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GENERALIZED HERMITE -HADAMARD TYPE INTEGRAL INEQUALITIES FOR FRACTIONAL INTEGRALS

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Abstract. In this paper, we have established Hermite-Hadamard type inequalities for fractional integrals depending on a parameter.

1. Introduction

Definition 1. *The function $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[15, p.137], [9]). These inequalities state that if $f:I\rightarrow\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 9, 10, 15]) and the references cited therein.

In [10], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1. *Let $f:I^{\circ}\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

Theorem 1. *Let $f:I^{\circ}\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.3}$$

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Meanwhile, Sarikaya et al.[18] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

(1.4)
$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] = \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right] f'(ta+(1-t)b)\,dt.$$

It is remarkable that Sarikaya et al.[18] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.5}$$

with $\alpha > 0$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [11, 12, 14, 16].

Definition 2. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

For some recent results connected with fractional integral inequalities see ([3, 4, 5, 6, 7, 8],[13],[17],[19],[20],[21])

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for Riemann-Liouville fractional integral and some other integral inequalities using the generalized identity is obtained for fractional integrals. The results presented in this paper provide extensions of those given in earlier works.

2. Main Results

We give a important fractional integral identity for differentiable convex functions:

Lemma 3. *Let* $f:[a,b]\rightarrow\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0\leq a<b$. *If* $f'\in L[a,b]$, *then the following equality for fractional integrals hold:*

$$
\begin{aligned}
& -\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}+\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \\
& \times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right] \\
= & \int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt
\end{aligned} \tag{2.1}
$$

where $\lambda\in[0,1]\backslash\{\frac{1}{2}\}$ *and* $\alpha>0$.

Proof. It suffices to note that

$$
\begin{aligned}
I &= \int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\
&= \int_0^1(1-t)^{\alpha}f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\
&\quad -\int_0^1 t^{\alpha}f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\
&= I_1-I_2
\end{aligned}
$$

Integrating by parts

$$
\begin{aligned}
I_1 &= \int_0^1(1-t)^{\alpha}f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\
&= \left.\frac{(1-t)^{\alpha}f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)(b-a)}\right|_0^1 \\
&\quad +\frac{\alpha}{(1-2\lambda)(b-a)}\int_0^1(1-t)^{\alpha-1}f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt
\end{aligned}
$$

$$\begin{aligned} &= -\frac{f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)} \\ &\quad +\frac{\alpha}{(1-2\lambda)(b-a)}\int_0^1 (1-t)^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= -\frac{f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}+\frac{\alpha}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}\left[(\lambda a+(1-\lambda)b)-x\right]f(x)dx \\ &= -\frac{f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}+\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b) \end{aligned}$$

and similarly we get

$$\begin{aligned} I_2 &= \int_0^1 t^{\alpha} f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= \frac{f(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}-\frac{\alpha}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}\left[x-(\lambda b+(1-\lambda)a)\right]f(x)dx \\ &= \frac{f(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a) \end{aligned}$$

From I_1 and I_2, it follows that

$$\begin{aligned} I &= I_1-I_2 \\ &= -\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)} \\ &\quad +\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]. \end{aligned}$$

This completes the proof. □

Remark 1. *If we take* $\lambda=0$ *in Lemma 3, then the identity (2.1) reduces the identity (1.4) which is proved in* [18]. *Similarly, if we take* $\lambda=1$ *in Lemma 3, then*
(2.2)

$$\frac{f(b)+f(a)}{(b-a)}+\frac{\Gamma(\alpha+1)}{(-1)^{\alpha+1}(b-a)^{\alpha+1}}\left[J^{\alpha}_{b^+}f(a)+J^{\alpha}_{a^-}f(b)\right]=\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(ta+(1-t)b\right)dt.$$

By using $J^{\alpha}_{b^+}f(a)+J^{\alpha}_{a^-}f(b)=(-1)^{\alpha}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]$ *in (2.2), it follows that*
(2.3)

$$\frac{f(b)+f(a)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]=\frac{(b-a)}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(ta+(1-t)b\right)dt.$$

