

GENERALIZED IMAGINARY VERMA AND WAKIMOTO MODULES

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ABSTRACT. We develop a general technique of constructing new irreducible weight modules for any affine Kac-Moody algebra using the parabolic induction, in the case when the Levi factor of a parabolic subalgebra is infinite-dimensional and the central charge is nonzero. Our approach uniforms and generalizes all previously known results with imposed restrictions on inducing modules. We also define generalized Imaginary Wakimoto modules which provide an explicit realization for generic generalized Imaginary Verma modules.

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CONTENTS

1. Introduction	1
2. Quase partitions and parabolic subalgebras	3
2.1. Quase partitions	4
2.2. Parabolic subalgebras	5
3. Generalized Imaginary Verma modules	5
3.1. Imaginary induction functor	5
3.2. Irreducibility of generalized loop modules	7
3.3. Irreducibility of generalized Imaginary Verma modules	11
3.4. Localization of generalized Imaginary Verma modules	14
4. Realization of induced modules	15
4.1. Generalized Imaginary Wakimoto modules	15
4.2. Imaginary Wakimoto functor	17
Acknowledgments	19
References	19

1. INTRODUCTION

Representation theory of Kac-Moody algebras has a vast range of applications in many areas of mathematics and physics. Affine Lie algebras is the most important family of infinite dimensional Kac-Moody algebras with important connections to quantum field

theory and integrable systems. That is why their representations attract an increasing attention from both mathematicians and physicists.

Parabolic induction plays a key role in the representation theory of Lie algebras. In particular, the induction functor is crucial for the study of weight modules for Affine Lie algebras. The induction functor is defined by a choice of a parabolic subalgebra which by definition contains a Borel subalgebra.

Borel subalgebras of Affine Lie algebras are essentially classified by the corresponding partitions or quase partitions of the root system [JK85], [Fut94, Fut97], [DFG09], [FK18]. Verma type modules induced from Borel subalgebras were studied in [Cox94], [Cox05], [FS93], [Fut92], [Fut96], [CF04], [CF06], [BBFK13] among the others. Parabolic subalgebras (different from Borel subalgebras) of Affine Kac-Moody algebras are divided into two types depending on the dimension of the Levi factor. We are interested in the case when the dimension of the Levi factor is infinite. Such parabolic subalgebras always contain certain infinite dimensional Heisenberg subalgebra. Modules induced from such parabolic subalgebras are called *generalized Imaginary Verma modules*. Previously, families of generalized Imaginary Verma modules were studied in [FK09], [FK14], [BBFK13], [FK18], [FKS19]. In all these cases the structure of the induced modules is well understood when the central charge (the scalar action of the central element) is nonzero. In particular, it allowed to construct new families of irreducible modules for Affine Lie algebras using the parabolic induction from certain irreducible modules over the Levi factor of a parabolic subalgebra. The largest known family of the representations of the Levi factor for which the corresponding induced module is irreducible was described in [FK18]. In particular, if the Levi factor of a parabolic subalgebra equals the sum of an infinite dimensional Heisenberg subalgebra and a Cartan subalgebra then the irreducibility is preserved for so-called admissible modules with nonzero central charge. Our first main result extends this statement to arbitrary weight modules over Heisenberg subalgebras.

For an Affine Lie algebra $\widehat{\mathfrak{g}}$ with a fixed Cartan subalgebra H consider a parabolic subalgebra $\widehat{\mathfrak{p}}$. Then $\widehat{\mathfrak{p}}$ can be decomposed as follows:

$$\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}},$$

where $\widehat{\mathfrak{l}}$ is the Levi factor and $\widehat{\mathfrak{u}}$ is the nilradical. Assume that $\widehat{\mathfrak{l}} = G + H$, where G is the Heisenberg subalgebra of $\widehat{\mathfrak{g}}$, that is the subalgebra generated by all imaginary root subspaces. The Heisenberg subalgebra G has a natural \mathbb{Z} -grading determined by imaginary roots, equivalently by the adjoint action of a certain element $d \in H$ (degree derivation). Let $c \in H$ be the central element of $\widehat{\mathfrak{g}}$. If V is $\widehat{\mathfrak{g}}$ -module (respectively G -module) with a scalar action of c then the corresponding scalar is the central charge of V .

For any $\lambda \in H^*$ and a \mathbb{Z} -graded G -module V with central charge $\lambda(c)$ we obtain a weight $(G + H)$ -module structure on V with the action of d respecting the grading. Defining $\widehat{\mathfrak{u}} \cdot V = 0$ we get a $\widehat{\mathfrak{p}}$ -module structure on V . Then the parabolic induction defines for each $\lambda \in H^*$ the functor \mathbb{I}^λ from the category of \mathbb{Z} -graded G -modules with central charge $\lambda(c)$ to the category of weight $\widehat{\mathfrak{g}}$ -modules. We have

Theorem 1.1. *If $\lambda(c) \neq 0$ then the functor \mathbb{I}^λ preserves the irreducibility.*

Theorem 1.1 serves as a key step to show our main result which gives the irreducibility of generalized Imaginary Verma modules with nonzero central charge. Let $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$ be a parabolic subalgebra of $\widehat{\mathfrak{g}}$ with an infinite dimensional Levi subalgebra $\widehat{\mathfrak{l}}$. Then $\widehat{\mathfrak{l}} = \widehat{\mathfrak{l}}^0 + G(\widehat{\mathfrak{l}})^\perp$, where $\widehat{\mathfrak{l}}^0$ is the Lie subalgebra generated by H and by all real root subspaces of $\widehat{\mathfrak{l}}$ and $G(\widehat{\mathfrak{l}})^\perp$ is the orthogonal complement to the imaginary part of $\widehat{\mathfrak{l}}^0$ with respect to the Killing form.

If M is a weight (with respect to H) $\widehat{\mathfrak{l}}^0$ -module with central charge a and S is a $G(\widehat{\mathfrak{l}})^\perp \oplus \mathbb{C}d$ -module with the same central charge and diagonalizable action of d , then $M \otimes S$ is an $\widehat{\mathfrak{l}}$ -module. Such $\widehat{\mathfrak{l}}$ -modules are called *tensor modules*. Let $\mathcal{T} = \mathcal{T}_a(\widehat{\mathfrak{l}})$ be the category of tensor $\widehat{\mathfrak{l}}$ -modules with central charge a . Any $V \in \mathcal{T}_a(\widehat{\mathfrak{l}})$ has a structure of a $\widehat{\mathfrak{p}}$ -module with trivial action of the radical $\widehat{\mathfrak{u}}$. Further induction from a $\widehat{\mathfrak{p}}$ -module to a $\widehat{\mathfrak{g}}$ -module defines the functor $\mathbb{I}_{a, \widehat{\mathfrak{p}}}^{\mathcal{T}}$ from the category $\mathcal{T}_a(\widehat{\mathfrak{l}})$ to the category of weight $\widehat{\mathfrak{g}}$ -modules. Our second main result is the following theorem which provides a general tool of constructing new irreducible modules for all Affine Kac-Moody algebras.

Theorem 1.2. *Let $a \in \mathbb{C} \setminus \{0\}$. Then the functor $\mathbb{I}_{a, \widehat{\mathfrak{p}}}^{\mathcal{T}}$ preserves the irreducibility.*

Free field realizations of induced modules from a parabolic subalgebra with infinite dimensional Levi factor $\widehat{\mathfrak{l}}$ were constructed in [FGM15], [Mar11], [Mar13], [FKS19]. Based on the ideas of [FKS19] and [FK21] we define the *generalized Imaginary Wakimoto modules* which realize explicitly generic generalized Imaginary Verma modules. We also define the *Imaginary Wakimoto functor* from the category of $\widehat{\mathfrak{l}}$ -modules to the category of $\widehat{\mathfrak{g}}$ -modules which sends an $\widehat{\mathfrak{l}}$ -module to the corresponding generalized Imaginary Wakimoto module.

2. QUASE PARTITIONS AND PARABOLIC SUBALGEBRAS

Let $\widehat{\mathfrak{g}}$ be an affine Kac-Moody algebra with a Cartan subalgebra H and 1-dimensional center $Z = \mathbb{C}c$. We have the root decomposition

$$\widehat{\mathfrak{g}} = H \oplus (\oplus_{\alpha \in \Delta} \widehat{\mathfrak{g}}_\alpha),$$

where Δ is the root system of $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}_\alpha = \{x \in \widehat{\mathfrak{g}} \mid [h, x] = \alpha(h)x \text{ for every } h \in H\}$. Let π be a fixed basis of the root system Δ . Then the root system Δ has a standard partition into positive and negative roots with respect to π , $\Delta_+(\pi) = \Delta_+$ and $\Delta_-(\pi) = \Delta_-$ respectively. Let $\delta \in \Delta_+$ be the indivisible imaginary root. Then the set of all imaginary roots is $\Delta^{\text{im}} = \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\}$ and the set of all real roots $\Delta^{\text{re}} = \Delta \setminus \Delta^{\text{im}}$. Denote by G the Heisenberg subalgebra of $\widehat{\mathfrak{g}}$ generated by all imaginary root subspaces of $\widehat{\mathfrak{g}}$. Hence $G = G_- \oplus \mathbb{C}c \oplus G_+$, where $G_\pm = \oplus_{k > 0} \widehat{\mathfrak{g}}_{\pm k\delta}$. We will denote by \mathfrak{g} the underlined finite-dimensional simple Lie algebra and by d the element of H such that $\delta(d) = 1$ and $\alpha(d) = 0$ for any root α of \mathfrak{g} .

2.1. Quase partitions. Recall that a subset $P \subset \Delta$ is called a *partition* of Δ if P is closed with respect to the sum of roots, $P \cap (-P) = \emptyset$ and $P \cup (-P) = \Delta$.

