# Generalized hypergeometric coherent states for special functions: <br> mathematical and physical properties 

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#### Abstract

In continuation of our previous works [J. Phys. A: Math. Gen. 35, 9355-9365 (2002)], [J. Phys. A: Math. Gen. 38, 7851 (2005)] and [Eur. Phys. J. D 72, 172 (2018)], we investigate a class of generalized coherent states for associated Jacobi polynomials and hypergeometric functions, satisfying the resolution of the identity with respect to a weight function expressed in terms of Meijer's G-function. We extend the state Hilbert space of the constructed states and discuss the property of the reproducing kernel and its analytical expansion. Further, we provide the expectation values of observables relevant to this quantum model. We also perform the quantization of the complex plane, compute and analyze the probability density and the temporal stability in these states. Using the completeness relation provided by the coherent states, we achieve the thermodynamic analysis in the diagonal $P$-representation of the density operator.


Key words: Hypergeometric coherent states; Meijer's G-functions; Bessel functions; Reproducing kernel; Polynomials; Quantization; Density probability.

## 1 Introduction

During the past few decades, the concept of coherent states (CSs) has aroused great scientific interest since their introduction, at the beginning of the 1960s, by Schrödinger [1] for the quantum harmonic oscillator ( HO ) as a specific quantum states which have the most similar dynamical behavior to that of classical HO.

Glauber [2] and Sudarshan [3] reconsidered the definition of these Schrödinger CSs, while the conditions any state must fulfill to be coherent, (i. e., continuity in complex label, normalization, non orthogonality, unity operator resolution with unique positive weight function of the integration measure, temporal stability and action identity), were elaborated by Klauder [4]. More details on the CSs and their different generalizations can be found in the literature. See, for example, [5, 6, 7, 8]. Different kinds of CSs have also been generalized for quantum systems. One can mention the Barut-Girardello CSs [9], Perelomov CSs [6], Gazeau-Klauder CSs [10], Penson-Solomon CSs [11] Klauder-Penson-Sixdeniers CSs [12], generalized hypergeometric CSs (GHCSs) introduced by Appl
and Schiller [13]. These have been taken into account in many works since their introduction. One can also consult 14 concerning nonclassical properties of CSs. Generalized hypergeometric photonadded and photon-depleted CSs, and deformed photon-added non-linear CSs were introduced, respectively, in [15] and [16]. Note that the photon-added CSs (PACSs) and the lower truncated CSs are the limiting cases of suitably deformed PACSs [17]. Photon-added Gazeau-Klauder and Klauder-Perelomov CSs for exactly solvable Hamiltonians were studied in [18], while photon-added Barut-Girardello CSs of the pseudoharmonic oscillator were constructed in [19]. Besides, in [19] the GHCSs were extended to mixed (thermal) states and applied, particularly, to the case of a pseudoharmonic oscillator.

In general, the GHCSs are given by the expression [13, 19]:

$$
\begin{equation*}
|z\rangle=\frac{1}{\sqrt{{ }_{p} F_{q}\left(\left\{a_{i}\right\}_{1}^{p} ;\left\{b_{j}\right\}_{1}^{q} ;|z|^{2}\right)}} \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{\rho_{p, q}(n)}}|n\rangle, \tag{1}
\end{equation*}
$$

where ${ }_{p} F_{q}\left(\left\{a_{i}\right\}_{1}^{p} ;\left\{b_{j}\right\}_{1}^{q} ;|z|^{2}\right)$ are the generalized hypergeometric functions [20]:

$$
\begin{equation*}
{ }_{p} F_{q}\left(\left\{a_{i}\right\}_{1}^{p} ;\left\{b_{j}\right\}_{1}^{q} ; x\right)=\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\left(a_{i}\right)_{n}}{\prod_{j=1}^{p}\left(b_{j}\right)_{n}} \frac{x^{n}}{n!} \equiv \sum_{n=0}^{\infty} \frac{1}{\rho_{p, q}(n)} x^{n} \tag{2}
\end{equation*}
$$

$p$ and $q$ are integer numbers; the $\rho_{p, q}(n)$ stand for generalized factorials expressed through Pochhammer symbols $(a)_{n}=\frac{\Gamma(n+a)}{\Gamma(a)}$ and Euler gamma functions $\Gamma(x)$. The appellation of generalized hypergeometric coherent state refers to the normalization function given by generalized hypergeometric functions.

Cotfas, in his work [21], provided a factorization method of associated hypergeometric operators, and deduced the associated algebra and corresponding CSs. These are eigenstates of the annihilation operator denoted $a_{m}$. Following Aleixo et al. [22, we introduced a right inverse operator $a_{m}^{-1}$ in [23] in order to define generalized associated hypergeometric CSs (GAH-CSs). These states fulfill the Klauder prescriptions required for a set of CSs and establish a close connection between the quantum and classical formulations of a given physical system. Recently [24], some of us applied the general procedure of CS quantization, (also known as the Berezin-Klauder-Toeplitz quantization), in the complex plane to a set of GPAH-CSs from the resolution of the identity obtained by a positive weight function expressed in terms of Meijer's G-functions. In addition, these authors discussed the nonclassical behaviour by investigating the Mandel Q-parameter expressed in terms of generalized hypergeometric functions. All these developments on generalizations of GHCSs and GPAH-CSs, which also put in light the applications of hypergemetric functions and Meijer's G-functions (see [13, 21, 23, 19, 15, 25], and references therein), motivate the present work. Indeed, following the method developed in [21]-[23], we built a family of generalized hypergeometric CSs (GHCSs) for associated Jacobi polynomials and hypergeometric functions. These CSs coincide with the GHCSs introduced by Appl and Schiller [13], (see (1) in this paper), and their expansion is provided in terms of the Fock basis states. All computations, including the weight function of the integration measure, are performed in terms of the Meijer's G-functions related to the hypergeometric functions, and modified Bessel functions of the first kind, highlighting the relevance of the usefulness of these functions.

The diagonal expansion of the density operator, known as the Glauber-Sudarshan (GS)-Prepresentation [26], plays a key role in the concepts of Husimi distribution [27] and Wehrl entropy [28] in various constructions (see for example [19, 25, 29, 30, 31] for more details). Also in [32], the density operator diagonal representation in the CSs basis was used to study harmonic oscillator quantum systems and models of spinless electrons moving in a two-dimensional noncommutative
space, subject to a magnetic field coupled with a harmonic oscillator. Relevant statistical properties such as the $Q$-Husimi distribution and the Wehrl entropy were also investigated. Besides, multi-matrix vector coherent states (VCSs) basis was successfully performed in the density operator representation and applied to Landau levels of an electron in an electromagnetic field coupled to an isotropic harmonic potential [33]. Main relevant statistical properties such as the Mandel $Q$-parameter and the signal-to-quantum-noise ratio were derived and discussed. In addition, more recently [34, in the context of supersymmetric harmonic oscillator, a matrix formulation of the density operator to construct a two-component VCS representation was achieved. Relevant statistical properties were described, with a link with quantum information given via an integral representation of a qubit.

The paper is organized as follows. Section 2 recalls the construction method of hypergeometric CSs as it is carried out in the literature. In section 3, we construct the GHCS and establish the resolution of the identity satisfied by these states by providing the appropriate weight function expressed by the Meijer G-functions [20], which solves the Stieltjes moment problem. Section 4 deals with the analytical insights of the GHCSs through the reproducing kernel and the analytic representation of a given function in the Hilbert space spanned by these states. In Section 5, the expectation values of the observables describing the quantum system are derived. The quantization of a complex plane, known as the Berezin-Klauder-Toeplitz quantization, (and also called coherent state or anti-Wick quantization), using these states is investigated in Section 6. In Section 7, the probability density and the time dependence of the GHCSs are discussed. Section 8 deals with the treament of thermodynamical properties of the quantum system, with the thermal expectations of the relevant observables determined from the Glauber-Sudarshan $P$-diagonal representation of the density operator.

## 2 General method of GHCSs construction

We start with the following definition.
Definition 2.1. The generalized associated hypergeometric type CSs (GAH-CSs) are the CSs corresponding to the $m^{\text {th }}$ derivative $\Phi_{l, m}=\kappa^{m} \Phi_{l}^{(m)}$ of the classical orthogonal polynomials $\Phi_{l}$ satisfying the second order differential equation of hypergeometric type:

$$
\begin{equation*}
\sigma(s) \Phi_{l}^{\prime \prime}(s)+\tau(s) \Phi_{l}^{\prime}(s)+\lambda_{l} \Phi_{l}(s)=0 \tag{3}
\end{equation*}
$$

where $\lambda_{l}=-\frac{1}{2} l(l-1) \sigma^{\prime \prime}-l \tau^{\prime}, \kappa=\sqrt{\sigma}$, with $\sigma$ a nonnegative function; $\sigma$ and $\tau$ are polynomials of at most second and exactly first degrees, respectively.