If we choose $\alpha=1$ *in (2.3), it follows that (2.3) reduces to (1.2).*

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $|f'|^q$, $q\geq 1$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\
&\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\right|\\
\leq\ & 2^{\frac{q-1}{q}}\left[\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]\left[|f'(\lambda a+(1-\lambda)b)|^{q}+|f'(\lambda b+(1-\lambda)a)|^{q}\right]^{\frac{1}{q}}.
\end{aligned}\tag{2.4}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ and $\alpha>0$.

Proof. Firstly, we suppose that $q=1$. Using Lemma 3 and convexity of $|f'|^q$, we find that

$$\begin{aligned}
&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\
&\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\right|\\
\leq\ & \int_{0}^{1}|(1-t)^{\alpha}-t^{\alpha}|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt\\
\leq\ & \int_{0}^{1}|(1-t)^{\alpha}-t^{\alpha}|\left[t\left|f'(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f'(\lambda b+(1-\lambda)a)\right|\right]dt\\
=\ & \int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[t\left|f'(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f'(\lambda b+(1-\lambda)a)\right|\right]dt\\
&+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]\left[t\left|f'(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f'(\lambda b+(1-\lambda)a)\right|\right]dt\\
=\ & K_1+K_2.
\end{aligned}\tag{2.5}$$

Hence, conculating K_1 ve K_2, we have

$$\begin{aligned}
K_1 &= \int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha\right]\left[t\left|f'(\lambda a+(1-\lambda)b)\right| + (1-t)\left|f'(\lambda b+(1-\lambda)a)\right|\right] dt \\
&= \left|f'(\lambda a+(1-\lambda)b)\right| \int_0^{\frac{1}{2}} \left[(1-t)^\alpha t - t^{\alpha+1}\right] dt \\
&\quad + \left|f'(\lambda b+(1-\lambda)a)\right| \int_0^{\frac{1}{2}} \left[(1-t)^{\alpha+1} - t^\alpha (1-t)\right] dt \\
&= \left|f'(\lambda a+(1-\lambda)b)\right| \left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right] \\
&\quad + \left|f'(\lambda b+(1-\lambda)a)\right| \left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right] \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned}
K_2 &= \int_{\frac{1}{2}}^{1} \left[t^\alpha - (1-t)^\alpha\right]\left[t\left|f'(\lambda a+(1-\lambda)b)\right| + (1-t)\left|f'(\lambda b+(1-\lambda)a)\right|\right] dt \\
&= \left|f'(\lambda a+(1-\lambda)b)\right| \left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right] \\
&\quad + \left|f'(\lambda b+(1-\lambda)a)\right| \left[\frac{1}{(\alpha+2)(\alpha+1)} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right]. \qquad (2.7)
\end{aligned}$$

Using (2.6) and (2.7) in (2.5), it follows that

$$\begin{aligned}
&\left| \frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \right. \\
&\quad \left. \times \left[J^\alpha_{(\lambda b+(1-\lambda)a)^+} f(\lambda a+(1-\lambda)b) + J^\alpha_{(\lambda a+(1-\lambda)b)^-} f(\lambda b+(1-\lambda)a)\right] \right| \\
&\le \frac{1}{\alpha+1}\left[1-\frac{1}{2^\alpha}\right]\left[\left|f'(\lambda a+(1-\lambda)b)\right| + \left|f'(\lambda b+(1-\lambda)a)\right|\right].
\end{aligned}$$

Secondly, we suppose that $q>1$. Using Lemma 3 and power mean inequality, we obtain

$$\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| \left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right| dt$$

$$\leq \left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| \left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}. \tag{2.8}$$

