighted Lebesgue measure. However, the monotonicity of l enables us to overcome this difficulty and approximate integrals with respect to $d\mu$ by integrals with respect to absolutely continuous measures.

LEMMA 2.5 *If μ is a measure on $[0,\infty)$ then there exists a sequence u_n of non-negative functions such that*

$$\int_0^\infty F(x)u_n(x)dx \text{ increases with } n \text{ to } \int_{0,\infty} F(x)d\mu(x) \text{ and} \tag{2.8}$$

$$\lim_{n\to\infty}\int_0^\infty F(x)\left(\int_x^\infty u_n(z)dz\right)^\beta u_n(x)dx \approx \int_{(0,\infty)} F(x)\left(\int_{[x,\infty)} d\mu(z)\right)^\beta d\mu(x) \tag{2.9}$$

for every $\beta > 0$ and every non-negative, non-decreasing, left continuous function F.

Proof Set $U(y) = \int_{(y,\infty)} d\mu(x)$ for $y \geq 0$ and note that $U\mathcal{X}_{(0,n)}$ is non-increasing and right continuous for each integer $n \geq 1$. Set

$$u_n(x) = n\big[U(x)\mathcal{X}_{(0,n)}(x) - U(x+1/n)\mathcal{X}_{(0,n)}(x+1/n)\big].$$

If $y < n-1$ then

$$\int_y^\infty u_n(x)dx = n\int_y^n U(x)dx - n\int_y^{n-1/n} U(x+1/n)dx = n\int_y^{y+1/n} U(x)dx.$$

Since U is non-increasing, this sequences of averages is non-decreasing and

$$U(y+1/n) \leq \int_y^\infty u_n(x)dx \leq U(y).$$

The right continuity of U shows that

$$\int_y^\infty u_n(x)dx \quad \text{increases with } n \text{ to } \int_{(y,\infty)} d\mu(x). \tag{2.10}$$

Suppose that F is non-negative, non-decreasing and left continuous. Standard arguments [10, p. 262ff] show that there exists a measure ϕ on the Borel subsets of $[0,\infty)$ such that $F(x) = \int_{[0,x)} d\phi(y)$ for $x > 0$. Now

(2.10) and the Monotone Convergence Theorem show that

$$\int_{[0,\infty)}\int_{(y,\infty)} u_n(x)dx\,d\phi(y) \text{ increases with } n \text{ to } \int_{[0,\infty)}\int_{(y,\infty)} d\mu(x)d\phi(y).$$

Interchange the order of integration and this becomes

$$\int_{(0,\infty)}\int_{[0,x)} d\phi(y)u_n(x)dx \text{ increases with } n \text{ to } \int_{(0,\infty)}\int_{[0,x)} d\phi(y)d\mu(x)$$

which establishes (2.8).

Now we repeat the last part of the above argument with $u_n(x)$ replaced by $(\int_x^\infty u_n(z)dz)^\beta u_n(x)$ and $d\mu(x)$ replaced by $(\int_{[x,\infty)} d\mu(z))^\beta d\mu(x)$. The conclusion (2.9) will follow once we show that

$$\int_y^\infty \left(\int_x^\infty u_n(z)dz\right)^\beta u_n(x)dx$$

increases with n to something equivalent to

$$\int_{(y,\infty)} \left(\int_{[x,\infty)} d\mu(z)\right)^\beta d\mu(x).$$

Performing the integration, we have

$$(\beta+1)\int_y^\infty \left(\int_x^\infty u_n(z)dz\right)^\beta u_n(x)dx = \left(\int_y^\infty u_n(x)dx\right)^{\beta+1}$$

which increases to $(\int_{(y,\infty)} d\mu(x))^{\beta+1}$ by (2.10). It remains to show that

$$\left(\int_{(y,\infty)} d\mu(x)\right)^{\beta+1} \approx \int_{(y,\infty)} \left(\int_{[x,\infty)} d\mu(z)\right)^\beta d\mu(x).$$

Replacing the interval $[x, \infty)$ by (y, ∞) in the right hand integral shows that the left hand integral dominates it. To prove the other direction, suppose that $\mu(y, \infty) < \infty$ and choose $y_0 > y$ such that

$$\int_{(y,\infty)} d\mu(x) \leq 2 \int_{(y,y_0]} d\mu(x) \text{ and } \int_{(y,\infty)} d\mu(x) \leq 2 \int_{[y_0,\infty)} d\mu(x).$$

It is easy to see that such a y_0 must exist. Now

$$\begin{aligned}\left(\int_{(y,\infty)} d\mu(x)\right)^{\beta+1} &\leq 2^{\beta+1} \int_{(y,y_0]} d\mu(x)\left(\int_{[y_0,\infty)} d\mu(x)\right)^{\beta} \\ &\leq 2^{\beta+1} \int_{(y,y_0]} \left(\int_{[x,\infty)} d\mu(z)\right)^{\beta} d\mu(x) \\ &\leq 2^{\beta+1} \int_{(y,\infty)} \left(\int_{[x,\infty)} d\mu(z)\right)^{\beta} d\mu(x).\end{aligned}$$

Although such a y_0 may not exist in the case $\mu(y, \infty) = \infty$, the conclusion remains valid. We omit the details.

Generally speaking, the result of the last lemma cannot be extended to include functions F which are not left continuous. This leads us to make the following technical restriction on the function b and the kernel k. If $0 < z < x$ then

$$\sup\{k(t,y) : z \leq y \leq b(t) < x\} = \sup\{k(t,y) : z \leq y \leq b(t) \leq x\}. \quad (2.11)$$

This will ensure that the kernel $l(x, z)$, defined by (2.3), is left continuous in x.

THEOREM 2.6 *Let $p, q \in (1, \infty)$ and v be a non-negative weight function on $(0, \infty)$. Suppose that (S, σ) is a measure space, $b : S \to [0, \infty)$ is σ-measurable, k satisfies the GHO condition on $\{(s, y) : 0 \leq y \leq b(s)\}$ and (2.11) holds whenever $0 < z < x$. Let C be the least constant, finite or infinite, for which the inequality*

$$\left(\int_S \left(\int_0^{b(s)} k(s,y) f(y) dy\right)^q d\sigma(s)\right)^{1/q} \leq C\left(\int_0^{\infty} f(y)^p v(y) dy\right)^{1/p}$$

holds for all non-negative functions f. *If* $1 < p \le q < \infty$ *then* $C \approx \max(A_0, A_1)$ *and if* $1 < q < p < \infty$ *then* $C \approx \max(B_0, B_1)$ *where*

$$A_0 = \sup_{y>0} \left(\int_{\{s:b(s)>y\}} k(s,y)^q d\sigma(s) \right)^{1/q} \left(\int_0^y v(z)^{1-p'} dz \right)^{1/p'}$$

$$A_1 = \sup_{s\in S} \left(\int_{\{t:b(t)\ge b(s)\}} d\sigma(t) \right)^{1/q} \left(\int_0^{b(s)} k(s,y)^{p'} v(y)^{1-p'} dy \right)^{1/p'}$$

$$B_0 = \left(\int_0^\infty \left(\int_{\{s:b(s)>y\}} k(s,y)^q d\sigma(s) \right)^{r/q} \left(\int_0^y v(z)^{1-p'} dz \right)^{r/q'} v(y)^{1-p'} dy \right)^{1/r}$$

$$B_1 = \left(\int_S \left(\int_{\{t:b(t)\ge b(s)\}} d\sigma(t) \right)^{r/p} \left(\int_0^{b(s)} k(s,y)^{p'} v(y)^{1-p'} dy \right)^{r/p'} d\sigma(s) \right)^{1/r}$$

Here r *is defined by* $1/r = 1/q - 1/p$.

Proof Define l and μ by (2.3) and (2.7) respectively. Let C' be the least constant, finite or infinite, such that

$$\left(\int_{(0,\infty)} \left(\int_0^x l(x,y) f(y) dy \right)^q d\mu(x) \right)^{1/q} \le C' \left(\int_0^\infty f(y)^p v(y) dy \right)^{1/p}$$

holds for all non-negative f. By Theorem 2.4, $C \approx C'$. Now let u_n be the sequence from Lemma 2.5 and define $C(n)$ to be the least constant, finite or infinite, such that

$$\left(\int_0^\infty \left(\int_0^x l(x,y) f(y) dy \right)^q u_n(x) dx \right)^{1/q} \le C(n) \left(\int_0^\infty f(y)^p v(y) dy \right)^{1/p}$$

holds for all non-negative f. The assumption (2.11) shows that $l(x,y)$ is left continuous in the variable x and it follows that $(\int_0^x l(x,y) f(y) dy)^q$ is non-negative, non-decreasing, and left continuous for each non-negative f. By Lemma 2.5

$$\int_0^\infty \left(\int_0^x l(x,y)f(y)dy\right)^q u_n(x)dx$$

increases to

$$\int_{(0,\infty)} \left(\int_0^x l(x,y)f(y)dy\right)^q d\mu(x)$$

as $n \to \infty$ so $C(n)$ is an increasing sequence and $\sup_n C(n) = \lim_{n\to\infty} C(n) = C'$.

