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GENERALIZED HELICAL IMMERSIONS

By

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Introduction.

In this paper, we assume that all geodesics are parametrized by the arclength. Let f be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\tilde{M}$. If geodesics in M are viewed as specific curves in $\tilde{M}$, what are the shape of $f(M)$? Several geometricians studied this problem. K. Sakamoto characterized an isometric immersion f of a complete connected Riemannian manifold M into a Euclidean space or a sphere such that every geodesic in M is viewed as a helix in the ambient space and that the order and the Frenet curvatures of the helix are independent of the choice of the geodesic (cf. [15], [16]). In [5], D. Ferus and S. Schirrmacher investigated an isometric immersion f of a compact connected Riemannian manifold M into a Euclidean space $\boldsymbol{R}^m$ satisfying the following condition:

(A) Almost every geodesic in M is viewed as a generic helix in $\boldsymbol{R}^m$.

Here "almost every geodesic" means that the tangent vectors of such geodesics fill the unit tangent bundle of M up to a closed set of measure zero and a generic helix means a helix of even order such that the closure of the image coincides with the lowest dimensional Clifford torus containing it. In [4] and [5], they showed that the condition (A) is equivalent to the following two conditions, respectively:

(B) f is extrinsic symmetric in the sense of [4].

(C) The second fundamental form of f is parallel.

In this paper, we consider an isometric immersion f of a Riemannian manifold M into a Riemannian manifold $\tilde{M}$ such that every geodesic in M is viewed as a helix in $\tilde{M}$, where the order of the helix may depend on the choice of the geodesic. We call such a immersion a *generalized helical immersion* and the highest order of those helices the *order of f*. First, we show that all isometric immersions with parallel second fundamental form are generalized helical. Conversely, it is very interesting to investigate in what case a generalized helical immersion has the

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parallel second fundamental form. We tackle this problem for a generalized helical immersion of a compact Riemannian manifold into a Euclidean space. Concretely, we can obtain the following result.

THEOREM. *Let f be a generalized helical immersion of order $2d$ of a compact connected Riemannian manifold M into a Euclidean space $\boldsymbol{R}^m$. Assume that the following condition $(*)$ hold:*

$(*)$ *For each $p \in M$, there is at least one geodesic in M through p which is viewed as a generic helix of order $2d$ in $\boldsymbol{R}^m$.*

Then f has the parallel second fundamental form and hence f is congruent to the composition of the standard isometric embedding of a symmetric R-space M_0 and a totally geodesic embedding.

Furthermore, we can show that the symmetric R-space M_0 is of rank d. Note that this condition $(*)$ is very weaker than the above condition (A) in a sense.

In Sect. 1 and 2, we prepare basic notations, definitions and lemmas. In Sect. 3, we show that all isometric immersions with parallel second fundamental form are generalized helical, where the ambient space may be a general Riemannian manifold. In Sect. 4, we investigate the order of the standard isometric embedding of a symmetric R-space into a Euclidean space. In Sect. 5, we characterize a generalized helical immersion f of a compact connected Riemannian manifold M into a Euclidean space satisfying the above condition $(*)$, where we use results in Sect. 2 and 4. In Sect. 6, we obtain results analogous to those of Sect. 5 in the case where the ambient space is a sphere. In Sect. 7, in the case where M is a Riemannian homogeneous space G/K and f is a G-equivariant, we state results deduced from those in Sect. 5 and 6.

Throughout this paper, unless otherwise mentioned, we assume that all geometric objects are of class C^∞ and all manifolds are connected ones without boundary.

1. Notations and definitions.

In this section, we shall state basic notations and definitions. Let $\sigma : I \to M$ be a curve in a Riemannian manifold M parametrized by the arclength s, where I is an open interval of the real line $\boldsymbol{R}$. Denote by v_0 the velocity vector field $\dot{\sigma}$ of σ. Set $\lambda_1 := \|\nabla_{v_0} v_0\|$, where ∇ is the Levi-Civita connection of M. If λ_1 is not identically zero, then we define v_1 by $\nabla_{v_0} v_0 = \lambda_1 v_1$ on $I_1 := \{s \in I \mid \lambda_1(s) \neq 0\}$. Set $\lambda_2 := \|\nabla_{v_0} v_1 + \lambda_1 v_0\|$. If λ_2 is not identically zero, then we define v_2 by $\nabla_{v_0} v_1 +$

$\lambda_1 v_0 = \lambda_2 v_2$ on $I_2 := \{s \in I_1 \mid \lambda_2(s) \neq 0\}$. Inductively, we define λ_i, I_i and v_i $(i \geq 3)$. If λ_1 is identically zero on I, that is, σ is a geodesic, then σ is said to be of order 1. If λ_{d-1} is not identically zero on I_{d-2} and λ_d is identically zero on I_{d-1}, then σ is said to be of order d, where $d \geq 2$. If σ is of order d, then we have a matrix equation

$$\nabla_{v_0}(v_0, v_1, \dots, v_{d-1}) = (v_0, v_1, \dots, v_{d-1})\Lambda \tag{1.1}$$

on I_{d-1}, where Λ is a matrix of type (d, d) defined by

$$\Lambda = \begin{pmatrix} 0 & -\lambda_1 & 0 & \cdots & \cdots & 0 \\ \lambda_1 & 0 & -\lambda_2 & \ddots & & \vdots \\ 0 & \lambda_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\lambda_{d-1} \\ 0 & \cdots & \cdots & 0 & \lambda_{d-1} & 0 \end{pmatrix}.$$

Let I_{d-1}^0 be a component of I_{d-1}. Then the restriction of the relation (1.1) to I_{d-1}^0, $\lambda_i|_{I_{d-1}^0}$ and $v_i|_{I_{d-1}^0}$ $(1 \leq i \leq d-1)$ are called the Frenet formula, the i-th Frenet curvature and the i-th Frenet normal vector of $\sigma|_{I_{d-1}^0}$, respectively. Also, if $\lambda_i|_{I_{d-1}^0}$ $(1 \leq i \leq d-1)$ are constant along $\sigma|_{I_{d-1}^0}$, then $\sigma|_{I_{d-1}^0}$ is called a *helix of order d*. Then we note that $I_i^0 = I$ $(1 \leq i \leq d-1)$. In particular, a helix σ_1 (resp. σ_2) of order $2d$ (resp. $2d+1$) in an m-dimensional Euclidean space $\boldsymbol{R}^m$ is expressed as follows:

$$\sigma_1(s) = c_0 + \sum_{i=1}^{d} r_i(e_{2i-1}\cos a_i s + e_{2i}\sin a_i s) \tag{1.2}$$

$$(\text{resp. } \sigma_2(s) = c_0 + \sum_{i=1}^{d} r_i(e_{2i-1}\cos a_i s + e_{2i}\sin a_i s) + bse_{2d+1}),$$

where c_0 is a constant vector of $\boldsymbol{R}^m$, $e_1, \dots, e_{2d+1}$ is an orthonormal system of $\boldsymbol{R}^m$, r_i $(1 \leq i \leq d)$ and b are positive constants and a_i $(1 \leq i \leq d)$ are mutually distinct positive constant. Thus $\operatorname{Im}\sigma_1$ is contained in the d-dimensional Clifford torus

$$T := \left\{ c_0 + \sum_{i=1}^{d} r_i(e_{2i-1}\cos\theta_i + e_{2i}\sin\theta_i) \,\middle|\, 0 \leq \theta_i < 2\pi(i = 1, \dots, d) \right\}.$$