Hence, using convexity of $|f'|^q$ and (2.8) we obtain

$$\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J^\alpha_{(\lambda b+(1-\lambda)a)^+} f(\lambda a+(1-\lambda)b) + J^\alpha_{(\lambda a+(1-\lambda)b)^-} f(\lambda b+(1-\lambda)a)\right]\right|$$

$$\leq \left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| \left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha\right] dt + \int_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha\right] dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| \left[t\left|f'(\lambda a+(1-\lambda)b)\right|^q + (1-t)\left|f'(\lambda b+(1-\lambda)a)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{2}{\alpha+1}\left[1-\frac{1}{2^\alpha}\right]\right)^{1-\frac{1}{q}} \left(\frac{1}{\alpha+1}\left[1-\frac{1}{2^\alpha}\right]\right)^{\frac{1}{q}}$$

$$\times \left[\left|f'(\lambda a+(1-\lambda)b)\right|^q + \left|f'(\lambda b+(1-\lambda)a)\right|^q\right]^{\frac{1}{q}}$$

$$\leq 2^{\frac{q-1}{q}} \left[\frac{1}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right] \left[\left|f'(\lambda a+(1-\lambda)b)\right|^q + \left|f'(\lambda b+(1-\lambda)a)\right|^q\right]^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 1. *Under assumptation Theorem 3 with $\lambda = 0$ and $\lambda = 1$, we have*

$$\left|\frac{f(a)+f(b)}{(b-a)} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]\right| \tag{2.9}$$
$$\leq 2^{\frac{q-1}{q}}\left[\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]\left[|f'(a)|^{q} + |f'(b)|^{q}\right]^{\frac{1}{q}}.$$

where $q \geq 1$.

Proof. By using $J_{b+}^{\alpha}f(a) + J_{a-}^{\alpha}f(b) = (-1)^{\alpha}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]$ in Theorem 3 with $\lambda = 1$, we obtain the inequality (2.9). □

Remark 2. *If we take $\alpha = 1$ in Corollary 1, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}2^{\frac{q-1}{q}}\left[|f'(a)|^{q} + |f'(b)|^{q}\right]^{\frac{1}{q}}$$

where $q \geq 1$. Choosing $q = 1$ in last inequality, it follows that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[|f'(a)| + |f'(b)|\right]$$

which are proved Dragomir and Agarwal in [10].

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \leq a < b$. If $|f'|^{q}$ convex on $[a,b]$ for same fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(\lambda a+(1-\lambda)b) + f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J_{(\lambda b+(1-\lambda)a)^{+}}^{\alpha}f(\lambda a+(1-\lambda)b) + J_{(\lambda a+(1-\lambda)b)^{-}}^{\alpha}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq \left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left(\frac{|f'(\lambda a+(1-\lambda)b)|^{q} + |f'(\lambda b+(1-\lambda)a)|^{q}}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$ and $\lambda \in [0,1]\setminus\{\frac{1}{2}\}$.

Proof. Proof: Using Lemma 3, convexity of $|f|^q$ and well-known Hölder's inequality, we obtain

$$\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq \int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)]|\,dt$$
$$\leq \left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)]|^q\,dt\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]^p\,dt+\int_{\frac{1}{2}}^1[t^{\alpha}-(1-t)^{\alpha}]^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left[t\,|f'(\lambda a+(1-\lambda)b)|^q+(1-t)\,|f'(\lambda b+(1-\lambda)a)|^q\right]dt\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^{\frac{1}{2}}[(1-t)^{\alpha p}-t^{\alpha p}]\,dt+\int_{\frac{1}{2}}^1[t^{\alpha p}-(1-t)^{\alpha p}]\,dt\right)^{\frac{1}{p}}$$
$$\times\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(\lambda b+(1-\lambda)a)|^q}{2}\right)^{\frac{1}{q}}$$
$$\leq \left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(\lambda b+(1-\lambda)a)|^q}{2}\right)^{\frac{1}{q}}.$$