Following [FK18] we will say that a subset $P \subset \Delta$ is a *quase partition* of Δ if the following conditions are satisfied:

- $P \cap (-P) = \emptyset$ and $P \cup (-P) = \Delta$.
- Let $\widehat{\mathfrak{b}}_P$ be a Lie subalgebra of $\widehat{\mathfrak{g}}$ generated by H and by the root spaces $\widehat{\mathfrak{g}}_\alpha$ with $\alpha \in P$. Then for any $\widehat{\mathfrak{g}}_\alpha \subset \widehat{\mathfrak{b}}_P$ follows $\alpha \in P$.

In particular, for any quase partition P , the subset of real roots of P is closed with respect to the sum of roots and hence $P \cap \Delta^{\text{re}}$ is a partition of real roots. Indeed, $(P \cap \Delta^{\text{re}}) \cap (-P \cap \Delta^{\text{re}}) = \emptyset$ and $(P \cap \Delta^{\text{re}}) \cup (-P \cap \Delta^{\text{re}}) = \Delta^{\text{re}}$. Moreover, the second condition of a quase partition implies that $P \cap \Delta^{\text{re}}$ is closed with respect to the addition of roots. On the contrary, the subset of imaginary roots of a quase partition P is not necessarily a partition of Δ^{im} .

We will call a subalgebra $\widehat{\mathfrak{b}}_P$ the *Borel subalgebra* of $\widehat{\mathfrak{g}}$ corresponding to the quase partition P (cf. [DFG09], [FK18]). By the results of [JK85] and [Fut94] there are finitely many conjugacy classes of $P \cap \Delta^{\text{re}}$ by the Weyl group. On the other hand the number of conjugacy classes of quase partitions is infinite [BBFK13].

Example 2.1. (1) The partition $P = \Delta_+$ corresponds to the standard Borel subalgebra.

(2) The *natural partition*

$$\Delta_{+, \text{nat}} = \{\alpha + n\delta \in \Delta \mid \alpha \in \dot{\Delta}_+, n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{N}\}$$

corresponds to the natural Borel subalgebra

$$\widehat{\mathfrak{b}}_{\text{nat}} = H \oplus \left(\bigoplus_{\alpha \in \Delta_{+, \text{nat}}} \widehat{\mathfrak{g}}_\alpha \right),$$

where $\dot{\Delta}$ is the root system of \mathfrak{g} and $\dot{\Delta}_+ = \dot{\Delta} \cap \Delta_+$.

(3) Each function $\phi : \mathbb{N} \rightarrow \{\pm\}$ defines a Borel subalgebra $\widehat{\mathfrak{b}}_{\text{nat}}^\phi$ corresponding to the quase partition

$$\{\alpha + k\delta \in \Delta \mid \alpha \in \dot{\Delta}_+, k \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{N}, \phi(n) = +\} \cup \{-m\delta \mid m \in \mathbb{N}, \phi(m) = -\}.$$

The subalgebras $\widehat{\mathfrak{b}}_P$ do not exhaust all non-conjugated Borel subalgebras of $\widehat{\mathfrak{g}}$. To get the full classification one needs to consider all possible triangular decompositions of the Heisenberg subalgebra G . Using the Killing form, in each subspace $\widehat{\mathfrak{g}}_{k\delta}$ we can choose a commuting basis $x_k^1, \dots, x_k^{m_k}$ such that $[x_k^i, x_n^j] = k\delta_{ij}\delta_{k, -n}c$ for all possible k, n, i, j . A triangular decomposition $G = G_- \oplus \mathbb{C}c \oplus G_+$, where $G_\pm = \sum_{\pm k \in \mathbb{Z}_{>0}} \widehat{\mathfrak{g}}_{k\delta}$, corresponds to a partition of imaginary roots. On the other hand, each function $\phi : \mathbb{N} \rightarrow \{\pm\}$ defines a quase partition of imaginary roots and the triangular decomposition $G = G_-^\phi \oplus \mathbb{C}c \oplus G_+^\phi$, where

$$G_\pm^\phi = \sum_{\phi(k)=+} \widehat{\mathfrak{g}}_{\pm k\delta} \oplus \sum_{\phi(k)=-} \widehat{\mathfrak{g}}_{\mp k\delta}.$$

We can obtain different triangular decompositions of G by dividing x_k^i arbitrarily between G_+ and G_- for each integer $k > 0$ and each $i = 1, \dots, m_k$ (if $x_k^i \in G_\pm$ then $x_{-k}^i \in G_\mp$).

Such triangular decompositions of G are more general than those corresponding to quase partitions. They lead to new families of Borel subalgebras of $\widehat{\mathfrak{g}}$ (cf. [FK18]).

For simplicity from now on we will consider the Borel subalgebras $\widehat{\mathfrak{b}}_P$ corresponding to quase partitions P of the root system, though everything can be easily extended to an arbitrary Borel subalgebra of $\widehat{\mathfrak{g}}$.

2.2. Parabolic subalgebras. A subalgebra of $\widehat{\mathfrak{g}}$ is called a *parabolic subalgebra* if it contains properly a Borel subalgebra. There are two types of parabolic subalgebras: those containing the standard Borel subalgebra and those containing the real part of the natural Borel subalgebra. These parabolic subalgebras are called *type I* and *type II* respectively (cf. [Fut97]). Examples of parabolic subalgebras of $\widehat{\mathfrak{g}}$ can be constructed from parabolic subsets $P \subset \Delta$, which are closed subsets of Δ such that $P \cup -P = \Delta$. Given such a parabolic subset P , the corresponding parabolic subalgebra of $\widehat{\mathfrak{g}}$ is generated by H and by all root subspaces $\widehat{\mathfrak{g}}_\alpha$ with $\alpha \in P$. A classification of parabolic subsets of Δ was obtained in [Fut92], [Fut97].

Every parabolic subalgebra $\widehat{\mathfrak{p}}$ of $\widehat{\mathfrak{g}}$ containing a Borel subalgebra $\widehat{\mathfrak{b}}$ has a Levi decomposition $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$, with the Levi factor $\widehat{\mathfrak{l}}$ and $\widehat{\mathfrak{u}} \subset \widehat{\mathfrak{b}}$. If $\widehat{\mathfrak{p}}$ is of *type II* then $\widehat{\mathfrak{l}}$ is infinite-dimensional. In this case the subalgebra $\widehat{\mathfrak{l}}$ contains a sum of certain affine Lie subalgebras and a subalgebra of the Heisenberg algebra G . Further suppose $\widehat{\mathfrak{p}}$ is such parabolic subalgebra of $\widehat{\mathfrak{g}}$ which is not a Borel subalgebra. Then its real part can be described as follows [Fut97], [FK09]. Assume for simplicity that a parabolic subalgebra $\widehat{\mathfrak{p}}$ of $\widehat{\mathfrak{g}}$ contains the whole natural Borel subalgebra (not just its real part). Then such parabolic subalgebra $\widehat{\mathfrak{p}}$ has the Levi decomposition $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$, where $\widehat{\mathfrak{l}}$ is the infinite dimensional Levi factor. Denote by $\widehat{\mathfrak{l}}^0$ the Lie subalgebra of $\widehat{\mathfrak{l}}$ generated by all its real root subspaces and by H . Set $G(\widehat{\mathfrak{l}})$ to be a subalgebra of $\widehat{\mathfrak{l}}^0$ spanned by its imaginary root subspaces. Let $G(\widehat{\mathfrak{l}})^\perp \subset G$ be the orthogonal complement of $G(\widehat{\mathfrak{l}})$ in G with respect to the Killing form, that is $G = G(\widehat{\mathfrak{l}}) + G(\widehat{\mathfrak{l}})^\perp$, $[G(\widehat{\mathfrak{l}})^\perp, \widehat{\mathfrak{l}}^0] = 0$ and $\widehat{\mathfrak{l}}^0 \cap G(\widehat{\mathfrak{l}})^\perp = \mathbb{C}c$. The Levi subalgebra $\widehat{\mathfrak{l}}$ may contain the whole $G(\widehat{\mathfrak{l}})^\perp$ or a part of it. In the former case $\widehat{\mathfrak{l}}$ contains G and $\widehat{\mathfrak{l}} = \widehat{\mathfrak{l}}^0 + G(\widehat{\mathfrak{l}})^\perp$.

Let n be the rank of the underlined simple finite dimensional Lie algebra \mathfrak{g} and $\{\alpha_1, \dots, \alpha_n\} \subset \pi$ the set of simple roots of \mathfrak{g} . Set $I = \{1, \dots, n\}$ and consider a subset $\omega \subset I$. Denote by $\dot{\Delta}^\omega$ the root system generated by the roots $\alpha_i, i \in \omega$ and set

$$\Delta^\omega = \{\alpha + n\delta \in \Delta \mid \alpha \in \dot{\Delta}^\omega, n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\},$$

$\Delta^{\omega, \text{re}} = \Delta^\omega \cap \Delta^{\text{re}}$. Then the real roots of $\widehat{\mathfrak{l}}$ are conjugated to $\Delta^{\omega, \text{re}}$ for some choice of ω (cf. [Fut94]).

3. GENERALIZED IMAGINARY VERMA MODULES

3.1. Imaginary induction functor. Let $a \in \mathbb{C}$ and V be a $\widehat{\mathfrak{p}}$ -module with central charge a and with $\widehat{\mathfrak{u}} \cdot V = 0$. Define the *generalized Verma type module*

$$M_{a, \widehat{\mathfrak{p}}}(V) = \text{Ind}_{\widehat{\mathfrak{p}}}^{\widehat{\mathfrak{g}}}(V) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}})} V.$$

When $\widehat{\mathfrak{p}} = \widehat{\mathfrak{b}}$ is a Borel subalgebra and $V \simeq \mathbb{C}$ is a one-dimensional $\widehat{\mathfrak{b}}$ -module such that $hv = \lambda(h)v$ for all $v \in V$ and $h \in H$, we get a *Verma type module* $M_{\widehat{\mathfrak{b}}}(\lambda) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{b}})} \mathbb{C}$ ([Fut97]).