If $\overline{\operatorname{Im}\sigma_1} = T$ holds, then σ_1 is said to be *generic*, where $\overline{\operatorname{Im}\sigma_1}$ is the closure of the image of σ_1. Note that σ_1 is generic if and only if $a_1, \dots, a_d$ are linearly independent over the rational number field $\boldsymbol{Q}$. Let σ be a helix in an m-dimensional sphere S^m and ι a totally umbilical embedding of S^m into $\boldsymbol{R}^{m+1}$. Then, since ι is extrinsic spherical, $\iota \circ \sigma$ is a helix in $\boldsymbol{R}^{m+1}$ by Corollary 3.3 of [17]. Furthermore, since $\mathrm{Im}(\iota \circ \sigma)$ is contained in a compact set $\iota(S^m)$, the order of $\iota \circ \sigma$ is even. Let $2d$ be the order of $\iota \circ \sigma$. It is shown that the order of σ is $2d-1$ (resp. $2d$) if the centroid of the d-dimensional Clifford torus T containing $\mathrm{Im}(\iota \circ \sigma)$ coincides (resp. does not coincide) with the center of S^m. If $\iota \circ \sigma$ is generic, then we shall call σ a *generic helix* (*in* S^m).

Let f be an isometric immersion of an n-dimensional Riemannian manifold M^n into an m-dimensional Riemannian manifold $\tilde{M}^m$. We shall identify the tangent space T_pM of M at p with the subspace $f_*(T_pM)$ of $T_{f(p)}\tilde{M}$, where f_* is the differential of f. Denote by ∇ (resp. $\tilde{\nabla}$) the Levi-Civita connection on M (resp. $\tilde{M}$) and A, h and $\nabla^{\perp}$ the shape operator, the second fundamental form and the normal connection of f, respectively. Denote by $\overline{\nabla}$ both $\nabla^* \otimes \cdots \otimes \nabla^* \otimes \nabla^{\perp}$ and $\nabla^{\perp^*} \otimes \nabla^* \otimes \cdots \otimes \nabla^* \otimes \nabla$, where ∇^* is the dual connection of ∇. Also, we shall denote the i-th order derivative of h (resp. A) with respect to $\overline{\nabla}$ by $\overline{\nabla}^i h$ (resp. $\overline{\nabla}^i A$). If, for every geodesic σ in M, $f \circ \sigma$ is a helix of order d and the Frenet curvatures of $f \circ \sigma$ do not depend on the choice of σ, then f is called a *helical immersion of order* d. In this paper, if, for every geodesic σ in M, $f \circ \sigma$ is a helix of order at most d and there is at least one geodesic σ_0 in M such that $f \circ \sigma_0$ is a helix of order d, then we shall call f *a generalized helical immersion of order* d.

2. Basic lemmas.

In this section, we prepare basic lemmas which are used in Sect. 5. Let f be an isometric immersion of an n-dimensional Riemannian manifold M^n into an m-dimensional Riemannian manifold $\tilde{M}^m$. Take a geodesic $\sigma : I \to M^n$. Denote by v_0 the velocity vector field $\dot{\sigma}$ of σ. Assume that $\tilde{\sigma} := f \circ \sigma$ is a helix of order d in $\tilde{M}^m$. Let λ_i (resp. v_i) be the i-th Frenet curvature (resp. the i-th Frenet normal vector) of $\tilde{\sigma}$ $(i = 1, \ldots, d-1)$. For convenience, let $\lambda_i = 0$ and $v_i = 0$ $(i \geq d)$. In terms of the Gauss formula and the Weingarten formula of f and the Frenet formula of $\tilde{\sigma}$, we can deduce the following relations.

LEMMA 2.1. *The vector fields* $\lambda_1 \cdots \lambda_i v_i$ $(i \geq 1)$ *along* σ *are expressed as follows*:

$$(\mathrm{F}_1) \quad \lambda_1 v_1 = h(v_0, v_0),$$

$$(\mathrm{F_i})\quad \lambda_1\cdots\lambda_i v_i = \alpha_i v_0 + \sum_{j=0}^{i-2}(\overline{\nabla}^j A)_{\xi_{ij}}(v_0,\ldots,v_0) + \sum_{j=0}^{i-1}(\overline{\nabla}^j h)(v_0,\ldots,v_0,w_{ij})\quad (i\ge 2),$$

where α_i $(i\ge 2)$, ξ_{ij} $(i\ge 2, 0\le j\le i-2)$ *and* w_{ij} $(i\ge 2, 0\le j\le i-1)$ *are given by*

$$\begin{cases}\alpha_2=\lambda_1{}^2, \alpha_3=0, \alpha_i=\lambda_{i-1}{}^2\alpha_{i-2}\quad (i\ge 4),\\ w_{20}=0, w_{21}=v_0, \xi_{20}=-h(v_0,v_0)\\ \xi_{ij}=\lambda_{i-1}{}^2\xi_{i-2,j}+\xi_{i-1,j-1}+\nabla^{\perp}_{v_0}\xi_{i-1,j}\quad (i\ge 3, 1\le j\le i-2)\\ \xi_{i0}=\lambda_{i-1}{}^2\xi_{i-2,0}+\nabla^{\perp}_{v_0}\xi_{i-1,0}-\sum_{j=0}^{i-2}(\overline{\nabla}^j h)(v_0,\ldots,v_0,w_{i-1,j})\quad (i\ge 3)\\ w_{ij}=\lambda_{i-1}{}^2 w_{i-2,j}+w_{i-1,j-1}+\nabla_{v_0}w_{i-1,j}\quad (i\ge 3, 1\le j\le i-1)\\ w_{i0}=\alpha_{i-1}v_0+\lambda_{i-1}{}^2 w_{i-2,0}+\nabla_{v_0}w_{i-1,0}\\ \qquad +\sum_{j=0}^{i-3}(\overline{\nabla}^j A)_{\xi_{i-1,j}}(v_0,\ldots,v_0)\quad (i\ge 3).\end{cases}$$

Here let $\xi_{ii}=\xi_{i,i-1}=0$, $w_{ii}=w_{i,i+1}=0$ $(i\ge 1)$ *and* $w_{10}=v_0$.

For each unit tangent vector w of M, we denote the maximal geodesic in M parametrized by the arclength s whose velocity vector field at $s=0$ is equal to w by σ_w and the osculating order of $f\circ\sigma_w$ at $s=0$ by $o(w)$. For each $p\in M$, set $V_{p,i}:=\{w\in S_pM \mid o(w)=i\}$ $(i\ge 1)$, where S_pM is the unit tangent sphere of M at p. We define a function $\hat{\lambda}_i$ $(i\ge 1)$ on the unit tangent bundle SM of M by

$$\hat{\lambda}_i(w):=\begin{cases}\lambda_i^w(0) & (w\in\bigcup_{i+1\le j}\bigcup_{p\in M}V_{p,j})\\ 0 & (w\in\bigcup_{1\le j\le i}\bigcup_{p\in M}V_{p,j}),\end{cases}$$

where λ_i^w is the i-th Frenet curvature of the restriction $f\circ\sigma_w|_{I^0}$ of $f\circ\sigma_w$ to a sufficiently small neighbourhood I^0 of $0\left(w\in\bigcup_{i+1\le j}\bigcup_{p\in M}V_{p,j}\right)$. Also, we define a map $\hat{v}_i : SM\to T\tilde{M}$ $(i\ge 1)$ by