The $\Phi_{l, m}$, called associated hypergeometric-type functions (AHF), are solutions of the eigenvalue problem $H_{m} \Phi_{l, m}=\lambda_{l} \Phi_{l, m}$ where the Hamiltonian operator $H_{m}$ is expressed as a second order differential operator as follows:

$$
\begin{equation*}
H_{m}=-\sigma \frac{d^{2}}{d s^{2}}-\tau \frac{d}{d s}+\frac{m(m-2)}{4} \frac{\sigma^{\prime 2}}{\sigma}+\frac{m}{2} \tau \frac{\sigma^{\prime}}{\sigma}-\frac{1}{2} m(m-2) \sigma^{\prime \prime}-m \tau^{\prime} \tag{4}
\end{equation*}
$$

The $\Phi_{l, m}$ are orthogonal

$$
\begin{equation*}
\int_{a}^{b} \Phi_{l, m} \Phi_{k, m} \rho d s=0, \quad l \neq k, \quad l, k \in\{m, m+1, m+2, \ldots\} \tag{5}
\end{equation*}
$$

with respect to the positive weight function $\rho$ related to the polynomial functions $\sigma$ and $\tau$ by the Pearson's equation $(\sigma \rho)^{\prime}=\tau \rho$, over the interval $(a, b)$, which can be finite or infinite. The operator $H_{m}$ factorizes as:

$$
H_{m}-\lambda_{m}=A_{m}^{\dagger} A_{m}, \quad H_{m+1}-\lambda_{m}=A_{m} A_{m}^{\dagger}
$$

and fulfills the intertwining relations

$$
H_{m} A_{m}^{\dagger}=A_{m}^{\dagger} H_{m+1} \quad \text { and } \quad A_{m} H_{m}=H_{m+1} A_{m} .
$$

The mutually formal adjoint first-order differential operators

$$
A_{m}: \mathscr{H}_{m} \longrightarrow \mathscr{H}_{m+1} \quad \text { and } \quad A_{m}^{\dagger}: \mathscr{H}_{m+1} \longrightarrow \mathscr{H}_{m}
$$

are defined as [21]:

$$
\begin{align*}
A_{m} & =\kappa(s) \frac{d}{d s}-m \kappa^{\prime}(s) \quad \text { and }  \tag{6}\\
A_{m}^{\dagger} & =-\kappa(s) \frac{d}{d s}-\frac{\tau(s)}{\kappa(s)}-(m-1) \kappa^{\prime}(s) \tag{7}
\end{align*}
$$

with $\kappa=\sqrt{\sigma} . \mathscr{H}_{m}$ is the Hilbert space of $\left\{\Phi_{k, m}\right\}_{k \geq m}$, for $m \in \mathbb{N}$, with respect to the inner product (5). We restrict ourselves to the case when for each $m \in \mathbb{N}, \mathscr{H}_{m}$ is dense in the Hilbert space $\mathscr{H}=\left\{\varphi \in L^{2}(\rho(s) d s)\right\}$ where $L^{2}$ is the space of square integrable functions. The following shape invariance relations are satisfied

$$
\begin{align*}
A_{m} A_{m}^{\dagger} & =A_{m+1}^{\dagger} A_{m+1}+r_{m+1} \\
r_{m+1} & =\lambda_{m+1}-\lambda_{m}=-m \sigma^{\prime \prime}-\tau^{\prime} \tag{8}
\end{align*}
$$

where eigenvalues $\lambda_{l}$ and eigenfunctions $\Phi_{l, m}$ are:

$$
\begin{align*}
\lambda_{l} & =\sum_{k=1}^{l} r_{k}, \\
\Phi_{l, m} & =\frac{A_{m}^{\dagger}}{\lambda_{l}-\lambda_{m}} \frac{A_{m+1}^{\dagger}}{\lambda_{l}-\lambda_{m+1}} \cdots \frac{A_{l-2}^{\dagger}}{\lambda_{l}-\lambda_{l-2}} \frac{A_{l-1}^{\dagger}}{\lambda_{l}-\lambda_{l-1}} \Phi_{l, l} \tag{9}
\end{align*}
$$

for all $l \in \mathbb{N}$ and $m \in\{0,1,2, \ldots, l-1\}, \Phi_{l, l}$ satisfying the relation $A_{l} \Phi_{l, l}=0$.
The annihilation and creation operators are defined as:

$$
\begin{equation*}
a_{m}, a_{m}^{\dagger}: \mathscr{H}_{m} \longrightarrow \mathscr{H}_{m}, \quad a_{m}=U_{m}^{\dagger} A_{m} \quad \text { and } a_{m}^{\dagger}=A_{m}^{\dagger} U_{m} \tag{10}
\end{equation*}
$$

within the unitary operator

$$
\begin{equation*}
U_{m}: \mathscr{H}_{m} \longrightarrow \mathscr{H}_{m}, \quad U_{m}|l \cdot m\rangle=|l+1, m+1\rangle . \tag{11}
\end{equation*}
$$

The states $|l, m\rangle=\frac{\Phi_{l, m}}{\left\|\Phi_{l, m}\right\|}$ are defined for all $l \geq m$ and for each $m \in \mathbb{N}$. The mutually formal adjoint operators $a_{m}$ and $a_{m}^{\dagger}$ act on the states $|l, m\rangle$ as

$$
\begin{align*}
a_{m}|l, m\rangle & =\sqrt{\lambda_{l}-\lambda_{m}}|l-1, m\rangle \quad \text { and } \\
a_{m}^{\dagger}|l, m\rangle & =\sqrt{\lambda_{l+1}-\lambda_{m}}|l+1, m\rangle, \quad l \geq m, \tag{12}
\end{align*}
$$

and satisfy the commutation relations:

$$
\begin{equation*}
\left[a_{m}, a_{m}^{\dagger}\right]=\mathscr{R}_{m},\left[a_{m}^{\dagger}, \mathscr{R}_{m}\right]=\sigma^{\prime \prime} a_{m}^{\dagger},\left[a_{m}, \mathscr{R}_{m}\right]=-\sigma^{\prime \prime} a_{m} \tag{13}
\end{equation*}
$$

where $\mathscr{R}_{m}=-\sigma^{\prime \prime} N_{m}-\tau^{\prime}, N_{m}: \mathscr{H}_{m} \longrightarrow \mathscr{H}_{m}$ is the number operator defined as $N_{m} \Phi_{l, m}=l \Phi_{l, m}$. Remark that, when $\operatorname{deg}(\sigma)=1$, the algebra defined by the generators in (13) is isomorphic to the Heisenberg-Weyl algebra [21]. In addition to the commutation relations (13), we have

$$
\begin{equation*}
A_{m} \mathscr{R}_{m}=\mathscr{R}_{m+1} A_{m} \tag{14}
\end{equation*}
$$

and the similarity transformation

$$
\begin{equation*}
U_{m} \mathscr{R}_{m} U_{m}^{\dagger}=\mathscr{R}_{m+1}+\sigma^{\prime \prime}, \quad \text { for all } \quad m \in \mathbb{N} \tag{15}
\end{equation*}
$$

Setting for all $m \in \mathbb{N},|n\rangle=|m+n, m\rangle, e_{n}=\lambda_{m+n}-\lambda_{m}, \quad m \in \mathbb{N}$, we obtain, with $N_{m}=a_{m}^{\dagger} a_{m}$ :

$$
\begin{align*}
a_{m}|n\rangle & =\sqrt{e_{n}}|n-1\rangle \\
a_{m}^{\dagger}|n\rangle & =\sqrt{e_{n+1}}|n+1\rangle  \tag{16}\\
\left(H_{m}-\lambda_{m}\right)|n\rangle & =e_{n}|n\rangle
\end{align*}
$$

The CSs for AHF were provided by Cotfas [21] as:

$$
\begin{equation*}
|z\rangle=\mathscr{N}\left(|z|^{2}\right) \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{\varepsilon_{n}}}|n\rangle, \mathscr{N}\left(|z|^{2}\right)=\left[\sum_{n=0}^{\infty} \frac{|z|^{2 n}}{\varepsilon_{n}}\right]^{-1 / 2} \tag{17}
\end{equation*}
$$

for any $z$ in the open $\operatorname{disc} \mathscr{C}(O, \mathscr{R})$ with centre $O$ and radius

$$
\begin{align*}
\mathscr{R} & =\limsup _{n \rightarrow \infty} \sqrt[n]{\varepsilon_{n}} \neq 0  \tag{18}\\
\text { with } \varepsilon_{n} & = \begin{cases}1 & \text { if } n=0 \\
e_{1} e_{2} \cdots e_{n} & \text { if } n>0\end{cases} \tag{19}
\end{align*}
$$

These CSs are eigenstates of the annihilation operator $a_{m}$, i.e., $a_{m}|z\rangle=z|z\rangle$.
Introducing the right-inverse operators $A_{m}^{-1}, a_{m}^{-1}$, it was established in [23] that the CSs 17 ] can be rewritten as

$$
\begin{equation*}
|z\rangle=\mathscr{N}\left(|z|^{2}\right) \sum_{n=0}^{\infty}\left(z a_{m}^{-1}\right)^{n}|0\rangle \tag{20}
\end{equation*}
$$

with their generalization given by

$$
\begin{equation*}
\left|z ; \mathscr{R}_{m}\right\rangle=\sum_{n=0}^{\infty}\left(z \mathscr{R}_{m} a_{m}^{-1}\right)^{n}|0\rangle=\sum_{n=0}^{\infty} \frac{z^{n}}{h_{n}\left(\mathscr{R}_{m}\right)}|n\rangle \tag{21}
\end{equation*}
$$

where

$$
\begin{align*}
h_{0}\left(\mathscr{R}_{m}\right)= & 1  \tag{22}\\
h_{n}\left(\mathscr{R}_{m}\right)= & \frac{\sqrt{\varepsilon_{n}}}{} \prod_{k=0}^{n-1}\left(\mathscr{R}_{m}+k \sigma^{\prime \prime}\right) \tag{23}
\end{align*} \text { for } n \geq 1
$$

The states 21 are eigenstates of $a_{m}$,

$$
\begin{equation*}
a_{m}\left|z ; \mathscr{R}_{m}\right\rangle=z\left(\mathscr{R}_{m}-\sigma^{\prime \prime}\right)\left|z ; \mathscr{R}_{m}\right\rangle \tag{24}
\end{equation*}
$$

and satisfy the second order differential equation

$$
\begin{equation*}
\left\{a_{m}-z\left(\mathscr{R}_{m}-\sigma^{\prime \prime}\right)\right\} \frac{d}{d z}\left|z ; \mathscr{R}_{m}\right\rangle=\left(\mathscr{R}_{m}-\sigma^{\prime \prime}\right)\left|z ; \mathscr{R}_{m}\right\rangle \tag{25}
\end{equation*}
$$

Furthermore, we generalized the CSs 21 as:

$$
\begin{equation*}
\left|z ; \mathscr{R}_{m}\right\rangle=\sum_{n=0}^{\infty}\left(z f\left(\mathscr{R}_{m}\right) a_{m}^{-1}\right)^{n}|0\rangle=\sum_{n=0}^{\infty} \frac{z^{n}}{h_{n}\left(\mathscr{R}_{m}\right)}|n\rangle \tag{26}
\end{equation*}
$$

for any analytical function $f$, where

$$
\begin{align*}
& h_{0}\left(\mathscr{R}_{m}\right)= 1 \\
& \text { and } h_{n}\left(\mathscr{R}_{m}\right)= \sqrt{\varepsilon_{n}}  \tag{27}\\
& \prod_{k=0}^{n-1} f\left(\mathscr{R}_{m}+k \sigma^{\prime \prime}\right)
\end{align*} \text { for } n \geq 1 .
$$

