GENERALIZED HOPF BIFURCATION AND h-ASYMPTOTIC STABILITY

by

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1. Introduction

The prevalent approach to the Hopf bifurcation problem is to prove directly the existence of the bifurcating periodic orbits by using such standard procedures as the implicit function theorem, the Liapunov-Schmidt method and its known variants, and topological degree arguments (see [7]). The phenomenon of Hopf bifurcation often occurs because of exchange of stability properties of the equilibrium under perturbations (see for instance, Chafee in [7] p. 85-88, Andronov et. al. [1], Marchetti et. al. [6] and Negrini and Salvadori [8]). This connection between the exchange of stability of the equilibrium and the appearance of bifurcating periodic orbits can be carefully investigated in order to develop a different approach for obtaining existence results and qualitative properties of these orbits. Now we want to provide a systematic development of the procedure sketched in [6] and [8] by considering the generalized Hopf bifurcation as was studied by Chafee [3] who used the alternative method as described by Hale [4].

In particular consider an n dimensional system of differential equations

$$\dot{x} = f_0(x) \tag{1.1}$$

where $f_0 \in C^\infty[B^n(r_0),R^n]$, $f_0(0) = 0$, and $B^n(r_0) = \{x \quad R^n : ||x|| < r_0\}$.

Assume the Jacobian matrix $f_0'(0)$ has a complex conjugate pair of eigenvalues $\pm i$ and that all other eigenvalues λ satisfy $\lambda \neq mi$, $m = 0,\pm 1,\pm 2,\ldots$.

For those f close to f_0 consider the perturbed differential equation

$$\dot{x} = f(x) \quad . \tag{1.2}$$

We are interested in determining the number of nonzero periodic orbits of (1.2) lying near the origin and having period T close to 2π for each f close to f_0 (with respect to an appropriate topology).

In considering this problem, Chafee [3] constructed a determining equation $\psi(\xi,f) = 0$ where $\xi = \rho^2$, in which ρ is a measure of the amplitude of the periodic solution being sought and f is the function appearing in (1.2). Letting k (finite) be the multiplicity of the root $\xi = 0$ of the equation $\psi(\cdot,f_0) = 0$ $(\psi(0,f_0) = 0)$, Chafee proved among other results the following two properties: (a) there exists a neighborhood N of f_0 and a number $r_1 > 0$ such that for any $f \in N$ equation (1.2) has no more than k nontrivial periodic orbits in $B^n(r_1)$ with period close to 2π; (b) for any integer j, $0 \leq j \leq k$, for any neighborhood N^* of f_0, $N^* \subseteq N$, and for any $r_2 \in (0,r_1]$ there exists an $f \in N^*$ such that equation (1.2) has exactly j nontrivial periodic orbits in $B^n(r_2)$ with periodic close to 2π.

Since the construction of the determining equation requires the use of fixed point theorems its form is only known implicitly. Thus the determination of the number k is a new problem that needs to be solved. We are interested in solving this new problem.

In this paper we illustrate the case $n = 2$. The general problem is much more difficult and a paper devoted to the general case in collaboration with P. Negrini is in preparation. The techniques in R^n are extensions of the ideas of this paper in R^2. Indeed the use of appropiate Liapunov functions used in this paper with some delicate modifications allows to a treatment in R^n. We prove that properties (a) and (b) occur if and only if the origin of (1.1) is either $(2k+1)$ asymptotically stable or $(2k+1)$ completely unstable; that is the origin is asymptotically stable in the future or in the past and this property is recognizable in a suitable sense by the terms of $f_0(\cdot)$ of degree $\leq 2k+1$. This result gives a complete answer to the problem since we provide a constructive procedure to determine k. Indeed this is exactly the classical method of Poincaré which is an algebraic procedure to construct either a power series satisfying formally the condition of being a first integral of (1.1) or a positive definite function whose derivatives along the solution of (1.1) is definite in sign.

Now let S_j, $0 \leq j \leq k$, be those $f \in N$ for which system (2.2) has exactly j nontrivial periodic orbits lying in $B^2(r_1)$. Our techniques allow us to construct a broad class of functions lying in the interior of S_j. Indeed these functions can be found by determining polynomials of order k that have j simple roots. In this way we provide a method for determining a class of functions that enjoy the structural property of preserving the number of periodic orbits near the origin under small perturbation (see the Remarks after the necessity part of the proof of Theorem 3.1).

Using different techniques Andronov and Leontovich in Chapter IX of [1] previously obtained the properties (a) and (b) occur if the origin of (1.1) is either $(2k+1)$-asymptotically stable or $(2k+1)$-completely unstable. Their techniques do not seem to provide a constructive procedure for the determination of those f which have the above structural property.

For the case $k=1$ we have provided an answer to a problem posed by Chafee [3, Sec. 7]. Namely, we have given an explicit condition, based on the Poincaré procedure to determine whether there exists one periodic or no periodic orbits of (1.2).

We also consider the case $k=\infty$ and obtain an infinite dimensional analogue of our previous results.

The methods we employ here involve the relationship between the sign of the first non-zero derivative of the displacement function of (1.1) at the origin and the above stability properties of the origin. In contrast to Andronov and Leontovich we find it useful to put (1.2) into a type of normal form (see Takens [10]) so that the polynomial part of the displacement function is in a 1-1 correspondence with the first $2k+1$ terms of the Maclaurin expansion of the right-hand side of the normal form.

In fact our paper shows that the generalized Hopf bifurcation in R^2 reduces to a study of a finite dimensional varying vector field. Thus if the origin is $(2k+1)$-asymptotically stable or $(2k+1)$-completely unstable for system (1.1) then in the language of Takens [10] our problem reduces to a study of $2k+1$ universal unfoldings.

Some of our results without proofs were given at a conference in Trieste [2].

Finally we remark that this paper illustrates the importance of the concepts and arguments of stability theory in approaching Hopf bifurcation.

2. Notation and Preliminaries

Under the hypotheses given on f_0, by applying a coordinate transformation, equation (1.1) in R^2 can be written as

$$\dot{x} = -y + X_0(x,y) \quad , \qquad\qquad \dot{y} = x + Y_0(x,y) \quad , \tag{2.1}$$

where $X_0(0,0) = Y_0(0,0) = 0$ and $X_0'(0,0) = Y_0'(0,0) = 0$. We shall again refer to the right-hand side of (2.1) as f_0. Similarly equation (1.2) can be written as

$$\dot{x} = \alpha x - \beta y + X(x,y) \quad , \qquad\qquad \dot{y} = \alpha y - \beta x + Y(x,y) \quad , \tag{2.2}$$

where $X(0,0) = Y(0,0) = 0$ and $X'(0,0) = Y'(0,0) = 0$ and again refer to the right-hand side of (2.2) as f.

We now define the topology on the space $C^\infty[B^2(r_0),R^2]$, where r_0 is a fixed real positive number and $B^2(r_0)$ is the open ball in R^2 centered at the origin having radius r_0. As in [3] define a function $||\ ||$ mapping $C^\infty[B^2(r_0),R^2]$ into R by

$$||f|| = \sum_{\ell=0}^{\infty} \frac{1}{2^\ell} \frac{||f||^{(\ell)}}{1+||f||^{(\ell)}}$$

where $||f||^{(\ell)}$ denotes the usual C^ℓ-supremum norm of f on $B^2(r_0)$.

Then $C^\infty[B^2(r_0),R^2]$ is a metric linear space under $\|\ \|$.

Converting (2.2) into polar form by letting $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\dot{r} = \alpha r + X^*(r,\theta)\cos\theta + Y^*(r,\theta)\sin\theta \quad ,$$

$$r\dot{\theta} = \beta r + Y^*(r,\theta)\cos\theta - X^*(r,\theta)\sin\theta \quad ,$$

where $X^*(r,\theta) = X(r\cos\theta, r\sin\theta)$ and $Y^*(r,\theta) = Y(r\cos\theta, r\sin\theta)$. Since β has value close to one and $X^*(r,\theta)$, $Y^*(r,\theta)$ are $O(r)$ we have the existence of $\bar{r} > 0$ and $b > 0$ such that $\dot{\theta} > b$ for all $r \in [0,\bar{r}]$. For every $r^0 \in [0,\bar{r})$ and $\theta_0 \in R$ the orbit of (2.2) passing through (r^0,θ_0) will be represented by the solution $r(f,\theta,r^0,\theta_0)$ of

$$\frac{dr}{d\theta} = R(f,r,\theta) \ , \quad r(\theta_0) = r^0 \quad . \tag{2.3}$$

where $R(f,r,\theta) = \dfrac{\alpha r + X^*(r,\theta)\cos\theta + Y^*(r,\theta)\sin\theta}{\Theta(f,r,\theta)}$

with $\Theta(f,r,\theta) = \quad + \dfrac{Y^*(r,\theta)\cos\theta - X^*(r,\theta)\sin\theta}{r}$

for $r > 0$, $\Theta(f,0,\theta) = \beta$. By continuous dependence arguments it follows that for a sufficiently small neighborhood M of f_0 and for sufficiently small $\bar{c} > 0$ the solution $r(r,\theta,c,0)$ of (2.3) exists on $[0,2\pi]$ for all $f \in M$ and all $c \in [0,\bar{c})$.