$$\hat{v}_i(w) := \begin{cases} v_i^w(0) & (w \in \bigcup_{i+1\le j} \bigcup_{p\in M} V_{p,j}) \\ 0 & (w \in \bigcup_{1\le j\le i} \bigcup_{p\in M} V_{p,j}), \end{cases}$$

where v_i^w is the i-th Frenet normal vector of the restriction $f \circ \sigma_w|_{I^0}$ of $f \circ \sigma_w$ to a sufficiently small neighbourhood I^0 of 0 ($w \in \bigcup_{i+1\le j} \bigcup_{p\in M} V_{p,j}$). It is easy to show that $\hat{\lambda}_i$ is continuous on $\bigcup_{i\le j} \bigcup_{p\in M} V_{p,j}$ ($i \ge 1$). Here we shall give datas of $V_{p,i}$ and $\hat{\lambda}_i$ for the Clifford embedding $f_0 : S^1(1) \times S^1(1) \hookrightarrow \boldsymbol{R}^4$. The sets $V_{p,i}$ are as follows:

$$V_{p,1} = \emptyset,\ V_{p,2} = \left\{ \pm e_1, \pm e_2, \pm \frac{e_1+e_2}{\sqrt{2}}, \pm \frac{e_1-e_2}{\sqrt{2}} \right\},$$

$$V_{p,3} = \emptyset,\ V_{p,4} = S_pM \setminus V_{p,2},\ V_{p,i} = \emptyset\ (i \ge 5)$$

and the functions $\hat{\lambda}_i$ are as follows:

$$\hat{\lambda}_1(e_1 \cos\theta + e_2 \sin\theta) = \sqrt{\cos^4\theta + \sin^4\theta}$$

$$\hat{\lambda}_2(e_1 \cos\theta + e_2 \sin\theta) = \frac{|\sin 4\theta|}{4\sqrt{\cos^4\theta + \sin^4\theta}}$$

$$\hat{\lambda}_3(e_1 \cos\theta + e_2 \sin\theta) = \begin{cases} \dfrac{|\sin 2\theta|}{2\sqrt{\cos^4\theta + \sin^4\theta}} & \left(\theta \ne \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right) \\ 0 & \left(\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right), \end{cases}$$

$$\hat{\lambda}_i(e_1 \cos\theta + e_2 \sin\theta) = 0 \quad (i \ge 4),$$

where (e_1, e_2) is an orthonormal tangent frame at p such that e_1 (resp. e_2) is tangent to the fibre of the projection of $S^1(1) \times S^1(1)$ onto the first (resp. the second) component and $0 \le \theta < 2\pi$. This implies that f_0 is a generalized helical embedding of order 4. Also, we see that $\hat{\lambda}_3$ is contionuos on $\bigcup_{j\ge 3} V_{p,j} (= V_{p,4})$ but so is not on $\bigcup_{j\le 2} V_{p,j} (= V_{p,2} \cup V_{p,4})$.

From Lemma 2.1, we can prove the following lemma.

LEMMA 2.2. *Assume that f is a generalized helical and $V_{p,d} \ne \emptyset$ and $V_{p,i} = \emptyset$ $(i \ge d+1)$ for $p \in M$. Then the set $V_{p,i}$ $(1 \le i \le d-1)$ are closed sets of measure zero in S_pM and $V_{p,d}$ is a dense open set is S_pM.*

PROOF. According to Lemma 2.1, for each i $(\le d-1)$, there exist non-zero polynomial functions P_i and Q_i on T_pM such that $P_i\hat{\lambda}_1^2 \cdots \hat{\lambda}_i^2 = Q_i$ on S_pM and that P_i has no zero point on $\bigcup_{i-1\le j\le d} V_{p,j}$. Hence we have

$$V_{p,1} = \{v \in S_pM \mid \hat{\lambda}_1(v) = 0\} = \{v \in S_pM \mid Q_1(v) = 0\},$$

$$V_{p,2} = \{v \in S_pM \backslash V_{p,1} \mid \hat{\lambda}_2(v) = 0\} = \{v \in S_pM \backslash V_{p,1} \mid Q_2(v) = 0\},$$

$$\vdots$$

$$V_{p,d-1} = \left\{ v \in S_pM \backslash \left(\bigcup_{1 \le i \le d-2} V_{p,i} \right) \middle| \hat{\lambda}_{d-1}(v) = 0 \right\}$$

$$= \left\{ v \in S_pM \backslash \left(\bigcup_{1 \le i \le d-2} V_{p,i} \right) \middle| Q_{d-1}(v) = 0 \right\}.$$

Thus we see that $V_{p,i}$ $(1 \le i \le d-1)$ are closed sets of measure zero in S_pM. Therefore, $V_{p,d}$ is a dense open set in S_pM. $\square$

3. Isometric immersions with parallel second fundamental form.

In this section, we shall show that all isometric immersions with parallel second fundamental form are generalized helical. Let f be an isometric immersion of an n-dimensional Riemannian manifold M^n into an m-dimensional Riemannian manifold $\tilde{M}^m$ with parallel second fundamental form and $\sigma : I \to M$ a geodesic in M. Assume that the osculating order of $f \circ \sigma$ at each point is d. Set $v_0 := \dot{\sigma}$ and denote by λ_i (resp. v_i) the i-th Frenet curvature (resp. the i-th Frenet normal vector) of $f \circ \sigma$ $(1 \le i \le d-1)$. Then we can obtain the following fact.

LEMMA 3.1. *The Frenet curvatures λ_i $(1 \le i \le d-1)$ are constant along σ (i.e., $f \circ \sigma$ is a helix) and the following relations hold*:

$$(F_i') \quad \lambda_1 \cdots \lambda_i v_i = \alpha_i v_0 + A_{\xi_i} v_0 + h(w_i, v_0) \quad (1 \le i \le d-1),$$

where α_i, ξ_i and w_i $(1 \le i \le d-1)$ are given by

$$\begin{cases} \alpha_1 = 0, \alpha_2 = {\lambda_1}^2, \alpha_i = {\lambda_{i-1}}^2 \alpha_{i-2} \ (3 \le i \le d-1), \\ w_1 = v_0, w_2 = 0, \xi_1 = 0, \xi_2 = -h(v_0, v_0) \\ w_i = \alpha_{i-1} v_0 + {\lambda_{i-1}}^2 w_{i-2} + A_{\xi_{i-1}} v_0 \ (3 \le i \le d-1), \\ \xi_i = {\lambda_{i-1}}^2 \xi_{i-2} - h(w_{i-1}, v_0) \ (3 \le i \le d-1). \end{cases}$$

PROOF. We shall prove in case of $d \ge 4$. First, by using the Gauss formula and the Frenet formula, we have

$$\lambda_1 v_1 = \tilde{\nabla}_{v_0} v_0 = h(v_0, v_0),$$

which implies (F_1'). Also, from this relation, we have $\lambda_1{}^2 = \langle h(v_0, v_0), h(v_0, v_0)\rangle$. Differentiating this relation in the direction v_0 and using $\overline{\nabla} h = 0$, we have $v_0(\lambda_1{}^2) = 0$. Thus λ_1 is constant along σ. By operating $\tilde{\nabla}_{v_0}$ to (F_1') and using $\overline{\nabla} h = 0$, we have

$$\lambda_1\lambda_2 v_2 = \lambda_1{}^2 v_0 - A_{h(v_0,v_0)} v_0,$$

which implies (F_2'). From this relation, we have

$$\lambda_1{}^2\lambda_2{}^2 = -\lambda_1{}^4 + \langle A_{h(v_0,v_0)} v_0, A_{h(v_0,v_0)} v_0\rangle.$$