The CSs (26) are eigenstates of $a_{m}$,

$$
\begin{equation*}
a_{m}\left|z ; \mathscr{R}_{m}\right\rangle=z f\left(\mathscr{R}_{m}-\sigma^{\prime \prime}\right)\left|z ; \mathscr{R}_{m}\right\rangle \tag{28}
\end{equation*}
$$

and satisfy the condition

$$
\begin{equation*}
\left\{a_{m}-z f\left(\mathscr{R}_{m}-\sigma^{\prime \prime}\right)\right\} \frac{d}{d z}\left|z ; \mathscr{R}_{m}\right\rangle=f\left(\mathscr{R}_{m}-\sigma^{\prime \prime}\right)\left|z ; \mathscr{R}_{m}\right\rangle \tag{29}
\end{equation*}
$$

Taking into account the fact that $\mathscr{R}_{m}$ is an operator which acts on the states $|n\rangle$ as

$$
\begin{equation*}
\mathscr{R}_{m}|n\rangle=\left[-(m+n) \sigma^{\prime \prime}-\tau^{\prime}\right]|n\rangle=r_{m+n+1}|n\rangle, \tag{30}
\end{equation*}
$$

we rewrite the CSs 26 under the form:

$$
\begin{equation*}
|z ; m\rangle=\sum_{n=0}^{\infty} \frac{z^{n}}{h_{n}(m)}|n\rangle \tag{31}
\end{equation*}
$$

where

$$
\begin{align*}
h_{0}(m) & =1 \\
\text { and } h_{n}(m) & =\frac{\sqrt{\varepsilon_{n}}}{} \prod_{k=0}^{n-1} f\left(r_{m+n+1-k}\right) \tag{32}
\end{align*} \text { for } n \geq 1
$$

The properties $(28)$ and 29 become

$$
\begin{align*}
a_{m}|z ; m\rangle & =z f\left(r_{m+n+2}^{\prime \prime}\right)|z ; m\rangle  \tag{33}\\
\left\{a_{m}-z f\left(r_{m+n+2}\right)\right\} \frac{d}{d z}|z ; m\rangle & =f\left(r_{m+n+2}\right)|z ; m\rangle
\end{align*}
$$

respectively. We established in [23] that the generalized coherent states (31) verify the properties of label continuity, overcompleteness, temporal stability and action identity.

## 3 Generalized hypergeometric coherent states for associated Jacobi polynomials and hypergeometric functions

### 3.1 The construction

In this section, we construct CSs for Jacobi associated fuctions and hypergeometric functions. The resolution of the identity satisfied by these states is discussed through the Stieltjes moment problem and solved using the Mellin transform. The appropriate solution is given in terms of Meijer G functions.

The polynomial functions $\sigma(x)$ and $\tau(x)$ corresponding to Jacobi associated functions polynomials are

$$
\begin{equation*}
\sigma(x)=1-x^{2}, \quad \tau(x)=(\zeta-\gamma)-(\gamma+\zeta+2) x \tag{34}
\end{equation*}
$$

while for the hypergeometric functions polynomials they are given by

$$
\begin{equation*}
\sigma(x)=(1-x) x, \quad \tau(x)=(\zeta+1)-(\gamma+\zeta+2) x \tag{35}
\end{equation*}
$$

The derivatives of first order of $\tau(x)$ and second order of $\sigma(x)$ are provided as:

$$
\begin{equation*}
\tau^{\prime}(x)=-(\gamma+\zeta+2), \quad \sigma^{\prime \prime}(x)=-2 \tag{36}
\end{equation*}
$$

respectively. For commodity, set

$$
\begin{equation*}
\mu=\gamma+\zeta+2 \tag{37}
\end{equation*}
$$

The eigenvalues of the Hamiltonian $H_{m}$, given in 4 , with associated eigenvectors $\left\{\Phi_{l, m}\right\}_{l \geq 0}$, are obtained from (36) and (37) as follows:

$$
\begin{equation*}
\lambda_{l}=-\frac{1}{2} l(l-1) \sigma^{\prime \prime}-l \tau^{\prime}=l(l+\mu-1) \tag{38}
\end{equation*}
$$

such that we get [25]:

$$
\begin{equation*}
e_{n}=\lambda_{m+n}-\lambda_{m}=n(2 m+n+\mu-1) \tag{39}
\end{equation*}
$$

The quantity $h_{n}\left(R_{m}\right)$ yields

$$
\begin{equation*}
h_{n}\left(R_{m}\right)=\sqrt{\Gamma(n+1) \frac{\Gamma(2 m+n+\mu)}{c^{2 n} \Gamma(2 m+\mu)}} \tag{40}
\end{equation*}
$$

In order to obtain $h_{n}\left(R_{m}\right)$ for any value of $c$, set

$$
\begin{equation*}
f\left(R_{m}\right)=f\left(R_{m}+\sigma^{\prime \prime}\right)=\sqrt{\left(-\frac{1}{2} R_{m}\right)\left(-\frac{1}{2} R_{m}\right)} \tag{41}
\end{equation*}
$$

with

$$
R_{m}=-(m+n) \sigma^{\prime \prime}-\tau^{\prime}=2\left(m+n+\frac{\mu}{2}\right)
$$

Then, we obtain the product

$$
\begin{equation*}
\prod_{k=0}^{n-1} f\left(R_{m}+k \sigma^{\prime \prime}\right)=\sqrt{\frac{\Gamma\left(n-\frac{R_{m}}{2}\right) \Gamma\left(n-\frac{R_{m}}{2}\right)}{\Gamma\left(-\frac{R_{m}}{2}\right) \Gamma\left(-\frac{R_{m}}{2}\right)}}=c^{n} \tag{42}
\end{equation*}
$$

Fixing $\nu=\frac{\mu}{2}$, from 42 it comes :

$$
\begin{equation*}
h_{n}\left(R_{m}\right)=\left[\Gamma(n+1) \frac{\Gamma(n+2 m+2 \nu)}{\Gamma(2 m+2 \nu)} \frac{\Gamma\left(-\frac{R_{m}}{2}\right)}{\Gamma\left(n-\frac{R_{m}}{2}\right)} \frac{\Gamma\left(-\frac{R_{m}}{2}\right)}{\Gamma\left(n-\frac{R_{m}}{2}\right)}\right]^{-\frac{1}{2}} \tag{43}
\end{equation*}
$$

Thereby

$$
\begin{equation*}
h_{n}(m)=\left[\Gamma(n+1) \frac{(2 m+2 \nu)_{n}}{\left[(-(m+n)-2 \nu)_{n}\right]^{2}}\right]^{-\frac{1}{2}} \tag{44}
\end{equation*}
$$

Then, the related CSs $|z ; m\rangle$ for both associated Jacobi polynomials and hypergeometric functions are provided as follows:

$$
\begin{equation*}
|z ; m\rangle=\left[N\left(|z|^{2}, m\right)\right]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{\frac{\Gamma(n+1)(2 m+2 \nu)_{n}}{\left[(-m-n-\nu)_{n}\right]^{2}}}}|n\rangle \tag{45}
\end{equation*}
$$

where the normalization function is given in terms of Meijer's G function as

$$
N\left(|z|^{2}, m\right)=\frac{\Gamma(2 m+2 \nu)}{[\Gamma(-m-n-\nu)]^{2}} G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu  \tag{46}\\
0, & 1-2 m-2 \nu
\end{array}\right.\right)
$$

Remark 3.1. 1. The constructed GHCSs (45) coincide with the GHCSs introduced by Appl and Schiller (1), and correspond to the GPAH-CSs for $c \neq 1$ and for the number of added quanta (or photons) $p=0$ [25].
2. In the case $c=1$, the GHCSs $|z ; m\rangle$ are delivered by

$$
\begin{equation*}
|z ; m\rangle=\frac{1}{\sqrt{{ }_{0} F_{1}\left(2 m+2 \nu ;|z|^{2}\right)}} \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{\Gamma(n+1)(2 m+2 \nu)_{n}}}|n\rangle \tag{47}
\end{equation*}
$$

with the normalization constant given as follows:

$$
\begin{align*}
N\left(|z|^{2} ; m\right) & =\sum_{n=0}^{\infty} \frac{|z|^{2 n}}{\left|h_{n}(m)\right|^{2}}={ }_{0} F_{1}\left(2 m+2 \nu ;|z|^{2}\right) \\
& =\Gamma(2 m+2 \nu)|z|^{1-2 m-2 \nu} I_{2 m+2 \nu-1}(2|z|) . \tag{48}
\end{align*}
$$

The GHCSs (47) coincide with the GPAH-CSs obtained in [23] for the number of added quanta (or photons) $p=0$.

Moreover, they are not orthogonal to each other since

### 3.2 Continuity in the labeling

The GHCSs (47) are continuous in labeling $z$. Indeed, the transformation of CSs parameter $z^{\prime} \longrightarrow z$ leads to the transformation of GHCSs $\left|z^{\prime} ; m\right\rangle \longrightarrow|z ; m\rangle$ :

$$
\begin{equation*}
\text { If }\left|z-z^{\prime}\right| \longrightarrow 0 \quad \text { then } \||z, m\rangle-\left|z^{\prime}, m\right\rangle \|^{2}=2\left[1-\operatorname{Re}\left(\left\langle z^{\prime}, m \mid z, m\right\rangle\right)\right] \longrightarrow 0, \tag{50}
\end{equation*}
$$

where (48) and (49) together have been used.