To emphasize that we will consider the case of a parabolic subalgebra $\widehat{\mathfrak{p}}$ of type II we call $M_{a,\widehat{\mathfrak{p}}}(V)$ *generalized Imaginary Verma module* with central charge a . We have the following well known particular cases of the construction above.

- (1) In the particular case when $\widehat{\mathfrak{l}} = G + H$ and V is a $(G + H)$ -module, the module $M_{a,\widehat{\mathfrak{p}}}(V)$ is a *generalized loop module* with central charge a .
- (2) Taking V to be a Verma module for $G + H$ with highest weight $\lambda \in H^*$ with respect to the triangular decomposition $G = G_- \oplus \mathbb{C}c \oplus G_+$, the corresponding module $M_{a,\widehat{\mathfrak{p}}}(V)$ is an *imaginary Verma module* [Fut94].
- (3) Finally, taking V to be a Verma module for $G + H$ with highest weight $\lambda \in H^*$ with respect to the triangular decomposition $G = G_-^\phi \oplus \mathbb{C}c \oplus G_+^\phi$, the corresponding module $M_{a,\widehat{\mathfrak{p}}}(V)$ is a ϕ -*Imaginary Verma module* [BBFK13].

For any Lie algebra \mathfrak{a} with a Cartan subalgebra \mathfrak{h} denote by $\mathcal{M}(\mathfrak{a})$ the category of all weight (i.e. \mathfrak{h} -diagonalizable) \mathfrak{a} -modules, and by $\mathcal{M}(a, \mathfrak{a})$ its subcategory of all weight \mathfrak{a} -modules with central charge a . Then we have the following *Imaginary (induction) functor*

$$\mathbb{I}_{a,\widehat{\mathfrak{p}}}: \mathcal{M}(a, \widehat{\mathfrak{l}}) \rightarrow \mathcal{M}(a, \widehat{\mathfrak{g}}),$$

which sends an $\widehat{\mathfrak{l}}$ -module V to the $\widehat{\mathfrak{g}}$ -module $M_{a,\widehat{\mathfrak{p}}}(V)$.

For any positive integer k , denote $G_k = \widehat{\mathfrak{g}}_{k\delta} \oplus \mathbb{C}c \oplus \widehat{\mathfrak{g}}_{-k\delta}$. We say that a G_k -module M is $U(\widehat{\mathfrak{g}}_{k\delta})$ -surjective (respectively $U(\widehat{\mathfrak{g}}_{-k\delta})$ -surjective) if for any two elements $v_1, v_2 \in M$ there exist $v \in M$ and $u_1, u_2 \in U(\widehat{\mathfrak{g}}_{k\delta})$ (respectively, $u_1, u_2 \in U(\widehat{\mathfrak{g}}_{-k\delta})$) such that $v_i = u_i v$, $i = 1, 2$. A $G(\widehat{\mathfrak{l}})^\perp$ -module M is admissible if for any positive integer k , any its cyclic $G_k \cap G(\widehat{\mathfrak{l}})^\perp$ -submodule $M' \subset M$ is $U(\widehat{\mathfrak{g}}_{k\delta} \cap G(\widehat{\mathfrak{l}})^\perp)$ -surjective or $U(\widehat{\mathfrak{g}}_{-k\delta} \cap G(\widehat{\mathfrak{l}})^\perp)$ -surjective. Recall the following irreducibility criterion for admissible tensor modules which follows from [FK18, Theorem 1, Corollary 3].

Theorem 3.1. [FK18] *Let $a \in \mathbb{C}$, $\widehat{\mathfrak{l}} = \widehat{\mathfrak{l}}^\perp + G(\widehat{\mathfrak{l}})^\perp$ and $V \simeq M \otimes S$, where M is an irreducible weight $\widehat{\mathfrak{l}}^\perp$ -module and S is an admissible irreducible \mathbb{Z} -graded $G(\widehat{\mathfrak{l}})^\perp$ -module. Then $\mathbb{I}_{a,\widehat{\mathfrak{p}}}(V) = \mathbb{I}_{a,\widehat{\mathfrak{p}}}^T(V)$ is irreducible $\widehat{\mathfrak{g}}$ -module if $a \neq 0$.*

Example 3.2. Let $a \in \mathbb{C}$, $\widehat{\mathfrak{p}}$ is a parabolic subalgebra of $\widehat{\mathfrak{g}}$ with the Levi factor $\widehat{\mathfrak{l}} = G + H$. Let $L(a)$ be the highest weight G -module with respect to the decomposition $G = G_- \oplus \mathbb{C}c \oplus G_+$ and with central charge a (irreducible if $a \neq 0$). Define an $\widehat{\mathfrak{l}}$ -module structure on $L(a)$ by fixing the scalar action of $H \cap \mathfrak{g}$ and defining the weight structure on $L(a)$ with respect to the derivation d . Then $\mathbb{I}_{a,\widehat{\mathfrak{p}}}(L(a))$ is irreducible $\widehat{\mathfrak{g}}$ -module if $a \neq 0$. It is isomorphic to an imaginary Verma module [Fut97].

Let $L(a)^-$ be the lowest weight G -module with respect to the same triangular decomposition and with central charge a (irreducible if $a \neq 0$), and $V = L(a) \otimes L(a)^-$. Then V is neither diagonal (in the sense of [BBFK13]) nor admissible.

We will show that the condition of admissibility can be lifted in Theorem 3.1. In the following subsection we will prove it for an arbitrary generalized loop module.

3.2. Irreducibility of generalized loop modules. Let $I = \{1, \dots, n\}$ and $\alpha_1, \dots, \alpha_n$ be simple roots of \mathfrak{g} . For any subset $\omega \subset I$, let Q_{\pm}^{ω} denote a semigroup of H^* generated by $\pm\alpha_i$, $i \in \omega$. Set $Q^{\omega} = \bigoplus_{j \in \omega} \mathbb{Z}\alpha_j \oplus \mathbb{Z}\delta$ and if $\omega = I$, denote Q^I by Q . We can see that $U(\widehat{\mathfrak{g}})$ is a Q -graded algebra, then

$$U(\widehat{\mathfrak{g}}) = \bigoplus_{\phi \in Q} U(\widehat{\mathfrak{g}})_{\phi},$$

and the elements of $U(\widehat{\mathfrak{g}})_{\phi}$ are called homogeneous elements of $U(\widehat{\mathfrak{g}})$ of homogeneous degree $\phi \in Q$.

Let $\alpha \in Q_{-}^{\omega}$ and $\alpha = -\sum_{j \in \omega} k_j \alpha_j$, where each k_j is in $\mathbb{Z}_{\geq 0}$. We define the ω -height of α to be $\text{ht}_{\omega}(\alpha) = \sum_{j=1}^n k_j$.

For a parabolic subalgebra $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$, the subalgebra $\widehat{\mathfrak{u}}$ satisfying $\widehat{\mathfrak{g}} = \widehat{\mathfrak{p}} \oplus \widehat{\mathfrak{u}}$ is called *the opposite radical* of $\widehat{\mathfrak{p}}$. Observe that for any $\widehat{\mathfrak{p}}$ -module V the induced module $M_{a, \widehat{\mathfrak{p}}}(V) \simeq U(\widehat{\mathfrak{u}}) \otimes_{\mathbb{C}} V$ as a vector space.

Assume $\omega = \omega_{\widehat{\mathfrak{u}}}$ consists of all indices i for which α_i is a simple root of $\widehat{\mathfrak{g}}$ with $\widehat{\mathfrak{g}}_{\alpha} \in \widehat{\mathfrak{u}}$. Let $v \in M_{a, \widehat{\mathfrak{p}}}(V)$ be a nonzero weight element ($v \in M_{a, \widehat{\mathfrak{p}}}(V)_{\mu}$, for $\mu \in H^*$). Then

$$v = \sum_i u_i v_i,$$

for some finite set of indices, where for all i , $u_i \in U(\widehat{\mathfrak{u}})$ are linearly independent elements and $v_i \in V$. Since v is a weight vector then each u_i is a homogeneous element of $U(\widehat{\mathfrak{u}})$ of homogeneous degree $\phi_i \in Q$. Each ϕ_i can be written as $\phi_i = \psi_i + \tau_i$, where $\psi_i \in Q_{-}^{\omega_{\widehat{\mathfrak{u}}}}$ and $\tau_i \in Q^I \setminus \omega_{\widehat{\mathfrak{u}}}$. Moreover, due to homogeneity we have $\psi_i = \psi_j$ for all i, j . Denote by ψ this common restriction to $Q_{-}^{\omega_{\widehat{\mathfrak{u}}}}$.

The $\widehat{\mathfrak{u}}$ -height of v will be defined as the ω -height of ψ and will be denoted by $\text{ht}_{\widehat{\mathfrak{u}}}(v)$.

We assume now that $\omega = \emptyset$ and $\widehat{\mathfrak{l}} = G + H$. We will say that V is a weight $(G + H)$ -module if V is a \mathbb{Z} -graded G -module with diagonalizable action of H , where the action of d is compatible with the \mathbb{Z} -grading. An irreducible weight $(G + H)$ -module is determined by an irreducible \mathbb{Z} -graded G -module V and a weight $\lambda \in H^*$ with the action of $h \in \mathfrak{h} = H \cap \mathfrak{g}$ and c on V by $\lambda(h)$ and $\lambda(c)$ respectively, while the action of d is given by $\lambda(d)$ on a chosen graded component of V .