We now define the displacement function for (2.2), $V(f,c)$, given by

$$V(f,c) = r(f,2\pi,c) - c \quad ,$$

where $r(f,2\pi,c) \equiv r(f,2\pi,c,0)$. Since $R(f,\cdot,\cdot) \in C^\infty$ we have

$$r(f,\theta,c) = u_1(f,\theta)c + u_2(f,\theta)c^2 + \dots u_k(f,\theta)c^k + \eta(f,\theta,c) \quad , \qquad (2.4)$$

where k is any positive integer and $\eta(f,\theta,\cdot)$ is of order greater than k; moreover, $u_1(f,0) = 1$, $u_i(f,0) = 0$, $i = 2,3,\dots,k$, and $\eta(f,0,c) = 0$. Inserting (2.4) into (2.3), we have for each fixed f the following system of differential equations for $u_i(f,\theta)$;

$$\begin{aligned} \frac{\partial u_1}{\partial\theta} &= \frac{\alpha}{\beta} u_1 \\ \frac{\partial u_i}{\partial\theta} &= \frac{\alpha}{\beta} u_i + V_i(u_1,u_2,\dots,u_{i-1},\theta) \, , \quad i = 2,3,\dots \, . \end{aligned} \qquad (2.5)$$

Defining $\tilde{V}(f,c) \equiv \frac{V(f,c)}{c}$ for $c \neq 0$ and $\tilde{V}(f,0) = u_1(f,2\pi) - 1$, we see that the orbit of (2.3) through $(c,0)$ is closed if and only if $V(f,c) = 0$; that is either $c = 0$ or $\tilde{V}(f,c) = 0$. Notice that $V(f,\cdot)$ and $\tilde{V}(f,\cdot)$ are C^∞.

We now introduce the concept of h-asymptotic stability which plays a major role in the formulation of our results. The following terminology and results were given in the paper of Negrini and Salvadori [8].

Definition. Let h be an integer ≥ 2. The solution $x \equiv y \equiv 0$ of (2.1) is said to be h-asymptotically stable (resp. h-completely unstable) if

(i) for every $\tau_1, \tau_2 \in C^\infty[B^2(r_0), R]$ of order greater than h the solution of the system

$$\dot{x} = -y + X_{02}(x,y) + X_{03}(x,y) + \dots X_{0h}(x,y) + \tau_1(x,y)$$

$$\dot{y} = x + Y_{02}(x,y) + Y_{03}(x,y) + \dots Y_{0h}(x,y) + \tau_2(x,y)$$

is asymptotically stable (resp. completely unstable). Here $X_{0i}, Y_{0i}, 2 \leq i \leq h$, represent the i^{th} terms of the Maclaurin expansion of X_0, Y_0 respectively (recall that $X_{01}(x,y) \equiv Y_{01}(x,y) \equiv 0$);

(ii) property (i) is not satisfied when h is replaced by any integer $m \in \{2,3, \dots, h-1\}$.

The properties in the above definition can occur only when h is odd.

Moreover it has been shown that the h-asymptotic stability or h-complete instability can be recognized by means of a classical procedure due to Poincaré [9]. In particular consider a polynomial of the form

$$F(x,y) = x^2 + y^2 + F_3(x,y) + \dots F_m(x,y) \quad , \tag{2.6}$$

where m is an even integer and F_j is a homogeneous polynomial of degree j. Let $\dot{F}_{(2.1)}(x,y)$ be the derivative of F along solutions of (2.1). The Poincaré procedure is an algebraic method for the determination of m, F, and a constant G_m such that

$$\dot{F}_{(2.1)}(x,y) = G_m(x^2+y^2)^{m/2} + \chi(x,y) \tag{2.7}$$

where χ is of order greater than m. Clearly (2.7) is satisfied for $m=2$ since $G_2=0$. In addition if (2.7) is satisfied for $m=\overline{m}$ with $G_{\overline{m}}=0$ then (2.7) can be satisfied for $m=\overline{m}+2$ and G_m is uniquely determined.

We now have the following relationship between the Poincaré procedure and the h-asymptotic stability and h-complete stability of the origin of system (2.1):

Proposition 2.1 [8]. Let $h \geq 3$ be an odd integer. The solution $x \equiv y \equiv 0$ of (2.1) is h-asymptotically stable (resp. h-completely unstable) if and only if $G_r = 0$ for $r \in \{2,4,\ldots,h-1\}$ and $G_{h+1} < 0$ (resp. $G_{h+1} > 0$).

Another result in [8] establishes a relationship between h-asymptotic stability or h-complete instability of the origin of (2.1) and the displacement function of (2.1) evaluated at the origin, $V(f_0,0)$. Namely

Proposition 2.2 [8]. Let $h \geq 3$ be an odd integer. Then the solution $x \equiv y \equiv 0$ of (2.1) is h-asymptotically stable (resp. h-completely unstable) if and only if $\frac{\partial^i V}{\partial c^i}(f_0,0) = 0$ for $i \in \{1,2,\ldots,h-1\}$ and $\frac{\partial^h V}{\partial c^h}(f_0,0) < 0$ (resp. $\frac{\partial^h V}{\partial c^h}(f_0,0) > 0$).

Thus the displacement function for (2.1) when the origin is h-asymptotically stable or h-completely unstable has the form

$$V(f_0,c) = gc^h + \eta(f_0,c)$$

where $\eta(f_0,c)$ is of degree greater than h in c and the sign of g is the same as the sign of the Poincaré constant G_{h+1}.

3. Main Results

In this section we present our main results and give the proofs in the next section. Our first result is an equivalence between the h-asymptotic stability or h-complete instability of the origin of (2.1) and the number of non-trivial periodic solutions of (2.2) for those f which are sufficiently close to f_0.

Theorem 3.1. Let k be any integer such that $k \geq 1$. Then a necessary and sufficient condition that the solution $x \equiv y \equiv 0$ of (2.1) is either h-asymptotically stable or h-completely unstable with $h = 2k+1$ is

(i) there exists a neighborhood N of f_0 and a number $r_1 > 0$ such that for any $f \in N$ equation (2.2) has no more than k nontrivial periodic orbits in $B^2(r_1)$;

(ii) for any integer j, $0 \leq j \leq k$, for any neighborhood N^* of f, $N^* \subseteq N$, and for any $r \in (0,r_1]$ there exists an $f \in N^*$ such that equation (2.2) has exactly j nontrivial periodic orbits in $B^2(r_2)$.

In addition if the solution $x \equiv y \equiv 0$ of (2.1) is either h-asymptotically stable or h-completely unstable with $h = 2k+1$ then

(iii) letting $S_j \equiv S_j(N, r_1)$, $0 \le j \le k$, be the set of functions $f \in N$ for which there exist exactly j nontrivial periodic orbits of (2.2) in $B^2(r_1)$, we have that S_j has a nonempty interior with f_0 lying on its boundary and $N = \bigcup_{j=0}^{k} S_j$;

(iv) for any $r \in (0, r_1]$, there exists a neighborhood $N_r \subseteq N$ of f such that if $f \in N_r$ and if Γ if a periodic orbit of (2.2) lying in $B^2(r_1)$ then Γ lies in $B^2(r)$.

Remarks. Chafee [3] obtained the conclusions (i)-(iv) under the hypotheses that the determining equation $\psi(\xi, f) = 0$ (see the Introduction) satisfies the condition that $\xi = 0$ is a zero of multiplicity k of $\psi(\cdot, f)$. Moreover he proved the conclusions held in R^n (see the Introduction) for those orbits whose period is near 2π. In R^2 we have by continuous dependence that all the periodic orbits near the origin have periodic near 2π. Moreover it is sufficient that f_0, $f \in C^k[B^2(r_0), R^2]$.