### 3.3 Resolution of the unity operator

Next, a fundamental property of any CSs is the resolution of unity operator. We have the following proposition:

Proposition 3.2. The GHCSs $|z ; m\rangle$ satisfy the following resolution of the identity

$$
\begin{equation*}
\int_{\mathbb{C}} \frac{d^{2} z}{\pi}|z ; m\rangle\langle z ; m| W\left(|z|^{2}, m\right)=\mathbb{I}_{\mathfrak{H}} \tag{51}
\end{equation*}
$$

with the Hilbert space $\mathfrak{H}=\operatorname{span}\{|n\rangle\}_{n=0}^{\infty}$ identical to the Fock basis, where the weight function $W\left(|z|^{2}, m\right)$, obtained through the Mellin transform, is given by

$$
\begin{align*}
W\left(|z|^{2}, m\right)= & \frac{1}{[\Gamma(-m-\nu) \Gamma(-m-n-\nu)]^{-}} G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right) \\
& \times G_{2,2}^{2,0}\left(|z|^{2} \left\lvert\, \begin{array}{c}
m+\nu ; m+\nu \\
0 ; 2 m+2 \nu-1
\end{array}\right.\right) . \tag{52}
\end{align*}
$$

Proof. See the Appendix.

In Figure 1. we plot the weight function (52) versus $x=|z|^{2}$ for different values of $m, n$ and $\nu$. All the curves are positive, this confirms the positivity of the weight function for the parameter $\nu>0$. We also notice that the polynome parameter $\nu$ does not affect the general behaviour of the curves but increases their asymptotic behaviour by taking smaller values.


Figure 1: Plots of the weight function (52) versus $x=|z|^{2}$ : with the parameters $m=1, n=2$ and for different values of $\nu$; with the parameters $m=2, \nu=0.7$ and for different values of $n$; with the parameters $n=2, \nu=0.7$ and for different values of $m$.

## 4 Reproducing kernel and analytic representation

### 4.1 Reproducing kernel

The overcompleteness of the constructed GHCSs $|z, m\rangle$ allows the study of their relation with the reproducing kernels [8].

Define the quantity $K\left(z, z^{\prime}\right):=\left\langle z^{\prime}, m \mid z, m\right\rangle$, by using the connection between the Meijer's Gfunctions and the modified Bessel functions of the first kind [35]-[38], as follows:

$$
\begin{align*}
K\left(z, z^{\prime}\right) & =\frac{G_{2,2}^{1,2}\left(-z \bar{z}^{\prime} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)}{\left[G_{2,2}^{1,2}\left(-\left.\left|z^{\prime}\right|^{2}\right|_{|c|} ^{1+m+n+\nu,} \begin{array}{l}
1+m+n+\nu \\
0, \\
1-2 m-2 \nu
\end{array}\right)\right]^{\frac{1}{2}}}\left[G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)\right]^{\frac{1}{2}} \\
& =\left(\frac{\left|z z^{\prime}\right|}{z \bar{z}^{\prime}}\right)^{m+\nu-\frac{1}{2}} \frac{I_{2 m+2 \nu-1}\left(2 \sqrt{\left.z \bar{z}^{\prime}\right)}\right.}{\sqrt{I_{2 m+2 \nu-1}(2|z|) I_{2 m+2 \nu-1}\left(2\left|z^{\prime}\right|\right)}} . \tag{53}
\end{align*}
$$

$K\left(z, z^{\prime}\right)$ is a reproducing kernel. Indeed, we have the following result.
Proposition 4.1. The following properties
(i) hermiticity $\overline{K\left(z, z^{\prime}\right)}=K\left(z^{\prime}, z\right)$,
(ii) positivity $K(z, z)>0$, and
(iii) idempotence $\int_{\mathbb{C}} K\left(z, z^{\prime \prime}\right) K\left(z^{\prime \prime}, z^{\prime}\right) \frac{W\left(\left|z^{\prime \prime}\right|^{2}\right) d^{2} z^{\prime \prime}}{\pi}=K\left(z, z^{\prime}\right)$
are satisfied by $K$ on the Hilbert spce $\mathfrak{H}$.
Proof. See the Appendix.

### 4.2 Analytic representation in the GHCSs basis

From the resolution of the identity property (51), given $|\Psi\rangle \in \mathfrak{H}$, we have

$$
\begin{equation*}
|\Psi\rangle=\int_{\mathbb{C}} \frac{d^{2} z}{\pi} W\left(|z|^{2}, m\right) \Psi(z)|z ; m\rangle \tag{54}
\end{equation*}
$$

where $\Psi(z):=\langle z ; m \mid \Psi\rangle$. Then, the following reproducing property

$$
\begin{equation*}
\Psi(z)=\int_{\mathbb{C}} \frac{d^{2} z^{\prime}}{\pi} W\left(\left|z^{\prime}\right|^{2}, m\right) \Psi\left(z^{\prime}\right) K\left(z^{\prime}, z\right) \tag{55}
\end{equation*}
$$

is also satisfied. The Hilbert space $\mathfrak{H}$ can be represented as the Hilbert space of analytic functions in the variable $z$.

Given a normalized state $|\Phi\rangle=\sum_{k=0}^{\infty} C_{k}|k\rangle, C_{k} \in \mathbb{C}$ on $\mathfrak{H}$, we obtain

$$
\begin{align*}
\langle\bar{z} ; m \mid \Phi\rangle= & {\left[\frac{\Gamma(2 m+2 \nu)}{[\Gamma(-m-n-\nu)]^{2}} G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)\right]^{-\frac{1}{2}} } \\
& \times \sum_{n=0}^{\infty} C_{n} \frac{(-m-n-\nu)_{n} z^{n}}{\sqrt{\Gamma(n+1)(2 m+2 \nu)_{n}}} \tag{56}
\end{align*}
$$

such that the entire functions

$$
\left.\left.\begin{array}{rl}
f(z, m) & =\left[\frac { \Gamma ( 2 m + 2 \nu ) } { [ \Gamma ( - m - n - \nu ) ] ^ { 2 } } G _ { 2 , 2 } ^ { 1 , 2 } \left(-|z|^{2}\right.\right.
\end{array} \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu \tag{57}
\end{array}\right)\right]^{\frac{1}{2}}\langle\bar{z} ; m \mid \Phi\rangle
$$

are analytic over the whole $z$ plane. Then, from the resolution of the identity (51), we can write

$$
\begin{equation*}
|\Phi\rangle=\int_{\mathbb{C}} \frac{d^{2}}{\pi} W\left(|z|^{2}, m\right)\left[N\left(|z|^{2}, m\right)\right]^{-1} f(\bar{z}, m)|z ; m\rangle \tag{58}
\end{equation*}
$$

and express the scalar product of two states $\left|\Phi_{1}\right\rangle$ and $\left|\Phi_{2}\right\rangle$ given on $\mathfrak{H}$ by the formula

$$
\begin{equation*}
\left\langle\Phi_{1} \mid \Phi_{2}\right\rangle=\int_{\mathbb{C}} \frac{d^{2} z}{\pi} W\left(|z|^{2}, m\right)\left[N\left(|z|^{2}, m\right)\right]^{-1} \overline{f_{1}(\bar{z}, m)} f_{2}(\bar{z}, m) \tag{59}
\end{equation*}
$$

where

$$
\begin{align*}
& \overline{f_{1}(\bar{z}, m)}=\sum_{n=0}^{\infty} \overline{C_{n}} \frac{(-m-n-\nu)_{n}}{\sqrt{\Gamma(n+1)(2 m+2 \nu)_{n}}} z^{n} \\
& f_{2}(\bar{z}, m)=\sum_{k=0}^{\infty} C_{k} \frac{(-m-k-\nu)_{k}}{\sqrt{\Gamma(k+1)(2 m+2 \nu)_{k}}} \bar{z}^{k} \tag{60}
\end{align*}
$$

## 5 Expectation values

The constructed CSs can be used in different physical applications to calculate the expectation (mean) values of any significant physical observable $\mathscr{O}$ which characterizes the quantum system embedded in the considered potential. We have the following statement.

Proposition 5.1. From the definition of the CSs in (45), the mean value of a physical observable $\mathscr{O}$ in the generalized CSs for hypergeometric type functions $|z ; m\rangle$ is obtained as

$$
\begin{align*}
\langle A\rangle_{z, m}= & {\left[\frac{\Gamma(2 m+2 \nu)}{[\Gamma(-m-n-\nu)]^{2}} G_{2,2}^{1,2}\left(-\left.|z|^{2}\right|_{1} ^{1+m+n+\nu,} \begin{array}{c}
1+m+n+\nu \\
0, \\
1-2 m-2 \nu
\end{array}\right)\right]^{-1} } \\
& \times \sum_{n=0}^{\infty} \frac{z^{2 n}\langle n| A|n\rangle}{\frac{\Gamma(n+1)(2 m+2 \nu)_{n}}{\left[(-m-n-\nu)_{n}\right]^{2}}} . \tag{61}
\end{align*}
$$

Then, we get

$$
\left.\langle N\rangle_{z, m}=-C_{1}^{(n)} \frac{G_{2,2}^{1,2}\left(-|z|^{2}\right.}{} \begin{array}{c}
1+m+n+\nu, 1+m+n+\nu  \tag{62}\\
1, \quad 1-2 m-2 \nu
\end{array}\right), ~\left(-|z|^{2} \begin{array}{cc}
1+m+n+\nu, 1+m+n+\nu \\
0, \quad 1-2 m-2 \nu
\end{array}\right),
$$

and

$$
\begin{align*}
\left\langle N^{2}\right\rangle_{z, m}= & \frac{1}{G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{c}
1+m+n+\nu, 1+m+n+\nu \\
0,
\end{array}\right.\right)}\left[-C_{1}^{(n)} G_{2,2}^{1,2 m-2 \nu}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, 1+m+n+\nu \\
1, & 1-2 m-2 \nu
\end{array}\right.\right)\right. \\
& \left.+C_{2}^{(n)} G_{2,2}^{1,2}\left(\begin{array}{l|l|}
-|z|^{2} & \begin{array}{c}
1+m+n+\nu, 1+m+n+\nu \\
2, \\
1-2 m-2 \nu
\end{array}
\end{array}\right)\right] \tag{63}
\end{align*}
$$

providing the intensity correlation

$$
\left.\left(g^{(2)}\right)_{z, m}=G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{c}
1+m+n+\nu,  \tag{64}\\
1+m+n+\nu \\
2, \\
1-2 m-2 \nu
\end{array}\right.\right) \frac{G_{2,2}^{1,2}\left(-|z|^{2}\right.}{1+m+n+\nu, 1+m+n+\nu} \begin{array}{cc}
1+1-2 m-2 \nu
\end{array}\right)
$$

with $C_{1}^{(n)}=C_{2}^{(n)}=1$.
Proof. See the Appendix.