If $\lambda(c) = a$, we will denote the corresponding generalized Imaginary Verma module $M_{a, \widehat{\mathfrak{p}}}(V)$ by $M_a(V)$. For each $\lambda \in H^*$ we have a functor \mathbb{I}^{λ} from the category of \mathbb{Z} -graded G -modules with central charge $\lambda(c)$ to the category of $\widehat{\mathfrak{g}}$ -modules, which sends a G -module V to $M_{\lambda(c)}(V)$ with a natural weight structure with respect to H as described above. Our goal is to show that Theorem 3.1 holds for modules $M_a(V)$ in full generality without the assumption of admissibility. Namely, we have

Theorem 3.3. *Let V be an irreducible weight $G+H$ -module and $a \neq 0$. Then the $\widehat{\mathfrak{g}}$ -module $M_a(V)$ is irreducible. In particular, for any $\lambda \in H^*$ the functor \mathbb{I}^λ preserves irreducibility if $\lambda(c) \neq 0$.*

The following is the key lemma which generalizes [FK18, Lemma 2].

Lemma 3.4. *Let V be a weight $(G+H)$ -module with nonzero scalar action of c : $cv = av$ for some $a \in \mathbb{C} \setminus \{0\}$ and any $v \in V$. Also, let $w_1, \dots, w_m \in V$ be nonzero elements such that $w_i \in V_{\mu_i}$ for $\mu_i \in H^*$ and $\mu_i - \mu_{i+1} \in \mathbb{Z}_{>0}\delta$ for all i . Let k_i be such that $\mu_1 - \mu_i = k_i\delta$, $i = 1, \dots, m$. Then for a sufficiently large N and for any choice of nonzero elements $x_k \in \widehat{\mathfrak{g}}_{k\delta}$ such that $[x_k, x_{-k}] = kc \neq 0$, at least one of the sums*

$$\sum_{i=1}^m x_{N-k_i} w_i, \quad \sum_{i=1}^m x_{-N-k_i} w_i$$

is nonzero.

Proof. Suppose that

$$\sum_{i=1}^m x_{N-k_i} w_i = \sum_{i=1}^m x_{-N-k_i} w_i = 0$$

for all sufficiently large N . We proceed by the induction on m . First assume that $m = 1$, that is $x_N w = x_{-N} w = 0$ for all sufficiently large N and some nonzero $w \in V$. Then we immediately get a contradiction as for any such N we have

$$0 = [x_N, x_{-N}]w = Ncw = Naw \neq 0.$$

To explain the idea of the induction step consider the case $m = 2$ and assume that

$$x_N w_1 + x_{N-k_2} w_2 = x_{-N} w_1 + x_{-N-k_2} w_2 = 0$$

for all sufficiently large N . Introduce the following operators for all pairs $(r, s) \in \mathbb{Z}^2$:

$$\Omega_{N-r, N-s} := x_{N-r} x_{-N-s} - x_{N-s} x_{-N-r}.$$

Note that if $N-r \neq N+s$, then $\Omega_{N-r, N-s} = x_{-N-s} x_{N-r} - x_{N-s} x_{-N-r}$. Applying x_{-N} and x_N respectively to the equalities above we get

$$x_{-N} x_N w_1 + x_{-N} x_{N-k_2} w_2 = x_N x_{-N} w_1 + x_N x_{-N-k_2} w_2 = 0.$$

Subtracting these equalities we obtain

$$(x_{-N} x_N - x_N x_{-N}) w_1 + (x_{-N} x_{N-k_2} - x_N x_{-N-k_2}) w_2 = -Naw_1 + \Omega_{N-k_2, N}(w_2) = 0,$$

implying $\Omega_{N-k_2, N}(w_2) = Naw_1$. Let us show that $\Omega_{N-k_2, N}(w_1) = 0$. Indeed,

$$\begin{aligned} \Omega_{N-k_2, N}(w_1) &= (x_{-N} x_{N-k_2} - x_N x_{-N-k_2}) w_1 = x_{N-k_2} (x_{-N} w_1) - x_{-N-k_2} (x_N w_1) = \\ &= x_{N-k_2} (-x_{-N-k_2} w_2) - x_{-N-k_2} (-x_{N-k_2} w_2) = 0, \end{aligned}$$

since $k_2 > 0$. Choose $N_1 \gg N_2 \gg 0$. Then we have

$$0 = \Omega_{N_1-k_2, N_1}(x_{N_2} w_1 + x_{N_2-k_2} w_2) = x_{N_2} \Omega_{N_1-k_2, N_1}(w_1) + x_{N_2-k_2} \Omega_{N_1-k_2, N_1}(w_2) = x_{N_2-k_2} (N_1 a w_1)$$

and $x_{N_2-k_2}w_1 = 0$ for any sufficiently large N_2 . Similarly,

$$0 = \Omega_{N_1-k_2, N_1}(x_{-N_2}w_1 + x_{-N_2-k_2}w_2) = x_{-N_2}\Omega_{N_1-k_2, N_1}(w_1) + x_{-N_2-k_2}\Omega_{N_1-k_2, N_1}(w_2) = x_{-N_2-k_2}(N_1aw_1)$$

and $x_{-N_2-k_2}w_1 = 0$ for any sufficiently large N_2 . We conclude that $x_Nw_1 = x_{-N}w_1 = 0$ for all sufficiently large N , which a contradiction as $a \neq 0$.

Suppose that the statement holds for any set of $< m$ elements. Applying x_{-N} and x_N respectively to the equalities above we get

$$x_{-N} \sum_{i=1}^m x_{N-k_i}w_i = x_N \sum_{i=1}^m x_{-N-k_i}w_i = 0.$$

Subtracting these equalities we obtain

$$(x_{-N}x_N - x_Nx_{-N})w_1 + \sum_{i=2}^m \Omega_{N-k_i, N}w_i = -Naw_1 + \sum_{i=2}^m \Omega_{N-k_i, N}(w_i) = 0$$

and

$$\sum_{i=2}^m \Omega_{N-k_i, N}(w_i) = Naw_1.$$

We also have for each $i = 2, \dots, m$

$$\begin{aligned} \Omega_{N-k_i, N}(w_1) &= (x_{-N}x_{N-k_i} - x_Nx_{-N-k_i})w_1 = x_{N-k_i}(x_{-N}w_1) - x_{-N-k_i}(x_Nw_1) = \\ &= x_{N-k_i}\left(-\sum_{j=2}^m x_{-N-k_j}w_j\right) - x_{-N-k_i}\left(-\sum_{j=2}^m x_{N-k_j}w_j\right) = \sum_{j=2, j \neq i}^m (-x_{N-k_i}x_{-N-k_j} + x_{-N-k_i}x_{N-k_j})w_j \end{aligned}$$

and

$$(1) \quad \Omega_{N-k_i, N}(w_1) = \sum_{j=2, j \neq i}^m \Omega_{N-k_j, N-k_i}(w_j), \quad i = 2, \dots, m.$$

Note that

$$\Omega_{N-k_j, N-k_i} + \Omega_{N-k_i, N-k_j} = 0$$

for all i, j and $[\Omega_{N-k_j, N-k_i}, \Omega_{N-k_s, N-k_r}] = 0$ for all $i, j, r, s = 2, \dots, m$. We can view the equalities (1) as a linear system of $m - 1$ equations with operator commuting coefficients $\Omega_{N-k_j, N-k_i}$. Note that $m - 2$ equations contain w_m .

Choose arbitrary sufficiently large $t_1 \gg t_2 \gg \dots \gg t_{m-2} \gg N$ and multiply the j -th equation by $x_{t_j-k_m}$ for each $j = 1, \dots, m - 2$. Recall that we have

$$\sum_{i=1}^m x_{t_j-k_i}w_i = \sum_{i=1}^m x_{-t_j-k_i}w_i = 0, \quad j = 1, \dots, m - 2,$$

by our assumption. Then we can replace $x_{t_j-k_m}w_m$ in the j -th equation by $-\sum_{i=1}^{m-1} x_{t_j-k_i}w_i$. We obtain $m - 1$ equations with parameters t_j , $j = 1, \dots, m - 2$. Using multiplication and addition with will eliminate w_2, \dots, w_{m-1} and obtain the equality $\Omega(w_1) = 0$, where

Ω is a certain polynomial operator spanned by the products of $x_{t_j-l_j}$ for some $l_j \geq 0$, $j = 1, \dots, m-2$, and $\Omega_{N-k_r, N-k_s}$, $r, s = 1, \dots, m$. Let $K \gg t_1$. Then

$$0 = \Omega \sum_{i=1}^m x_{\pm K-k_i} w_i = \sum_{i=1}^m x_{\pm K-k_i} \Omega(w_i) = \sum_{i=2}^m x_{\pm K-k_i} \Omega(w_i).$$

If $\Omega(w_i) \neq 0$ for at least one index $i = 2, \dots, m$, then we can apply the induction hypothesis and obtain the contradiction. It remains to consider the case when $\Omega(w_i) = 0$ for all i .