As we mentioned in the Introduction the determination of the multiplicity k of the zeros of $\psi(\cdot, f_0)$ for $\xi = 0$ is a new problem in itself since the construction of $\psi(\xi, f)$ is given implicitly. In order to verify the conclusions (i)-(iv) we have provided an explicit method, which requires the determination of h for which the origin of (2.1) is either h-asymptotically stable or h-completely unstable. From Proposition 2.1 the odd integer h (and thus k since $k = \frac{h-1}{2}$) can be found by utilizing the classical algebraic procedure of Poincaré.

Due to the equivalence of h-asymptotic stability or h-complete instability of the origin for system (2.1) and the conclusion (i) and (ii), we see that our result provides a percise relationship between the existence of small periodic orbits of the perturbed equation (2.2) and the stability properties of the origin of system (2.1).

Now we present a result which extends Theorem 3.1 to the case when $k = \infty$.

Theorem 3.2. A necessary and sufficient condition that the solution $x \equiv y \equiv 0$ of (2.1) is neither h-asymptotically stable nor h-completely unstable for any integer h is that for any neighborhood N of f_0, for any integer $j \geq 0$, and for any number $r \in (0, r_1]$ there exists $f \in N$ such that equation (2.2) has j nontrivial periodic solutions lying in $B^2(r)$.

For the case $h = 3$ our next result gives a criterion, in terms of the Poincaré constant G_4, for establishing whether f is contained in S_0 or in S_1.

Theorem 3.3. Assume that the solution $x \equiv y \equiv 0$ of (2.1) is either 3-asymptotically stable or 3-completely unstable. Then there exists a neighborhood N of f_0, $N \subseteq \mathcal{N}$ and an $\varepsilon \in (0, \varepsilon_1)$ such that we have $f \in S_0(N, \varepsilon)$ if $\alpha G_4 \geq 0$ and $f \in S_1(N, \varepsilon)$ if $\alpha G_4 < 0$.

Remarks. A similar result was given by Chafee [3] who proved that $f \in S_0(N, \varepsilon)$ if $\alpha\psi'(0, f_0) \geq 0$ and $f \in S_1(N, \varepsilon)$ if $\alpha\psi'(0, f_0) < 0$. Chafee pointed out in Section 7 of [3] that he has provided no method for determining the sign of $\psi'(0, f_0)$. From Theorem 3.3 we observe that the sign of $\psi'(0, f_0)$ is the same as that of G_4 and we thus have provided an explicit method in terms of f_0 for the determination of the conclusions of Theorem 3.3.

In the case of the Hopf bifurcation, when the perturbed system f if restricted to a one parameter family of functions f_μ, $-1 < \mu < 1$ then we find Theorem 3.3 yields results that complement and in a sense improve some of the work done by Negrini and Salvadori [8]. In this case (2.2) can be written as

$$\begin{aligned} \dot{x} &= \alpha(\mu)x - \beta(\mu)y + X(\mu,x,y) \quad , \\ \dot{y} &= \alpha(\mu)y + \beta(\mu)x + Y(\mu,x,y) \quad , \end{aligned} \tag{3.1}$$

where $\alpha(0) = 0$, $X(0,x,y) = X_0(x,y)$ and $Y(0,x,y) = Y_0(x,y)$. From the C^∞ version of the Hopf bifurcation theorem we have that if the "transversality condition" $\alpha'(0) \neq 0$ is satisfied then there exists a bifurcation function $\mu(c)$ defined for c nonegative and small such that for each c and μ the orbit of (3.1) through $(c,0)$ is closed if and only if $\mu = \mu(c)$. If in addition to the transversality condition one assumes that the origin of (3.1) for $\mu = 0$ is either h-asymptotically stable or h-completely unstable then Negrini and Salvadori [8] have proved that the inverse function $c(\mu)$ exists. If, for example, $\alpha'(0) > 0$ and the origin is h-asymptotically stable they prove that for each μ sufficiently small there exists a periodic orbit of (3.1). In Theorem 3.3 we prove for $h = 3$ the same result without requiring the condition $\alpha'(0) > 0$. That is if we only assume $\alpha(\mu) > 0$ for $\mu > 0$ and that the origin of (3.1) for $\mu = 0$ is 3-asymptotically stable then $\alpha(\mu)G_4 < 0$, and thus for each $\mu > 0$ there exists a periodic orbit of (3.1) according to Theorem 3.3. Hence the function $c(\mu)$ exists although $\mu(c)$ may not exist.

4. Proofs

We indicate a procedure that transforms (2.1) into a normal form developed by Takens [10]. Consider a positive odd integer $h = 2k + 1$ and let $w = (w_1, \dots, w_d)$ be any real d-vector, $d = h^2 + 3h$. Regard the w_i's as the coefficients of a two dimensional polynomial $Y(w,\cdot,\cdot)$ of degree h and satisfying $Y(w,0,0) \equiv 0$. Now consider the function $Z(w,\cdot,\cdot) : B^2(r_0) \to R^2$ defined by

$$Z(w,x,y) = f_0(x,y) + Y(w,z,y) \quad ,$$

and the associated differential system

$$\begin{aligned} \dot{x} &= Z_1(w,x,y) \\ \dot{y} &= Z_2(w,x,y) \end{aligned} \tag{*}$$

We notice that $Z(0,0,0) = 0$ and the eigenvalues of the Jacobian matrix $D_{x,y}Z(0,0,0)$ are $\pm i$. Then, according to the result in [10], there exists a C^∞ diffeomorphism $T : R^d \times R^2 \to R^d \times R^2$ in which $T_i(w,x,y) = w_i$, $1 \leq i \leq d$, $T_{d+1}(w,x,y) = x_1$, $T_{d+2}(w,x,y) = y_1$, such that system (*) can be transformed into a system of the form

$$\begin{aligned} \dot{x}_1 &= -y_1 + x_1 g_1(w, x_1^2+y_1^2) + x_1 g_2(w,x_1,y_1) \\ \dot{y}_1 &= x_1 + y_1 g_1(w, x_1^2+y_1^2) + y_1 g_2(w,x_1,y_1) \quad , \end{aligned}$$

for w and (x,y) sufficiently small, where $g_1, g_2 \in C^\infty[B^2(r_0),R]$, $g_1(0,0) = 0$ and g_2 is flat at all points $(w,0,0)$.

We wish to implement this transformation on the set of all f, $f(0,0) = 0$, in a neighborhood of f_0. For a given f let $w = (w_1,\ldots,w_d)$ be the set of coefficients of the terms of degree $\leq h$ in the Maclaurin expansion of $f - f_0$. Thus we write

$$f(x,y) = f_0(x,y) + Y(w,x,y) + \tilde{Y}(f,x,y) \quad ,$$

where $Y(w,\cdot,\cdot)$ is a polynomial of degree h satisfying $Y(w,0,0) \equiv 0$ and $\tilde{Y}$ is of order $> h$ in (x,y). There by using the above transformation T, we easily recognize that (2.2) can be written (we replace (x_1,y_1) with (x,y)):

$$\begin{aligned} \dot{x} &= a_0 x - y + a_1 x(x^2+y^2) + a_2 x(x^2+y^2)^2 + \ldots a_k x(x^2+y^2)^k + \Phi(f,x,y) , \\ \dot{y} &= a_0 y + x + a_1 y(x^2+y^2) + a_2 y(x^2+y^2)^2 + \ldots a_k y(x^2+y^2)^k + \Psi(f,x,y) , \end{aligned} \tag{4.1}$$

where $a_0, \ldots, a_k$ are constants depending on f (in fact depending only on w). Moreover $\operatorname{sgn} a_0 = \operatorname{sgn} \alpha$ and Φ,Ψ are of order $> 2k+1$ in (x,y).

Suppose the origin for (2.1) is either h-asymptotically stable or h-completely unstable; then

$$a_j(f_0) = 0 , \quad j = 0, \ldots, k-1 , \quad a_k(f_0) = a \quad , \tag{4.2}$$

with $a < 0$, [resp. $a > 0$] in the case of h-asymptotic stability [resp.

h-complete instability]. The relations (4.2) are a consequence of the equivalence proved in [8] between the $(s,-)$ [resp. $(s,+)$] Takens' singularity and the assumed h-asymptotic stability [resp. h-complete instability] of the origin for $f = f_0$. Thus from (4.1) and (4.2) the system (2.1) under the Takens transformation becomes

$$\begin{aligned} \dot{x} &= -y + ax(x^2+y^2)^{\frac{h-1}{2}} + \Phi(f_0,x,y) \quad , \\ \dot{y} &= x + ay(x^2+y^2)^{\frac{h-1}{2}} + \Psi(f_0,x,y) \quad . \end{aligned} \tag{4.3}$$

For each f in a neighborhood of f_0, where f defines the right-hand side of (2.2), let f^N denote the right-hand side of (4.1). Since T is a C^∞ diffeomorphism the mapping $T_*: C^\infty[B^2(0,r_0),R^2] \to C^\infty[T(B^2(0,r_0)),R^2]$ defined by $T_*(f) = f^N$ is easily seen to be continuous in the topology given at the beginning of Section 2. Hence $\Phi(\cdot,\cdot,\cdot)$ and $\Psi(\cdot,\cdot,\cdot)$ are continuous.