## 6 Quantization with the generalized hypergeometric coherent states

In this paragraph, we deal with the general procedure described in [7] and used, for example, in our previous works [39]-41]. For more details, one may also consult references quoted therein.

### 6.1 Coherent State Quantization: General Scheme

Let $X$ be a set of parameters equipped with a measure $\mu$ and let $L^{2}(X, \mu)$ be its associated Hilbert space of complex-valued square integrable functions with respect to $\mu$. Let us choose in $L^{2}(X, \mu)$ a finite or countable orthonormal set $\mathscr{O}=\left\{\phi_{n}, n=0,1,2, \ldots\right\}$,

$$
\begin{equation*}
\left\langle\phi_{m} \mid \phi_{n}\right\rangle=\int_{X} \overline{\phi_{m}(x)} \phi_{n}(x) \mu(d x)=\delta_{m n}, \tag{65}
\end{equation*}
$$

obeying the (crucial) condition:

$$
\begin{equation*}
0<\sum_{n}\left|\phi_{n}(x)\right|^{2}:=\mathscr{N}(x)<\infty \quad \text { a.e. . } \tag{66}
\end{equation*}
$$

Let $\mathscr{H}:=\overline{\operatorname{span}(\mathscr{O})}$ in the Hilbert space $L^{2}(X, \mu)$ be a separable complex Hilbert space with orthonormal basis $\left\{\left|e_{n}\right\rangle, n=0,1,2, \ldots\right\}$, in one-to-one correspondence with the elements of $\mathscr{O}=$ $\left\{\phi_{n}, n=0,1,2, \ldots\right\}$. One defines the family of states $\mathscr{F}_{\mathscr{H}}=\{|x\rangle, x \in X\}$ in $\mathscr{H}$ as:

$$
\begin{equation*}
|x\rangle=\frac{1}{\sqrt{\mathscr{N}(x)}} \sum_{n} \overline{\phi_{n}(x)}\left|e_{n}\right\rangle \in \mathscr{H} . \tag{67}
\end{equation*}
$$

From conditions (65) and (66) these CS are normalized, $\langle x \mid x\rangle=1$ and resolve the identity in $\mathscr{H}$ :

$$
\begin{equation*}
\int_{X} \mathscr{N}(x)|x\rangle\langle x| \mu(d x)=\mathbb{I}_{\mathscr{H}} . \tag{68}
\end{equation*}
$$

The relation (68) allows us to implement a coherent state quantization of the set of parameters $X$ by associating to a function $X \ni x \mapsto f(x)$ that satisfies appropriate conditions the operator $A_{f}$ in $\mathscr{H}$ as:

$$
\begin{equation*}
f(x) \mapsto A_{f}:=\int_{X} \mathscr{N}(x) f(x)|x\rangle\langle x| \mu(d x) . \tag{69}
\end{equation*}
$$

The matrix elements of $A_{f}$ with respect to the basis $\left|e_{n}\right\rangle$ are given by

$$
\begin{align*}
\left(A_{f}\right)_{n m} & =\left\langle e_{n}\right| A_{f}\left|e_{m}\right\rangle \\
& =\int_{X} f(x) \overline{\phi_{n}(x)} \phi_{m}(x) \mu(d x) \tag{70}
\end{align*}
$$

The operator $A_{f}$ is

1. symmetric if $f(x)$ is real valued,
2. bounded if $f(x)$ is bounded, and
3. self-adjoint if $f(x)$ is real semi-bounded (through

Friedrich's extension or self adjoint extension).

### 6.2 Quantization of elementary classical observables

The resolution of the identity provided by (51) allows us to implement the CS quantization (also named Berezin-Klauder-Toeplitz or anti-Wick quantization) of the complex plane by associating a function $\mathbb{C} \ni z \mapsto f(z)$. For this purpose, let us define the operator on the Hilbert space $\mathfrak{H}$

$$
\begin{equation*}
f(z) \mapsto A_{f}=\int_{\mathbb{C}} f(z)|z ; m\rangle\langle z, m| \frac{d^{2} z}{\pi} W\left(|z|^{2}, m\right) \tag{71}
\end{equation*}
$$

such that

$$
\begin{equation*}
A_{f}:=\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{|n\rangle\langle k|}{\sqrt{\frac{\Gamma(n+1)(2 m+2 \nu)_{n}}{\left[\left(-N_{m}-\nu\right)_{n}\right]^{2}} \frac{\Gamma(k+1)(2 m+2 \nu)_{k}}{\left[\left(-N_{m}^{\dagger}-\nu\right)_{k}\right]^{2}}}} \int_{\mathbb{C}}\left[N\left(|z|^{2}, m\right)\right]^{-1} f(z) z^{n} \bar{z}^{k} \frac{d^{2} z}{\pi} W\left(|z|^{2}, m\right) \tag{72}
\end{equation*}
$$

where the operators $N_{m}, N_{m}^{\dagger}$ act on the Fock Hilbert space $\{|n\rangle\}_{n=0}^{\infty}$ as delivered in 16.
The Berezin-Klauder-Toeplitz quantization of the elementary classical variables $z$ and $\bar{z}$ is realized via the maps $z \mapsto A_{z}$ and $\bar{z} \mapsto A_{\bar{z}}$ defined on the Hilbert $\mathfrak{H}$. Then, after some algebra, we obtain in the complex plane

$$
\begin{gather*}
A_{z}=\sum_{n=0}^{\infty}(-m-n-1-\nu) \sqrt{(n+1)(2 m+2 \nu+n)}|n\rangle\langle n+1|,  \tag{73}\\
A_{\bar{z}}=\sum_{n=0}^{\infty}(-m-n-\nu) \sqrt{n(2 m+2 \nu+n-1)}|n\rangle\langle n-1|, \tag{74}
\end{gather*}
$$

where the matrix elements

$$
\begin{align*}
& \left(A_{z}\right)_{k, n}=\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \int_{0}^{2 \pi} \int_{0}^{\infty} \frac{r d r d \theta}{\pi}\left[N\left(r^{2}, m\right)\right]^{-1} \frac{e^{i(n+1-k) \theta} r^{n+1+k} W\left(r^{2}, m\right)}{\sqrt{\frac{\Gamma(n+1)(2 m+\mu)_{n}}{\left[\left(-N_{m}-\nu\right)_{n}\right]^{2}} \frac{\Gamma(k+1)(2 m+\mu)_{k}}{\left[\left(-N_{m}^{\dagger}-\nu\right)_{k}\right]^{2}}}}|n\rangle\langle k|,  \tag{75}\\
& \left(A_{\bar{z}}\right)_{k, n}=\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \int_{0}^{2 \pi} \int_{0}^{\infty} \frac{r d r d \theta}{\pi}\left[N\left(r^{2}, m\right)\right]^{-1} \frac{e^{i(n-k-1) \theta} r^{n+1+k} W\left(r^{2}, m\right)}{\sqrt{\frac{\Gamma(n+1)(2 m+\mu)_{n}}{\left[(-m-n-\nu)_{n}\right]^{2}} \frac{\Gamma(k+1)(2 m+)_{k}}{\left((-m-k-\nu)_{k}\right]^{2}}}}|n\rangle\langle k|, \tag{76}
\end{align*}
$$

are together obtained from $(72)$, and the following relations

$$
z=r e^{i \theta}, \quad \bar{z}=r e^{-i \theta} \quad \text { and } \quad \frac{d^{2} z}{\pi}=\frac{r d r d \theta}{\pi}
$$

$$
\int_{0}^{2 \pi} e^{i(n-m) \theta} d \theta=\left\{\begin{array}{lll}
0 & \text { if } & m \neq n  \tag{77}\\
2 \pi & \text { if } & m=n
\end{array}\right.
$$

are used.
The commutator of the operators $A_{z}$ and $A_{\bar{z}}$ takes the form

$$
\begin{align*}
{\left[A_{z}, A_{\bar{z}}\right]=} & \sum_{\substack{n=0 \\
\\
\\
\\
\times|n\rangle\langle n|}}\left\{(-m-n-1-\nu)^{2}(n+1)(2 m+2 \nu+n)-(-m-n-\nu)^{2} n(2 m+2 \nu+n-1)\right\}
\end{align*}
$$

For the quantization of $f(z)=|z|^{2}$, using the integral formula (72), we obtain the corresponding operator

$$
\begin{align*}
A_{|z|^{2}} & =\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \int_{0}^{2 \pi} \int_{0}^{\infty} \frac{r d r d \theta}{\pi}\left[N\left(r^{2}, m\right)\right]^{-1} \frac{e^{i(n-k-1) \theta} r^{n+1+k}}{\sqrt{\frac{\Gamma(n+1)(2 m+\mu) n}{\left[(-m-n-\nu)_{n}\right]^{2}} \frac{\Gamma(k+1)(2 m+\mu)_{k}}{\left[(-m-k-\nu)_{k}\right]^{2}}}} W\left(r^{2}, m\right)|n\rangle\langle k| \\
& =\sum_{n=0}^{\infty}(n+1)(2 m+\mu+n)|n\rangle\langle n| . \tag{79}
\end{align*}
$$

Its commutators with $A_{z}$ and $A_{\bar{z}}$ yield

$$
\begin{gather*}
{\left[A_{z}, A_{|z|^{2}}\right]=-2 \sum_{n=0}^{\infty}(m+\nu+n+1)^{2} \sqrt{(n+1)(2 m+2 \nu+n)}|n\rangle\langle n+1|,}  \tag{80}\\
{\left[A_{\bar{z}}, A_{|z|^{2}}\right]=2 \sum_{n=0}^{\infty}(m+\nu+n)^{2} \sqrt{n(2 m+2 \nu+n-1)}|n\rangle\langle n-1|,} \tag{81}
\end{gather*}
$$

respectively.