Differentiating this relation in the direction v_0, we have $\lambda_1{}^2 v_0(\lambda_2{}^2) = 0$, where we use $\overline{\nabla} A = 0$ and $\overline{\nabla} h = 0$. Thus λ_2 is constant along σ. Assume that (F_{i-1}') and (F_i') hold and $\lambda_1, \ldots, \lambda_i$ are constant along σ, where $2 \le i \le d - 2$. By operating $\tilde{\nabla}_{v_0}$ to (F_i'), we have

$$\lambda_1 \cdots \lambda_{i+1} v_{i+1} = \lambda_i{}^2(\lambda_1 \cdots \lambda_{i-1} v_{i-1}) + A_{(\nabla^{\perp}_{v_0}\xi_i - h(w_i,v_0))} v_0 + h(\alpha_i v_0 + A_{\xi_i} v_0 + \nabla_{v_0} w_i, v_0). \tag{3.1}$$

On the other hand, it follows from $\overline{\nabla} A = 0$ and $\overline{\nabla} h = 0$ that $\nabla^{\perp}_{v_0}\xi_i = 0$ and $\nabla_{v_0} w_i = 0$. Hence, by substituting (F_{i-1}') in (3.1), we can obtain

$$\lambda_1 \cdots \lambda_{i+1} v_{i+1} = \lambda_i{}^2 \alpha_{i-1} v_0 + A_{(\lambda_i{}^2\xi_{i-1} - h(w_i,v_0))} v_0 + h(\alpha_i v_0 + \lambda_i{}^2 w_{i-1} + A_{\xi_i} v_0, v_0),$$

which implies (F_{i+1}'). From this relation, we have

$$\lambda_1{}^2 \cdots \lambda_{i+1}{}^2 = \alpha_i{}^2 + 2\alpha_i\langle A_{\xi_i} v_0, v_0\rangle + \langle A_{\xi_i} v_0, A_{\xi_i} v_0\rangle + \langle h(w_i, v_0), h(w_i, v_0)\rangle.$$

Differentiating this relation in the direction v_0, we can obtain $\lambda_1{}^2 \cdots \lambda_i{}^2 (v_0 \lambda_{i+1}{}^2) = 0$. Thus λ_{i+1} is constant along σ. Therefore, by the induction, the proof is completed. □

REMARK. It is clear that $\alpha_{2i+1} = 0$, $\xi_{2i+1} = 0$ $(1 \le i \le [d/2] - 1)$ and $w_{2j} = 0$ $(1 \le j \le [(d-1)/2])$, where [ ] is the Gauss symbol. Hence we have $v_{2i} \in TM$ $(0 \le i \le [(d-1)/2])$ and $v_{2j+1} \in T^{\perp}M$ $(0 \le j \le [d/2] - 1)$.

From this lemma, we can obtain the following result.

PROPOSITION 3.2. *Let $f : M^n \hookrightarrow \tilde{M}^m$ be an isometric immersion with parallel second fundamental form. Then f is generalized helical immersion of order at most* $\min\{2n, 2(m-n) + 1\}$.

PROOF. Let $\sigma : I \to M$ be a geodesic in M parametrized by the arclength s. Denote by $d(s)$ the osculating order of $f \circ \sigma$ at $(f \circ \sigma)(s)$ and set $d_0 := \max_{s \in I} d(s)$. Also, set $I_k := \{s \in I \mid d(s) = k\}$ $(1 \leq k \leq d_0)$. It is clear that I_{d_0} is open. By the previous lemma, $\hat{\lambda}_i$ $(1 \leq i \leq d_0 - 1)$ are constant on $\dot{\sigma}(I_{d_0})$, where $\hat{\lambda}_i$ $(1 \leq i \leq d_0 - 1)$ are functions on SM defined in Sect. 2. Hence, it follows from the continuity of $\hat{\lambda}_i$ on $\bigcup_{i \leq j} \bigcup_{p \in M} V_{p,j}$ $(1 \leq i \leq d_0 - 1)$ that $I = I_{d_0}$ holds, that is, $f \circ \sigma$ is a helix of order d_0. Therefore, by the arbitrarity of σ, f is generalized helical. Furthermore, by the above remark, we see that f is of order at most $\min\{2n, 2(m-n)+1\}$. □

4. The order of the standard isometric embedding of a symmetric *R*-space.

At the beginning of this section, we shall recall the characterizing theorems of isometric immersions into a Euclidean space and a sphere with parallel second fundamental form, which are used in Sect. 5 and 6.

THEOREM 4.1 ([3]). *Let f be a full isometric immersion of a complete Riemannian manifold M into a Euclidean space with parallel second fundamental form. Then f is congruent to $\phi \circ \pi$ or $(\phi \times id) \circ \pi$, where ϕ is the standard isometric embedding of a symmetric R-space M_0, id is the identity map of an l-dimensional Euclidean space $\boldsymbol{R}^l$ and π is a Riemannian covering of M onto M_0 or $M_0 \times \boldsymbol{R}^l$.*

THEOREM 4.2 ([18]). *Let f be a full isometric immersion of an n-dimensional complete Riemannian manifold M into a sphere with parallel second fundamental form. Then the following statements* (i) *and* (ii) *hold*:

(i) *If f is minimal, then f is congruent to $\bar{\phi} \circ \pi$, where $\bar{\phi}$ is the umbilical reduction of the standard isometric embedding ϕ of a symmetric R-space M_0 to a hypersphere containing $\phi(M_0)$ and π is a Riemannian covering of M onto M_0,*

(ii) *If f is not minimal, then f is congruent to $\psi \circ \bar{\phi} \circ \pi$, where $\bar{\phi}$ and π are as in the above and ψ is a totally umbilical* (*but non-totally geodesic*) *embedding of codimension-*1 *into a sphere and π is a Riemannian covering of M onto M_0.*

