GENERALIZED INTERPOLATION IN H^{∞} (1)

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1. Introduction.

1.1. The problem considered in the present paper is that of determining the commutants of certain Hilbert space operators and of some related operator algebras. The first theorem to be proved contains as special cases two classical interpolation theorems due to Carathéodory and Pick, and it was this that inspired the title of the paper. In actuality, the "generalized interpolation" is an operator dilation.

Before any results can be stated precisely, it will be necessary to introduce some notations. Let C be the unit circle and D the open unit disk in the complex plane. Lebesgue measure on C will be denoted by m. The spaces $L^p(m)$ will be denoted simply by L^p, and the corresponding Hardy classes by H^p. The functions in H^p have natural analytic extensions into D, and when desirable we shall regard these functions as so extended.

The *shift operator* is the operator U on L^2 defined by $(Uf)(z)=zf(z)$. The operators we shall study first are projections of U. Let ψ be a nonconstant inner function, and let K be the subspace $H^2 \ominus \psi H^2$. The orthogonal projection in L^2 with range K will be denoted by P. Let S be the projection of U onto K, that is, the operator $PU|K$. For ϕ a function in H^{∞} let $\phi(S)$ denote the projection onto K of the operator on L^2 of multiplication by ϕ. When an operator T on K can be written as $\phi(S)$ for a ϕ in H^{∞}, we shall say that this ϕ *interpolates* T.

The operators $\phi(S)$ are precisely the operators that commute with S. It is easy to show that these operators do in fact commute with S; the converse is given by

THEOREM 1. *If T is an operator on K that commutes with S, then there is a function ϕ in H^{∞} such that $\|\phi\|_{\infty}=\|T\|$ and $T=\phi(S)$.*

This theorem is proved in §2. In §§3 and 4 its relation with the Carathéodory and Pick theorems is discussed. These correspond to the special cases where ψ is a power of z and a finite Blaschke product with distinct zeros. §5 contains some incomplete results concerning the question of when the function ϕ of Theorem 1 is unique. In §6 the condition under which $\phi(S)$ is completely continuous is determined. §7 pertains to another special case, that where ψ is the function $\exp[(z+1)/(z-1)]$. The operator S for this case is closely related to the Volterra operator on $L^2[0, 1]$.

The remainder of the paper, §§8–10, is devoted to obtaining a generalization

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of Theorem 1 in which K is replaced by a space of operator valued functions. The generalization is stated in §8 and its proof is outlined in §9. Not all the details will be given, because the arguments are formally the same as those used to prove Theorem 1.

Our proofs are functional analytic in character, and in particular they exploit the duality between L^1 and L^∞. Duality relations have of course been used by many authors in studying H^p spaces. The main analytic fact needed in the proof of Theorem 1 is a classical theorem of F. Riesz on the factorization of H^1 functions. To obtain the generalized version of Theorem 1, a generalization of Riesz's theorem to operator valued functions is needed. A proof of this result is presented in the concluding §10.

I should like to express my warm thanks to Henry Helson for many valuable discussions. In particular, the idea for the proof of Proposition 2.1 below arose from one of these discussions.

1.2. It is perhaps appropriate at this point to present a little background material. The following remarks will, I hope, suggest some of the considerations that led me to this study.

Suppose $\mathscr{L}$ is a Hilbert space and Σ is a semigroup of operators on $\mathscr{L}$. If $\mathscr{K}$ is a subspace of $\mathscr{L}$ and Q is the orthogonal projection onto $\mathscr{K}$, then $\mathscr{K}$ is called *semi-invariant* under Σ provided $QW_1QW_2Q=QW_1W_2Q$ for all W_1 and W_2 in Σ. When this happens, the family of operators $\{QW|\mathscr{K} : W\in\Sigma\}$ is a semigroup—called the *projection* of Σ onto $\mathscr{K}$—and the natural map of Σ onto its projection is a homomorphism. Of special interest is the case where Σ consists of the positive powers of some fixed operator W; we then speak simply of a semi-invariant subspace of W. If $\mathscr{K}$ is semi-invariant under W, then W is called a *dilation* of its projection onto $\mathscr{K}$.