## $7 \quad$ Probability density and time evolution

This section deals with the semi-classical character of the GHCSs. We analyse how these states do evolve in time under the action of the time evolution operator provided by the physical Hamiltonian describing the quantum system.

We start with the overlap (49) expressed in terms of the modified Bessel functions of the first kind as follows:

$$
\begin{equation*}
\left\langle z^{\prime}, m \mid z, m\right\rangle=\left(\frac{\left|z z^{\prime}\right|}{z \overline{z^{\prime}}}\right)^{m+\nu-\frac{1}{2}} \frac{I_{2 m+2 \nu-1}\left(2 \sqrt{z \bar{z}^{\prime}}\right)}{\sqrt{I_{2 m+2 \nu-1}(2|z|) I_{2 m+2 \nu-1}\left(2\left|z^{\prime}\right|\right)}} \tag{82}
\end{equation*}
$$

Then, taking a normalized state $\left|z_{0} ; m\right\rangle$, the related phase space distribution is provided through the probability density:

$$
\begin{align*}
z \rightarrow \varrho_{z_{0}}(z) & :=\left|\left\langle z ; m \mid z_{0} ; m\right\rangle\right|^{2} \\
& =\frac{I_{2 m+2 \nu-1}\left(2 \sqrt{z \bar{z}^{\prime}}\right) I_{2 m+2 \nu-1}\left(2 \sqrt{\bar{z} z^{\prime}}\right)}{I_{2 m+2 \nu-1}(2|z|) I_{2 m+2 \nu-1}\left(2\left|z^{\prime}\right|\right)} . \tag{83}
\end{align*}
$$

Remark 7.1. The expression 83) is analogue to the probability density defined for BGCSs

$$
\begin{equation*}
|z\rangle_{m}=\frac{|z|^{m / 2}}{\sqrt{I_{m}(2|z|)}} \sum_{n=m}^{+\infty} \frac{z^{n-m}}{\sqrt{\Gamma(n-m+1) \Gamma(n+1)}}|n, m\rangle \tag{84}
\end{equation*}
$$

built in [41], given as follows:

$$
\begin{equation*}
\overbrace{\varrho_{z_{0}}}(z):=\left.\left.\right|_{m}\left\langle z \mid z_{0}\right\rangle_{m}\right|^{2}=\frac{I_{m}\left(2 \sqrt{z_{0} \bar{z}}\right) I_{m}\left(2 \sqrt{\bar{z}_{0} z}\right)}{I_{m}(2|z|) I_{m}\left(2\left|z_{0}\right|\right)} \tag{85}
\end{equation*}
$$

also provided through modified Bessel functions of the first kind. Thus, we can emphasize that the GHCSs (47) show similar time evolution behaviour as the BGCSs (84).

Then, the associated time evolution behaviour is supplied by

$$
\begin{equation*}
\left.z \rightarrow \varrho_{z_{0}}(z, t)=\left|\langle z ; m| e^{-i H_{m} t}\right| z_{0} ; m\right\rangle\left.\right|^{2} \tag{86}
\end{equation*}
$$

After acting the evolution operator $U(t)=e^{-i H_{m} t}$, with $H_{m}$ provided by (4) and its eigenvalues $e_{n}=n(n+2 \nu+2 m-1)$ with $\mu=2 \nu$ (see (39), on the GHCSs $\left|z_{0} ; m\right\rangle$, we obtain

$$
\begin{align*}
\left|z_{0} ; t ; m\right\rangle & =e^{-i H_{m} t}\left|z_{0} ; m\right\rangle \\
& =\left[N\left(\left|z_{0}\right|^{2}, m\right)\right]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-m-n-\nu)_{n}\left(z_{0}\right)^{n}}{\sqrt{\Gamma(n+1)(2 m+2 \nu)_{n}}} e^{-i H_{m} t}|n\rangle \\
& =\left[N\left(\left|z_{0}(t)\right|^{2}, m\right)\right]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-m-n-\nu)_{n}\left(z_{0} e^{-i(n+2 \nu+2 m-1) t}\right)^{n}}{\sqrt{\Gamma(n+1)(2 m+2 \nu)_{n}}}|n\rangle . \tag{87}
\end{align*}
$$

Then, in the basis $\left\{\left|e^{-i n^{2} t} n\right\rangle:=\left|\Phi_{n}(t)\right\rangle=\left|e^{-i \theta_{n}(t)} n\right\rangle\right.$, $\left.\theta_{n}(t)=n^{2} t\right\}_{n=0}^{\infty}$, the equation (87) becomes

$$
\begin{align*}
\overbrace{\left|z_{0} ; t ; m\right\rangle} & =: e^{-i H_{m} t}\left|z_{0} ; m\right\rangle \\
& =\left[N\left(\left|z_{0}(t)\right|^{2}, m\right)\right]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-m-n-\nu)_{n}\left(z_{0}(t)\right)^{n}}{\sqrt{\Gamma(n+1)(2 m+2 \nu)_{n}}}\left|\Phi_{n}(t)\right\rangle \\
& =\left|z_{0}(t) ; m\right\rangle \tag{88}
\end{align*}
$$

where $z_{0}(t):=z_{0} e^{-i(2 \nu+2 m-1) t}$. Then, by recasting the GHCSs in the basis $\left\{\left|\Phi_{n}(t)\right\rangle\right\}_{n=0}^{\infty}$ as follows:

$$
\begin{equation*}
\overbrace{|z ; m\rangle}=\left[N\left(|z|^{2}, m\right)\right]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-m-n-\nu)_{n} z^{n}}{\sqrt{\Gamma(n+1)(2 m+2 \nu)_{n}}}\left|\Phi_{n}(t)\right\rangle \tag{89}
\end{equation*}
$$

we get from (83)

$$
\begin{align*}
\varrho_{z_{0}}(z, t) & :=|\overbrace{\langle z ; m|} \overbrace{\left.z_{0} ; t ; m\right\rangle}|^{2} \\
& =\frac{I_{2 m+2 \nu-1}\left(2 \sqrt{\bar{z} z_{0}(t)}\right) I_{2 m+2 \nu-1}\left(2 \sqrt{z \overline{z_{0}(t)}}\right)}{I_{2 m+2 \nu-1}(2|z|) I_{2 m+2 \nu-1}\left(2\left|z_{0}(t)\right|\right)} . \tag{90}
\end{align*}
$$

Thereby, the time dependence of a given GHCSs $|z ; m\rangle$ is realized as

$$
\begin{align*}
\overbrace{|z ; t ; m\rangle} & =e^{-i H_{m} t}|z ; m\rangle \\
& =|z(t) ; m\rangle, z(t):=e^{-i(2 \nu+2 m-1) t} z . \tag{91}
\end{align*}
$$

The relation (91) shows that the time evolution of the GHCSs $|z ; m\rangle$ reduces to a rotation in the complex plane given by $z \mapsto z(t)=e^{-i(2 \nu+2 m-1) t} z$. Therefore, the semi-classical feature of the GHCSs is given by (86), while the temporal stability property is highlighted by the relation (91). The latter asserts that the temporal evolution of any GHCS always remains a GHCS, and fixes the phase behaviour of the GHCSs $|z ; m\rangle$ with the factor $e^{-i(2 \nu+2 m-1) t}$.

## 8 Thermal properties of the GHCSs

This section furnishes a description of statistical properties of the GHCSs for the model in the situation of a thermal equilibrium. Consider a quantum gas of the system in the thermodynamic equilibrium with a reservoir at temperature $T$, which satisfies a quantum canonical distribution. The corresponding normalized density operator is given by

$$
\begin{equation*}
\rho^{(m)}=\frac{1}{Z(\beta)} \sum_{n=0}^{\infty} e^{-\beta E_{n}}|n\rangle\langle n|, E_{n}=n(n+\mu+1), \tag{92}
\end{equation*}
$$

where the partition function denoted $Z(\beta)$ is taken as the normalization constant with its expression:

$$
\begin{equation*}
Z(\beta)=\sum_{n=0}^{\infty} e^{-\beta n(n+\mu+1)} \tag{93}
\end{equation*}
$$

ensuring the normalization condition: $\operatorname{Tr}\left(\rho^{(m)}\right)=1$.
The density operator GS- $P$-representation (also known as the diagonal expansion) is given by

$$
\begin{equation*}
\rho=\int_{\mathbb{C}} \frac{d^{2} z}{\pi} W\left(|z|^{2}, m\right)|z ; m\rangle P\left(|z|^{2}, m\right)\langle z ; m|, \tag{94}
\end{equation*}
$$

with the quasi-distribution function $P\left(|z|^{2}, m\right)$ given, by the normalization condition

$$
\begin{equation*}
\frac{1}{\pi} \int W\left(|z|^{2}, m\right) P\left(|z|^{2}, m\right) d^{2} z=1 \tag{95}
\end{equation*}
$$

as

$$
P\left(|z|^{2}, m\right)=e^{\beta \mu} \sum_{k=0}^{\infty} \frac{\beta^{k}}{k!}\left(\frac{d}{d a}\right)^{2 k} \frac{\left[e^{a} G_{2,2}^{2,0}\left(e^{a}|z|^{2} \left\lvert\, \begin{array}{cc}
m+\nu, & m+\nu  \tag{96}\\
0, & 2 m+\mu-1
\end{array}\right.\right)\right]}{G_{2,2}^{2,0}\left(|z|^{2} \left\lvert\, \begin{array}{cc}
m+\nu, & m+\nu \\
0, & 2 m+\mu-1
\end{array}\right.\right)},
$$

where $a=\beta(\mu+1)$.
Using the identity (see [19, 29])

$$
\begin{equation*}
e^{-\beta E_{n}}=e^{-n \beta(\mu+1)} \sum_{k=0}^{\infty} \frac{\beta^{k}}{k!}(n)^{2 k} \tag{97}
\end{equation*}
$$

we obtain the thermal average of the integer powers of number operator $\left\langle N^{s}\right\rangle$ in terms of Meijer's G-functions:

$$
\left.\left\langle e^{\varepsilon N}\right\rangle_{|z ; m\rangle}=\frac{G_{2,2}^{1,2}\left(-e^{\varepsilon}|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu  \tag{98}\\
0, & 1-2 m-\mu
\end{array}\right.\right)}{G_{2,2}^{1,2}\left(-\left.|z|^{2}\right|_{\mid c c} ^{1+m+n+\nu,} 1+m+n+\nu\right.} \begin{array}{l}
0, \\
1-2 m-\mu
\end{array}\right) .
$$