Choose the largest index n for which x_n appears in Ω and consider the equality

$$\sum_{i=1}^m x_{-t_1-k_i} w_i = 0.$$

Applying Ω we obtain

$$0 = [\Omega, \sum_{i=1}^m x_{-t_1-k_i}] w_i = [\Omega, x_{-t_1}] w_1,$$

implying that $\Omega^1 w_1 = 0$ where Ω^1 is a polynomial operator with degree of x_{t_1} reduced by 1 (as $[x_{t_1}, x_{-t_1}] = at_1$). Again we can assume that $\Omega^1 w_i = 0$ for all i (otherwise apply the induction hypothesis). Continuing this way we eliminate all x_{t_j} , $j = 1, \dots, m-2$ and obtain a polynomial operator Ω^2 which is spanned by the products of $\Omega_{N-k_r, N-k_s}$, $r, s = 1, \dots, m$, and $\Omega^2 w_1 = 0$. Again we can assume that $\Omega^2 w_i = 0$ for all i . Note that each monomial in Ω^2 contains exactly one factor of type $\Omega_{N-k_i, N}$, $i > 2$ and these factors do not commute. All other factors that appear in Ω^2 commute. Hence, each monomial in Ω^2 contains at most one factor x_N and $\Omega^2 = f x_N + g$, where f and g do not contain x_N and contain only factors x_k with $k < N$. Consider the equality

$$\sum_{i=1}^m x_{-N-k_i} w_i = 0$$

and apply Ω . Then we get

$$0 = [\Omega^2, \sum_{i=1}^m x_{-N-k_i}] w_i = [\Omega^2, x_{-N}] w_1 = f[x_N, x_{-N}] w_1 = N a f w_1.$$

We conclude that $f w_1 = 0$ and hence $f w_i = 0$ for all i . Choose the factor in f with the largest index and repeat the argument. Continuing we will obtain $w_1 = 0$ which is a contradiction. This completes the proof. \square

Now we proceed with the proof of Theorem 3.3. Let V be an irreducible weight $(G+H)$ -module. Consider the $\widehat{\mathfrak{g}}$ -module $M_a(V)$ and assume $a \neq 0$. Let $v \in M_a(V)$ be a nonzero element. We will show that v generates $M_a(V)$. We can assume that v is a weight element and

$$v = \sum_i u_i w_i,$$

where $u_i \in U(\widehat{\mathfrak{u}})$ are linearly independent homogeneous elements and $w_i \in V_{\mu_i}$ for all i . Let $R = \text{ht}_{\widehat{\mathfrak{u}}}(v)$. We will proceed by the induction on R . Suppose first that $R = 1$. Then $u_i = x_{-\beta+r_i\delta} \in \widehat{\mathfrak{g}}_{-\beta+r_i\delta}$ for some simple root β of $\widehat{\mathfrak{g}}$. For an integer N consider a nonzero $x_{\beta+N\delta} \in \widehat{\mathfrak{g}}_{\beta+N\delta}$. Then

$$x_{\beta+N\delta}v = x_{\beta+N\delta} \sum_i u_i w_i = \sum_i [x_{\beta+N\delta}, u_i] w_i = \sum_i x_{(N+r_i)\delta} w_i \in V.$$

By Lemma 3.4, $x_{\beta+N\delta}v \neq 0$ if $|N|$ is sufficiently large. Since V is irreducible then v generates $M_a(V)$.

Assume now that $R > 1$ and assume that any nonzero weight element of height $< R$ generates $M_a(V)$. We follow the proof of [BBFK13, Lemma 5.3]. Suppose that $u_i = x_{-\beta_{i1}+r_{i1}\delta} \cdots x_{-\beta_{is_i}+r_{is_i}\delta}^{p_{is_i}}$ for some simple roots $\beta_{i1}, \dots, \beta_{is_i}$ and some integers r_{i1}, \dots, r_{is_i} . We can assume without loss of generality that the summands are indexed in such a way that $r_{1s_1} \leq r_{2s_2} \leq \dots$ and if $r_{ij} = r_{1s_1}$ then p_{1s_1} is minimal. Applying $x_{\beta_{11}+N\delta}$ with sufficiently large $|N|$ we get $x_{\beta_{11}+N\delta}v \neq 0$ again by Lemma 3.4. Since $\text{ht}_{\widehat{\mathfrak{u}}}(x_{\beta_{11}+N\delta}v) = R - 1$, the element $x_{\beta_{11}+N\delta}v$ generates $M_a(V)$ by induction. If u_i contains a factor $x_{-\beta+r\delta}$ with non-simple positive root β for some i , then choose a simple root α such that $\alpha - \beta$ is a root and apply $x_{\alpha+N\delta}$ with sufficiently large $|N|$. Then $x_{\alpha+N\delta}v \neq 0$ and $\text{ht}_{\widehat{\mathfrak{u}}}(x_{\alpha+N\delta}v) = R - 1$. Again the induction completes the proof.

Corollary 3.5. *Let $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$ be a parabolic subalgebra of $\widehat{\mathfrak{g}}$ of type II and $\widehat{\mathfrak{l}} = G' + H$, where G' is a subalgebra of G . Let $a \in \mathbb{C} \setminus \{0\}$ and let V be an irreducible weight $\widehat{\mathfrak{l}}$ -module of central charge a . Then $M_{a, \widehat{\mathfrak{p}}}(V)$ is irreducible, and hence the restriction of the functor $\mathbb{I}_{a, \widehat{\mathfrak{p}}}$ to the category of weight $\widehat{\mathfrak{l}}$ -modules with central charge a preserves the irreducibility.*

Proof. Follows from Theorem 3.3 and from the fact that a highest weight module of an infinite dimensional Heisenberg subalgebra is irreducible if the central charge is nonzero. \square

Theorem 3.3 and is a key step to approach the irreducibility of generalized Imaginary Verma modules, which we are going to do in the following subsection.

3.3. Irreducibility of generalized Imaginary Verma modules. Recall that $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$ is a parabolic subalgebra of $\widehat{\mathfrak{g}}$ of type II, $\widehat{\mathfrak{l}} = \widehat{\mathfrak{l}}^0 + G(\widehat{\mathfrak{l}})^\perp$, where $\widehat{\mathfrak{l}}^0$ is the Lie subalgebra generated by H and by all real root subspaces of $\widehat{\mathfrak{l}}$ and $G(\widehat{\mathfrak{l}})^\perp$ is the orthogonal complement to the imaginary part of $\widehat{\mathfrak{l}}^0$ with respect to the Killing form.

Tensor $\widehat{\mathfrak{l}}$ -modules have the form $M \otimes S$, where M is a weight (with respect to H) $\widehat{\mathfrak{l}}^0$ -module with central charge a and S is a $G(\widehat{\mathfrak{l}})^\perp \oplus \mathbb{C}d$ -module with the same central charge and diagonalizable action of d . Modules over $G(\widehat{\mathfrak{l}})^\perp \oplus \mathbb{C}d$ correspond to \mathbb{Z} -graded modules (with grading compatible with the \mathbb{Z} -grading of $G(\widehat{\mathfrak{l}})^\perp$) with a fixed scalar action of d on a chosen nonzero component. We will say that a $G(\widehat{\mathfrak{l}})^\perp \oplus \mathbb{C}d$ -module S is strongly irreducible if it is irreducible as a (non-graded) $G(\widehat{\mathfrak{l}})^\perp$ -module. Verma modules with nonzero central charge and irreducible diagonal modules [BBFK13] are examples of such modules.

Though we do not have a classification of all irreducible tensor $\widehat{\mathfrak{l}}$ -modules, the following easy statement shows how to construct families of such modules.

Proposition 3.6. *Let M be an irreducible weight $\widehat{\mathfrak{l}^0}$ -module with central charge a , S a strongly irreducible \mathbb{Z} -graded $G(\widehat{\mathfrak{l}})^\perp \oplus \mathbb{C}d$ -module with the same central charge. Then $M \otimes S$ is an irreducible weight $\widehat{\mathfrak{l}}$ -module with diagonalizable tensor product action of d .*

We have the following generalization of Theorem 3.1.

Theorem 3.7. *Let $a \in \mathbb{C} \setminus \{0\}$ and let $V = M \otimes S$ be an irreducible weight tensor $\widehat{\mathfrak{l}}$ -module, where M is a weight $\widehat{\mathfrak{l}^0}$ -module and S is a $G(\widehat{\mathfrak{l}})^\perp \oplus \mathbb{C}d$ -module with diagonalizable action of d . Consider V as a $\widehat{\mathfrak{p}}$ -module with trivial action of the radical $\widehat{\mathfrak{u}}$. Then the generalized Imaginary Verma $\widehat{\mathfrak{g}}$ -module $\mathbb{I}_{a, \widehat{\mathfrak{p}}}(V) = M_{a, \widehat{\mathfrak{p}}}(V)$ is irreducible.*

Proof. Consider an arbitrary nonzero element of $v \in M_{a, \widehat{\mathfrak{p}}}(V)$. We can assume that v is a weight element. Write v as the following finite sum

$$v = \sum_i d_i(v_i \otimes w_i),$$

where $d_i \in U(\widehat{\mathfrak{u}})$, $v_i \in M$, $w_i \in S$, and we assume that $v_i \otimes w_i$ are linearly independent. Since v is a weight vector then each d_i is a homogeneous element of $U(\widehat{\mathfrak{u}})$. Let $R = \text{ht}_{\widehat{\mathfrak{u}}}(v)$. We proceed by the induction on R . Suppose that $R = 1$. As in the proof of Theorem 3.3 this is the most important case. Then for all i we have $d_i = x_{-\beta + \alpha_i + r_i \delta} \in \widehat{\mathfrak{g}}_{-\beta + \alpha_i + r_i \delta}$ for some simple root β of $\widehat{\mathfrak{g}}$ and some $\alpha_i \in Q(\mathfrak{l})$, where $Q(\mathfrak{l})$ is the root lattice of the underlined finite dimensional subalgebra \mathfrak{l} of $\widehat{\mathfrak{l}}$. Write α_i as a linear combination of simple roots of \mathfrak{l} with integer coefficients. The \mathfrak{l} -height $ht_{\mathfrak{l}}(d_i)$ of d_i is defined as a sum of all coefficients of such linear combination (note that $ht_{\mathfrak{l}}(d_i) \leq 0$ for all i). Choose i such that d_i has the least \mathfrak{l} -height which equals t . There can be several d_j 's with the same \mathfrak{l} -height t . Among all of them choose one with the least r_i . Without loss of generality we may assume that $i = 1$ satisfies these conditions (not necessarily unique). We claim that there exists an integer N with sufficiently large $|N|$ and a nonzero $x_{\beta - \alpha_1 + (N - r_1)\delta} \in \widehat{\mathfrak{g}}_{\beta - \alpha_1 + (N - r_1)\delta}$ (we assume that $\beta - \alpha_1 + (N - r_1)\delta$ is a root and that $|N| \gg |r_1|$) such that $x_{\beta - \alpha_1 + (N - r_1)\delta} v \neq 0$. For simplicity write $x_{\beta - \alpha_1 + (N - r_1)\delta}$ by x_N . Suppose this is not the case and we have

$$x_N v = x_N \sum_i d_i(v_i \otimes w_i) = \sum_i [x_N, d_i](v_i \otimes w_i) = 0$$

for all such N .