The semi-invariant subspaces of a semigroup Σ are related in a simple way to its invariant subspaces. Namely, every semi-invariant subspace of Σ is equal to the orthogonal complement of one invariant subspace of Σ with respect to a larger one, and every subspace of the latter form is semi-invariant under Σ [32, Lemma 0]. In particular, the subspaces K of L^2 introduced above are semi-invariant under the shift operator U, and also under the semigroup of multiplication operators on L^2 induced by the functions in H^∞.

Now Sz.-Nagy [36] has proved the following remarkable theorem: If A is an arbitrary contraction operator acting on a Hilbert space $\mathscr{H}$, then there is a unitary operator W, acting on a Hilbert space that contains $\mathscr{H}$ as a subspace, such that $\mathscr{H}$ is semi-invariant under W and A is the projection of W onto $\mathscr{H}$. In other words, every contraction has a unitary dilation. Moreover, W can be taken to be minimal, i.e., such that no proper reducing subspace of W contains $\mathscr{H}$, and with this added restriction W is essentially unique. The theorem of Sz.-Nagy presents the possibility of using unitary operators to study other kinds of operators, and this possibility has been pursued by many authors.

It is instructive to approach these matters from a somewhat different direction by starting with a unitary operator W and trying to identify the operators having it as their minimal unitary dilation. The above mentioned characterization of semi-invariant subspaces implies that in order for there to be any such operators other than W itself, it is necessary and sufficient that W have a nonreducing invariant subspace. The simplest unitary operator of the latter sort is the shift operator U. On the basis of the known structure of the invariant subspaces of U, it is easy to see that the only operators having U as their minimal unitary dilation are, to within unitary equivalence, the operator U itself, the operator $U|H^2$ and its adjoint, and the operators S of Theorem 1. Thus, from the point of view of the theory of unitary dilations, the operators S are among the simplest contractions, and it is natural to try to analyze them.

Just how intimate is the relation between a contraction and its minimal unitary dilation? The answer may depend on the contraction. Theorem 1 indicates that for the operators S the relation is a very intimate one indeed, in that every operator commuting with S is the projection of an operator commuting with U and having K as a semi-invariant subspace. The theorem in §8 establishes the same result for a wider class of operators. Whether these are but special instances of a general fact about unitary dilations I would not venture to guess.

2. **Proof of Theorem 1.**

2.1. Let $H^\infty(S)$ denote the family of operators $\phi(S)$ with ϕ in H^∞. From the above remarks on semi-invariant subspaces, it is clear that $H^\infty(S)$ is an algebra, and that the map of H^∞ onto $H^\infty(S)$ that sends ϕ onto $\phi(S)$ is a homomorphism. The kernel of this homomorphism is ψH^∞. We therefore get a natural (algebraic) isomorphism from the quotient space $H^\infty/\psi H^\infty$ onto $H^\infty(S)$. The first step in the proof of Theorem 1 will be to show that this natural isomorphism preserves norms and that it is a homeomorphism relative to the weak-star topology of $H^\infty/\psi H^\infty$ and the weak operator topology of $H^\infty(S)$. For this it is necessary to identify the space whose dual is $H^\infty/\psi H^\infty$.

The annihilator of H^∞ in L^1 is the space H_0^1, the subspace of H^1 consisting of the functions that vanish at the origin. Thus H^∞ is the dual of L^1/H_0^1. Moreover, the annihilator of ψH^∞ in L^1 is $\bar{\psi} H_0^1$. Hence the annihilator of ψH^∞ in L^1/H_0^1 is $H\bar{\psi}_0^1/H_0^1$, and we may conclude that the latter space has $H^\infty/\psi H^\infty$ as its dual.

The following lemma forms the basis for the first part of the proof.