Thereby

$$
\begin{align*}
\lim _{\varepsilon \rightarrow 0} \frac{\partial^{s}}{\partial \varepsilon^{s}}\left\langle e^{\varepsilon N}\right\rangle_{|z ; m\rangle} & =\left\langle N^{s}\right\rangle_{|z ; m\rangle} \lim _{\varepsilon \rightarrow 0}\left\langle e^{\varepsilon N}\right\rangle \\
& =\left\langle N^{s}\right\rangle_{|z ; m\rangle} \tag{99}
\end{align*}
$$

Then, from

$$
\left\langle N^{s}\right\rangle=\int_{\mathbb{C}} \frac{d^{2} z}{\pi} W\left(|z|^{2}, m\right) P\left(|z|^{2}, m\right)\left\langle N^{s}\right\rangle_{|z ; m\rangle}
$$

$$
\begin{equation*}
=\lim _{\varepsilon \rightarrow 0} \frac{\partial^{s}}{\partial \varepsilon^{s}} \int_{\mathbb{C}} \frac{d^{2} z}{\pi} W\left(|z|^{2}, m\right) P\left(|z|^{2}, m\right)\left\langle e^{\varepsilon N}\right\rangle_{|z ; m\rangle} \tag{100}
\end{equation*}
$$

we finally get

$$
\begin{equation*}
\left\langle N^{s}\right\rangle=\sum_{n=0}^{\infty} e^{\beta \mu} \sum_{k=0}^{\infty} \frac{\beta^{k}}{k!}\left(\frac{d}{d a}\right)^{2 k} \lim _{\varepsilon \rightarrow 0} \frac{\partial^{s}}{\partial \varepsilon^{s}}\left(e^{\varepsilon}\right)^{n}\left(e^{-a}\right)^{n} . \tag{101}
\end{equation*}
$$

In this manner, the thermal average of the first two powers of the number operator are

$$
\begin{equation*}
\langle N\rangle=1+2 e^{\beta(2-\mu)}, \quad\left\langle N^{2}\right\rangle=1+4 e^{\beta(2-\mu)} . \tag{102}
\end{equation*}
$$

Therefore, the thermal second-order correlation function $g^{(2)}$ and the thermal Mandel parameter are obtained as follows:

$$
\begin{gather*}
g^{(2)}=\frac{\left\langle N^{2}\right\rangle-\langle N\rangle}{\left\langle N^{2}\right\rangle}=\frac{2 e^{\beta(2-\mu)}}{\left(1+2 e^{\beta(2-\mu)}\right)^{2}},  \tag{103}\\
Q=\langle N\rangle\left(g^{(2)}-1\right)=-\left[1+4 \frac{e^{2 \beta(2-\mu)}}{1+2 e^{\beta(2-\mu)}}\right] . \tag{104}
\end{gather*}
$$

## 9 Concluding remarks

In this work, we have first explored the building method, developed in [21]-[25], for generalized associated hypergeometric coherent states, and, then, proposed a construction method for generalized hypergeometric CSs (GHCSs) for associated Jacobi polynomials and hypergeometric functions. The constructed GHCSs coincide with the GHCSs introduced by Appl and Schiller, and correspond to the GPAH-CSs for $c \neq 1$ and for the number of added quanta (or photons) $p=0$ [25]. The resolution of the identity property is established through a Stieltjes moment problem solved by an appropriate weight function, in terms of product of Meijer's G functions, by using the Mellin transform. Then, the analytical features of the GHCSs are discussed through the reproducing kernel and the analytic representation of a given function in the Hilbert space spanned by these CSs. Next, the expectation values of the observables describing the quantum model have been derived in the constructed GHCSs basis. Besides, the CS quantization procedure, known as the Berezin-KlauderToeplitz quantization, (and also called coherent state, or anti-Wick quantization), has been applied in the complex plane by using the basis of the GHCSs. The study of the properties of the GHCSs has been also carried out by the analysis of their time dependence under the action of the time evolution operator elaborated from the quantum Hamiltonian and a probability density. Using the GS- $P$-representation also known as the diagonal representation of the density operator, the relevant thermodynamical properties of the quantum system have been investigated and discussed in the GHCSs basis.

## Appendix

## Proof of Proposition 3.2

From the definition of the CSs (45), we have

$$
\begin{equation*}
\int_{\mathbb{C}} \frac{d^{2} z}{\pi}|z ; m\rangle\langle z ; m| W\left(|z|^{2}, m\right)=\sum_{n=0}^{\infty} \int_{0}^{\infty}[N(x, m)]^{-1} \frac{x^{n} W(x, m) d x}{\frac{\Gamma(n+1)(2 m+2 \nu)_{n}}{\left[(-m-n-\nu)_{n}\right]^{2}}}|n\rangle\langle n| \tag{105}
\end{equation*}
$$

where the following relations

$$
\begin{equation*}
z=r e^{i \theta}, \frac{d^{2} z}{\pi}=\frac{r d r d \theta}{\pi} \text { and }|z|^{2}=x \tag{106}
\end{equation*}
$$

are used. The second member in the last line of 105 leads to

$$
\begin{align*}
& \int_{0}^{\infty}[N(x, m)]^{-1} \frac{x^{n}}{\frac{\Gamma(n+1)(2 m+2 \nu)_{n}}{\left[(-m-n-\nu)_{n}\right]^{2}}} W(x, m) d x \\
= & \int_{0}^{\infty} \frac{[\Gamma(-m-\nu)]^{2} \Gamma(2 m+2 \nu)[N(x, m)]^{-1} W(x, m)}{[\Gamma(-m-n-\nu)]^{2} \Gamma(n+1) \Gamma(2 m+2 \nu+n)} x^{n} d x \\
= & 1 . \tag{107}
\end{align*}
$$

Then, setting

$$
\begin{aligned}
n & =s-1, \\
g^{(m)}(x) & =[\Gamma(-m-\nu)]^{2} \Gamma(2 m+2 \nu)[N(x, m)]^{-1} W(x, m)
\end{aligned}
$$

and by use of the Meijer's G-functions and the Mellin inversion theorem [42, 43]

$$
\begin{align*}
& \int_{0}^{\infty} d x x^{s-1} G_{p, q}^{m, n}\left(\alpha x \left\lvert\, \begin{array}{l}
a_{1}, \ldots, a_{n} ; a_{n+1}, \ldots, a_{p} \\
b_{1}, \ldots, b_{m} ; b_{m+1}, \ldots, b_{q}
\end{array}\right.\right) \\
= & \frac{1}{\alpha^{s}} \frac{\prod_{j=1}^{m} \Gamma\left(b_{j}+s\right)}{\prod_{j=m+1}^{q} \Gamma\left(1-b_{j}-s\right)} \frac{\prod_{j=1}^{n} \Gamma\left(1-a_{j}-s\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_{j}+s\right)}, \tag{108}
\end{align*}
$$

the Eq. 107 leads to

$$
\int_{0}^{\infty} \frac{g^{(m)}(x)}{[\Gamma(-m-s+1-\nu)]^{2} \Gamma(s) \Gamma(2 m+2 \nu+s-1)} x^{s-1} d x=1
$$

implying

$$
\begin{equation*}
\int_{0}^{\infty} g^{(m)}(x) x^{s-1} d x=[\Gamma(1-\nu-m-s)]^{2} \Gamma(s) \Gamma(2 m+2 \nu+s-1) \tag{109}
\end{equation*}
$$

Using the connection between the hypergeometric functions and the Meijer's G-functions [20, 35], it follows:

$$
\begin{gathered}
g^{(m)}(x)=G_{2,2}^{2,0}\left(x \left\lvert\, \begin{array}{c}
m+\nu ; m+\nu \\
0 ; 2 m+2 \nu-1
\end{array}\right.\right), \\
{[\Gamma(-m-\nu)]^{2} \Gamma(2 m+2 \nu)[N(x, m)]^{-1} W(x, m)=G_{2,2}^{2,0}\left(\left.x\right|_{0 ; 2 m+2 \nu-1} ^{m+\nu ; m+\nu}\right),}
\end{gathered}
$$

such that we get

$$
W(x, m)=\frac{N(x, m)}{[\Gamma(-m-\nu)]^{2} \Gamma(2 m+2 \nu)} G_{2,2}^{2,0}\left(\left.x\right|^{m+\nu ; m+\nu} \begin{array}{c}
0 ; 2 m+2 \nu-1
\end{array}\right) .
$$

Thereby, by replacing $x$ by $|z|^{2}$, it comes

$$
W\left(|z|^{2}, m\right)=\frac{G_{2,2}^{2,0}\left(|z|^{2} \left\lvert\, \begin{array}{c}
m+\nu ; m+\nu  \tag{110}\\
0 ; 2 m+2 \nu-1
\end{array}\right.\right)}{[\Gamma(-m-\nu) \Gamma(-m-n-\nu)]^{2}} G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right) .
$$