Since the standard isometric embedding of a symmetric R-space and the umbilical reduction of one to hypersphere containing the image have the parallel second fundamental form, they are generalized helical by Proposition 3.2. Now we shall investigate the orders of those embeddings. Let $\phi : M_0 \hookrightarrow \boldsymbol{R}^m$ be the standard isometric embedding of a symmetric R-space M_0 of rank d, where the rank of M_0 is the maximal dimension of a flat totally geodesic submanifold in M_0. Take an arbitrary unit tangent vector v of M_0. Since M_0 is of rank d, there is a d-dimensional flat torus T tangent to v totally geodesically embedded into M_0 (see [6, Chapter V, Theorem 6.2]). Let σ_v be a maximal geodesic in M_0 with $\dot{\sigma}_v(0) = v$. Since T is totally geodesic in M_0, σ_v is a geodesic in T, that is, $\phi \circ \sigma_v$ is a curve in $\phi(T)$. On the other hand, since the second fundamental form of ϕ is parallel, for almost every geodesic σ in M_0, $\phi \circ \sigma$ is a generic helix. From this fact, we can show that, for almost every geodesic σ in T, $\phi \circ \sigma$ is a generic helix. So, $\phi|_T$ is extrinsic symmetric. Hence, by Theorem 3 of [4], $\phi(T)$ is a Clifford torus. Therefore, since helices in $\boldsymbol{R}^m$ are given as (1.2), the order of a helix $\phi \circ \sigma_v$ is at most $2d$. Also, we see that, for almost every geodesic σ in T, $\sigma \circ \sigma$ is a helix of order $2d$. This implies that ϕ is of order $2d$. Furthermore, this fact implies that the umbilical reduction $\bar{\phi}$ of ϕ to the hypersphere S^{m-1} containing $\phi(M_0)$ is of order $2d-1$ or $2d$. Let ψ be a totally umbilical (but non-totally geodesic) embedding of S^{m-1} into an m-dimensional sphere S^m and ι a totally umbilical embedding of S^m into $\boldsymbol{R}^{m+1}$. Set $\tilde{\phi} := \psi \circ \bar{\phi}$. It is clear that $\tilde{\phi}$ is generalized helical. Let σ_0 be a geodesic in M_0 such that $\bar{\phi} \circ \sigma_0$ is a helix of order $2d-1$ or $2d$ and T_0 be the d-dimensional flat torus tangent to $\dot{\sigma}_0(0)$ totally geodesically embedded into M_0. It is clear that the centroid of the Clifford torus $(\iota \circ \tilde{\phi})(T_0)$ does not coincide with the center of S^m. Hence the order of $\tilde{\phi} \circ \sigma_0$ is $2d$. This implies that $\tilde{\phi}$ is of order $2d$.

5. Generalized helical immersions into a Euclidean space.

In this section, we shall characterize a generalized helical immersion of a compact Riemannian manifold into a Euclidean space satisfying the condition $(*)$ stated in Introduction. In the sequel, we assume that all geodesics are maximal and denote the maximal geodesic in M parametrized by the arclength s whose velocity vector at $s = 0$ is v $(\in SM)$ by σ_v. First we shall prepare the following lemma.

LEMMA 5.1. *Let f be an isometric immersion of an $n(\geq 2)$-dimensional compact Riemannian manifold M into a Euclidean space. Assume that $f_*(T_qM) = f_*(T_pM)$ holds for every $q \in f^{-1}(f(p))$ and furthermore, for every great circle C in S_pM through a point v_0 of S_pM, there are four unit tangent vectors $u_1, \dots, u_4 \in C$ with $u_i \neq \pm u_j$ $(1 \leq i \neq j \leq 4)$ such that $f \circ \sigma_{u_i}$ is a generic helix $(1 \leq i \leq 4)$. Then $\overline{\nabla} h_p = 0$ holds.*

PROOF. Take an arbitrary $w_0 \in S_pM \backslash \{\pm v_0\}$. Let C be a great circle in S_pM through v_0 and w_0. From the assumption, there exist unit tangent vectors $u_1, \dots, u_4 \in C$ with $u_i \neq \pm u_j$ $(1 \leq i \neq j \leq 4)$ such that $f \circ \sigma_{u_i}$ is a generic helix

$(1 \leq i \leq 4)$. Then, we can show $\overline{\nabla} h(u_i, u_i, u_i) = 0$ $(1 \leq i \leq 4)$ because $f_*(T_q M) = f_*(T_p M)$ holds for every $q \in f^{-1}(f(p))$ and M is compact (see [5, the proof of Theorem]). Hence, since $\overline{\nabla} h$ is symmetric by the Codazzi equation, we see that $\overline{\nabla} h = 0$ on C^3. In particular, we have $(\overline{\nabla} h)(v_0, v_0, v_0) = (\overline{\nabla} h)(w_0, w_0, w_0) = 0$. Thus, from the arbitrarity of w_0, we see that $\overline{\nabla} h(w, w, w) = 0$ holds for every $w \in S_p M$. Therefore, we obtain $\overline{\nabla} h_p = 0$. □

For simplicity, we shall denote the fact that two isometric immersions f_1 and f_2 are congruent by $f_1 \approx f_2$.

In case of $\dim M = 2$, we can show the following result in terms of the previous lemma.

Proposition 5.2. *Let f be a full isometric immersion of a 2-dimensional compact Riemannian manifold M into a Euclidean space. Assume that, for each $p \in M$, there are at least four geodesics in M through p which are viewed as generic helices in the ambient Euclidean space. Then the following* (i), (ii) *or* (iii) *holds*:

(i) *$f \approx \phi_1$, where ϕ_1 is a totally umbilical embedding of a 2-dimensional sphere into a 3-dimensional Euclidean space $\boldsymbol{R}^3$,*

(ii) *$f \approx \phi_2 \circ \pi$, where ϕ_2 is the Veronese embedding of a 2-dimensional real projective space $\boldsymbol{RP}^2$ into a 5-dimensional Euclidean space $\boldsymbol{R}^5$ and π is a Riemannian covering of M onto $\boldsymbol{RP}^2$,*

(iii) *$f \approx \phi_3 \circ \pi'$, where ϕ_3 is the Clifford embedding of a 2-dimensional flat torus T into a 4-dimensional Euclidean space $\boldsymbol{R}^4$ and π' is a Riemannian covering of M onto T.*

Proof. Let $U := \{p \in M \mid (\forall q \in f^{-1}(f(p)))[f_*(T_q M) = f_*(T_p M)]\}$. From Lemma 5.1, we have $\overline{\nabla} h = 0$ on U. It is clear that U is dense in M. Therefore, $\overline{\nabla} h = 0$ holds on M. Hence, according to Theorem 4.1, f is congruent to the composition $\phi \circ \pi$ of the standard isometric embedding ϕ of a 2-dimensional symmetric R-space M_0 and a Riemannian covering π of M onto M_0. It follows from $\dim M_0 = 2$ that M_0 is of rank 1 or 2. If M_0 is of rank 1, then M_0 is a 2-dimensional sphere or a 2-dimensional real projective space. Also, if M_0 is of rank 2, then M_0 is a 2-dimensional flat torus. Hence we can obtain the conclusion. □

In the sequel, we assume that f is generalized helical. For each geodesic σ in M, $f \circ \sigma$ is contained in a compact set $f(M)$. Hence, $f \circ \sigma$ is a helix of even order. Thus f is of even order. Let the order of f be $2d$. As in Sect. 2, we define

$V_{p,i}$, $\hat{\lambda}_i$ and $\hat{v}_i$, where $i \geq 1$ and $p \in M$. Also, we define a matrix-valued function $\hat{\Lambda}$ on SM by

$$\hat{\Lambda} := \begin{pmatrix} 0 & -\hat{\lambda}_1 & 0 & \cdots & \cdots & 0 \\ \hat{\lambda}_1 & 0 & -\hat{\lambda}_2 & \ddots & & \vdots \\ 0 & \hat{\lambda}_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\hat{\lambda}_{2d-1} \\ 0 & \cdots & \cdots & 0 & \hat{\lambda}_{2d-1} & 0 \end{pmatrix}.$$