Now we apply the results of [12] to get $C(n) \approx \max(A_0(n), A_1(n))$ when $1 < p \le q < \infty$ and $C(n) \approx \max(B_0(n), B_1(n))$ when $1 < q < p < \infty$ where

$$A_0(n) = \sup_{y>0} \left(\int_y^\infty l(x,y)^q u_n(x)dx\right)^{1/q} \left(\int_0^y v(z)^{1-p'}dz\right)^{1/p'}$$

$$A_1(n) = \sup_{x>0} \left(\int_x^\infty u_n(z)dz\right)^{1/q} \left(\int_0^x l(x,y)^{p'} v(y)^{1-p'}dy\right)^{1/p'}$$

$$B_0(n) = \left(\int_0^\infty \left(\int_y^\infty l(x,y)^q u_n(x)dx\right)^{r/q} \left(\int_0^y v(z)^{1-p'}dz\right)^{r/q'} v(y)^{1-p'}dy\right)^{1/r}$$

$$B_1(n) = \left(\int_0^\infty \left(\int_x^\infty u_n(z)dz\right)^{r/p} \left(\int_0^x l(x,y)^{p'} v(y)^{1-p'}dy\right)^{r/p'} u_n(x)dx\right)^{1/r}$$

We show $\sup_n A_0(n) \approx A_0$, $\sup_n A_1(n) \approx A_1$, $\sup_n B_0(n) \approx B_0$, and $\sup_n B_1(n) \approx B_1$ to complete the proof.

For each fixed y, $\mathcal{X}_{(y,\infty)}(x)l(x,y)^q$ is non-negative, non-decreasing, and left continuous so, by Lemma 2.5,

$$\int_y^\infty l(x,y)^q u_n(x)dx = \int_0^\infty \mathcal{X}_{(y,\infty)}(x)l(x,y)^q u_n(x)dx$$

increases with n to

$$\int_{(0,\infty)} \mathcal{X}_{(y,\infty)}(x) l(x,y)^q \, d\mu(x) = \int_{[0,\infty)} \mathcal{X}_{(y,\infty)}(x) l(x,y)^q \, d\mu(x).$$

Lemma 2.3 shows that the last expression is equal to

$$\int_S \mathcal{X}_{(y,\infty)}(b(s)) l(b(s),y)^q \, d\sigma(s) = \int_{\{s:b(s)>y\}} l(b(s),y)^q \, d\sigma(s)$$

which is equivalent, by Lemma 2.2, to

$$\int_{\{s:b(s)>y\}} k(s,y)^q \, d\sigma(s).$$

Thus, $\sup_n A_0(n) \approx A_0$ and, by the Monotone Convergence Theorem, $\sup_n B_0(n) \approx B_0$.

The proof that $\sup_n A_1(n) \approx A_1$ also relies on the left continuity in x of $l(x,y)$. As above we find that $\int_x^\infty u_n(z)dz$ increases to

$$\int_{(0,\infty)} \mathcal{X}_{(x,\infty)}(z) d\mu(z) = \int_{[0,\infty)} \mathcal{X}_{(x,\infty)}(z) d\mu(z) = \int_{\{t:b(t)>x\}} d\sigma(t).$$

Observe that since $\{t : b(t) > x\} \subset \{t : b(t) \geq \inf(b(S) \cap [x,\infty))\}$ we have

$$\int_{\{t:b(t)>x\}} d\sigma(t) \leq \sup_{\{s:b(s)\geq x\}} \int_{\{t:b(t)\geq b(s)\}} d\sigma(t).$$

Now

$$\begin{aligned}
&\sup_{x>0}\left(\int_{b(t)>x} d\sigma(t)\right)^{1/q}\left(\int_0^x l(x,y)^{p'} v(y)^{1-p'}\,dy\right)^{1/p'}\\
&\le \sup_{x>0}\sup_{b(s)\ge x}\left(\int_{b(t)\ge b(s)} d\sigma(t)\right)^{1/q}\left(\int_0^{b(s)} l(b(s),y)^{p'} v(y)^{1-p'}\,dy\right)^{1/p'}\\
&\le \sup_{b(s)>0}\left(\int_{b(t)\ge b(s)} d\sigma(t)\right)^{1/q}\left(\int_0^{b(s)} l(b(s),y)^{p'} v(y)^{1-p'}\,dy\right)^{1/p'}\\
&\le \sup_{b(s)>0}\lim_{x\to b(s)^-}\left(\int_{b(t)>x} d\sigma(t)\right)^{1/q}\left(\int_0^x l(x,y)^{p'} v(y)^{1-p'}\,dy\right)^{1/p'}\\
&\le \sup_{x>0}\left(\int_{b(t)>x} d\sigma(t)\right)^{1/q}\left(\int_0^x l(x,y)^{p'} v(y)^{1-p'}\,dy\right)^{1/p'}.
\end{aligned} \tag{2.12}$$

Because the first and last expressions coincide all the inequalities above are equalities and since Lemma 2.2 shows that the expression (2.12) is equivalent to A_1 we have $\sup_n A_1(n) \approx A_1$ as required.

For the proof of $\sup_n B_1(n) \approx B_1$ we apply Lemma 2.5 with $\beta = r/p$ to see that $\sup_n B_1(n)$ is equivalent to

$$\left(\int_{(0,\infty)}\left(\int_{[x,\infty)} d\mu(z)\right)^{r/p}\left(\int_0^x l(x,y)^{p'} v(y)^{1-p'}\,dy\right)^{r/p'} d\mu(x)\right)^{1/r}$$

which Lemma 2.3, applied twice, shows to be just

$$\left(\int_S\left(\int_{\{t:b(t)\ge b(s)\}} d\sigma(t)\right)^{r/p}\left(\int_0^{b(s)} l(b(s),y)^{p'} v(y)^{1-p'}\,dy\right)^{r/p'} d\sigma(s)\right)^{1/r}.$$

By Lemma 2.2 the last expression is equivalent to B_1.

When the kernel $k \equiv 1$ the weight conditions simplify and the result extends to include the case $0 < q < 1$.

COROLLARY 2.7. *Suppose* $0 < q < \infty, 1 < p < \infty, v$ *is a non-negative weight function on* $(0,\infty), (S,\sigma)$ *is a measure space, and*

$b : S \to [0,\infty)$ is σ-measurable. Let C be the least constant, finite or infinite, for which the inequality

$$\left(\int_S\left(\int_0^{b(s)} f(y)dy\right)^q d\sigma(s)\right)^{1/q} \le C\left(\int_0^\infty f(y)^p v(y)dy\right)^{1/p}$$

holds for all $f \ge 0$. If $1 < p \le q < \infty$ then *$C \approx A$ and if $0 < q < p < \infty$ then $C \approx B$ where*

$$A = \sup_{y>0}\left(\int_{\{s:b(s)>y\}} d\sigma(s)\right)^{1/q}\left(\int_0^y v(z)^{1-p'}dz\right)^{1/p'}$$

$$B = \left(\int_S\left(\int_{\{t:b(t)\ge b(s)\}} d\sigma(t)\right)^{r/p}\left(\int_0^{b(s)} v(y)^{1-p'}dy\right)^{r/p'} d\sigma(s)\right)^{1/r}.$$

Here $1/r = 1/q - 1/p$. Also, if $q > 1$ or $0 < q < 1$ and $v^{1-p'}$ is locally integrable then

$$B \approx \left(\int_0^\infty\left(\int_{\{s:b(s)>y\}} d\sigma(s)\right)^{r/q}\left(\int_0^y v(z)^{1-p'}dz\right)^{r/q'} v(y)^{1-p'}dy\right)^{1/r}.$$

Proof The case $1 < p \le q < \infty$ follows from Theorem 2.6 by taking $k \equiv 1$ since in this case $A = A_0$ and it is not difficult to see that $A_1 \le A$. In the case $0 < q < p < \infty$ we define $C(n)$ as in Theorem 2.6. We still have $\lim_{n\to\infty} C(n) \approx C$. Using [11, Theorem 2.4] we have

$$C(n) \approx \left(\int_0^\infty\left(\int_x^\infty u_n(z)dz\right)^{r/p}\left(\int_0^x v(y)^{1-p'}dy\right)^{r/p'} u_n(x)dx\right)^{1/r}.$$

In the same way that we showed $\sup B_1(n) \approx B_1$ in Theorem 2.6 we see that the right hand side converges to B. The final assertion follows from the remark on page 93 of [11]. This completes the proof.