## Proof of Proposition 4.1

The proof of (i) and (ii) can be easily obtained. Indeed, after direct calculations, we arrive at

$$
\left.\left.\left.\left.\overline{K\left(z, z^{\prime}\right)}=\frac{1}{\left[G _ { 2 , 2 } ^ { 1 , 2 } \left(-\left.\left|z^{\prime}\right|^{2}\right|_{\substack{1+m+n+\nu, 0, 0, 1+2 m-2 \nu}} ^{1+m+n+\nu}\right.\right.}\right)\right]^{\frac{1}{2}} \frac{G_{2,2}^{1,2}\left(-z^{\prime} \bar{z} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)}{\left[G _ { 2 , 2 } ^ { 1 , 2 } \left(-\left.|z|^{2}\right|_{\substack{1+m+n+\nu, 0, 1+m+n+\nu}} ^{1-2 m-2 \nu}\right.\right.}\right)\right]^{\frac{1}{2}}
$$

$$
\begin{equation*}
=K\left(z^{\prime}, z\right) \tag{111}
\end{equation*}
$$

and

$$
\begin{equation*}
K(z, z)=1>0 \tag{112}
\end{equation*}
$$

We start the proof of the idempotence property by first using the expression of the reproducing kernel 53), and then writing:

$$
\begin{align*}
& \int_{\mathbb{C}} K\left(z, z^{\prime \prime}\right) K\left(z^{\prime \prime}, z\right) W\left(\left|z^{\prime \prime}\right|^{2}\right) \frac{d^{2} z^{\prime \prime}}{\pi}=\int_{\mathbb{C}} G_{2,2}^{2,0}\left(\left|z^{\prime \prime}\right|^{2} \left\lvert\, \begin{array}{c}
m+\nu ; m+\nu \\
0 ; 2 m+2 \nu-1
\end{array}\right.\right) \frac{G_{2,2}^{1,2}\left(-z \bar{z}^{\prime \prime} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)}{\left[G_{2,2}^{1,2}\left(-\left.|z|^{2}\right|_{\left.\begin{array}{c}
1+m+n+\nu, \\
1+m+n+\nu \\
0, \\
1-2 m-2 \nu
\end{array}\right)}\right)\right]^{\frac{1}{2}}} \\
& \times \frac{1}{[\Gamma(-m-\nu) \Gamma(-m-n-\nu)]^{2}} \\
& \left.\times \frac{G_{2,2}^{1,2}\left(-z^{\prime \prime} \bar{z}^{\prime}\right.}{} \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right), ~ d^{2} z^{\prime \prime} . \tag{113}
\end{align*}
$$

Let

$$
\mathfrak{S}\left(z, z^{\prime}\right)=G_{2,2}^{1,2}\left(-z \overline{z^{\prime}} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu  \tag{114}\\
0, & 1-2 m-2 \nu
\end{array}\right.\right) G_{2,2}^{1,2}\left(\begin{array}{l|cc}
-z^{\prime \prime} \bar{z}^{\prime} & \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}
\end{array}\right)
$$

and

$$
\begin{equation*}
\left.\mathfrak{X}\left(z^{\prime \prime}\right)=\int_{\mathbb{C}} \frac{G_{2,2}^{2,0}\left(\left|z^{\prime \prime}\right|^{2}\right.}{\substack{m+\nu ; m+\nu \\ 0 ; 2 m+2 \nu-1}}\right)\left(\mathbb{\Gamma ( - m - \nu ) \Gamma ( - m - n - \nu ) ] ^ { 2 }} \mathfrak{S}\left(z, z^{\prime}\right) \frac{d^{2} z^{\prime \prime}}{\pi}\right. \tag{115}
\end{equation*}
$$

From

$$
\begin{equation*}
\mathfrak{S}\left(z, z^{\prime}\right)=\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{[\Gamma(-m-\nu)]^{4}}{\Gamma(2 m+2 \nu+n) \Gamma(2 m+2 \nu+k)} \frac{\left(\overline{z^{\prime \prime}} z\right)^{n}\left(\overline{z^{\prime}} z^{\prime \prime}\right)^{k}}{\Gamma(n+1) \Gamma(k+1)} \tag{116}
\end{equation*}
$$

we get

$$
\begin{align*}
\mathfrak{X}\left(z^{\prime \prime}\right)= & \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{[\Gamma(-m-\nu)]^{4}}{\Gamma(2 m+2 \nu+n) \Gamma(2 m+2 \nu+k)} \frac{(z)^{n}\left(\overline{z^{\prime}}\right)^{k}}{\Gamma(n+1) \Gamma(k+1)} \\
& \times \int_{\mathbb{C}} \frac{G_{2,2}^{2,0}\left(\left|z^{\prime \prime}\right|^{2} \left\lvert\, \begin{array}{c}
m+\nu ; m+\nu \\
0 ; 2 m+2 \nu-1
\end{array}\right.\right)}{[\Gamma(-m-\nu) \Gamma(-m-n-\nu)]^{2}}\left(\overline{z^{\prime \prime}}\right)^{n}\left(z^{\prime \prime}\right)^{k} \frac{d^{2} z^{\prime \prime}}{\pi} \\
= & G_{2,2}^{1,2}\left(-z \bar{z}^{\prime} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right) . \tag{117}
\end{align*}
$$

Thereby

$$
\begin{align*}
\int_{\mathbb{C}} K\left(z, z^{\prime \prime}\right) K\left(z^{\prime \prime}, z\right) W\left(\left|z^{\prime \prime}\right|^{2}\right) \frac{d^{2} z^{\prime \prime}}{\pi}= & \frac{1}{\left[G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)\right]^{\frac{1}{2}}} \\
& \times \frac{G_{2,2}^{1,2}\left(-z \overline{z^{\prime}} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)}{\left[G_{2,2}^{1,2}\left(-\left|z^{\prime}\right|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)\right]^{\frac{1}{2}}}  \tag{118}\\
= & K\left(z, z^{\prime}\right) .
\end{align*}
$$

Proof of Proposition 5.1

From the relations (45), the expectation of a given observable $A$ in the basis of the CSs $|z ; m\rangle$ is obtained as follows:

$$
\begin{align*}
\langle z ; m| A|z ; m\rangle=\langle A\rangle_{z, m}= & {\left[\frac{\Gamma(2 m+2 \nu)}{[\Gamma(-m-k-\nu)]^{2}} G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+k+\nu, & 1+m+k+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)\right.} \\
& \left.\times \frac{\Gamma(2 m+2 \nu)}{[\Gamma(-m-n-\nu)]^{2}} G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)\right]^{-\frac{1}{2}} \\
& \times \sum_{n, k=0}^{\infty} \frac{z^{n} \bar{z}^{k}}{\sqrt{\frac{\Gamma(n+1)(2 m+2 \nu)_{n}}{\left[(-m-n-\nu)_{n}\right]^{2}} \frac{\Gamma(k+1)(2 m+2 \nu)_{k}}{\left[(-m-k-\nu)_{k}\right]^{2}}}\langle k| A|n\rangle .} \text {. } \tag{119}
\end{align*}
$$

Thereby, if $n=k$, it comes

$$
\begin{equation*}
\langle A\rangle_{z, m}=\left[N\left(|z|^{2}, m\right)\right]^{-1} \sum_{n=0}^{\infty} \frac{\left[(-m-n-\nu)_{n}\right]^{2}}{(2 m+2 \nu)_{n}} \frac{z^{2 n}}{\Gamma(n+1)}\langle n| A|n\rangle \tag{120}
\end{equation*}
$$

For the number operator $N$, we have in the Fock basis $\{|n\rangle\}_{n=0}^{\infty}:\langle n| N^{i}|n\rangle=n^{i}$ and $\left\langle n^{\prime}\right| N^{i}|n\rangle=0$. Using the ansatz [19], set

$$
\begin{equation*}
S_{i}=\sum_{n=0}^{\infty} \frac{\left[(-m-n-\nu)_{n}\right]^{2}}{(2 m+2 \nu)_{n}} \frac{x^{n}}{\Gamma(n+1)} n^{i} \text { with } x=|z|^{2} \tag{121}
\end{equation*}
$$

For $i>0$, we obtain the following relation for $n^{i}$

$$
\begin{equation*}
n^{i}=\sum_{l=1}^{i} C_{l}^{(i)} \frac{n!}{(n-l)!}=\sum_{l=0}^{i}(-1)^{l} C_{l}^{(i)} \frac{n!}{(n-l)!} \tag{122}
\end{equation*}
$$

Then, we get

$$
\begin{equation*}
S_{i}=\left(x \frac{d}{d x}\right)^{i} S_{0}=\sum_{l=1}^{i} C_{l}^{(n)} x^{l}\left(\frac{d}{d x}\right)^{l} S_{0} \tag{123}
\end{equation*}
$$

where

$$
S_{0}=\frac{\Gamma(2 m+2 \nu)}{[\Gamma(-m-n-\nu)]^{2}} G_{2,2}^{1,2}\left(-x \left\lvert\, \begin{array}{c}
1+m+n+\nu, 1+m+n+\nu  \tag{124}\\
0,1-2 m-2 \nu
\end{array}\right.\right)
$$

such that, by applying the $n t h$ derivative of the Meijer's G-functions, we get:

$$
\left.\begin{array}{rl}
S_{i} & =\sum_{l=1}^{i}(-1)^{l} C_{l}^{(n)} \frac{\Gamma(2 m+2 \nu)}{[\Gamma(-m-n-\nu)]^{2}} G_{3,3}^{2,2}\left(-x \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu, \\
0, & l, \\
\hline
\end{array}\right.\right) \\
& =\sum_{l=1}^{i}(-1)^{l} C_{l}^{(n)} \frac{\Gamma(2 m-2 \nu}{} \tag{125}
\end{array}\right) .
$$

From 125 , the expectations of the operators $N$ and $N^{2}$ are:

$$
\begin{equation*}
\left.\langle N\rangle_{z, m}=\left[N\left(|z|^{2}, m\right)\right]^{-1} S_{1}=-C_{1}^{(n)} \frac{G_{2,2}^{1,2}\left(-|z|^{2}\right.}{\substack{1+m+n+\nu, 1+m+n+\nu \\ 1, 1-2 m-2 \nu}}\right) \tag{126}
\end{equation*}
$$

and

$$
\begin{align*}
\left\langle N^{2}\right\rangle_{z, m}= & {\left[N\left(|z|^{2}, m\right)\right]^{-1} S_{2} } \\
= & \frac{1}{G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, & 1+m+n+\nu \\
0, & 1-2 m-2 \nu
\end{array}\right.\right)}\left[-C_{1}^{(n)} G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{c}
1+m+n+\nu, 1+m+n+\nu \\
1, \quad 1-2 m-2 \nu
\end{array}\right.\right)\right. \\
& \left.+C_{2}^{(n)} G_{2,2}^{1,2}\left(-|z|^{2} \left\lvert\, \begin{array}{cc}
1+m+n+\nu, 1+m+n+\nu \\
2, & 1-2 m-2 \nu
\end{array}\right.\right)\right] . \tag{127}
\end{align*}
$$

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