Thus in order to prove Theorems 3.1 and 3.3 it is sufficient to assume that systems (2.1) and (2.3) are in the normal form given by (4.3) and (4.1) respectively. In the proof of Theorem 3.1 (ii) we need to elaborate further since the mapping T_* is not a homeomorphism.

Unless otherwise stated we shall assume that the origin for system (2.1) is either h-asymptotically stable or h-completely unstable where $h = 2k+1$.

The displacement function for (4.1), again denoted by $V(f,c)$ is given by

$$V(f,c) = (u_1(f,2\pi) - 1)c + u_2(f,2\pi)c^2 + u_3(f,2\pi)c^3 + \dots u_{2k+1}(f,2\pi)c^{2k+1}$$
$$+ \rho(f,c) \quad ,$$

where $\rho(f,\cdot)$ has degree $> 2k+1$. In case $f = f_0$ then recall $u_1(f_0,2\pi) = 1$ and $u_{2i+1}(f_0,2\pi) = 0$, $i = 2, \dots, k-1$. We shall again denote by $\tilde{V}(f,c)$ the function defined in Section 2.

Proposition 4.1. Consider the system (4.1). Then the displacement function is given by

$$V(f,c) = (u_1(a_0,2\pi) - 1)c + u_3(a_0,a_1,2\pi)c^3$$
$$+ \dots u_{2i+1}(a_0,a_1, \dots ,a_i,2\pi)c^{2i+1}$$
$$+ \dots u_{2k+1}(a_0,a_1, \dots ,a_k,2\pi)c^{2k+1} + \rho(f,c) \qquad (4.4)$$

and the relationship between $(a_0,a_1, \dots ,a_k)$ and $(u_1(a_0,2\pi)$, $u_3(a_0,a_1,2\pi), \dots ,u_{2i+1}(a_0,a_1, \dots ,a_i,2\pi), \dots ,u_{2k+1}(a_0,a_1, \dots ,a_k,2\pi)$ is a homeomorphism (actually a C^∞ diffeomorphism) in a neighborhood of $(0,0, \dots ,a)$ such that $u_1(0,2\pi) = 1$, $u_{2i+1}(0,0, \dots ,2\pi) = 0$, $1 < i < 2k-1$ and $u_{2k+1}(0,0, \dots ,a,2\pi) = 2\pi a$.

Proof: Writing equation (4.1) in polar form we find that

$$\frac{dr}{d\theta} = a_0 r + a_1 r^3 + \dots a_i r^{2i+1} + \dots a_k r^{2k+1} + \gamma(\theta,r) \quad ,$$

where $\gamma(\theta,r)$ has order greater than $2k+1$ in r. By assuming r has the form (2.4) then $u_i(\cdot,\theta)$ satisfies (2.5), which in the case when i is odd reduces to

$$\frac{\partial u_1}{\partial\theta} = a_0 u_1$$

$$\frac{\partial u_{2i+1}}{\partial\theta} = a_0 u_{2i+1} + w_i(a_1,a_2,\ldots,a_{i-1}, u_1,u_3,\ldots,u_{2i+1}) + a_i u_1^{\,2i+1} \qquad i = 1,2,\ldots,k \ , \tag{4.5}$$

where w_i is linear in each of the a_j, $j = 1,\ldots,i-1$. In the case when i is even we find that the solution of (2.5) satisfying the initial conditions $u_{2j}(f,0) = 0$ is given by $u_{2j}(f,\theta) \equiv 0$ $j = 1,\ldots,k$. We thus see $V(f,c)$ is given by (4.4).

By a standard continuous dependence argument it follows that $(u_1(\cdot,2\pi), u_3(\cdot,2\pi),\ldots,u_{2k+1}(\cdot,2\pi))$ is C^∞ in $(a_0,a_1,\ldots,a_k)$ since the solutions of (4.5) are uniquely determined by initial conditions. Since $u_1(\cdot,0) = 1$, $u_{2i+1}(\cdot,0) = 0$ $i = 1,\ldots,k$ and $g_i(0,\ldots,0,u_1,u_3,\ldots,u_{2i-1}) = 0$ then from (4.5) $u_1(0,\theta) \equiv 1$, $u_{2i+1}(0,0,\ldots,0,\theta) \equiv 0$ and $u_{2k+1}(0,0,\ldots,0,a,\theta) = a\theta$. To complete the proof, the solution of (4.5) with the given above initial conditions is

$$u_1(2\pi) = e^{2\pi a}$$

$$u_{2i+1}(2\pi) = e^{2\pi a}\int_0^{2\pi} e^{-a\theta} f_i(a_1,a_2,\ldots,a_{i-1}, u_1(\theta),u_3(\theta),\ldots, u_{2i-1}(\theta))\,d\theta + e^{2\pi a}\int_0^{2\pi} e^{-a\theta} a_i u_1^{\,2i+1}(\theta)\,d\theta \ ,$$

$$1 \le i \le k \ . \tag{4.6}$$

Moreover by continuous dependence arguments there exist C^∞ functions g_i such that the solutions of (4.5) can be written as

$$u_{2i+1}(\theta) = g_i(\theta, a_0, a_1, \dots, a_i(2\pi), u_2(2\pi), \dots, u_{2i+1}(2\pi)),$$

$$1 \le i \le k$$

and

$$u_1(\theta) = e^{a_0\theta}.$$

Define the functions h_i as

$$h_i(\theta, a_0, a_1, \dots, a_{i-1}, u_1(2\pi), u_3(2\pi), \dots, u_{2i+1}(2\pi))$$

$$= (f_i(a_1, a_2, \dots, a_{i-1}, e^{a_0\theta}, g_1(\theta, a_0, a_1, u_1(2\pi), u_3(2\pi)),$$

$$\dots, g_{i-1}(\theta, a_0, \dots, a_{i-1}, u_1(2\pi), u_3(2\pi), \dots, u_{2i-1}(2\pi))$$

$$1 \le i \le k \ .$$

Substituting the above in (4.6) we have

$$u_1(2\pi) = e^{2\pi a}$$

$$u_{2i+1}(2\pi) = e^{2\pi a_0}\int_0^{2\pi} e^{-a_0\theta}\, h_i(\theta, a_0, a_1, \dots, a_{i-1}, u_1(2\pi),$$

$$u_3(2\pi), \dots, u_{2i-1}(2\pi))d\theta + e^{2\pi a_0}\int_0^{2\pi} a_i e^{2ia_0\theta}\, d\theta,$$

$$1 \le i \le k \ . \qquad (4.7)$$

Now (4.7) gives an implicit relationship between the set $(u_1(2\pi), u_3(2\pi), \ldots, u_{2k+1}(2\pi))$ and the set $(a_0, a_1, \ldots, a_k)$. Thus in order to show $(a_0, a_1, \ldots, a_k)$ is a continuous function of $(u_1(2\pi), u_3(2\pi), \ldots, u_{2k+1}(2\pi))$ in a neighborhood of $(1,0,0,\ldots,2\pi a)$ we apply the inverse mapping theorem to (4.7). Denoting the right side of (4.7) by F_{2i+1}, we must prove that the determinant of the Jacobian

$$\left(\frac{\partial F_{2i+1}}{\partial a_j}\right) \text{ evaluated at } \begin{cases} a_i = 0, \quad 0 \le i \le k-1, \quad a_k = a \\ u_1(2\pi) = 1, \quad u_{2i+1}(2\pi) = 0, \quad 1 \le i \le k-1, \\ \qquad u_{2k+1}(2\pi) = a \end{cases} \tag{4.8}$$

is not zero. Since the matrix in (4.8) is lower triangular (from (4.7)) we easily have that its determinant evaluated at the point given in (4.8) is equal to $(2\pi)^{k+1}$. In view of the smoothness of the functions F_{2i+1} the proof of Proposition 4.1 is complete.