Let ${}^t(\eta_0(v,s),\dots,\eta_{2d-1}(v,s))$ be the first column of the matrix $\int_0^s \exp s\hat{\Lambda}(v)\,ds$, where $v \in SM$ and $s > 0$. Since f is of order $2d$, there is $p \in M$ with $V_{p,2d} \neq \emptyset$. In the sequel, we assume that $v \in V_{p,2d}$. From the Frenet formula (1.1), we can obtain the following expression of $f \circ \sigma_v$:

$$f(\sigma_v(s)) = f(p) + \sum_{i=0}^{2d-1} \eta_i(v,s)\hat{v}_i(v) \tag{5.1}$$

for $s > 0$. By the straightforward computation,

$$\det \hat{\Lambda}(v) = \hat{\lambda}_1(v)^2 \hat{\lambda}_3(v)^2 \cdots \hat{\lambda}_{2d-1}(v)^2 \neq 0$$

is shown. Hence, since $\hat{\Lambda}(v)$ is skew-symmetric and non-singular, the normal form of $\hat{\Lambda}(v)$ is given by

$$T(v)^{-1}\hat{\Lambda}(v)T(v) = \bigoplus_{i=1}^{d} B(a_i(v))$$

with some orthogonal matrix $T(v)$, where

$$B(a_i(v)) = \begin{pmatrix} 0 & a_i(v) \\ -a_i(v) & 0 \end{pmatrix} \quad (0 < a_1(v) \leq \cdots \leq a_d(v)).$$

In the same method as the proof of Lemma 3.1 in [16], we can show that $a_1(v),\dots,a_d(v)$ are mutually distinct and that the following relations hold:

$$\eta_{2i}(v,s) = \sum_{k=1}^{d} b_{2i,k}(v)(1 - \cos(a_k(v)s)) \quad (0 \leq i \leq d-1), \tag{5.2}$$

$$\eta_{2i+1}(v,s) = \sum_{k=1}^{d} b_{2i+1,k}(v)\sin(a_k(v)s) \quad (0 \leq i \leq d-1),$$

where $b_{i,k}$ $(0 \le i \le 2d-1, 1 \le k \le d)$ are functions on $V_{p,2d}$ determined by $\hat{\lambda}_1, \dots, \hat{\lambda}_{2d-1}$. From (5.1) and (5.2), we can obtain the following expression of $f \circ \sigma_v$:

$$\begin{aligned} f(\sigma_v(s)) = & f(p) + \sum_{i=0}^{d-1}\left(\sum_{k=1}^{d} b_{2i,k}(v)\right)\hat{v}_{2i}(v) \\ & + \sum_{k=1}^{d}\left\{\left(-\sum_{i=0}^{d-1} b_{2i,k}(v)\hat{v}_{2i}(v)\right)\cos(a_k(v)s)\right. \\ & \left. + \left(\sum_{i=0}^{d-1} b_{2i+1,k}(v)\hat{v}_{2i+1}(v)\right)\sin(a_k(v)s)\right\}. \end{aligned} \tag{5.3}$$

On the other hand, we can prove the following lemma.

Lemma 5.3. *The functions a_i $(1 \le i \le d)$ on $V_{p,2d}$ are analytic.*

Proof. Denote by $\rho(z,v)$ the characteristic polynomial $\det(zE - \hat{\Lambda}(v))$ of $\hat{\Lambda}(v)$, where $v \in V_{p,2d}$ and E is the identity matrix. Since $\hat{\lambda}_i|_{V_{p,2d}}$ $(1 \le i \le 2d-1)$ are analytic by Lemma 2.1, $\rho(z,v)$ is analytic with respect to v on $V_{p,2d}$. Also, we have

$$\rho(z,v) = (z^2 + a_1(v)^2)\cdots(z^2 + a_d(v)^2) \tag{5.4}$$

for every $v \in V_{p,2d}$. Hence, we see that a_i $(1 \le i \le d)$ are continuous on $V_{p,2d}$. Fix $v_0 \in V_{p,2d}$ and $i_0 \in \{1, \dots, d\}$. Since $a_1(v_0), \dots, a_d(v_0)$ are mutually distinct, we can take a closed curve K in the complex plane such that $a_{i_0}(v_0)\sqrt{-1}$ positions inside K and $a_i(v_0)\sqrt{-1}$ $(i \ne i_0)$ position outside K. It follows from the continuity of a_i $(1 \le i \le d)$ that there is a neighbourhood U of v_0 in $V_{p,2d}$ such that, for every $v \in U$, $a_{i_0}(v)\sqrt{-1}$ positions inside K and $a_i(v)\sqrt{-1}$ $(i \ne i_0)$ position outside K. From (5.4), we have

$$a_{i_0}(v)\sqrt{-1} = \frac{1}{2\pi\sqrt{-1}}\int_K z\frac{d\rho(z,v)/dz}{\rho(z,v)}dz$$

for every $v \in U$. Hence, it follows from the analyticity of $\rho(z,v)$ with respect to v that a_{i_0} is analytic on U. Therefore, from the arbitrarity of v_0, we see that so is a_{i_0} on $V_{p,2d}$. □

From Lemma 5.1, 5.3 and (5.3), we can prove the following characterizing theorem.

THEOREM 5.4. *Let f be a full generalized helical immersion of order $2d$ of an $n(\geq 2)$-dimensional compact Riemannian manifold M into a Euclidean space. Assume that the following condition $(*)$ holds:*

$(*)$ For each $p \in M$, there is at least one geodesic in M through p which is viewed as a generic helix of order $2d$ in the ambient Euclidean space.
Then $f \approx \phi \circ \pi$, where ϕ is the standard isometric embedding of a symmetric R-space M_0 of rank d and π is a Riemannian covering of M onto M_0.

PROOF. In case of $d = 1$, f is a planar geodesic immersion. Hence, by [14], $f \approx \phi \circ \pi$, where ϕ is the standard isometric embedding of a symmetric R-space M_0 of rank 1 and π is a Riemannian covering of M onto M_0. Assume that $d \geq 2$. Set $U := \{p \in M \mid (\forall q \in f^{-1}(f(p)))[f_*(T_qM) = f_*(T_pM)]\}$. Fix $p_0 \in U$. From the assumption, there exists $v_0 \in V_{p_0,2d}$ such that $f \circ \sigma_{v_0}$ is generic. Since $V_{p_0,2d}$ is a dense open set in $S_{p_0}M$ by Lemma 2.2, there is a convex neighbourhood W of v_0 in $S_{p_0}M$ contained in $V_{p_0,2d}$. Take an arbitrary $w_0 \in W \backslash \{\pm v_0\}$. Let C_{w_0} be the great circle in $S_{p_0}M$ through v_0 and w_0. Let $F : [0, \pi/2] \to \boldsymbol{R}^d$ be a curve in $\boldsymbol{R}^d$ defined by $F(\theta) := (a_1((v_0 \cos\theta + w_0 \sin\theta)/(\|v_0 \cos\theta + w_0 \sin\theta\|)), \dots, a_d((v_0 \cos\theta + w_0 \sin\theta)/(\|v_0 \cos\theta + w_0 \sin\theta\|)))$ for $\theta \in [0, \pi/2]$, where a_i $(1 \leq i \leq d)$ are the above functions on $V_{p_0,2d}$. This curve F is an analytic curve by Lemma 5.3. Let $P_{r_1 \cdots r_d}$ be the hyperplane in $\boldsymbol{R}^d$ through the origin and with the normal vector $(r_1, \dots, r_d)$. Since $f \circ \sigma_{v_0}$ is generic, $F(0) \in \boldsymbol{R}^d \backslash \bigcup_{(r_1,\dots,r_d) \in \boldsymbol{Q}^d \backslash \{(0,\dots,0)\}} P_{r_1 \cdots r_d}$ by (5.3). Suppose that $J := \{\theta \in (0, \pi/2] \mid F(\theta) \in \boldsymbol{R}^d \backslash \bigcup_{(r_1,\dots,r_d) \in \boldsymbol{Q}^d \backslash \{(0,\dots,0)\}} P_{r_1 \cdots r_d}\}$ is finite. Set $\theta_0 := \min J$. Since $F((0, \theta_0)) \subset \bigcup_{(r_1,\dots,r_d) \in \boldsymbol{Q}^d \backslash \{(0,\dots,0)\}} P_{r_1 \cdots r_d}$, there is $(r_1^0, \dots, r_d^0) \in \boldsymbol{Q}^d \backslash \{(0, \dots, 0)\}$ such that $F([\theta_1, \theta_1 + \varepsilon]) \subset P_{r_1^0 \cdots r_d^0}$ for some $\theta_1 \in (0, \theta_0)$ and a sufficiently small positive number ε. Hence, since F is an analytic curve, we have $F([0, \pi/2]) \subset P_{r_1^0 \cdots r_d^0}$. In particular, we have $F(0) \in P_{r_1^0 \cdots r_d^0}$. This contradicts $F(0) \in \boldsymbol{R}^d \backslash \bigcup_{(r_1,\dots,r_d) \in \boldsymbol{Q}^d \backslash \{(0,\dots,0)\}} P_{r_1 \cdots r_d}$. Therefore, J is infinite. This implies that