COROLLARY 2.8 *Suppose $0 < q < \infty$, $1 < p < \infty$, v is a non-negative weight function on $(0,\infty)$, (S,σ) is a measure space, and*

$a : S \longrightarrow [0, \infty)$ is *σ-measurable. Let C be the least constant, finite or infinite, for which the inequality*

$$\left(\int_S \left(\int_{a(s)}^{\infty} f(y)dy\right)^q d\sigma(s)\right)^{1/q} \leq C\left(\int_0^{\infty} f(y)^p v(y)dy\right)^{1/p}$$

holds for all $f \geq 0$. If $1 < p \leq q < \infty$ *then* $C \approx A'$ *and if* $0 < q < p < \infty$ *then* $C \approx B'$ *where*

$$A' = \sup_{y>0}\left(\int_{\{s:a(s)<y\}} d\sigma(s)\right)^{1/q}\left(\int_y^{\infty} v(z)^{1-p'}dz\right)^{1/p'}$$

$$B' = \left(\int_S \left(\int_{\{t:a(t)\leq a(s)\}} d\sigma(t)\right)^{r/p}\left(\int_{a(s)}^{\infty} v(y)^{1-p'}dy\right)^{r/p'} d\sigma(s)\right)^{1/r}.$$

Here $1/r = 1/q - 1/p$. *Also, if* $q > 1$ *or* $0 < q < 1$ *and* $v^{1-p'}$ *is locally integrable then*

$$B' \approx \left(\int_0^{\infty} \left(\int_{\{s:a(s)\leq y\}} d\sigma(s)\right)^{r/q}\left(\int_y^{\infty} v(z)^{1-p'}dz\right)^{r/q'} v(y)^{1-p'}dy\right)^{1/r}.$$

Proof Make the change of variable $y \longrightarrow 1/y$ and apply Corollary 2.7 with $b(s) = 1/a(s)$. We omit the details.

3 DECOMPOSITION OF NEARLY BLOCK DIAGONAL OPERATORS

Block diagonal matrices are well understood. There are direct sum decompositions of both the domain and codomain spaces so that the action of the whole matrix is broken down into the action of the blocks on their individual summands. A similar process can be carried out for more general linear operators whose domain and codomain can be decomposed in such a fashion. We restrict our attention to positive linear operators, those that take non-negative functions to non-negative functions. This restriction allows us to consider operators which do not have a strictly block diagonal decomposition but which decompose into blocks whose natural domains (and codomains) may overlap to some extent. Our decomposition theorem for these nearly block diagonal operators is Theorem 3.3.

DEFINITION 3.1 *If K is a linear operator taking non-negative ν-measurable functions to non-negative σ-measurable functions we define the norm of K to be*

$$\|K\|_{L^p_\nu \to L^q_\sigma} = \sup\left\{\int_S Kf(s)g(s)d\sigma(s) : f \geq 0, g \geq 0, \|f\|_{L^p_\nu} \leq 1, \|g\|_{L^{q'}_\sigma} \leq 1\right\}.$$

We identify a function φ on the measure space (X, ξ) with the multiplication operator $f \mapsto \varphi f$ so that if $\varphi : X \to [0, \infty)$ then

$$\|\varphi\|_{L^p_\xi \to L^q_\xi} = \sup\left\{\int_X \varphi(x)f(x)g(x)d\xi(x) : f \geq 0, g \geq 0, \|f\|_{L^p_\xi} \leq 1, \|g\|_{L^{q'}_\xi} \leq 1\right\}.$$

DEFINITION 3.2 *A non-negative, linear operator K is* nearly block diagonal *provided there exists a measure space (X, ξ), σ-measurable subsets S_x* of *(S, σ), ν-measurable subsets Y_x of (Y, ν), and a positive constant M such that*

$$(1/M)Kf(s) \leq \int_X \mathcal{X}_{S_x}(s)K(f\mathcal{X}_{Y_x})(s)d\xi(x) \leq MKf(s), \quad s \in S, f \geq 0; \tag{3.1}$$

$$\begin{aligned} M^{-1} &\leq \int_{\{x:s\in S_x\}} d\xi(x) \leq M, s \in S; \text{ and} \\ M^{-1} &\leq \int_{\{x:y\in Y_x\}} d\xi(x) \leq M, y \in Y. \end{aligned} \tag{3.2}$$

In this case we say that

$$(\xi, \{(S_x, Y_x) : x \in X\})$$

is a nearly block diagonal decomposition of K.

The assertion of (3.1) is that the action of the operator K can be expressed in terms of the action of the blocks and (3.2) controls the extent of the overlap of the decompositions of the spaces Y and S.

THEOREM 3.3 *Suppose that* (X, ξ) *is a measure space and* $(\xi, \{(S_x, Y_x) : x \in X\})$ *is a nearly block diagonal decomposition of* K. *If* $K_x f = \mathcal{X}_{S_x} K(f \mathcal{X}_{Y_x})$ *then*

$$\|K\|_{L^p_\nu(Y) \to L^q_\sigma(S)} \leq M^{1+1/p+1/q'} \| \|K_x\|_{L^p_\nu(Y_x) \to L^q_\sigma(S_x)} \|_{L^p_\xi(X) \to L^q_\xi(X)}. \tag{3.3}$$

If ξ *is counting measure on a subset of* X *then*

$$\| \|K_x\|_{L^p_\nu(Y_x) \to L^q_\sigma(S_x)} \|_{L^p_\xi(X) \to L^q_\xi(X)} \leq M^{1+1/p'+1/q} \|K\|_{L^p_\nu(Y) \to L^q_\sigma(S)}. \tag{3.4}$$

Here M *is the constant from Definition 3.2.*

Proof Fix non-negative functions f and g with $\|f\|_{L^p_\nu(Y)} \leq 1$ and $\|g\|_{L^{q'}_\sigma(S)} \leq 1$. Set $F(x) = M^{-1/p} \|f \mathcal{X}_{Y_x}\|_{L^p_\nu(Y_x)}$ and $G(x) = M^{-1/q'} \|g \mathcal{X}_{S_x}\|_{L^{q'}_\sigma(S_x)}$. Note that

$$\begin{aligned}
\|F(x)\|_{L^p_\xi(X)} &= M^{-1/p} \left(\int_X \int_Y f(y)^p \mathcal{X}_{Y_x(y)} d\nu(y) d\xi(x) \right)^{1/p} \\
&= M^{-1/p} \left(\int_Y f(y)^p \int_X \mathcal{X}_{Y_x(y)} d\xi(x) d\nu(y) \right)^{1/p} \\
&\leq M^{-1/p} M^{1/p} \|f\|_{L^p_\nu(Y)} \leq 1
\end{aligned}$$

by (3.2). In a similar way we see that $\|G(x)\|_{L^{q'}_\xi(X)} \leq 1$.

To establish (3.3) we use Definition 3.1.

$$\begin{aligned}\int_S Kf(s)g(s)d\sigma(s) &\le M\int_S\int_X K_xf(s)d\xi(x)g(s)d\sigma(s)\\ &= M\int_X\int_S K_x(f\mathcal{X}_{Y_x})(s)g(s)\mathcal{X}_{S_x}(s)d\sigma(s)d\xi(x)\\ &\le M\int_X \|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|f\mathcal{X}_{Y_x}\|_{L^p_v(Y_x)}\\ &\quad\times \|g\mathcal{X}_{S_x}\|_{L^{q'}_\sigma(S_x)}d\xi(x)\\ &= M^{1+1/p+1/q'}\int_X \|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}F(x)G(x)d\xi(x)\\ &\le M^{1+1/p+1/q'}\|\,\|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^p_\xi(X)\to L^q_\xi(X)}.\end{aligned}$$

Taking the supremum over all choices of f and g we have

$$\|K\|_{L^p_v(Y)\to L^q_\sigma(S)} \le M^{1+1/p+1/q'}\|\,\|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^p_\xi(X)\to L^q_\xi(X)}$$

which is (3.3).

Suppose now that ξ is counting measure on some subset of X. Inequality (3.4) is trivial if $\|K\|_{L^p_v(Y)\to L^q_\sigma(S)}$ is infinite so we assume that it is finite. It is clear from the definition of K_x that $K_xf(s)\le Kf(s)$ for all $x\in X$, all $s\in S$ and all non-negative f. It follows that $\|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}<\infty$ for all $x\in X$.

Fix $\lambda\in(0,1)$. For each $x\in X$ choose non-negative functions f_x and g_x such that $\|f_x\|_{L^p_v(Y_x)}\le 1$, $\|g_x\|_{L^{q'}_\sigma(S_x)}\le 1$ and

$$\lambda\|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\le\int_S K_xf_x(s)g_x(s)d\sigma(s). \tag{3.5}$$

Replacing f_x by $f_x\mathcal{X}_{Y_x}$ and g_x by $g_x\mathcal{X}_{S_x}$ does not affect (3.5) and cannot increase the norms of f_x and g_x so we may assume henceforth that $f_x=f_x\mathcal{X}_{Y_x}$ and $g_x=g_x\mathcal{X}_{S_x}$.