Proposition 4.2. Let j,k be integers with $0 \le j \le k$. Assume $\bar{a} > 0$ and $\lambda \ne 0$ and consider the function $f : [0,\bar{a}] \times [0,\bar{a}] \to R$ defined as

$$f(\mu,x) = (x-\mu)(x-2\mu)\ldots(x-j\mu)(x+\mu)^{k-j} + \phi(\mu,x) \; ,$$

where $\phi(\cdot,\cdot) \in C^{k+1}[[0,\bar{a}] \times [0,\bar{a}],R]$. Suppose that for each $\mu \in [0,\bar{a}]$, $\phi(\mu,\cdot) = O(x^k)$ and that the coefficient of the x^{k+1} term of the Maclaurin expansion of $\phi(0,\cdot)$ is not zero. Then there exists

$x_1 > 0$, $\mu_1 > 0$ such that for $0 < \mu < \mu_1$, $f(\mu,\cdot)$ has exactly j zeros on the interval $(0,x_1]$ and they are all simple. Moreover these zeros lie on the interval $[0,(j+1)\mu]$.

Proof: Without loss of generality we may assume that the coefficient of the x^{k+1} term of the Maclaurin expansion of $\phi(0,\cdot)$ is positive, for otherwise we may replace $f(\mu,x)$ with $-f(\mu,x)$. There exists $\mu_2 > 0$, $x_2 > 0$ such that $\phi(\mu,\cdot)$ is strictly increasing on $[0,x_2]$ for all $\mu \in (0,\mu_2]$. Set

$$\bar{f}(\mu,x) = \frac{f(\mu,x)}{(x+\mu)^{k-j}} \quad \text{and} \quad \bar{\phi}(\mu,x) = \frac{\phi(\mu,x)}{(x+\mu)^{k-j}} . \tag{4.9}$$

In order to prove our result it suffices to find an $x_1 > 0$ and $\mu_1 > 0$ such that for $\mu \in (0,\mu_1)$, $\bar{f}(\mu,\cdot)$ has exactly j zeros on $(0,x_1]$ which are all simple. We then show that these zeros lie on the interval $(0,(j+1)\mu]$.

Write

$$\bar{f}(\mu,x) = \psi(\mu,x) + \bar{\phi}(\mu,x) ,$$

where

$$\psi(\mu,x) = \lambda(x-\mu)(x-2\mu)\ldots(x-j\mu) .$$

It follows from the hypotheses on $\phi(\mu,x)$ and (4.9) that $\dfrac{\partial^i \bar{\phi}(0,0)}{\partial x^i} = 0$

for $0 \le i \le j$ and that $\bar{\phi} \in C^j[[0,\bar{a}] \times [0,\bar{a}],R]$; thus $\bar{f} \in C^j[[0,\bar{a}] \times [0,\bar{a}],R]$. Since $\frac{\partial^j \psi(0,0)}{\partial x^j} \neq 0$ then $\frac{\partial^j \bar{f}(0,0)}{\partial x^j} \neq 0$; hence there exists $\mu_3 > 0$, $x_3 > 0$ such that $\frac{\partial^j \bar{f}(\mu,x)}{\partial x^j} \neq 0$ for $\mu \in [0,\mu_3]$ and $x \in [0,x_3]$. By Rolle's Theorem we immediately conclude that for $\mu \in [0,\mu_3]$, $\bar{f}(\mu,\cdot)$ has no more than j zeros (counting multiplicity) on the interval $[0,x_3]$.

It thus suffices to show that there exist $\mu_1 \le \mu_3$ and $x_1 \le x_3$ such that for $\mu \in [0,\mu_1]$, $\bar{f}(\mu,\cdot)$ has at least j distinct zeros on $(0,x_1)$ in order to conclude that $\bar{f}(\mu,\cdot)$ for $\mu \in [0,\mu_1]$ has exactly j roots on $(0,x_1]$, which are all simple. From the conditions on $\phi(\mu,x)$ there exists an $x_4 > 0$ and $\mu_4 > 0$ with $x_4 \le x_3$ and $\mu_4 \le \mu_3$ such that $\bar{\phi}(\mu,\cdot)$ is strictly increasing on $[0,x_4]$ for each $\mu \in (0,\mu_4]$. Since $\psi(\mu,x)$ is a homogeneous polynomial in μ,x it follows that

$$\psi(\mu,x) = \lambda(x^j + c_1\mu x^{j-1} + c_2\mu^2 x^{j-2} + \dots c_j\mu^j)$$

for appropriate constants c_i , $1 \le i \le j$. We now obtain estimates on the ordinates of the j relative extremum of $\psi(\mu,\cdot)$. Consider the roots of

$$\frac{\partial\psi(\mu,x)}{\partial x} = \lambda(jx^{j-1} + c_1(j-1)\mu x^{j-2} + \dots c_{k-1}\mu^{k-1}) = 0 \quad . \qquad (4.10)$$

For $\mu = 1$ let b_s $1 \le s \le j-1$ be the real solutions of (4.10) since $\psi(\mu,\cdot)$ has $j-1$ distinct relative extremum. Thus b_s satisfy

$$jb_s^{j-1} + c\,(j-1)b_s^{j-2} + \ldots c_{j-1} = 0 \tag{4.11}$$

and b_s satisfies (4.10); indeed

$$\frac{\partial\psi}{\partial x}(\mu, b_s\mu) = \lambda(\mu^{j-1}\, jb_s^{j-1} + \mu^{j-1}\, c\,(j-1)b_s^{j-2} + \ldots \mu^{j-1}\, c_{j-1})$$
$$= \lambda\mu^{j-1}(\lambda b_s^{j-1} + c\,(j-1)b_s^{j-2} + \ldots c_{j-1}) = 0 ,$$

in view of (4.11). Thus for each μ the relative extremum of $\psi(\mu,\cdot)$ occur at $x = b_s\mu$ for $1 \le s \le j$ and

$$\psi(\mu, b_s\mu) = \lambda\mu^j(b_s^j + c_1 b_s^{j-1} + \ldots c_j) \quad . \tag{4.12}$$

From the properties of $\overline{\phi}(\mu,x)$ we observe that $\overline{\phi}(\mu,m\mu) = O(\mu^j)$ for any number m. Let us now pick a number $\mu_1 > 0$ satisfying several conditions. First of all select $\mu_1 > 0$ such that $\mu_1 < \mu_4$ and $(j+1)\mu_1 \le x_4$. In addition let μ_1 have the property that all $\mu \in [0,\mu_1]$

$$\begin{aligned} &\text{(i)} \quad |\psi(\mu, b_s\mu)| > |\overline{\phi}(\mu, j\mu)| \qquad s = 1,2,\ldots,j \\ &\text{(ii)} \quad |\psi(\mu,(j+1)\mu| > |\overline{\phi}(\mu,(j+1)\mu)| \quad . \end{aligned} \tag{4.13}$$

Such a μ_1 exists in view of (4.12) and the fact that $\overline{\phi}(\mu,m\mu) = O(\mu^j)$. For $\mu \in (0,\mu_1]$, $\overline{\phi}(\mu,\cdot)$ is increasing for $x \in [0,(j+1)\mu_1]$ (since $(j+1)\mu_1 \le x_4$) and thus from (4.13) (i) $|\psi(\mu,b_s\mu)| > |\overline{\phi}(\mu,b_s\mu)|$.

It thus follows for $0 < u \leq \mu_1$ that on any interval $(\ell\mu, (\ell+1)\mu)$, ℓ a positive integer $< j$, in which $\psi(\mu,\cdot)$ is negative then $\bar{f}(\mu,\cdot) = \psi(\mu,\cdot) + \bar{\phi}(\mu,\cdot)$ has at least two zeros. Thus one of three possiblities can occur:

(a) $\bar{f}(\mu,\cdot)$ has at least $j-2$ zeros on $(\mu, j\mu)$ and $\psi(\mu,\cdot)$ is negative on $[0,\mu)$ and on $(j\mu,(j+1)\mu]$. But then $\bar{f}(\mu,\cdot)$ has at least one zero on $(0,\mu)$ since $\bar{\phi}(\mu,0) = 0$ and $\bar{\phi}(\mu,\cdot)$ is positive on $(0,\mu]$. Moreover, in view of (4.13) (ii), $\bar{f}(\mu,\cdot)$ has at least one zero on $(j\mu,(j+1)\mu)$.