Lemma 2.1. *If f is a function in H_0^1, then there are functions g_1 and g_2 in K, with $\|g_1\|_2^2 \leq \|f\|_1$ and $\|g_2\|_2^2 \leq \|f\|_1$, such that*

$$\int \phi\bar{\psi} f\, dm = (\phi(S)g_1, g_2) \tag{1}$$

for all ϕ in H^∞. Conversely, if g_1 and g_2 are in K, then there is an f in H_0^1 such that (1) *holds for all ϕ in H^∞.*

Proof. Let f be in H_0^1. By a well-known theorem of F. Riesz [30], there is a factorization $f=f_1f_2$, where f_1 and f_2 are in H^2 and H_0^2 respectively, and $|f_1|^2=|f_2|^2=|f|$ almost everywhere. For any ϕ in H^∞,

$$\int \phi\bar{\psi} f\,dm = (\phi f_1, \psi \bar{f}_2). \tag{2}$$

As $\bar{f}_2$ is in $H^{2\perp}$, the function $\psi\bar{f}_2$ is in $(\psi H^2)^\perp = K \oplus H^{2\perp}$. Hence $\psi\bar{f}_2 - P\psi\bar{f}_2$ is in $H^{2\perp}$, and, setting $g_2=P\psi\bar{f}_2$, we have

$$(\phi f_1, \psi\bar{f}_2) = (\phi f_1, g_2). \tag{3}$$

Moreover, the function f_1-Pf_1 is in ψH^2, and therefore so is the function $\phi(f_1-Pf_1)$. Hence, setting $g_1=Pf_1$, we have

$$(\phi f_1, g_2) = (\phi g_1, g_2) = (\phi(S)g_1, g_2). \tag{4}$$

Combining equalities (2), (3), and (4), we see that (1) holds for all ϕ in H^∞. As obviously $\|g_1\|_2^2 \leqq \|f\|_1$ and $\|g_2\|_2^2 \leqq \|f\|_1$, the proof of the first part of the lemma is complete.

To prove the second part of the lemma, suppose g_1 and g_2 are functions in K. Then $\bar{\psi} g_2$ is in $H^{2\perp}$, and therefore $\psi\bar{g}_2$ is in H_0^2. Hence we can achieve (1) simply by setting $f=\psi g_1\bar{g}_2$.

PROPOSITION 2.1. *The natural isomorphism of $H^\infty/\psi H^\infty$ onto $H^\infty(S)$ is norm preserving.*

Proof. It is obvious that the map in question never increases norms; we must show that it never decreases norms. Let ϕ be a function in H^∞ such that the coset $\phi+\psi H^\infty$ has unit norm in $H^\infty/\psi H^\infty$. Let ε be any positive number. As $H^\infty/\psi H^\infty$ is the dual of $\bar{\psi}H_0^1/H_0^1$, there is an f in H_0^1 such that $\|f\|_1=1$ and

$$\left|\int \phi\bar{\psi} f\,dm\right| > 1-\varepsilon.$$

By Lemma 2.1, there are functions g_1 and g_2 in K, with $\|g_1\|_2 \leqq 1$ and $\|g_2\|_2 \leqq 1$, such that $\int \phi\bar{\psi} f\,dm = (\phi(S)g_1, g_2)$. It obviously follows that $\|\phi(S)\| > 1-\varepsilon$. As ε is arbitrary we have $\|\phi(S)\|=1$, and the proof is complete.

A standard compactness argument shows that each coset in $H^\infty/\psi H^\infty$ contains a function whose H^∞-norm achieves the coset norm. Thus the preceding proposition implies that whenever an operator on K can be interpolated by a function in H^∞, it can be interpolated by a function whose H^∞-norm equals the norm of the operator. The remainder of the proof of Theorem 1 is devoted to showing that the interpolation is in fact possible for any operator commuting with S.

It should be pointed out that Proposition 2.1 is all one really needs to obtain the interpolation theorems of Carathéodory and Pick. In the cases corresponding to these theorems the subspace K is finite dimensional, and it is a triviality to determine the operators that commute with S and to show they can all be interpolated. The problem is to show that the interpolations can be carried out without increasing norms.

PROPOSITION 2.2. *The natural isomorphism of $H^\infty/\psi H^\infty$ onto $H^\infty(S)$ is a homeomorphism relative to the weak-star topology on $H^\infty/\psi H^\infty$ and the weak operator topology on $H^\infty(S)$.*

Proof. Suppose $\{\phi_j\}$ is a net in H^∞ and ϕ_0 a function in H^∞ such that $\phi_j(S)\to\phi_0(S)$ in the weak operator topology. By Lemma 2.1, for any f in H_0^1 we can find functions g_1 and g_2 in K such that (1) holds for all ϕ in H^∞. It follows that $\int\phi_j\bar{\psi}f\,dm\to\int\phi_0\bar{\psi}f\,dm$ for all f in H_0^1, and this means that $\phi_j+\psi H^\infty\to\phi_0+\psi H^\infty$ in the weak-star topology of $H^\infty/\psi H^\infty$.