Set $y_N^i = [x_N, d_i]$. Note that $y_N^1 = [x_N, d_1] \in \widehat{\mathfrak{g}}_{N\delta}$ is nonzero and recall that $\widehat{\mathfrak{l}} = \widehat{\mathfrak{l}^0} + G(\widehat{\mathfrak{l}})^\perp$ and $[z, G(\widehat{\mathfrak{l}})^\perp] = 0$ for all root elements $z \in \widehat{\mathfrak{l}^0}$. Write each y_N^i as follows: $y_N^i = D_N^i + Z_N^i$, where $D_N^i \in \widehat{\mathfrak{l}^0}$ and $Z_N^i \in G(\widehat{\mathfrak{l}})^\perp$. Clearly, $[D_N^i, Z_N^i] = 0$ for all i and $Z_N^1 \neq 0$ (it is obvious for any N if $\widehat{\mathfrak{g}}$ is an untwisted affine Lie algebra as $d_1 \in U(\widehat{\mathfrak{u}})$. In the case of twisted affine Lie algebras N can be chosen to guarantee this condition).

Then

$$x_N v = v_1 \otimes Z_N^1 w_1 + D_N^1 v_1 \otimes w_1 + \sum_{i>1} (v_i \otimes Z_N^i w_i + D_N^i v_i \otimes w_i) = 0.$$

Because of the \mathbb{Z} -grading on S , the equality above implies

$$v_1 \otimes Z_N^1 w_1 + \sum_{i>1} v_i \otimes Z_N^i w_i = 0.$$

If v_i 's are linearly independent then we get a contradiction as either $Z_N^1 w_1 \neq 0$ or $Z_{-N}^1 w_1 \neq 0$. If they are linearly dependent then choose a maximal linearly independent subset indexed for simplicity as follows v_1, v_2, \dots, v_k , and rewrite the sum above as

$$v_1 \otimes \left(\sum_{j \in Y_1} a_j^1 Z_N^j w_j \right) + \sum_{i=2}^k v_i \otimes \left(\sum_{j \in Y_i} a_j^i Z_N^j w_j \right) = 0,$$

where Y_1, \dots, Y_k are certain subsets of the set of indices and a_j^i are nonzero scalars. Using now the linear independence of v_1, \dots, v_k we get that

$$\sum_{j \in Y_i} a_j^i Z_N^j w_j = 0$$

for all $i = 1, \dots, k$. Consider for example $i = 1$. Rewrite the sum above for $i = 1$ in the following form:

$$\sum_{j \in \tilde{Y}_1} a_j^1 Z_N^j \tilde{w}_j = 0,$$

where all \tilde{w}_j are weight elements of S with different weights and \tilde{Y}_1 is a subset of Y_1 . Then we get a contradiction with Lemma 3.4. Hence we can assume that for all $j \in Y_1$, the elements w_j have the same weight in S and $Z_N^j \in \hat{\mathfrak{g}}_{N\delta}$. Moreover, as $\alpha_r = \alpha_s$ for all $r, s \in Y_1$, then Z_N^r and Z_N^s are the same up to a scalar. Suppose that $w_j, j \in Y_1$ are linearly independent. Then we get

$$0 = Z_N \sum_{j \in Y_1} \hat{w}_j = Z_{-N} \sum_{j \in Y_1} \hat{w}_j,$$

for some nonzero $Z_{\pm N} \in \hat{\mathfrak{g}}_{\pm N\delta}$ and some linearly independent elements $\hat{w}_j, j \in Y_1$. As the central charge nonzero we have $\sum_{j \in Y_1} \hat{w}_j = 0$, which is a contradiction. Finally, if $w_j, j \in Y_1$ are linearly dependent then $v_1 \otimes w_j$ are linearly dependent which contradicts our original assumption. This completes the proof for $R = 1$.

Suppose now that $R > 1$. Since the proof of the induction step in [BBFK13, Lemma 5.3] does not depend on the module V , the same argument applies in our case. Hence there exists $x \in \hat{\mathfrak{u}}$ such that $xv \neq 0$. Since $\text{ht}_{\hat{\mathfrak{u}}}(xv) = R - 1$, the proof is completed by induction. \square

Denote by $\mathcal{T}_a(\widehat{\mathfrak{l}})$ the category of tensor $\widehat{\mathfrak{l}}$ -modules with central charge a . The restriction of the imaginary induction functor $\mathbb{I}_{a,\widehat{\mathfrak{p}}}$ on $\mathcal{T}_a(\widehat{\mathfrak{l}})$ defines the functor $\mathbb{I}_{a,\widehat{\mathfrak{p}}}^{\mathcal{T}}$ from the category of tensor $\widehat{\mathfrak{l}}$ -modules with central charge a to the category of weight $\widehat{\mathfrak{g}}$ -modules.

As a consequence of Theorem 3.7 we immediately obtain the following corollary which implies Theorem 1.2.

Corollary 3.8. *Let $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$ be a parabolic subalgebra of $\widehat{\mathfrak{g}}$ of type II, $V \in \mathcal{T}_a(\widehat{\mathfrak{l}})$ an irreducible $\widehat{\mathfrak{l}}$ -module and $a \in \mathbb{C} \setminus \{0\}$. Then the functor $\mathbb{I}_{a,\widehat{\mathfrak{p}}}^{\mathcal{T}}$ preserves the irreducibility.*

3.4. Localization of generalized Imaginary Verma modules. Let $\widehat{\mathfrak{g}}$ be an affine Kac-Moody algebra. For a real positive root $\alpha \in \Delta_+^{\text{re}}$ of $\widehat{\mathfrak{g}}$, we define the *twisting functor*

$$T_\alpha = T_{f_\alpha}: \mathcal{M}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{M}(\widehat{\mathfrak{g}}) \quad \text{and} \quad T_{-\alpha} = T_{e_\alpha}: \mathcal{M}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{M}(\widehat{\mathfrak{g}})$$

by

$$T_\alpha(M) = U(\widehat{\mathfrak{g}})_{(f_\alpha)}/U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}})} M, \quad T_{-\alpha}(M) = U(\widehat{\mathfrak{g}})_{(e_\alpha)}/U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}})} M$$

for $\alpha \in \Delta_+^{\text{re}}$ and $M \in \mathcal{M}(\widehat{\mathfrak{g}})$.

Consider a parabolic subalgebra $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$ of $\widehat{\mathfrak{g}}$ of type II. Denote by $\Delta^{\text{re}}(\widehat{\mathfrak{l}}) \subset \Delta^{\text{re}}$ the set of real roots of $\widehat{\mathfrak{l}}$. The link between the twisting functor T_α for $\alpha \in \Delta^{\text{re}}(\widehat{\mathfrak{l}})$ and the Imaginary induction functor $\mathbb{I}_{a,\widehat{\mathfrak{p}}}$ is given in the following theorem.

Theorem 3.9. *Let $\alpha \in \Delta^{\text{re}}(\widehat{\mathfrak{l}}) \subset \Delta^{\text{re}}$. Then there exists a natural isomorphism*

$$\eta_\alpha: T_\alpha \circ \mathbb{I}_{a,\widehat{\mathfrak{p}}} \rightarrow \mathbb{I}_{a,\widehat{\mathfrak{p}}} \circ T_\alpha^{\widehat{\mathfrak{l}}}$$

of functors, where $T_\alpha^{\widehat{\mathfrak{l}}}: \mathcal{M}(\widehat{\mathfrak{l}}) \rightarrow \mathcal{M}(\widehat{\mathfrak{l}})$ is the twisting functor for $\widehat{\mathfrak{l}}$ assigned to α .

Proof. We will follow the proof of [FK21, Theorem 4.16] as the argument works for any parabolic subalgebra of $\widehat{\mathfrak{g}}$. Let $\alpha \in \Delta^{\text{re}}(\widehat{\mathfrak{l}})$ be a positive root and $f_\alpha \in \widehat{\mathfrak{g}}_\alpha$ a nonzero element. Case of negative roots can be treated similarly. By the Poincaré–Birkhoff–Witt theorem we have the following isomorphism

$$U(\widehat{\mathfrak{g}})_{(f_\alpha)} \simeq U(\widehat{\mathfrak{u}})_{(f_\alpha)} \otimes_{\mathbb{C}} U(\widehat{\mathfrak{p}})$$

of $U(\widehat{\mathfrak{u}})$ -modules. Then for a $\widehat{\mathfrak{g}}$ -module V we have

$$U(\widehat{\mathfrak{g}})_{(f_\alpha)} \otimes_{U(\widehat{\mathfrak{g}})} \mathbb{I}_{a,\widehat{\mathfrak{p}}}(V) \simeq U(\widehat{\mathfrak{g}})_{(f_\alpha)} \otimes_{U(\widehat{\mathfrak{g}})} U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}})} V \simeq U(\widehat{\mathfrak{u}})_{(f_\alpha)} \otimes_{\mathbb{C}} V.$$

Hence we get an isomorphism

$$(T_\alpha \circ \mathbb{I}_{a,\widehat{\mathfrak{p}}})(V) \simeq (U(\widehat{\mathfrak{u}})_{(f_\alpha)}/U(\widehat{\mathfrak{u}})) \otimes_{\mathbb{C}} V$$

of $U(\widehat{\mathfrak{u}})$ -modules. But we also have an isomorphism of $U(\widehat{\mathfrak{u}})$ -modules

$$(\mathbb{I}_{a,\widehat{\mathfrak{p}}} \circ T_\alpha^{\widehat{\mathfrak{l}}})(V) \simeq U(\widehat{\mathfrak{u}}) \otimes_{\mathbb{C}} T_\alpha^{\widehat{\mathfrak{l}}}(V).$$

The correspondence

$$f_\alpha^{-n}u \otimes v \mapsto \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} \text{ad}(f_\alpha)^k(u) \otimes f_\alpha^{-n-k}v$$

defines the isomorphism of $U(\widehat{\mathfrak{u}})$ -modules and of $U(\widehat{\mathfrak{g}})$ -modules with the inverse

$$u \otimes f_\alpha^{-n}v \mapsto \sum_{k=0}^{\infty} \binom{n+k-1}{k} f_\alpha^{-n-k} \text{ad}(f_\alpha)^k(u) \otimes v$$

for all $u \in U(\widehat{\mathfrak{u}})$, $v \in V$ and any $n \in \mathbb{N}$. \square

4. REALIZATION OF INDUCED MODULES

4.1. Generalized Imaginary Wakimoto modules. Let \mathfrak{g} be a simple finite dimensional Lie algebra with Cartan decomposition $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$. From now on we assume that $\widehat{\mathfrak{g}}$ is untwisted affine Kac-Moody algebra, that is

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is the central element and d is a degree derivation: $[d, x \otimes t^n] = n(x \otimes t^n)$. It will be convenient to replace the ring of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ by the field $\mathbb{C}((t))$.