Let $F(x)$ and $G(x)$ be non-negative functions on (X,ξ) with $\|F\|_{L^p_\xi(X)}\le 1$ and $\|G\|_{L^{q'}_\xi(X)}\le 1$ and set

$$\mathcal{F}(y)=M^{-1/p'}\int_X F(x)f_x(y)\,d\xi(x)$$

and

$$\mathcal{G}(s) = M^{-1/q} \int_X G(x) g_x(s)\, d\xi(x).$$

Since ξ is counting measure, it is clear that

$$F(x) f_x(y) \le M^{1/p'} \mathcal{F}(y) \quad \text{and} \quad G(x) g_x(s) \le M^{1/q} \mathcal{G}(s)$$

for all $y \in Y, s \in S$ and x in the support of ξ.

We use duality to estimate the norm of $\mathcal{F}$ in $L^p_\nu(Y)$. Suppose H is non-negative and $\|H\|_{L^{p'}_\nu(Y)} \le 1$. Then

$$\begin{aligned} \| \|H\mathcal{X}_{Y_x}\|_{L^{p'}_\nu(Y_x)} \|_{L^{p'}_\xi(X)} &= \left(\int_X \int_Y H(y)^{p'} \mathcal{X}_{Y_x}(y) d\nu(y) d\xi(x) \right)^{1/p'} \\ &= \left(\int_Y H(y)^{p'} \int_X \mathcal{X}_{Y_x}(y) d\xi(x) d\nu(y) \right)^{1/p'} \le M^{1/p'} \end{aligned}$$

so we have

$$\begin{aligned} \int_Y \mathcal{F}(y) H(y) d\nu(y) &= M^{-1/p'} \int_Y \int_X F(x) f_x(y) d\xi(x) H(y) d\nu(y) \\ &= M^{-1/p'} \int_X F(x) \int_Y f_x(y) H(y) d\nu(y) d\xi(x) \\ &= M^{-1/p'} \int_X F(x) \int_Y f_x(y) H(y) \mathcal{X}_{Y_x}(y) d\nu(y) d\xi(x) \\ &\le M^{-1/p'} \int_X F(x) \|f_x\|_{L^p_\nu(Y_x)} \|H\mathcal{X}_{Y_x}\|_{L^{p'}_\nu(Y_x)} d\xi(x) \\ &\le M^{-1/p'} \int_X F(x) \|H\mathcal{X}_{Y_x}\|_{L^{p'}_\nu(Y_x)} d\xi(x) \\ &\le M^{-1/p'} \|F\|_{L^p_\xi(X)} \| \|H\mathcal{X}_{Y_x}\|_{L^{p'}_\nu(Y_x)} \|_{L^{p'}_\xi(X)} \\ &\le M^{-1/p'} M^{1/p'} = 1. \end{aligned}$$

Taking the supremum over the functions H we have $\|\mathcal{F}\|_{L^p_\nu(Y)} \le 1$.

A similar argument shows that $\|\mathcal{G}\|_{L^{q'}_\sigma(S)} \leq 1$.
Now

$$\begin{aligned}
\lambda \int_X & \|K_x\|_{L^p_\nu(Y_x)\to L^q_\sigma(S_x)} F(x)G(x)d\xi(x) \\
&\leq \int_X \int_S K_x f_x(s) g_x(s) d\sigma(s) F(x) G(x) d\xi(x) \\
&= \int_S \int_X K_x(F(x)f_x)(s) G(x) g_x(s) d\xi(x) d\sigma(s) \\
&\leq M^{1/p'+1/q} \int_S \int_X K_x \mathcal{F}(s) G(s) d\xi(x) d\sigma(s) \\
&\leq M^{1+1/p'+1/q} \int_S K\mathcal{F}(s) G(s) d\sigma(s) \\
&\leq M^{1+1/p'+1/q} \|K\|_{L^p_\nu(Y)\to L^q_\sigma(S)}.
\end{aligned}$$

Taking the supremum over all non-negative $F(x)$ and $G(x)$ with $\|F\|_{L^p_\xi(X)} \leq 1$ and $\|G\|_{L^{q'}_\xi(X)} \leq 1$ and letting $\lambda \to 1^-$ we have

$$\| \|K_x\|_{L^p_\nu(Y_x)\to L^q_\sigma(S_x)} \|_{L^p_\xi(X)\to L^q_\xi(X)} \leq M^{1+1/p'+1/q} \|K\|_{L^p_\nu(Y)\to L^q_\sigma(S)}.$$

This completes the proof.

To use the above theorem we must understand the norm $\| \; \|_{L^p_\xi(X)\to L^q_\xi(X)}$. This is not difficult. A proof of the following simple proposition may be found in [6].

PROPOSITION 3.4 *If (X, ξ) is a measure space, $1 \leq q < p \leq \infty$ and $1/r = 1/q - 1/p$ then*

$$\|\phi\|_{L^p_\xi(X)\to L^q_\xi(X)} = \|\phi\|_{L^r_\xi(X)}$$

for any non-negative ϕ. If ξ is counting measure on a subset of X and $1 \leq p \leq q \leq \infty$ then

$$\|\phi\|_{L^p_\xi(X)\to L^q_\xi(X)} = \|\phi\|_{L^\infty_\xi(X)}.$$

4 CONDITIONS FOR BOUNDEDNESS OF K

To give necessary and sufficient conditions for the boundedness of the operator

$$Kf(s) = \int_{a(s)}^{b(s)} k(s,y)f(y)dy \tag{4.1}$$

from $L^p_v[0,\infty)$ to $L^q_\sigma(S)$ we apply the decomposition theorem of the previous section. The action of the operator on the resulting blocks is handled using the results of Section 2. The necessary and sufficient conditions for boundedness on the blocks combine to give integral conditions similar in form to those of Theorem 2.6.

The values of f off $Y = \cup_{s\in S}[a(s), b(s)]$ have no effect on the values of Kf so it is natural to consider the functions f to be defined on Y. It is easy to see that $K : L^p_v[0,\infty) \to L^q_\sigma(S)$ if and only if $K : L^p_v(Y) \to L^q_\sigma(S)$.

We begin by introducing the concept of a normalizing measure which provides us with a nearly block diagonal decomposition of the operator K.

DEFINITION 4.1 *Let (S,σ) be a measure space and suppose that a and b are non-negative σ-measurable functions on S such that $a(s) \le b(s)$ for all s. A measure ξ on $[0,\infty]$ is called a* normalizing measure *for (a,b) provided there exist positive constants c_1 and c_2 such that*

$$c_1 \le \int_{a(s)}^{b(s)} d\xi(x) \le c_2 \tag{4.2}$$

for all $s \in S$. If, in addition, ξ is counting measure on a subset of $[0,\infty)$ then ξ is called a discrete normalizing measure.

Next we show that a normalizing measure is all that is required for the operator K of (4.1) to be nearly block diagonal.

LEMMA 4.2 *Let (S,σ) be a measure space and suppose that a and b are non-negative σ-measurable functions on S such that $a(s) \le b(s)$ for all s. If ξ is a normalizing measure for (a,b) then*

$(\xi, \{(S_x, Y_x) : x \in X\})$ is a nearly block diagonal decomposition of K where $X = Y = \cup_{s\in S}[a(s), b(s)], S_x = \{s \in S : a(s) \le x \le b(s)\}$, and $Y_x = \{y \in [0,\infty) : S_y \cap S_x \ne \emptyset\}$.

Proof Let c_1 and c_2 be positive constants for which ξ satisfies (4.2) and set $M = \max(1/c_1, 2c_2)$. Since

$$\int_{\{x:s\in S_x\}} d\xi(x) = \int_{a(s)}^{b(s)} d\xi(x)$$

for each $s \in S$, the first inequality in (3.2) follows from (4.2).

Note that $Y_x = \bigcup_{s\in S_x}[a(s), b(s)]$ which is a union of intervals containing x so Y_x is an interval. The symmetry in the definition of Y_x shows that $\{x : y \in Y_x\} = Y_y$ and since Y_y is an interval there exist sequences s_n and s'_n of points in S_y such that

$$\xi(Y_y) = \lim_{n\to\infty} \xi[a(s_n), b(s'_n)].$$

Since y is in both $[a(s_n), b(s_n)]$ and $[a(s'_n), b(s'_n)]$ the last expression is no greater than

$$\lim_{n\to\infty} \xi[a(s_n), b(s_n)] + \xi[a(s'_n), b(s'_n)] \le 2c_2 \le M.$$

For $y \in X$, there exists some s with $a(s) \le y \le b(s)$ so we have $[a(s), b(s)] \subset Y_y$ and hence

$$1/M \le c_1 \le \xi[a(s), b(s)] \le \xi(Y_y).$$

We have shown that $1/M \le \xi(Y_y) \le M$ which establishes the second inequality in (3.2).