Suppose on the other hand that $\{\phi_j\}$ is a net in H^∞ and ϕ_0 a function in H^∞ such that $\phi_j+\psi H^\infty\to\phi_0+\psi H^\infty$ in the weak-star topology of $H^\infty/\psi H^\infty$. By the second part of Lemma 2.1, for any functions g_1 and g_2 in K we can find an f in H_0^1 such that (1) holds for all ϕ in H^∞. This implies that $(\phi_j(S)g_1, g_2)\to(\phi_0(S)g_1, g_2)$ for all g_1 and g_2 in K, so that $\phi_j(S)\to\phi_0(S)$ in the weak operator topology. The proof is complete.

PROPOSITION 2.3. *The algebra $H^\infty(S)$ is the weakly closed algebra generated by S and the identity.*

Proof. We first show that $H^\infty(S)$ is weakly closed. Suppose $\{\phi_j\}$ is a net in H^∞ such that the net $\{\phi_j(S)\}$ converges weakly to the operator T. If f is a function in H_0^1, then by Lemma 2.1 there are functions g_1 and g_2 in K, with $\|g_1\|_2^2\leqq\|f\|_1$ and $\|g_2\|_2^2\leqq\|f\|_1$, such that (1) holds for all ϕ in H^∞. It follows that

$$\lim\int\phi_j\bar{\psi}f\,dm \tag{5}$$

exists for all f in H_0^1 and is no larger in absolute value than $\|T\|\,\|f\|_1$. Moreover, the limit (5) depends only on the coset of $\bar{\psi}f$ in $\bar{\psi}H_0^1/H_0^1$. Hence (5) defines a bounded linear functional on $\bar{\psi}H_0^1/H_0^1$, and this functional is induced by a function ϕ_0 in H^∞. We thus have $\phi_j+\psi H^\infty\to\phi_0+\psi H^\infty$ weak-star, and therefore $\phi_j(S)\to\phi_0(S)$ weakly by Proposition 2.2. Consequently $\phi_0(S)=T$, and we may conclude that $H^\infty(S)$ is weakly closed.

It remains to show that the polynomials in S are weakly dense in $H^\infty(S)$. But this is immediate from the fact that the ordinary polynomials are weak-star dense in H^∞. The proof of the proposition is therefore complete.

2.2. We shall complete the proof of Theorem 1 by showing that every operator commuting with S belongs to the weak closure of the set of polynomials in S. For this we use some properties of muliple shifts.

For r a positive integer let $\boldsymbol{C}^r$ denote the Hilbert space of r-dimensional complex column vectors, and let $x_1,\dots,x_r$ denote the vectors in the usual orthonormal basis for $\boldsymbol{C}^r$. Let L_r^2 denote the L^2-space with respect to the measure m of $\boldsymbol{C}^r$-valued functions on C. For g a function in ordinary L^2 and x a vector in $\boldsymbol{C}^r$, we let gx

stand for the function in L_r^2 that at z takes the value $g(z)x$. Thus each G in L_r^2 has a unique representation of the form

$$G = g_1x_1 + \cdots + g_rx_r \tag{6}$$

with $g_1, \ldots, g_r$ in L^2. The space L_r^2 may obviously be regarded as the direct sum of r copies of L^2, and we shall think of it in these terms. The *shift of multiplicity r* is the operator on L_r^2 of multiplication by z. We denote this operator by U_r; it is the direct sum of r copies of U.

By H_r^2 we mean the subspace of L_r^2 consisting of those functions that can be written in the form (6) with $g_1, \ldots, g_r$ in H^2. We denote by K_r the subspace $H_r^2 \ominus \psi H_r^2$, which may obviously be identified with the direct sum of r copies of K; it consists of all G of the form (6) with $g_1, \ldots, g_r$ in K. For T an operator on K we let T_r denote the direct sum of r copies of T, regarded in the natural manner as an operator on K_r. In particular, S_r is the projection of U_r onto K_r.