(b) $\bar{f}(\mu,\cdot)$ has at least $(j-1)$ zeros on $(\mu, j\mu)$ and either $\psi(\mu,\cdot)$ is negative on $[0,\mu)$ or on $(j\mu,(j+1)\mu]$. Reasoning as in (a) $\bar{f}(\mu,\cdot)$ has at least one zero on either $(0,\mu)$ or on $(j\mu,(j+1)\mu)$.

(c) $\bar{f}(\mu,\cdot)$ has at least j zeros on $(\mu, j\mu)$.

In all three cases $\bar{f}(\mu,\cdot)$ has at least j distinct zeros on $(0,(j+1)\mu_1)$ for each $\mu \in (0,\mu_1]$ and in fact these zeros lie on $(0,(j+1)\mu)$. If we let $x_1 = (j+1)\mu_1$ we obtain the conclusions of Proposition 4.2.

We now provide information concerning the zeros of $\tilde{V}(f,c)$ in a neighborhood of $(f_0,0)$.

Proposition 4.3. Assume the solution $x \equiv y \equiv 0$ of the system (2.1) is either h-asymptotically stable of h-completely unstable. Then there exists a neighborhood N of f_0 and a number r_3, $0 < r_3 < r_0$ such that for $f \in N$, $\tilde{V}(f,\cdot)$ has at most k zeros on $[0,r_3]$ where $k = \frac{h-1}{2}$.

Proof: As indicated before we may assume systems (2.1) and (2.2) have the normal form (4.3) and (4.1) respectively. For $c \neq 0$, $\tilde{V}(f,c)$ is given by

$$\tilde{V}(f,c) = (u_1(f,2\pi) - 1) + u_3(f,2\pi)c^2 + \dots y_{2k+1}(f,2\pi)c^{2k} + \tilde{\rho}(f,c)$$

where $\tilde{\rho}(f,c) = \frac{\rho(f,c)}{c}$ and $\tilde{\rho}(f,\cdot)$ had degree $> 2k$. By substituting $z = c^2$ it then suffices to prove that $\tilde{V}(f,\sqrt{z}) = 0$ has at most k roots. Now

$$\tilde{V}(f_0,\sqrt{z}) = 2\pi a\, z^k + \tilde{\rho}(f_0,\sqrt{z})$$

where $a \neq 0$ and $\tilde{\rho}(f_0,\sqrt{z}) = O(z^k)$. By continuous dependence argument $\frac{\partial^i \tilde{V}}{\partial_z k}(f,\sqrt{z})$ is continuous for all f in a neighborhood of f_0, for all z in a right neighborhood of 0, and for all integers i. Since $\frac{\partial^k \tilde{V}}{\partial_z k}(f_0,0) = 2\pi a \neq 0$ then by continuity there exist a neighborhood N of f_0 and a number r_3 such that $\frac{\partial^k V}{\partial z^k}(f,\sqrt{z}) \neq 0$ for each $f \in N$ and each $z \in [0,r_3^2]$. An application of Rolle's Theorem tells us that for each $f \in N$ there are at most k roots of the equation $\tilde{V}(f,\sqrt{z}) = 0$ for $z \in [0,r_3^2]$; that is for each $f \in N$ there are at most k roots of the equation $\tilde{V}(f,c) = 0$ for $c \in [0,r_3^2]$. This concludes the proof of the proposition.

Proposition 4.4. Assume the solution $x \equiv y \equiv 0$ of (2.1) is either h-asymptotically stable or h-completely unstable. There exists a number r_4, $0 < r_4 < r_0$ such that for any $r \in (0,r_4]$ there exists a neighborhood $N_r \subseteq N$ so that for any $f \in N_r$, $\tilde{V}(f,\cdot)$ has no zeros for $c \in [r,r_4]$.

Proof: Consider

$$\tilde{V}(f_0,c) = 2\pi a c^{2k} + \tilde{\rho}(f_0,c) \quad ,$$

where $\tilde{\rho}(f_0,c) = 0(c^{2k})$. It follows immediately that there exists a number r_4, $0 < r_4 < r_0$ such that

$$|\tilde{V}(f_0,c)| > \pi|a|c^{2k} \quad \text{for} \quad c \in [0,r_4] \quad .$$

Now given any $r \in (0,r_4]$

$$|\tilde{V}(f_0,c)| > \pi|a|r^{2k} \quad \text{for} \quad c \in [r,r_4] \quad . \tag{4.14}$$

Because of the continuity of $\tilde{V}(\cdot,\cdot)$ and the compactness of $[r,r_4]$ it follows that there exists a neighborhood N_r of f_0 with $N_r \subseteq N$ such that for $f \in N_r$ and for all $c \in [r,r_4]$

$$|\tilde{V}(f,c) - \tilde{V}(f_0,c)| < \frac{1}{2}\pi|a|r^{2k} \quad . \tag{4.15}$$

Thus from (4.14) and (4.15) for $f \in N_r$ and $c \in [r,r_4]$, $V(f,c) > \frac{1}{2}\pi|a|r^{2k}$, completing the proof of the proposition.

We now prove the necessity part of Theorem 3.1, that is we prove that the h-asymptotic stability or h-complete instability of the solution $x \equiv y \equiv 0$ of (2.1) with $h = 2k+1$ implies (i)-(iv) hold. We then prove Theorem 3.2, the sufficiency part of the Theorem 3.1, and Theorem 3.3.

Proof:

(i) Choose $r_1 = \min\{r_3, r_4\}$. Then the result follows immediately from Proposition 4.3 since the nonzero roots of $\tilde{V}(f,\cdot) = 0$ correspond to the nontrivial periodic solutions of (4.1),

(ii) In order to prove the result we claim it is permissible to once again assume that (2.1) and (2.2) are in normal form. Indeed, we prove that for any integer j, $0 \leq j \leq k$ and for any neighborhood $B^2(r_2)$ or the origin, $r_2 \in (0, r_1]$, there exists a two parameter family of functions $f^N_{\mu,\tau}(x_1, y_1)$ (recall $f^N(x_1,y_1) = (T * f)(x_1,y_1)$ and we replace (x_1,y_1) with (x,y)) such that for μ,τ sufficiently small, system (4.2) has exactly j nontrivial periodic solution in $B^2(r_2)$ and no periodic solutions in $B^2(r_1)/B^2(r_2)$. Moreover $f^N_{\mu,\tau}(\cdot) \to f^N(\ )$ as $(\mu,\tau) \to (0,0)$. Since $f^N_{\mu,\tau}$ is a two parameter family than the corresponding Takens' transformation of this family can be considered as a C^∞ diffeomorphism $T : R^2 \times R^2 \to R^2 \times R^2$; that is $d = 2$. More percisely, $\begin{pmatrix} \mu \\ \tau \\ x \\ y \end{pmatrix} = T^{-1}(\mu,\tau,x_1,y_1)$ and $f_{\mu,\tau}(x,y) =$ $(T * (\mu,\tau,x_1,y_1)^{-1} f^N_{\mu,\tau})(x_1,y_1)$, and thus $f_{\mu,\tau} \to f$ in the topology described in the beginning of Section 2. Moreover for each (μ,τ),

$(T^{-1}(\mu,\tau,\cdot,\cdot)$ is a homeomorphism of a neighborhood of the origin in (x_1,y_1) space onto a neighborhood of the origin in the (x,y) space. From these considerations it follows that for each integer j, $0 \leq j \leq k$, for each neighborhood N^* of f_0, and for each $r \in (0,r_1]$ there exists $(\mu.\tau)$ sufficiently small such that $f_{\mu,\tau} \in N^*$ and system (2.1) has j periodic orbits in $B^2(r)$.