It remains to show that (3.1) holds. An interchange of the order of integration yields

$$\begin{aligned}\int_X \mathcal{X}_{S_x}(s)K(f\mathcal{X}_{Y_x})(s)d\xi(x) &= \int_X \mathcal{X}_{S_x}(s)\int_{a(s)}^{b(s)} k(s,y)f(y)\mathcal{X}_{Y_x}(y)dy\, d\xi(x)\\ &= \int_{a(s)}^{b(s)} k(s,y)f(y)\int_X \mathcal{X}_{S_x}(s)\mathcal{X}_{Y_x}(y)d\xi(x)dy.\end{aligned}$$

The inner integral in the last expression is just $\xi[a(s), b(s)]$ so (4.2) implies (3.1). This completes the proof.

The main results of the paper are presented in Theorems 4.3 and 4.4. It is convenient to split up the cases $1 < q < p < \infty$ and $1 < p \le q < \infty$.

THEOREM 4.3 *Let $1 < q < p < \infty$, v be a non-negative weight, (S, σ) be a measure space, a and b be σ-measurable functions on S, and k be a non-negative kernel satisfying the GHO condition on $\{(s,y) : 0 \le y \le b(s)\}$ and also (2.11). Suppose that ξ is a normalizing measure for (a,b). Let C be the least constant, finite or infinite, such that*

$$\left(\int_S \left(\int_{a(s)}^{b(s)} k(s,y)f(y)dy\right)^q d\sigma(s)\right)^{1/q} \le C\left(\int_0^\infty f(y)^p v(y)dy\right)^{1/p}$$

holds for all $f \ge 0$. Then C is bounded above by a multiple of $\max(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$ *where*

$$\mathcal{B}_1^r = \int_X \int_0^x \left(\int_{\substack{a(s)<y\\ x\le b(s)}} k(s,x)^q d\sigma(s)\right)^{r/q} \left(\int_y^x v(z)^{1-p'}dz\right)^{r/q'} v(y)^{1-p'}dy\,d\xi(x)$$

$$\mathcal{B}_2^r = \int_X \int_x^\infty \left(\int_{\substack{a(s)\le x\\ y<b(s)}} k(s,y)^q d\sigma(s)\right)^{r/q} \left(\int_x^y v(z)^{1-p'}dz\right)^{r/q'} v(y)^{1-p'}dy\,d\xi(x)$$

$$\mathcal{B}_3^r = \int_S \int_{a(s)}^{b(s)} \left(\int_{\substack{a(t)\le x\\ b(s)\le b(t)}} d\sigma(t)\right)^{r/p} \left(\int_x^{b(s)} k(s,y)^{p'} v(y)^{1-p'}dy\right)^{r/p'} d\xi(x)d\sigma(s)$$

$$\mathcal{B}_4^r = \int_S \int_{a(s)}^{b(s)} \left(\int_{\substack{a(t)\le a(s)\\ x\le b(t)}} d\sigma(t)\right)^{r/p} \left(\int_{a(s)}^x \overline{k}(x,y)^{p'} v(y)^{1-p'}dy\right)^{r/p'} d\xi(x)d\sigma(s).$$

Here $\overline{k}(x,y) = \sup\{k(t,y) : b(t) = x\}$.

If ξ is a discrete normalizing measure then C is also bounded below by a multiple of $\max(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$.

Proof Let X, Y, S_x and Y_x be as in Lemma 4.2. Then $(\xi, \{(S_x, Y_x) : x \in X\})$ is a nearly block diagonal decomposition of K. It follows from Theorem 3.3 and Proposition 3.4 that

$$\begin{aligned}\|K\|_{L^p_v(Y)\to L^q_\sigma(S)} &\le M^{1+1/p+1/q'} \| \|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)} \|_{L^p_\xi(X)\to L^q_\xi(X)} \\ &= M^{1+1/p+1/q'} \| \|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)} \|_{L^r_\xi(X)}\end{aligned}$$

where M depends only on the constants c_1 and c_2 in the definition of the normalizing measure ξ.

If ξ is a discrete normalizing measure, the inequality may be essentially reversed to give

$$\| \|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)} \|_{L^r_\xi(X)} \le M^{1+1/p'+1/q} \|K\|_{L^p_v(Y)\to L^q_\sigma(S)}.$$

Since $C = \|K\|_{L^p_v[0,\infty)\to L^q_\sigma(S)} = \|K\|_{L^p_v(Y)\to L^q_\sigma(S)}$, this reduces the problem to looking at the norms of K_x for each x in X. To work with K_x we decompose it into three operators and apply the results of Section 2. Fix $x \in X$ and take $f \ge 0$ to be supported in Y_x. Then

$$\begin{aligned}K_x f(s) &= \mathcal{X}_{S_x}(s) \int_{a(s)}^{b(s)} k(s,y) f(y) dy \\ &= \mathcal{X}_{S_x}(s) \int_{a(s)}^{x} k(s,y) f(y) dy + \mathcal{X}_{S_x}(s) \int_{x}^{b(s)} k(s,y) f(y) dy. \quad (4.3)\end{aligned}$$

Note that, according to the definition of S_x, $a(s) \le x \le b(s)$ whenever $\mathcal{X}_{S_x}(s) \ne 0$. We now use the GHO condition on k to further decompose the first summand. If $x \notin b(S)$ then $\overline{k}(x,y) = 0$ and if $x \in b(S)$, say $x = b(t)$, then it follows from the condition (2.1) on k that $k(t,y) \le \overline{k}(x,y) \le Dk(t,y)$. In either case we have (using (2.1) or (2.2) as appropriate)

$$D^{-1} k(s,y) \le k(s,x) + \overline{k}(x,y) \le D^2 k(s,y)$$

whenever $y \le x \le b(s)$. Applying this estimate to the kernel k in the first summand of (4.3) shows that $K_x f(s)$ is bounded above and below by multiples of

$$\mathcal{X}_{S_x}(s)k(s,x)\int_{a(s)}^{x} f(y)dy + \mathcal{X}_{S_x}(s)\int_{a(s)}^{x} \overline{k}(x,y)f(y)dy \\ + \mathcal{X}_{S_x}(s)\int_{x}^{b(s)} k(s,y)f(y)dy \equiv K_x^{(1)}f(s) + K_x^{(2)}f(s) + K_x^{(3)}f(s).$$

Since the operators $K_x^{(1)}$, $K_x^{(2)}$, and $K_x^{(3)}$, are all non-negative

$$\|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)} \approx \|K_x^{(1)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)} + \|K_x^{(2)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)} \\ + \|K_x^{(3)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}.$$

and hence

$$\|\,\|K_x\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^r_\xi(X)} \approx \|\,\|K_x^{(1)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^r_\xi(X)} \\ + \|\,\|K_x^{(2)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^r_\xi(X)} \\ + \|\,\|K_x^{(3)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^r_\xi(X)}.$$

To complete the proof we show that

$$\|\,\|K_x^{(1)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^r_\xi(X)} \approx \mathcal{B}_1,$$

$$\|\,\|K_x^{(2)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^r_\xi(X)} \approx \mathcal{B}_4, \text{ and}$$

$$\|\,\|K_x^{(3)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^r_\xi(X)} \approx \max(\mathcal{B}_2, \mathcal{B}_3).$$

The norm $\|K_x^{(1)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}$ is the least constant for which the inequality

$$\left(\int_{S_x}\left(\int_{a(s)}^{x} f(y)dy\right)^q k(s,x)^q d\sigma(s)\right)^{1/q} \le C\left(\int_{Y_x} f(y)^p v(y)dy\right)^{1/p}$$

holds for all $f \geq 0$. It is straightforward to see that it is also the least constant for which

$$\left(\int_S \left(\int_{a(s)}^{\infty} f(y)dy\right)^q d\sigma_x^{(1)}(s)\right)^{1/q} \leq C\left(\int_0^{\infty} f(y)^p v_x^{(1)}(y)dy\right)^{1/p}$$

holds for all $f \geq 0$ where $d\sigma_x^{(1)}(s) = \mathcal{X}_{S_x}(s)k(s,x)^q d\sigma(s)$, $v_x^{(1)}(y) = v(y)$ for $y \in [0,x] \cap Y_x$ and $v_x^{(1)}(y) = \infty$ otherwise. By Corollary 2.8 we have

$$\|K_x^{(1)}\|^r_{L_v^p(Y_x)\to L_\sigma^q(S_x)} \approx \int_0^x \mathcal{X}_{Y_x}(y)\left(\int_{a(s)<y} \mathcal{X}_{S_x}(s)k(s,x)^q d\sigma(s)\right)^{r/q} \times \left(\int_y^x v(z)^{1-p'}dz\right)^{r/q'} v(y)^{1-p'}dy.$$