Here, we use

$$(A-B)^p\leq A^p-B^p,$$

for any $A>B\geq 0$ and $p\geq 1$. □

Corollary 2. *Under assumptation Theorem 4 with $\lambda=0$ and $\lambda=1$, we have*

$$\left|\frac{f(a)+f(b)}{(b-a)}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J^{\alpha}_{a^{+}}f(b)+J^{\alpha}_{b^{-}}f(a)\right]\right| \leq \left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.10}$$

Proof. By using $J^{\alpha}_{b^{+}}f(a)+J^{\alpha}_{a^{-}}f(b)=(-1)^{\alpha}\left[J^{\alpha}_{a^{+}}f(b)+J^{\alpha}_{b^{-}}f(a)\right]$ in Theorem 4 with $\lambda=1$, we obtain the inequality (2.10). □

Remark 3. *If we take $\alpha = 1$ in Corollary 2, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left[\frac{2}{p+1}\left(1-\frac{1}{2^p}\right)\right]^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

which are proved Dragomir and Agarwal in [10].

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $|f'|^q$ convex on $[a,b]$ for same fixed $q\geq 1$,then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\&\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\ \leq\ &\left[\frac{1}{q\alpha+1}\left(1-\frac{1}{2^{q\alpha+1}}\right)\right]^{\dfrac{1}{q}}\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(\lambda b+(1-\lambda)a)|^q}{2}\right)^{\dfrac{1}{q}}\end{aligned}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ and $\alpha>0$.

Proof. Using Lemma 3, convexity of $|f'|^q$, and well-known Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\&\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\ \leq\ &\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt\\ \leq\ &\left(\int_0^1 1^p dt\right)^{\frac{1}{p}}\left(\int_0^1|(1-t)^{\alpha}-t^{\alpha}|^q\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]^q\right|dt\right)^{\frac{1}{q}}\\ =\ &\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]^q\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right.\\&\left.+\int_0^{\frac{1}{2}}\left[t^{\alpha}-(1-t)^{\alpha}\right]^q\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&\leq \left(|f'(\lambda a+(1-\lambda)b)|^q \int_0^{\frac{1}{2}} \left[(1-t)^{q\alpha}t - t^{q\alpha+1}\right] dt \right. \\
&\quad + |f'(\lambda b+(1-\lambda)a)|^q \int_0^{\frac{1}{2}} \left[(1-t)^{q\alpha+1} - t^{q\alpha}(1-t)\right] dt \\
&\quad + |f'(\lambda a+(1-\lambda)b)|^q \int_{\frac{1}{2}}^{1} \left[t^{q\alpha+1} - (1-t)^{q\alpha}t\right] dt \\
&\quad \left. + |f'(\lambda b+(1-\lambda)a)|^q \int_{\frac{1}{2}}^{1} \left[t^{q\alpha}(1-t) - (1-t)^{q\alpha+1}\right] dt \right)^{\frac{1}{q}} \\
&= \left(\frac{1}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{\frac{1}{q}} \left[|f'(\lambda a+(1-\lambda)b)| + |f'(\lambda b+(1-\lambda)a)|\right]^{\frac{1}{q}}.
\end{aligned}
$$

Here, we use $(A-B)^p \leq A^p - B^p$, for any $A > B \geq 0$ and $q \geq 1$. □

Corollary 3. *Under assumptation Theorem 5 with $\lambda = 0$ and $\lambda = 1$, we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{(b-a)} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right]\right| \\
&\leq \left[\frac{1}{q\alpha+1}\left(1-\frac{1}{2^{q\alpha+1}}\right)\right]^{\frac{1}{q}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
\tag{2.11}
$$

Proof. By using $J_{b^+}^{\alpha}f(a) + J_{a^-}^{\alpha}f(b) = (-1)^{\alpha}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right]$ in Theorem 5 with $\lambda = 1$, we obtain the inequality (2.11). □

Remark 4. *If we take $\alpha = 1$ in Corollary 3, we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{2}\left[\frac{1}{q+1}\left(1-\frac{1}{2^{q+1}}\right)\right]^{\frac{1}{q}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

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