It thus suffices to prove our assertions given in the beginning of the proof and we may assume that (2.1) and (2.2) are in normal form. Proceeding with the proof, pick any integer j, $0 \leq j \leq k$, any neighborhood N^* of f_0, $N^* \subseteq N$ and any number $r_2 \in (0,r_1]$. Now let μ be a positive constant and we claim we can select a one parameter family of functions f_μ such that $f_\mu \to f_0$ as $\mu \to 0$ and $\tilde{V}(f_\mu,c)$ can be written as

$$V(f_\mu,c) = 2\pi a(c^2-\mu)(c^2-2\mu), \ldots, (c^2-j\mu)(c^2+\mu)^{k-j} + \tilde{\rho}(\mu,c) \quad , \quad (4.16)$$

where $\tilde{\rho}(\cdot,\cdot) \in C^\infty$ and $\tilde{\rho}(\mu,c) = O(c^{2k})$. Indeed from Proposition 4.1 there exists $b_1 > 0$ such that for $\mu \in [0,b_1]$ we can select $a_i(\mu)$, $1 \leq i \leq k$, in (4.1) so that the $2k^{th}$ order Taylor expansion of $\tilde{V}(f_\mu,\cdot)$ has exactly the form

$$2\pi a(c^2-\mu)(c^2-2\mu), \ldots, (c^2-j\mu)(c^2+\mu)^{k-j} \quad .$$

Moreover we may select f_μ so that in (4.1)

$$\Phi(f_\mu,x,y) \equiv \Phi(f_0,x,y)$$
$$\Psi(f_\mu,x,y) \equiv \Psi(f_0,x,y) \quad . \tag{4.17}$$

In view of the Takens normal form given in (**) we may assume that

$$\Phi(f_\mu,x,y) \equiv \Phi(f_0,x,y) = a_{k+1}x(x^2+y^2)^{k+1} + \hat{\Phi}(f_0,x,y)$$
$$\Psi(f_\mu,x,y) \equiv \Psi(f_0,x,y) = a_{k+1}y(x^2+y^2)^{k+1} + \hat{\Psi}(f_0,x,y) \quad , \tag{4.18}$$

where $\hat{\Phi}(f_0,x,y)$, $\hat{\Psi}(f_0,x,y) = O((x^2+y^2)^{k+2})$. Then (4.16) takes the form

$$\tilde{V}(f_\mu,c) = 2\pi a(c^2-\mu)(c^2-2\mu), \ldots ,(c^2-j\mu)(c^2+\mu)^{k-j} +$$
$$u_{2k+3}(f_\mu,2\pi)c^{2k+3} + \rho^*(\mu,c) \quad , \tag{4.19}$$

where $\rho^*(\mu,c) = O(c^{2k+2})$ and $u_{2k+3}(f_\mu,0)$ corresponds to the coefficient of c^{2k+3} in the expansion of r (see (2.4)) for the system given by f_μ . Using (4.3) and (4.18) one find that $u_{2k+3}(f_0,\theta)$ $= a_{k+1}\theta$.

We wish to apply Proposition 4.2 to the function $\dfrac{\tilde{V}(f_\mu,c)}{(c^2+\mu)^{k-j}}$ with $x = c^2$. This requires $0 \neq u_{2k+3}(f_0,2\pi) = 2\pi a_{k+1}$. Hence if $a_{k+1} \neq 0$ we are in position to apply Proposition 4.2. If $a_{k+1} = 0$ then define for $\tau \in (0,1]$

$$f_{\mu,\tau}(x,y) = f_{\mu}(x,y) + \tau x(x^2+y^2)^{k+1} .$$

So as not to have to discuss the two cases $a_{k+1} = 0$ and $a_{k+1} \neq 0$ separately let us define for all $\tau \in (0,1]$

$$f_{\mu,\tau}(x,y) \equiv f(x,y)$$

in case $a_{k+1} \neq 0$. Now write the function $\tilde{V}(f_{\mu,\tau},c)$ as

$$\tilde{V}(f_{\mu,\tau},c) = 2\pi a(c^2-\mu)(c^2-2\mu), \ldots ,(c^2-j\mu)(c^2+\mu)^{k-j} + u_{2k+3}(f_{\mu,\tau},2\pi)c^{2k+2} + \hat{\rho}(\tau,\mu,c) \tag{4.20}$$

where

$$u_{2k+3}(f_{0,\tau},2\pi) = \begin{cases} 2\pi\tau & \text{if} \quad a_{k+1} = 0 \\ 2\pi a_{k+1} & \text{if} \quad a_{k+1} \neq 0 , \end{cases} \tag{4.21}$$

and $\hat{\rho}(\tau,\mu,c) = 0(c^{2k+2})$. Defining (as in the proof of Proposition 4.2)

$$\hat{V}(\mu,\tau,x) = \frac{\tilde{V}(f_{\mu,\tau},\sqrt{x})}{(x+\mu)^{k-j}} ,$$

we need to show that there exists (μ,τ) sufficiently small so that $f_{\mu,\tau} \in N^*$ and $\hat{V}(\mu,\tau,\cdot)$ has exactly j zeros on the interval $(0,r_2^2]$ (recall $x = c^2$). Now $\frac{\partial^j V(0,0,0)}{\partial x^j} \neq 0$ and by the continuity of

$\frac{\partial^j v}{\partial x^j}(\cdot,\cdot,\cdot)$ there exists $\bar{\mu},\bar{\tau},x$ with $x_1 \le r_2^2$, $\mu \le b$ such that $\frac{\partial^j \hat{v}(\mu,\tau,x)}{\partial x^j} \neq 0$ for $x \in (0,x_1]$, $\tau \in [0,\bar{\tau}]$, and $\mu \in [0,\bar{\mu}]$. As a consequence of Rolle's Theorem, for $\tau \in [0,\bar{\tau}]$, $\mu \in [0,\bar{\mu}]$ and $x \in [0,x_1]$, $\hat{V}(\mu,\tau,\cdot)$ has at most j zeros on the interval $[0,x_1]$.

From the continuity of $u_{2k+3}(f_{\mu,\tau},2\pi)$ in μ and τ and the fact that $u_{2k+3}(f_{0,\tau},2\pi) \neq 0$ (see (4.21)) for $\tau \in (0,\bar{\tau}]$, then for each $\tau \in (0,\bar{\tau}]$ there exists a number $\mu_1(\tau)$ such that for $\mu \in [0,\mu_1(\tau)]$ $u_{2k+3}(f_{\mu,\tau},2\pi) \neq 0$.

Since $f_{\mu,\tau} \to f_0$ as $\mu,\tau \to (0,0)$ we may pick $\tau_1 \le \bar{\tau}$ and $\mu^* \le \min(\mu_1(\tau_1),\mu)$ so that for $\mu \in [0,\mu^*]$

$$f_{\mu,\tau_1} \in N_{x_1} \cap N^*$$

where N_{x_1} is given in Proposition 4.4. Thus $\hat{V}(\mu,\tau_1,\cdot)$ has no zeros on $[x_1,r_1^2]$ for all $\mu \in [0,\mu^*]$.

Applying Proposition 4.2 to the function $\hat{V}(\mu,\tau,x)$, there exists $\mu_2 < \mu^*$ with $(j+1)\mu_2 \le x_1$ such that for $\mu \in [0,\mu_2]$, $\hat{V}(\mu,\tau_1,\cdot)$ has j distinct zeros on the interval $(0,(j+1)\mu]$. Since for $\mu \in [0,\mu_2]$, $V(\mu,\tau_1,\cdot)$ has at most j zeros on $(0,x_1]$ we conclude $\hat{V}(\mu,\tau_1,\cdot)$ has exactly j positive zeros on $(0,x_1]$ and no zeros on $[x_1,r_2^2]$. Since $x_1 \le r_2^2$, $f_{\mu,\tau_1} \in N^*$, and $V(f_{\mu,\tau_1},\cdot)$ has exactly j distinct zeros on $(0,r_2)$, the proof of (ii) is complete.

(iii) From (i) and (ii) it follows that $N = \bigcup_{j=0}^{k} S_j$ is nonempty for each j, $0 \le j \le k$. Now choose any $\bar{j}$ with $0 \le \bar{j} \le k$ and we shall show that the interior of $S_{\bar{j}}$ is not empty. Consider the functions $f_{\mu,\tau} \in S_{\bar{j}}$ constructed in (ii) and we show for μ and τ sufficiently small $f_{\mu,\tau}$ is in the interior of $S_{\bar{j}}$. As in the proof of (ii), an application of Rolle's Theorem implies that there exist $b_1 \le r_1$ and a neighborhood N_1 of $f_{\mu,\tau}$ for all μ,τ sufficiently small, $N_1 \subset \mathcal{N}$, such that for $f \in N_1$, $\tilde{V}(f,\cdot)$ has at most $\bar{j}$ zeros on $[0,b_1]$. From Proposition 4.4 there exists a neighborhood $\mathcal{N}_{b_1}$ of f_0, $\mathcal{N}_{b_1} \subset \mathcal{N}$ so that for $f \in \mathcal{N}_{b_1}$, $\tilde{V}(f,\cdot)$ has no zeros on $[b_1,r_1]$. Pick μ,τ so small that $f_{\mu,\tau} \in \mathcal{N}_{b_1} \cap N$ and the zeros of $V(f_{\mu,\tau},\cdot)$ lie in $[0,\frac{b_1}{2}]$. Since $V(f_{\mu,\tau},\cdot)$ has $\bar{j}$ simple zeros on $[0,\frac{b_1}{2}]$ there exists a neighborhood N_2 of $f_{\mu,\tau}$, $N_2 \subset \mathcal{N}$ so that for $f \in N_2$, $\tilde{V}(f,\cdot)$ has at least $\bar{j}$ zeros on $[0,b_1]$. Defining $N_3 = N_1 \cap \mathcal{N}_{b_1} \cap N_2$, we see that N_3 is a nonempty neighborhood of $f_{\mu,\tau}$ and for $f \in N_3$, $\tilde{V}(f,\cdot)$ has exactly j zeros on $[0,b_1]$ and no zeros on $[b_1,r_1]$; that is $f \in S_{\bar{j}}$. Thus $f_{\mu,\tau}$ is in the interior of $S_{\bar{j}}$ for all μ,τ sufficiently small. Since $f_{\mu,\tau} \to f_0$ as $(\mu,\tau) \to (0,0)$ then f_0 is in the closure of the interior of $S_{\bar{j}}$; that is f_0 is on the boundary of $S_{\bar{j}}$ for each $\bar{j}$, $0 \le \bar{j} \le k$. This concludes the proof.