From this it readily follows that

$$\| \|K_x^{(1)}\|_{L_v^p(Y_x)\to L_\sigma^q(S_x)}\|_{L_\xi^r(X)} \approx \mathcal{B}_1.$$

The norm $\|K_x^{(2)}\|_{L_v^p(Y_x)\to L_\sigma^q(S_x)}$ is the least constant for which the inequality

$$\left(\int_{S_x}\left(\int_{a(s)}^x \overline{k}(x,y)f(y)dy\right)^q d\sigma(s)\right)^{1/q} \leq C\left(\int_{Y_x} f(y)^p v(y)dy\right)^{1/p}$$

holds for all $f \geq 0$. Making the substitution $g(y) = \overline{k}(x,y)f(y)$, we see that it is also the
least constant for which

$$\left(\int_S\left(\int_{a(s)}^{\infty} g(y)dy\right)^q d\sigma_x^{(2)}(s)\right)^{1/q} \leq C\left(\int_0^{\infty} g(y)^p v_x^{(2)}(y)dy\right)^{1/p}$$

holds for all $g \geq 0$ where $d\sigma_x^{(2)}(s) = \mathcal{X}_{S_x}(s)d\sigma(s)$, $v_x^{(2)}(y) = \overline{k}(x,y)^{-p}v(y)$ for $y \in [0,x] \cap Y_x$ and $v_x^{(1)}(y) = \infty$ otherwise. Again we appeal to Corollary 2.8. We get

$$\|K_x^{(2)}\|^r_{L^p_v(Y_x)\to L^q_\sigma(S_x)} \approx \int_S \left(\int_{a(t)\leq a(s)} \mathcal{X}_{S_x}(t)d\sigma(t) \right)^{r/p} \times \left(\int_{a(s)}^x \overline{k}(x,y)^{p'} v(y)^{1-p'} dy \right)^{r/p'} \mathcal{X}_{S_x}(s)d\sigma(s)$$

and so, with an interchange in the order of integration,

$$\| \|K_x^{(2)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}\|_{L^r_\xi(X)} \approx \mathcal{B}_4.$$

The norm $\|K_x^{(3)}\|_{L^p_v(Y_x)\to L^q_\sigma(S_x)}$ is the least constant for which the inequality

$$\left(\int_{S_x} \left(\int_x^{b(s)} k(s,y)f(y)dy \right)^q d\sigma(s) \right)^{1/q} \leq C \left(\int_{Y_x} f(y)^p v(y)dy \right)^{1/p}$$

holds for all $f \geq 0$. It is also the least constant for which

$$\left(\int_S \left(\int_0^{b(s)} k(s,y)f(y)dy \right)^q d\sigma_x^{(3)}(s) \right)^{1/q} \leq C \left(\int_0^\infty f(y)^p v_x^{(3)}(y)dy \right)^{1/p}$$

holds for all $f \geq 0$ where $d\sigma_x^{(3)}(s) = \mathcal{X}_{S_x}(s)d\sigma(s)$, $v_x^{(3)}(y) = v(y)$ for $y \in [x,\infty) \cap Y_x$ and $v_x^{(3)}(y) = \infty$ otherwise. This time we apply Theorem 2.6 to see that $\|K_x^{(3)}\|^r_{L^p_v(Y_x)\to L^q_\sigma(S_x)}$ is comparable to the maximum of

$$\int_x^\infty \mathcal{X}_{Y_x}(y) \left(\int_{y<b(s)} k(s,y)^q \mathcal{X}_{S_x}(s)d\sigma(s) \right)^{r/q} \left(\int_x^y v(z)^{1-p'} dz \right)^{r/q'} v(y)^{1-p'} dy$$

and

$$\int_S \left(\int_{b(s) \le b(t)} \mathcal{X}_{S_x}(t) d\sigma(t) \right)^{r/p} \left(\int_x^{b(s)} k(s,y)^{p'} v(y)^{1-p'} dy \right)^{r/q'} \mathcal{X}_{S_x}(s) d\sigma(s).$$

From these we conclude that

$$\| \| K_x^{(3)} \|_{L_v^p(Y_x) \to L_\sigma^q(S_x)} \|_{L_\xi^r(X)} \approx \max(\mathcal{B}_2, \mathcal{B}_3)$$

to complete the proof.

THEOREM 4.4 *Let $1 < p \le q < \infty$, v be a non-negative weight, (S, σ) be a measure space, a and b be σ-measurable functions on S with $a(s) \le b(s)$, and k be a non-negative kernel satisfying the GHO condition on $\{(s,y) : 0 \le y \le b(s)\}$ and also (2.11). Suppose that (a,b) admits a discrete normalizing measure. Let C be the least constant, finite or infinite, such that*

$$\left(\int_S \left(\int_{a(s)}^{b(s)} k(s,y) f(y) dy \right)^q d\sigma(s) \right)^{1/q} \le C \left(\int_0^\infty f(y)^p v(y) dy \right)^{1/p}$$

holds for all $f \ge 0$. Then $C \approx \max(\mathcal{A}_1, \mathcal{A}_2)$ where

$$\mathcal{A}_1 = \sup_{\{(x,y):x<y\}} \left(\int_{\substack{a(s) \le x \\ y \le b(s)}} k(s,y)^q d\sigma(s) \right)^{1/q} \left(\int_x^y v(z)^{1-p'} dz \right)^{1/p'}$$

$$\mathcal{A}_2 = \sup_{\{(x,s):x<b(s)\}} \left(\int_{\substack{a(t) \le x \\ b(s) \le b(t)}} d\sigma(t) \right)^{1/q} \left(\int_x^{b(s)} k(s,z)^{p'} v(z)^{1-p'} dz \right)^{1/p'}$$

Proof Suppose that counting measure on X_0 is a discrete normalizing measure for (a,b). Then for any choice of x_1 and x_2, counting measure on $X_0 \cup \{x_1, x_2\}$ is also a discrete normalizing measure for (a, b). Choose x_1 and x_2 such that

$$\mathcal{A}_1/2 \leq \sup_{\{y: x_1 < y\}} \left(\int_{\substack{a(s) \leq x_1 \\ y \leq b(s)}} k(s,y)^q d\sigma(s) \right)^{1/q} \left(\int_{x_1}^{y} v(z)^{1-p'} dz \right)^{1/p'} \quad \text{and}$$

$$\mathcal{A}_2/2 \leq \sup_{\{s: x_2 < b(s)\}} \left(\int_{\substack{a(t) \leq x_2 \\ b(s) \leq b(t)}} d\sigma(t) \right)^{1/q} \left(\int_{x_2}^{b(s)} k(s,z)^{p'} v(z)^{1-p'} dz \right)^{1/p'}.$$

Let ξ be counting measure on $X_0 \cup \{x_1, x_2\}$. We decompose the operator K just as in Theorem 4.3 and apply the results of Section 3 to get

$$\begin{aligned} \|K\|_{L^p_v[0,\infty) \to L^q_\sigma(S)} &\approx \| \|K_x\|_{L^p_v(Y_x) \to L^q_\sigma(S_x)} \|_{L^\infty_\xi(X)} \\ &\approx \| \|K_x^{(1)}\|_{L^p_v(Y_x) \to L^q_\sigma(S_x)} \|_{L^\infty_\xi(X)} \\ &\quad + \| \|K_x^{(2)}\|_{L^p_v(Y_x) \to L^q_\sigma(S_x)} \|_{L^\infty_\xi(X)} \\ &\quad + \| \|K_x^{(3)}\|_{L^p_v(Y_x) \to L^q_\sigma(S_x)} \|_{L^\infty_\xi(X)}. \end{aligned}$$

As we have seen above, the norm $\|K_x^{(1)}\|_{L^p_v(Y_x) \to L^q_\sigma(S_x)}$ is the least constant for which the inequality

$$\left(\int_S \left(\int_{a(s)}^{\infty} f(y) dy \right)^q d\sigma_x^{(1)}(s) \right)^{1/q} \leq C \left(\int_0^{\infty} f(y)^p v_x^{(1)}(y) dy \right)^{1/p}$$

holds for all $f \geq 0$ where $d\sigma_x^{(1)}(s) = \mathcal{X}_{S_x}(s) k(s,x)^q d\sigma(s)$, $v_x^{(1)}(y) = v(y)$ for $y \in [0, x] \cap Y_x$ and $v_x^{(1)}(y) = \infty$ otherwise. Corollary 2.8 shows that

$$\| \|K_x^{(1)}\|_{L^p_v(Y_x) \to L^q_\sigma(S_x)} \|_{L^\infty_\xi(X)}$$

is comparable to

$$\sup_{\substack{x \in X \\ \xi(x) > 0}} \sup_{y > 0} \left(\int_{a(s) < y < x \leq b(s)} k(s,x)^q d\sigma(s) \right)^{1/q} \left(\int_y^x v(z)^{1-p'} dz \right)^{1/p'}. \qquad (4.4)$$