$$\{v \in C_{w_0} \mid f \circ \sigma_v : \text{a generic helix}\}$$

is infinite. Hence, from the arbitrarity of $w_0 \in W \backslash \{\pm v_0\}$, we can obtain $\overline{\nabla} h_{p_0} = 0$ in terms of Lemma 5.1. Thus, by the arbitrarity of p_0, $\overline{\nabla} h = 0$ holds on U. Furthermore, since U is dense in M, $\overline{\nabla} h = 0$ holds on M. Hence, from Theorem 4.1, we have $f \approx \phi \circ \pi$, where ϕ is the standard isometric embedding of a symmetric R-space M_0 and π is a Riemannian covering of M onto M_0. Furthermore, since f is of order $2d$, M_0 is of rank d (see Sect. 4). $\square$

Here we shall construct an example of a generalized helical immersion of a flat torus into a Euclidean space. Let $u_t = (1/\sqrt{m-1}, \dots, 1/\sqrt{m-1}, -1/\sqrt{m-1},$

$\ldots,-1/\sqrt{m-1})$, where $m=2^k+1(k\geq 1)$ and the numbers of $1/\sqrt{m-1}$ and $-1/\sqrt{m-1}$ are $t(t\geq 1)$. Let f be a map from an $n(=k+1)$-dimensional Euclidean space $\boldsymbol{R}^n$ to a $2m$-dimensional Euclidean space $\boldsymbol{R}^{2m}$ defined by

$$f(x_1,\ldots,x_n):=\left(\cos\left(\sum_{i=1}^n a_{i1}x_i\right),\sin\left(\sum_{i=1}^n a_{i1}x_i\right),\ldots\right.$$
$$\left.\cos\left(\sum_{i=1}^n a_{im}x_i\right),\sin\left(\sum_{i=1}^n a_{im}x_i\right)\right),$$

where $(x_1,\ldots,x_n)$ is a Euclidean coordinate system of $\boldsymbol{R}^n$ and a_{ij} $(1\leq i\leq n$, $1\leq j\leq m)$ are constants defined by

$$(a_{11},\ldots,a_{1m})=\left(\frac{1}{\sqrt{m(m-1)}},\ldots,\frac{1}{\sqrt{m(m-1)}},\sqrt{\frac{m-1}{m}}\right),$$
$$(a_{i1},\ldots,a_{im})=(u_{2^{i-2}},\ldots,u_{2^{i-2}},0)\quad(2\leq i\leq n).$$

It is clear that $\sum_{j=1}^m a_{i_1j}a_{i_2j}=\delta_{i_1i_2}(1\leq i_1,i_2\leq n)$, which assures that f is an isometric immersion. Clearly f deduces an isometric immersion $\bar{f}$ of an n-dimensional flat torus $T^n=S^1(\sqrt{m(m-1)})\times S^1(\sqrt{m-1})\times\cdots\times S^1(\sqrt{m-1})$ into $\boldsymbol{R}^{2m}$, where $S^1(\sqrt{m(m-1)})$ (resp. $S^1(\sqrt{m-1})$) is a 1-dimensional sphere of radius $\sqrt{m(m-1)}$ (resp. $\sqrt{m-1}$). Let G be the transformation group of T^n induced from that of all parallel translations of $\boldsymbol{R}^n$ and H the isotropy group of G at a point of T^n. It is easy to show that $\bar{f}$ is G-equivariant, where we note that the definition of a G-equivariant immersion will be stated in Sect. 7. Denote by π the covering map of $\boldsymbol{R}^n$ onto $T^n=G/H$. Take an arbitrary geodesic $\sigma(s):=\pi(b_1s,\ldots,b_ns)$ in T^n, where $\sum_{i=1}^n b_i^2=1$. Then we have

$$(\bar{f}\circ\sigma)(s)=\left(\cos\left(s\sum_{i=1}^n a_{i1}b_i\right),\sin\left(s\sum_{i=1}^n a_{i1}b_i\right),\ldots,\right.$$
$$\left.\cos\left(s\sum_{i=1}^n a_{im}b_i\right),\sin\left(s\sum_{i=1}^n a_{im}b_i\right)\right),$$

which implies that $\bar{f}\circ\sigma$ is a helix of order at most $2m$ in $\boldsymbol{R}^{2m}$. Let p_i $(1\leq i\leq n-1)$ be prime numbers with $3\leq p_1<p_2<\cdots<p_{n-1}$. In particular, if $(b_1,\ldots,b_n)=(1,\sqrt{p_1},\sqrt{p_2},\ldots,\sqrt{p_{n-1}})$, then $|\sum_{i=1}^n a_{ij}b_i|(1\leq j\leq m)$ are mutually distinct and hence $\bar{f}\circ\sigma$ is of order $2m$. Thus $\bar{f}$ is a generalized helical immersion of order $2m$. On the other hand, since $1\cdot\sum_{i=1}^n a_{i1}b_i+\cdots+1\cdot\sum_{i=1}^n a_{i,m-1}b_i+$

$(-1) \cdot \sum_{i=1}^n a_{im} b_i = 0$, $\bar{f} \circ \sigma$ cannot be a generic helix of order $2m$. This fact implies together with Lemma 2.2 that almost every geodesic in T^n is not viewed as a generic helix in $\boldsymbol{R}^{2m}$. Namely, the second fundamental form of $\bar{f}$ is not parallel. In particular, if $(b_1, \dots, b_n) = (0, \sqrt{3}, \sqrt{5}, 0, \dots, 0)$, then we have

$$\left| \sum_{i=1}^n a_{ij} b_i \right| = \begin{cases} \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{m-1}} & (j \equiv 0, 1 \ (\mathrm{mod}\, 4),\ j < m) \\[2ex] \dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{m-1}} & (j \equiv 2, 3 \ (\mathrm{mod}\, 4),\ j < m) \end{cases}$$

and $\sum_{i=1}^n a_{im} b_i = 0$ and hence $\bar{f} \circ \sigma$ is a generic helix of order 4. Therefore, since $\bar{f}$ is G-equivariant, we see that, for each $p \in T^n$, there is at least one geodesic in T^n through p which is viewed as a generic helix in $\boldsymbol{R}^{2m}$. From this example, we see that the condition $(*)$ in Theorem 5.4 cannot be replaced by the following condition:

For each $p \in M$, there is at least one geodesic in M through p which is viewed as a generic helix in the ambient Euclidean space.