(iv) This follows immediately from Proposition 4.4

Remarks. We observe that in order to construct f lying in the interior of S_j we can replace the polynomial part of order $2k$ in (4.16) with the more general expression

$$2\pi a(c^2-\mu_1)(c^2-\mu_2),\ \ldots\ ,(c^2-\mu_j)(c^2+\alpha)^{k-j} \quad ,$$

where μ_i, $1 \leq i \leq j$, and α are sufficiently small positive numbers and the μ_i's are distinct. From Proposition 4.1 we have a 1-1 correspondence between the coefficients of the polynomial part of order k of $\tilde{V}(f,\sqrt{c})$ and the coefficients of the right-hand side of (4.1) (up to order $2k+1$). Moreover small perturbations (of order $k+1$) of the polynomials of order k having j simple zeros will not cause any change in the number of zeros of $\tilde{V}(f,\sqrt{c})$. Thus functions f can easily be constructed that preserve the number of periodic orbits under perturbation.

We also see that if $f \in S_k$ then $\tilde{V}(f,\sqrt{c})$ has exactly k simple roots in a sufficiently small neighborhood of the origin. It then follows that f is in the interior of S_k, that is, S_k is open.

Proof of Theorem 3.2.

Necessity: Consider the following perturbation of (2.1)

$$\begin{aligned} \dot{x} &= -y + X_0(x,y) - bx(x^2+y^2)^j \\ \dot{y} &= x + Y_0(x,y) - by(x^2+y^2)^j \end{aligned} \qquad (4.22)$$

where b is a sufficiently small positive constant so that the right-hand side of system (4.22) lies in N. Let us now apply the Poincaré procedure to systems (2.1) and (4.22). Since the solution $x \equiv y \equiv 0$ of (2.1) is neither h-asymptotically stable nor h-completely unstable for any $h > 0$ there exists F of the form (2.6) with $m = 2j+2$ such that $[\dot{F}_{(2.1)}(x,y)]_i = 0$ for $2 \leq i \leq 2j+2$, where $[\dot{F}_{(2.1)}(x,y)]_i$ is the homogeneous polynomial of degree i in the expansion of $\dot{F}_{(2.1)}$. Thus we obtain

$$\dot{F}_{(4.22)}(x,y) = -b(x^2+y^2)^{j+1} + O(r^{2j+2}) \ .$$

From Proposition 3.1 we immediately conclude that the solution $x \equiv y \equiv 0$ of system (4.22) is $2j+1$-asymptotically stable. From (ii) of Theorem 3.1 applied to (4.22) there exists an $f \in N$ such that system (2.2) has j periodic orbits in $B^2(r)$. Thus we obtain the necessity part of Theorem 3.2.

Sufficiency: This follows immediately from the necessity part of Theorem 3.1 (i). Indeed if there exists an integer $\bar{h}$ (which must be odd) for which the origin of (2.1) is either $\bar{h}$-asymptotically stable or $\bar{h}$-completely unstable then Theorem 3.1 (i) implies that for all $f \in N$ there exist at most $\frac{\bar{h}-1}{2}$ nontrivial periodic solutions of (2.2) lying in $B^2(r_1)$. This is a contradiction and completes the proof of Theorem 3.2.

Proof of the Sufficiency Part of Theorem 3.1. We show that (i) and (ii) imply that the solution $x \equiv y \equiv 0$ of (2.1) is either h-asymptotically stable on h-completely unstable where $h = 2k+1$. Applying the Poincaré procedure to (2.1) we conclude that the solution $x \equiv y \equiv 0$ of (2.1) is either

(a) $\bar{h}$-asymptotically stable or $\bar{h}$-completely unstable, for some finite $\bar{h}$; or

(b) neither $\bar{h}$-asymptotically stable nor $\bar{h}$-completely unstable for any integer $\bar{h} > 0$.

In case (b), Theorem 3.2 implies that there exists an $f \in N$ such that equation (2.2) has $k+1$ nontrivial periodic solutions in $B^2(r_1)$.

This contradicts (i). It thus remains to prove $\bar{h} = h$ in case (a). Let $\bar{h} = 2\bar{k} + 1$ and first assume $\bar{k} > k$. By using the necessity part of Theorem 3.1 (ii) (which we have already proved) there exists an $r \leq r_1$ and an $f \in N$ such that (2.2) has $\bar{k}$ periodic orbits in $B^2(r)$. This contradicts (i). If $\bar{k} > k$ then again using the necessity part of Theorem 3.1 (i) there exists a neighborhood $\bar{N}$ of f_0 and a number $\bar{r} > 0$ such that for $f \in \bar{N}$, system (2.2) has at most $\bar{k}$ nontrivial periodic solutions in $B^2(r)$. But this contradicts Theorem 3.1 (ii) since there exist $f \in \bar{N}$ and an $r < \bar{r}$ for which system (2.2) has k nontrivial periodic solutions in $B^2(r)$. This completes the proof.

Proof of Theorem 3.3. Choose r_1 and N as in Theorem 3.1 (i) and without loss of generality assume $G_4 < 0$.

We now indicate a proof of the existence of a neighborhood N_2 of f_0, $N_2 \subseteq N$, and a number $\varepsilon_2 \in (0, r_1)$ such that for each $f \in N_2$ with $\alpha \leq 0$ there are no periodic orbits of (2.2) in $B^2(\varepsilon_2)$. Indeed it is sufficient to prove that N_2 and ε_2 can be found so that $B^2(\varepsilon_2)/(0,0)$ is a subset of the region of attraction of (2.2) for $f \in N_2$ and $\alpha \leq 0$. This easily follows by continuity arguments and an appropriate use of the Poincaré procedure in order to construct positive definite functions (depending on f) which are negative definite in $B^2(\varepsilon_1)$ along the solutions of (2.2) for $f \in N_2$. We leave the details to the reader.

Now choose $\bar{\varepsilon} \in (0, \varepsilon_2)$ and a neighborhood $\bar{N}$ of f_0, $\bar{N} \subseteq N$, such that the origin is the only critical point of (2.2) in $B^2(\bar{\varepsilon})$ for each $f \in \bar{N}$. Since the origin of system (2.1) is asymptotically stable then the origin is totally stable. Now pick any $\varepsilon_1 \in (0,\bar{\varepsilon})$. Then the total stability implies that there exist $\delta_1 = \delta_1(\varepsilon_1)$ and a neighborhood $N_1 \equiv N_1(\varepsilon_1)$ of f_0, $N_1 \subset \bar{N}$, such that for every $(x_0,y_0) \in B^2(\delta_1)$ and $f \in N_1$ the positive orbit of (2.2) passing through (x_0,y_0) is contained in $B^2(\varepsilon_1)$. Let us assume $f \in N_1$ and $\alpha > 0$. Then if $(x_0,y_0) \in B^2(\delta_1)$ the positive limit set of the orbit passing through (x_0,y_0) is nonempty, lies in $\overline{B^2(\varepsilon_1)}$, and contains no critical points (the origin is repelling for (2.2) since $\alpha > 0$). By the well known Bendixson Theorem [9] the positive limit set is a periodic orbit. It is the only periodic orbit contained in $B^2(r_1)$ in virtue of Theorem 3.1 (i).

The proof of Theorem 3.3 is complete if we define $N = N_1 \cap N_2$ and choose any $\varepsilon \in (\varepsilon_1,\bar{\varepsilon})$.

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