The norm $\|K_x^{(2)}\|_{L_v^p(Y_x)\to L_\sigma^q(S_x)}$ is the least constant for which the inequality

$$\left(\int_S \left(\int_{a(s)}^{\infty} g(y)dy\right)^q d\sigma_x^{(2)}(s)\right)^{1/q} \le C\left(\int_0^{\infty} g(y)^p v_x^{(2)}(y)dy\right)^{1/p}$$

holds for all $g \ge 0$ where $d\sigma_x^{(2)}(s) = \mathcal{X}_{S_x}(s)d\sigma(s)$, $v_x^{(2)}(y) = \overline{k}(x,y)^{-p}v(y)$ for $y \in [0,x] \cap Y_x$ and $v_x^{(1)}(y) = \infty$ otherwise. Corollary 2.8 shows that

$$\| \|K_x^{(2)}\|_{L_v^p(Y_x)\to L_\sigma^q(S_x)}\|_{L_\xi^\infty(X)}$$

is comparable to

$$\sup_{\substack{x\in X\\ \xi(x)>0}} \sup_{y>0} \left(\int_{a(s)<y<x\le b(s)} d\sigma(s)\right)^{1/q} \left(\int_y^x \overline{k}(x,z)^{p'} v(z)^{1-p'} dz\right)^{1/p'}.$$

Since $\overline{k}(x,z) = 0$ when $x \notin b(S)$ and $\overline{k}(x,z) \approx k(t,z)$ when $x = b(t)$ the last expression is comparable to

$$\sup_{\substack{t\in S\\ \xi(b(t))>0}} \sup_{y>0} \left(\int_{a(s)<y<b(t)\le b(s)} d\sigma(s)\right)^{1/q} \left(\int_y^{b(t)} k(t,z)^{p'} v(z)^{1-p'} dz\right)^{1/p'}. \tag{4.5}$$

The norm $\|K_x^{(3)}\|_{L_v^p(Y_x)\to L_\sigma^q(S_x)}$ is the least constant for which the inequality

$$\left(\int_S \left(\int_0^{b(s)} k(s,y)f(y)dy\right)^q d\sigma_x^{(3)}(s)\right)^{1/q} \le C\left(\int_0^{\infty} f(y)^p v_x^{(3)}(y)dy\right)^{1/p}$$

holds for all $f \ge 0$ where $d\sigma_x^{(3)}(s) = \mathcal{X}_{S_x}(s)d\sigma(s)$, $v_x^{(3)}(y) = v(y)$ for $y \in [x,\infty) \cap Y_x$ and $v_x^{(3)}(y) = \infty$ otherwise. By Theorem 2.6 the norm

$$\| \|K_x^{(3)}\|_{L_v^p(Y_x)\to L_\sigma^q(S_x)}\|_{L_\xi^\infty(X)}$$

is comparable to the maximum of

$$\sup_{\substack{x\in X\\ \xi(x)>0}} \sup_{y>0} \left(\int_{\{s: a(s)\le x<y<b(s)\}} k(s,y)^q \, d\sigma(s) \right)^{1/q} \left(\int_x^y v(z)^{1-p'} dz \right)^{1/p'} \tag{4.6}$$

and

$$\sup_{\substack{x\in X\\ \xi(x)>0}} \sup_{s\in S} \left(\int_{\{t: a(t)\le x<b(s)\le b(t)\}} d\sigma(t) \right)^{1/q} \left(\int_x^{b(s)} k(s,y)^{p'} v(y)^{1-p'} dy \right)^{1/p'}. \tag{4.7}$$

The maximum of the expressions (4.4) and (4.6) is comparable to

$$\sup_{\substack{\{(x,y):x<y\}\\ \xi(x)>0}} \left(\int_{\substack{a(s)\le x\\ y\le b(s)}} k(s,y)^q \, d\sigma(s) \right)^{1/q} \left(\int_x^y v(z)^{1-p'} dz \right)^{1/p'}$$

which is comparable to $\mathcal{A}_1$ because $\xi(x_1) > 0$.

In a similar way we see that the maximum of (4.5) and (4.7) is comparable to $\mathcal{A}_2$. This completes the proof.

5 NORMALIZING MEASURES

The results of the previous section depend on the existence of a discrete normalizing measure for the functions a and b. Here we prove that such a measure exists whenever a and b are similarly ordered in the following sense.

DEFINITION 5.1 *Let $\mathcal{I} = \{[c,d] : 0 \le c \le d \le \infty\}$ and define a partial order on $\mathcal{I}$ by $[c,d] \prec [\bar{c},\bar{d}]$ provided $c \le \bar{c}$ and $d \le \bar{d}$. We say that non-negative functions a and b on S are* similarly ordered *provided the set $\{[a(s),b(s)] : s \in S\}$ is a totally ordered subset of $\mathcal{I}$.*

To construct a discrete normalizing measure ξ, we need the set X_0 of atoms of ξ. This set is constructed in the next theorem.

THEOREM 5.2 *If $\mathcal{T}$ is a totally ordered subset of $\mathcal{I}$ then there exists a subset X_0 of $[0,\infty]$ such that $1 \leq (X_0 \cap [c,d]) \leq 3$ for all $[c,d] \in \mathcal{T}$.*

Proof A straightforward application of Zorn's Lemma shows that we may assume without loss of generality that $\mathcal{T}$ is a maximal totally ordered subset of $\mathcal{I}$. That is, we may assume that the only totally ordered subset of $\mathcal{I}$ which contains $\mathcal{T}$ is $\mathcal{T}$ itself. It follows from this assumption that $\cup\mathcal{T} = [0,\infty]$.

If $x \in [0,\infty]$ define $Lx = \inf\{\overline{c} : x \in [\overline{c},\overline{d}] \in \mathcal{T}\}$ and $Mx = \sup\{\overline{d} : x \in [\overline{c},\overline{d}] \in \mathcal{T}\}$. Clearly, $Lx \leq x \leq Mx$. If $x \leq y$ and $x \in [\overline{c},\overline{d}] \in \mathcal{T}$ then either $y \in [\overline{c},\overline{d}]$ or $y > \overline{d}$. In the former case $My \geq \overline{d}$ by definition and in the latter we have $My \geq y > \overline{d}$. Taking the supremum over all such $\overline{d}$ proves the first half of:

$$\text{If } x \leq y \text{ then } Mx \leq My \text{ and } Lx \leq Ly. \tag{5.1}$$

The other half is proved similarly.

We now establish the first half of:

$$\text{For each } x, [Lx,x] \in \mathcal{T} \text{ and } [x,Mx] \in \mathcal{T}. \tag{5.2}$$

Once again the second half may be proved similarly. By the maximality of $\mathcal{T}$, if we show that $\{[Lx,x]\} \cup \mathcal{T}$ is totally ordered then $[Lx,x] \in \mathcal{T}$ will follow. To do this we fix $[c,d] \in \mathcal{T}$ and show that either $[c,d] \prec [Lx,x]$ or $[Lx,x] \prec [c,d]$. If $x \leq c$ then $[Lx,x] \prec [c,d]$. If $c < x \leq d$ then $Lx \leq c$ by definition so again $[Lx,x] \prec [c,d]$. In the remaining case, when $d < x$, we see that whenever $x \in [\overline{c},\overline{d}] \in \mathcal{T}$ we have $d < x \leq \overline{d}$ so $[c,d] \prec [\overline{c},\overline{d}]$ because $\mathcal{T}$ is totally ordered. Thus $c \leq \overline{c}$ and, taking the infimum over all such $[\overline{c},\overline{d}]$, we conclude that $c \leq Lx$ so $[c,d] \prec [Lx,x]$. We have shown that $\{[Lx,x]\} \cup \mathcal{T}$ is totally ordered and hence $[Lx,x] \in \mathcal{T}$.

For each $x \in [0,\infty]$ define the subset E_x of $[0,\infty]$ as follows.

$$E_x = \left(\bigcup_{k=1}^{\infty} [L^k x, L^{k-1}x]\right) \cup \left(\bigcup_{k=1}^{\infty} [M^{k-1}x, M^k x]\right).$$

Here the exponents represent repeated application of the operator and $L^0 x = x = M^0 x$. It is clear that $E_x = \bigcup_{k=1}^{\infty} [L^k x, M^k x]$ and hence that

E_x is an interval (or a single point) containing x. It is important to observe that the operators L and M fix the sets E_x. That is:

$$\text{If } y \in E_x \text{ then } Ly \in E_x \text{ and } My \in E_x. \tag{5.3}$$

We prove the second half only. If $y \in [M^{k-1}x, M^k x]$ for some $k \geq 1$ then (5.1) shows that $My \in [M^k x, M^{k+1}x] \subset E_x$. If $y \in [L^k x, L^{k-1}x]$ for some $k \geq 1$ then $y \leq L^{k-1}x \leq x$ so $L^k x \leq y \leq My \leq Mx \leq M^k x$ and again we have $My \in E_x$.