The natural Borel subalgebra $\widehat{\mathfrak{b}}_{\text{nat}}$ is defined by $\widehat{\mathfrak{b}}_{\text{nat}} = H \oplus \widehat{\mathfrak{n}}_{\text{nat}}$, where

$$\begin{aligned} \widehat{\mathfrak{n}}_{\text{nat}} &= (\mathfrak{n} \otimes_{\mathbb{C}} \mathbb{C}((t))) \oplus (\mathfrak{h} \otimes_{\mathbb{C}} t\mathbb{C}[[t]]), \\ H &= (\mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}1) \oplus \mathbb{C}c \oplus \mathbb{C}d, \\ \widehat{\mathfrak{n}}_{\text{nat}} &= (\overline{\mathfrak{n}} \otimes_{\mathbb{C}} \mathbb{C}((t))) \oplus (\mathfrak{h} \otimes_{\mathbb{C}} t^{-1}\mathbb{C}[t^{-1}]). \end{aligned}$$

Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of \mathfrak{g} , where \mathfrak{l} is the Levi subalgebra and \mathfrak{u} is the radical of \mathfrak{p} . Let $\overline{\mathfrak{u}}$ be the opposite radical of \mathfrak{p} .

Then the *natural parabolic subalgebra* of $\widehat{\mathfrak{g}}$ associated with \mathfrak{p} is defined as

$$\widehat{\mathfrak{p}}_{\text{nat}} = \widehat{\mathfrak{l}}_{\text{nat}} \oplus \widehat{\mathfrak{u}}_{\text{nat}},$$

where $\widehat{\mathfrak{l}}_{\text{nat}} = (\mathfrak{l} \otimes_{\mathbb{C}} \mathbb{C}((t))) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\widehat{\mathfrak{u}}_{\text{nat}} = \mathfrak{u} \otimes_{\mathbb{C}} \mathbb{C}((t))$.

The opposite radical of $\widehat{\mathfrak{p}}_{\text{nat}}$ is $\widehat{\mathfrak{u}}_{\text{nat}} = \overline{\mathfrak{u}} \otimes_{\mathbb{C}} \mathbb{C}((t))$ and we have the decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{u}}_{\text{nat}} \oplus \widehat{\mathfrak{l}}_{\text{nat}} \oplus \widehat{\mathfrak{u}}_{\text{nat}}.$$

Note that $\widehat{\mathfrak{p}}_{\text{nat}}$ is a parabolic subalgebra of type II since $\widehat{\mathfrak{l}}_{\text{nat}}$ contains the Heisenberg subalgebra G . If $\mathfrak{p} = \mathfrak{b}$, that is, $\mathfrak{l} = \mathfrak{h}$ and $\mathfrak{u} = \mathfrak{n}$, we get $\widehat{\mathfrak{p}}_{\text{nat}} = \widehat{\mathfrak{b}}_{\text{nat}} \oplus (\mathfrak{h} \otimes_{\mathbb{C}} t^{-1}\mathbb{C}[t^{-1}])$.

On the other hand, if $\mathfrak{p} = \mathfrak{g}$ we have that $\widehat{\mathfrak{p}}_{\text{nat}} = \widehat{\mathfrak{g}}$.

Let $\mathcal{K} = \mathbb{C}((t))$ and let $\Omega_{\mathcal{K}} = \mathbb{C}((t)) dt$ be the module of Kähler differentials. For a finite-dimensional complex vector space U we define the infinite-dimensional complex vector spaces $\mathcal{K}(U) = U \otimes_{\mathbb{C}} \mathcal{K}$ and $\Omega_{\mathcal{K}}(U^*) = U^* \otimes_{\mathbb{C}} \Omega_{\mathcal{K}}$. Denote by $Pol \Omega_{\mathcal{K}}(U^*)$ the algebra of polynomial functions on $\Omega_{\mathcal{K}}(U^*)$.

Denote by $\dot{\Delta}(\mathbf{u})$ the set of roots α of \mathfrak{g} such that $\mathfrak{g}_\alpha \in \mathbf{u}$. Let $\{f_\alpha \mid \alpha \in \dot{\Delta}(\mathbf{u})\}$ be a basis of $\bar{\mathbf{u}}$ and let $\{x_\alpha \mid \alpha \in \dot{\Delta}(\mathbf{u})\}$ be the linear coordinate functions on $\bar{\mathbf{u}}$ with respect to given basis of $\bar{\mathbf{u}}$, that is, $x_\alpha(f_\beta) = \delta_{\alpha,\beta}$. Then the set $\{f_{\alpha,n} = f_\alpha \otimes t^n \mid \alpha \in \dot{\Delta}(\mathbf{u}), n \in \mathbb{Z}\}$ forms a topological basis of $\mathcal{K}(\bar{\mathbf{u}}) = \widehat{\mathbf{u}}_{\text{nat}}$, and the set $\{x_{\alpha,-n} = x_\alpha \otimes t^{n-1} dt \mid \alpha \in \dot{\Delta}(\mathbf{u}), n \in \mathbb{Z}\}$ forms a dual topological basis of $\Omega_{\mathcal{K}}(\bar{\mathbf{u}}^*) \simeq (\widehat{\mathbf{u}}_{\text{nat}})^*$.

Consider the infinite rank Weyl algebra $\mathcal{A}_{\mathcal{K}(\bar{\mathbf{u}})}$ topologically generated by

$$\{x_{\alpha,n}, \partial_{x_{\alpha,n}} \mid \alpha \in \dot{\Delta}(\mathbf{u}), n \in \mathbb{Z}\},$$

such that $a_{\alpha,n} = \partial_{x_{\alpha,n}}$ and $a_{\alpha,n}^* = x_{\alpha,-n}$, for $\alpha \in \dot{\Delta}(\mathbf{u}), n \in \mathbb{Z}$.

For $a \in \mathfrak{g}$ we define the formal distribution $a(z) \in \widehat{\mathfrak{g}}[[z^{\pm 1}]]$ by

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

where $a_n = a \otimes t^n$ for $n \in \mathbb{Z}$. Introduce the formal distributions $a_\alpha(z), a_\alpha^*(z) \in \mathcal{A}_{\mathcal{K}(\bar{\mathbf{u}})}[[z^{\pm 1}]]$ by

$$a_\alpha(z) = \sum_{n \in \mathbb{Z}} a_{\alpha,n} z^{-n-1} \quad \text{and} \quad a_\alpha^*(z) = \sum_{n \in \mathbb{Z}} a_{\alpha,n}^* z^{-n}$$

Inspired by the Feigin-Frenkel homomorphism for vertex algebras the following homomorphism of associative algebras was established in [FKS19].

Let $\sigma : \widehat{\mathfrak{p}}_{\text{nat}} \rightarrow \mathfrak{gl}(V)$ be a $\widehat{\mathfrak{p}}_{\text{nat}}$ -module such that $\sigma(c) = k \cdot \text{id}_V$ for $k \in \mathbb{C}$.

Theorem 4.1. [FKS19, Theorem 3.6] *There exists a homomorphism of associative algebras*

$$\pi_{\sigma(c), \mathfrak{g}} = \pi : U(\widehat{\mathfrak{g}}) \rightarrow \mathcal{A}_{\mathcal{K}(\bar{\mathbf{u}})} \widehat{\otimes}_{\mathbb{C}} U(\widehat{\mathfrak{h}})$$

given by $\pi(c) = \sigma(c)$ and

$$\begin{aligned} \pi(b(z)) &= - \sum_{\alpha \in \dot{\Delta}(\mathbf{u})} b_\alpha(z) \left[\frac{ad(u(z))e^{ad(u(z))}}{e^{ad(u(z))} - id} (e^{-ad(u(z))}(b))_{\bar{\mathbf{u}}} \right]_\alpha + (e^{-ad(u(z))}b(z))_{\mathfrak{p}} + \\ &\quad - \left(\frac{e^{ad(u(z))} - id}{ad(u(z))} \partial_z u(z), b \right) \sigma(c) \end{aligned}$$

for all $b \in \mathfrak{g}$, where

$$u(z) = \sum_{\alpha \in \dot{\Delta}(\mathbf{u})} b_\alpha^*(z) f_\alpha$$

and $[X]_\alpha$ is the α -th coordinate of $X \in \bar{\mathbf{u}}$ with respect to the basis $\{f_\alpha \mid \alpha \in \dot{\Delta}(\mathbf{u})\}$ of $\bar{\mathbf{u}}$.

Using Theorem 4.1 one gets explicit realization of generalized Imaginary Verma modules. Namely, we have the following result.