From Theorem 5.4, we can obtain the following result in the case where f is of order 4.

COROLLARY 5.5. *Let f be a full generalized helical immersion of order 4 of an $n(\geq 2)$-dimensional compact Riemannian manifold M into a Euclidean space. Assume that, for each $p \in M$, there is at least one non-periodic geodesic in M through p. Then $f \approx \phi \circ \pi$, where ϕ is the standard isometric embedding of a symmetric R-space M_0 of rank 2 and π is a Riemannian covering of M onto M_0.*

PROOF. Let σ be a non-periodic geodesic in M. From the assumption, $f \circ \sigma$ is a helix of order 2 or 4. Since σ is non-periodic, so is also $f \circ \sigma$ (see Proof of Theorem 2 of [5]). Hence $f \circ \sigma$ is a generic helix of order 4. Therefore, the conclusion is deduced from Theorem 5.4. □

6. Generalized helical immersions into a sphere.

In this section, we shall deduce some results for an isometric immersion into a sphere.

PROPOSITION 6.1. *Let f be a full isometric immersion of a 2-dimensional compact Riemannian manifold M into a sphere. Assume that, for each $p \in M$, there are at least four geodesics in M through p which are viewed as generic helices in the ambient sphere. Then, if f is minimal, then the following* (i), (ii) *or* (iii) *holds*:

(i) *$f \approx id$, where id is the identity transformation of a 2-dimensional sphere,*

(ii) *$f \approx \bar{\phi}_1 \circ \pi$, where $\bar{\phi}_1$ is the umbilical reduction of the Veronese embedding ϕ_1 of a 2-dimensional real projective space $\boldsymbol{R}P^2$ into a 5-dimensional Euclidean space to a hypersphere, π is a Riemannian covering of M onto $\boldsymbol{R}P^2$,*

(iii) *$f \approx \bar{\phi}_2 \circ \pi'$, where $\bar{\phi}_2$ is the Clifford embedding of a 2-dimensional flat torus T into a 3-dimensional sphere S^3, π' is a Riemannian covering of M onto T. Also, if f is not minimal, then $f \approx \psi \circ \bar{\phi}$, where $\bar{\phi}$ is the above immersion id, $\bar{\phi}_1 \circ \pi$ or $\bar{\phi}_2 \circ \pi'$ and ψ is a totally umbilical (but non-totally geodesic) embedding of codimension-1 into a sphere.*

PROOF. Let ι be a totally umbilical embedding of codimension-1 of the ambient sphere into a Euclidean space. It is easy to show that $\iota \circ f$ satisfies the conditions (other than the fullness) of Proposition 5.2. Hence, we can obtain the conclusion in terms of Proposition 5.2. □

Also, we can obtain the following characterizing theorem in terms of Theorem 5.4.

THEOREM 6.2. *Let f be a full generalized helical immersion of order $2d-1$ or $2d (d \geq 2)$ of an $n(\geq 2)$-dimensional compact Riemannian manifold M into a sphere. Assume that, for each $p \in M$, there is at least one geodesic in M through p which is viewed as a generic helix of order $2d-1$ or $2d$ in the ambient sphere. Then the following statements* (i) *and* (ii) *hold:*

(i) *If f is minimal, then $f \approx \bar{\phi} \circ \pi$, where $\bar{\phi}$ is the umbilical reduction of the standard isometric embedding ϕ of a symmetric R-space M_0 of rank d to a hypersphere and π is a Riemannian covering of M onto M_0.*

(ii) *If f is not minimal, then f is of order $2d$ and $f \approx \psi \circ \bar{\phi} \circ \pi$, where $\bar{\phi}$ and π are as in the above and ψ is a totally umbilical (but non-totally geodesic) embedding of codimension-1 into a sphere.*

PROOF. Let ι be a totally umbilical embedding of codimension-1 of the ambient sphere into a Euclidean space. It is clear that $\iota \circ f$ is a (not necessarily full) generalized helical immersion of order $2d$. Also, it follows from the assumption that, for each $p \in M$, there is at least one geodesic in M through p which is viewed as a generic helix of order $2d$ in the Euclidean space. Hence, we can obtain the conclusion in terms of Theorem 5.4. □

In particular, we can obtain the following result in the case where f is of order 3 or 4.

COROLLARY 6.3. *Let f be a full generalized helical immersion of order* 3 *or* 4 *of an $n(\geq 2)$-dimensional compact Riemannian manifold M into a sphere. Assume that, for each $p \in M$, there is at least one non-periodic geodesic in M through p. Then the following statements* (i) *and* (ii) *hold*:

(i) *If f is minimal, then $f \approx \bar{\phi} \circ \pi$, where $\bar{\phi}$ is the umbilical reduction of the standard isometric embedding ϕ of a symmetric R-space M_0 of rank* 2 *to a hypersphere and π is a Riemannian covering of M onto M_0.*

(ii) *If f is not minimal, then f is of order* 4 *and $f \approx \psi \circ \bar{\phi} \circ \pi$, where $\bar{\phi}$ and π are as in the above and ψ is a totally umbilical* (*but non-totally geodesic*) *embedding of codimension-*1 *into a sphere.*

PROOF. Let σ be a non-periodic geodesic in M. From the assumption, $f \circ \sigma$ is a helix of order at most 4. Since σ is non-periodic, $f \circ \sigma$ is a generic helix of order 3 or 4. Therefore, the conclusion is deduced from Theorem 6.2. □

7. Concluding remarks.

In this section, we shall state results deduced from those in Sect. 5 and 6 in the case where M is a Riemannian homogeneous space and f is equivariant. Let f be an isometric immersion of a Riemannian homogeneous space $M = G/K$ into a Riemannian manifold $\tilde{M}$. If there is a continuous homomorphism ρ of G into the isometry group of $\tilde{M}$ such that $f(g \cdot p) = \rho(g)(f(p))$ for every $p \in M$ and every $g \in G$, then f is said to be G-equivariant. The following result is deduced from Theorem 5.4.

THEOREM 7.1. *Let f be a G-equivariant full generalized helical immersion of order $2d$ of an $n(\geq 2)$-dimensional compact Riemannian homogeneous space $M = G/K$ into a Euclidean space. Assume that there is at least one geodesic in M which is viewed as a generic helix of order $2d$ in the ambient Euclidean space. Then $f \approx \phi \circ \pi$, where ϕ is the standard isometric embedding of a symmetric R-space M_0 of rank d and π is a Riemannian covering of M onto M_0.*

Also, the following result is deduced from Theorem 6.2.

THEOREM 7.2. *Let f be a G-equivariant full generalized helical immersion of order $2d-1$ or $2d$ of an $n(\geq 2)$-dimensional compact Riemannian homogeneous space $M = G/K$ into a sphere. Assume that there is at least one geodesic in M which is viewed as a generic helix of order $2d-1$ or $2d$ in the ambient sphere. Then the statement* (i) *and* (ii) *in Theorem* 6.2 *hold.*

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