It follows by induction from (5.3) that if $y \in E_x$ then $L^k y$ and $M^k y$ are in E_x. Since E_x is an interval, $[L^k y, M^k y] \subset E_x$ and hence $E_y \subset E_x$. Thus we have:

$$\text{If } y \in E_x \text{ then } E_y \subset E_x. \tag{5.4}$$

Next we improve this to:

$$\text{If } y \in E_x \text{ then } E_y = E_x. \tag{5.5}$$

Suppose first that $y \in [L^k x, L^{k-1}x]$ for some $k \geq 1$. We have $y \leq L^{k-1}x \leq x$ and since E_y is an interval it will follow that $x \in E_y$ if there is any point of E_y greater than or equal to x. Suppose for the sake of contradiction that $M^n y < x$ for all $n \geq 0$. Choose m as large as possible so that $M^n y < L^m x$ for all n. This is possible because the property holds for $m = 0$ and fails for $m = k$. Now choose $n \geq 0$ so that $L^{m+1}x \leq M^n y$ and we have $M^n y \in [L^{m+1}x, L^m x]$ so the definition of M yields $M^{n+1}y \geq L^m x$ contradicting the choice of m. This contradiction shows that $x \in E_y$. We may now apply (5.4) twice to get $E_y = E_x$. The proof in the case $y \in [M^{k-1}x, M^k x]$ is analogous.

Since $x \in E_x$ and (5.5) holds we see that the sets E_x partition $[0, \infty]$ so we may choose a set of representatives $\{x_j : j \in J\}$, for some index set J, such that $\bigcup_{j \in J} E_{x_j} = [0, \infty]$ and $E_{x_i} \cap E_{x_j} = \emptyset$ whenever $i, j \in J$ with $i \neq j$. Define the set X_0 to be

$$X_0 = \{M^k x_j, L^k x_j : j \in J, k = 0, 1, \ldots\}.$$

It remains to verify that X_0 has the desired property. If $[c, d] \in \mathcal{T}$ then choose $j \in J$ so that $c \in E_{x_j}$. We suppose that $c \in [M^{k-1}x_j, M^k x_j]$ for some $k \geq 1$ since if $c \in [L^{k+1}x_j, L^k x_j]$ for some $k \geq 1$ the argument is similar. Either $c \in X_0$ or $c \in (M^{k-1}x_j, M^k x_j)$. In the latter case we have $d \geq M^k x_j$ because (5.2) holds and $\mathcal{T}$ is totally ordered. In both cases there is at least one point of X_0 in $[c, d]$ so $1 \leq \#(X_0 \cap [c, d])$.

To show that $\#(X_0 \cap [c, d]) \leq 3$ it is enough to show that at most one point of X_0 is in (c, d). Since $[c, d] \subset [c, Mc] \subset E_c = E_{x_j}$ the only points of X_0 that may be in (c, d) are points of the form $M^k x_j$ or $L^k x_j$ for some $k \geq 0$. This is because all other points of X_0 are in some E_{x_i}, disjoint from E_{x_j}. If $M^k x_j \in (c, d)$ for some $k \geq 0$ then $M^{k+1} x_j \geq d$ and if $L^k x_j \in (c, d)$ for some $k \geq 0$ then $L^{k+1} x_j \leq c$ so at most one such point can be in (c, d). This completes the proof.

COROLLARY 5.3 *If non-negative functions a and b on S are similarly ordered then there is a discrete normalizing measure for* (a, b).

Proof Since $\{[a(s), b(s)] : s \in S\}$ is totally ordered, there exists a subset X_0 of $[0, \infty]$ satisfying $1 \leq \#(X_0 \cap [a(s), b(s)]) \leq 3$ for all $s \in S$. Let ξ be counting measure on the subset $X_0 \backslash \{\infty\}$ of $[0, \infty)$. Since $\infty \notin [a(s), b(s)]$ for any $s \in S$ we have

$$1 \leq \int_{a(s)}^{b(s)} d\xi \leq 3$$

for all $s \in S$.

While a discrete normalizing measure exists whenever a and b are similarly ordered, the construction can be somewhat complicated. In many cases, however, it is easy to discover normalizing measures.

EXAMPLE 5.4 Let $S = [0, \infty), a(s) = 0$, and $b(s) = s$. The Dirac measure at 0 is a discrete normalizing measure for a and b.

EXAMPLE 5.5 Let $S = [0, \infty), a(s) = s$, and $b(s) = s + L$. Lebesgue measure is a normalizing measure for a and b and counting measure on the set $\{n + L : n = 0, 1, \ldots\}$ is a discrete normalizing measure.

EXAMPLE 5.6. Fix A and B with $0 < A < B$. Let $S = [0, \infty), a(s) = As$, and $b(s) = Bs$. The measure dx/x is a normalizing measure for a and b and counting measure on the set $\{(B/A)^n : n = 0, \pm 1, \pm 2, \dots\}$ is a discrete normalizing measure.

EXAMPLE 5.7 (cf. [4, Theorem 2.5]) Let $S = [0, \infty)$. Suppose a and b are increasing, differentiable functions satisfying $a(0) = b(0) = 0$, $a(\infty) = b(\infty) = \infty$, and $0 < a(s) < b(s) < \infty$ for $0 < s < \infty$. Fix $x_0 \in (0, \infty)$ and define $x_n = (b \circ a^{-1})^n(x_0)$ for each $n \in \mathbf{Z}$. Then counting measure on $\{x_n : n \in \mathbf{Z}\}$ is a discrete normalizing measure for a and b. Also, ξ defined by

$$d\xi(x) = \sum_{n \in \mathbf{Z}} \mathcal{X}_{[x_n, x_{n+1}]} d(b \circ a^{-1})^{-n}(x)$$

is a normalizing measure for a and b.

6 APPLICATION TO TAYLOR APPROXIMATION

Suppose F is an $n+1$ times differentiable function on $(0, \infty)$. The nth degree Taylor polynomial of F, centred at a, is

$$P_{n,a}(F)(b) = F(a) + F'(a)(b-a) + \cdots + \frac{F^{(n)}(a)}{n!}(b-a)^n$$

and the remainder, $R_{n,a}(F)(b) \equiv F(b) - P_{n,a}(F)(b)$, may be expressed in the form

$$R_{n,a}(F)(b) = \frac{1}{n!} \int_a^b (b-y)^n F^{(n+1)}(y) dy. \tag{6.1}$$

If we let a and b vary with s we recognize the above remainder as an operator of the form (4.1) applied to $F^{(n+1)}$. Theorems 4.3 and 4.4 can therefore be used to control the accuracy of the approximation by a Taylor polynomial as *the centre and the point of evaluation vary.* The control is in terms of the size of the $n+1$ derivative of the function. Rather than state this as a general result, we provide a simple example in which $F(s)$ is approximated by its Taylor polynomial centred at $s/2$.

EXAMPLE 6.1 *Let n be a positive integer. There exists a positive constant C such that the inequality*

$$\left(\int_0^\infty R_{n,s/2}(F)(s)^2(s+1)^{-2n-3}ds\right)^{1/2} \leq C\left(\int_0^\infty F^{(n+1)}(y)^4 dy\right)^{1/4} \tag{6.2}$$

for all $n+1$ *times differentiable functions* F.

Proof We apply Theorem 4.3 with $p = 4, q = 2, k(s,y) = (s-y)^n/n!, a(s) = s/2, b(s) = s, v(y) = 1$, and $d\sigma(s) = (s+1)^{-2n-3}ds$. In view of (6.1), the conclusion of the Theorem 4.3, with f replaced by $F^{(n+1)}$, will yield (6.2).

To complete the proof we check the hypotheses of Theorem 4.3. As in Example 5.6 we have $\int_{s/2}^s dx/x = \log(2)$ so the measure dx/x is a normalizing measure for (a, b). Since

$$2^{-n}(s-y)^n \leq (s-t)^n + (t-y)^n \leq 2(s-y)^n \quad \text{for} \quad y \leq t \leq s,$$

the kernel k satisfies the GHO condition.

Simple-minded estimates show that $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$, and $\mathcal{B}_4$ are all finite. We show only the first.

$$\begin{aligned}
(n!)^4\mathcal{B}_1^4 &= \int_0^\infty \int_{x/2}^x \left(\int_x^{2y}(s-x)^{2n}(s+1)^{-2n-3}ds\right)^2 (x-y)^2 dy\, \frac{dx}{x} \\
&\leq \int_0^\infty \int_{x/2}^x ((2y-x)(2y-x)^{2n}(x+1)^{-2n-3})^2(x-y)^2 dy\, \frac{dx}{x} \\
&\leq \int_0^\infty (x/2)x^{4n+2}(x+1)^{-4n-6}(x/2)^2\, \frac{dx}{x} < \infty.
\end{aligned}$$

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