Theorem 4.2. [FKS19, Theorem 3.14] *Let V be a continuous $\widehat{\mathfrak{p}}_{\text{nat}}$ -module with central charge $a \in \mathbb{C}$ and with trivial action of $\widehat{\mathbf{u}}_{\text{nat}}$. Then the topological vector space $\text{Pol}\Omega_{\mathcal{K}}(\bar{\mathbf{u}}^*) \widehat{\otimes}_{\mathbb{C}}$*

V is a continuous $\widehat{\mathfrak{g}}$ -module with central charge a and with the module structure determined by Theorem 4.1.

Note that $\text{Pol } \Omega_{\mathcal{K}}(\overline{\mathbf{u}}^*) = \text{Pol} \left((\widehat{\mathbf{u}}_{\text{nat}})^* \right) \simeq S(\widehat{\mathbf{u}}_{\text{nat}}) \simeq \mathbb{C}[\partial_{x_{\alpha,n}} \mid \alpha \in \dot{\Delta}(\mathbf{u}), n \in \mathbb{Z}]$.

Set $W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(V) = \text{Pol } \Omega_{\mathcal{K}}(\overline{\mathbf{u}}^*) \otimes_{\mathbb{C}} V$. The $\widehat{\mathfrak{g}}$ -module $W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(V)$ will be called *generalized Imaginary Wakimoto module* due to its similarity with the generalized Imaginary Verma modules. In fact we have an isomorphism of $\widehat{\mathfrak{g}}$ -modules:

$$W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(V) \simeq M_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(V)$$

for any $\widehat{\mathfrak{l}}_{\text{nat}}$ -module V .

Families of generalized Imaginary Wakimoto modules were studied in [Cox05], [CF04], [CF06] under the name of *Intermediate Wakimoto modules*.

Applying Theorem 3.7 we immediately obtain

Corollary 4.3. *Let V be an irreducible tensor module over the Levi factor $\widehat{\mathfrak{l}}_{\text{nat}}$ with nonzero central charge. Then $W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(V)$ is irreducible.*

4.2. Imaginary Wakimoto functor. We define the following *Imaginary Wakimoto functor*

$$\mathbb{I}W_{a, \widehat{\mathfrak{p}}_{\text{nat}}} : \mathcal{M}(a, \widehat{\mathfrak{l}}_{\text{nat}}) \rightarrow \mathcal{M}(a, \widehat{\mathfrak{g}}),$$

which sends an $\widehat{\mathfrak{l}}_{\text{nat}}$ -module V to the $\widehat{\mathfrak{g}}$ -module $W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(V)$.

$$\begin{array}{ccc} & \mathcal{M}(a, \widehat{\mathfrak{l}}_{\text{nat}}) & \\ \text{Imaginary Wakimoto functor } \mathbb{I}W_{a, \widehat{\mathfrak{p}}_{\text{nat}}} & \left(\begin{array}{c} \downarrow \mathbb{I}_{a, \widehat{\mathfrak{p}}_{\text{nat}}} \\ \downarrow \end{array} \right) & \text{Imaginary induction functor} \\ & \mathcal{M}(a, \widehat{\mathfrak{g}}) & \end{array}$$

Consider the following particular cases.

Example 4.4. (1) Let $\widehat{\mathfrak{l}}_{\text{nat}} = G + H$ and $V \simeq L(a)$ a highest weight $G + H$ -module with highest weight λ , $\lambda(c) = a$. Then the Imaginary Verma module $M_{\widehat{\mathfrak{b}}_{\text{nat}}}(\lambda) = \mathbb{I}_{a, \widehat{\mathfrak{b}}_{\text{nat}}}(\mathbb{C})$ is isomorphic to $W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(L(a)) = \mathbb{I}W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(L(a))$.

(2) Similarly, for any Levi subalgebra $\widehat{\mathfrak{l}}_{\text{nat}}$ and a Verma $\widehat{\mathfrak{l}}_{\text{nat}}$ -module $M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)$ we have the isomorphisms

$$M_{\widehat{\mathfrak{p}}_{\text{nat}}}(M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)) = \mathbb{I}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)) \simeq \mathbb{I}W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)),$$

where $a = \lambda(c)$.

- (3) If in the previous item we take the Wakimoto module $\widehat{\mathfrak{l}}_{\text{nat}}$ -module $W_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)$ with highest weight λ then $\mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(W_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda))$ is an Intermediate Wakimoto module for the parabolic subalgebra $\widehat{\mathfrak{p}}_{\text{nat}}$. For a generic λ (that is, when $M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)$ is irreducible) we have an isomorphism $M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda) \simeq W_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)$ and hence

$$\mathbb{I}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)) \simeq \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(W_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)).$$

Moreover, if $\lambda(c) \neq 0$ then $\mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(W_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda))$ is irreducible by Theorem 3.7.

$$\begin{array}{ccc}
 & \mathbb{I}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)) & \text{Generalized Imaginary Verma module} \\
 \nearrow^{\mathbb{I}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}} & \vdots & \\
 M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda) & & \simeq \text{generically (for a generic } \lambda) \\
 \searrow_{\mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}} & & \\
 & \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(W_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)) & \text{Generalized Imaginary Wakimoto module}
 \end{array}$$

We also get from Theorem 3.7

Corollary 4.5. *Let $\widehat{\mathfrak{l}}_{\text{nat}} = \widehat{\mathfrak{l}}_{\text{nat}}^0 + G(\widehat{\mathfrak{l}}_{\text{nat}})^\perp$, where $[x, G(\widehat{\mathfrak{l}}_{\text{nat}})^\perp] = 0$, for any root element $x \in \widehat{\mathfrak{l}}_{\text{nat}}^0$. If N is an irreducible $\widehat{\mathfrak{l}}_{\text{nat}}^0$ -module and S is a strongly irreducible $G(\widehat{\mathfrak{l}}_{\text{nat}})^\perp \oplus \mathbb{C}d$ -module, both with the same nonzero central charge a , then*

$$\mathbb{I}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(N \otimes S) \simeq \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(N \otimes S)$$

is irreducible.

For any root $\alpha \in \dot{\Delta}_+(\mathfrak{l}) \subset \Delta_+^{\text{re}}(\widehat{\mathfrak{l}}_{\text{nat}}) \subset \Delta_+^{\text{re}}$ consider the twisting functor T_α . Then Theorem 3.9 and [FK21, Theorem 4.16] immediately imply

Corollary 4.6. $T_\alpha \circ \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}} = \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}} \circ T_\alpha^{\widehat{\mathfrak{l}}_{\text{nat}}}$. In particular, we have

$$T_\alpha(\mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda))) \simeq \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(T_\alpha^{\widehat{\mathfrak{l}}_{\text{nat}}}(M_{\widehat{\mathfrak{p}}}^{\mathfrak{l}}(M_{\mathfrak{l}}(\lambda)))) \simeq \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(M_{\widehat{\mathfrak{p}}}^{\mathfrak{l}}(T_\alpha^{\mathfrak{l}}(M_{\mathfrak{l}}(\lambda))))$$

and

$$\begin{aligned}
 T_\alpha(\mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(N \otimes S)) &\simeq \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(T_\alpha^{\widehat{\mathfrak{l}}_{\text{nat}}}(N \otimes S)) \simeq \\
 &\simeq \mathbb{I}\mathbb{W}_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(T_\alpha^{\widehat{\mathfrak{l}}_{\text{nat}}}(N) \otimes S),
 \end{aligned}$$

where $M_{\widehat{\mathfrak{l}}_{\text{nat}}}(\lambda)$ is the Verma $\widehat{\mathfrak{l}}_{\text{nat}}$ -module with highest weight λ , $M_{\mathfrak{l}}(\lambda)$ is the Verma \mathfrak{l} -module with highest weight λ , $\widehat{\mathfrak{p}}$ is the standard parabolic subalgebra of $\widehat{\mathfrak{l}}_{\text{nat}}$ with finite-dimensional Levi factor $\mathfrak{l} + H$, $M_{\widehat{\mathfrak{p}}}^{\mathfrak{l}}$ is the induction functor $\text{Ind}_{\widehat{\mathfrak{p}}}^{\widehat{\mathfrak{l}}_{\text{nat}}}$ and the modules N and S are as in Corollary 4.5.

Note that generically the modules in Corollary 4.6 are irreducible if $a \neq 0$. Hence Corollaries 4.5 and 4.6 provide technique to construct new irreducible modules together with explicit realizations by differential operators (see Theorem 4.2).

$$\begin{array}{ccc}
 M_{\widehat{\mathfrak{t}}_{\text{nat}}}(\lambda) & \simeq & M_{\widehat{\mathfrak{p}}}(M_{\mathfrak{t}}(\lambda)) \\
 \downarrow T_{\alpha}^{\widehat{\mathfrak{t}}_{\text{nat}}} & & \downarrow T_{\alpha}^{\widehat{\mathfrak{t}}_{\text{nat}}} \\
 T_{\alpha}^{\widehat{\mathfrak{t}}_{\text{nat}}}(M_{\widehat{\mathfrak{t}}_{\text{nat}}}(\lambda)) & \simeq & M_{\widehat{\mathfrak{p}}}(T_{\alpha}^{\mathfrak{t}}(M_{\mathfrak{t}}(\lambda))) \\
 \downarrow \mathbb{I}W_{a, \widehat{\mathfrak{p}}_{\text{nat}}} & & \\
 \mathbb{I}W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(T_{\alpha}^{\widehat{\mathfrak{t}}_{\text{nat}}}(M_{\widehat{\mathfrak{t}}_{\text{nat}}}(\lambda))) & \simeq & \mathbb{I}W_{a, \widehat{\mathfrak{p}}_{\text{nat}}}(M_{\widehat{\mathfrak{p}}}(T_{\alpha}^{\mathfrak{t}}(M_{\mathfrak{t}}(\lambda))))
 \end{array}$$

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