Let $L_{r\times r}^\infty$ be the space of all essentially bounded r-by-r matrix valued functions on C. Each function in $L_{r\times r}^\infty$ induces an operator on L_r^2 by means of multiplication from the left. By $H_{r\times r}^\infty$ we mean the space of those functions in $L_{r\times r}^\infty$ that send H_r^2 into itself. The subspace K_r is semi-invariant under the semigroup of multiplication operators on L_r^2 induced by the functions in $H_{r\times r}^\infty$. For Θ in $H_{r\times r}^\infty$ we denote by $\Theta(S_r)$ the projection onto K_r of the operator on L_r^2 of multiplication by Θ.

We shall regard the functions in H^∞ as also belonging to $H_{r\times r}^\infty$ by identifying any function ϕ in the former with the function ϕI_r in the latter, where I_r is the r-by-r identity matrix.

LEMMA 2.2. *If T is an operator on K that commutes with S, then T_r commutes with $\Theta(S_r)$ for all Θ in $H_{r\times r}^\infty$.*

The proof of this is routine and will therefore be omitted.

A function in $L_{r\times r}^\infty$ is called *rigid* if its values (regarded as operators on C^r) are partial isometries having a fixed initial space. We shall need the following theorem about the invariant subspaces of U_r.

The invariant subspaces of U_r contained in H_r^2 are precisely those of the form ΘH_r^2 with Θ a rigid function in $H_{r\times r}^\infty$.

This is a generalization of Beurling's theorem due originally to Lax [20]. Lax worked in a different setting from the present one; for a proof of Lax's theorem in the form stated above, see Halmos [14] or Helson [15, p. 61].

If Θ_1 and Θ_2 are rigid functions in $H_{r\times r}^\infty$, then Θ_1 is said to *divide* Θ_2 provided $\Theta_1 H_r^2 \supset \Theta_2 H_r^2$. This notion of divisibility is equivalent to a natural algebraic one, but that is not important here.

PROPOSITION 2.4. *The invariant subspaces of S_r are precisely those of the form $\Theta(S_r)K_r$ with Θ a rigid function in $H_{r\times r}^\infty$ dividing ψ.*

Proof. If Θ is a rigid function in $H_{r\times r}^\infty$ dividing ψ, then $\Theta H_r^2 \cap K_r$ is easily seen to be an invariant subspace of S_r. Conversely, if M is an invariant subspace of S_r, then $M \oplus \psi H_r^2$ is invariant under U_r and so by Lax's theorem has the form ΘH_r^2 for some rigid function Θ in $H_{r\times r}^\infty$ that divides ψ, and we have $M = \Theta H_r^2 \cap K_r$. So what we must show is this: if Θ is a rigid function in $H_{r\times r}^\infty$ dividing ψ, then $\Theta H_r^2 \cap K_r = \Theta(S_r)K_r$.

Let Θ be as described. Suppose F is a function in H_r^2 such that ΘF is in K_r. Let P_r denote the orthogonal projection of L_r^2 onto K_r. Since the subspace ψH_r^2 is invariant under multiplication by Θ, it contains the function $\Theta F - \Theta P_r F$. The latter function is therefore annihilated by P_r, and so

$$\Theta F = P_r \Theta F = P_r \Theta P_r F = \Theta(S_r) P_r F.$$

This means that ΘF is in $\Theta(S_r)K_r$, and we have proved the inclusion

$$\Theta H_r^2 \cap K_r \subset \Theta(S_r)K_r.$$

Suppose on the other hand that G is a function in K_r. Then $\Theta G - P_r \Theta G$ is in ψH_r^2 and therefore also in ΘH_r^2. Hence there is a G_1 in H_r^2 such that $\Theta G - P_r \Theta G = \Theta G_1$, i.e., such that $\Theta(S_r)G = \Theta(G - G_1)$. This means that $\Theta(S_r)G$ is in $\Theta H_r^2 \cap K_r$, and we have proved the inclusion $\Theta(S_r)K_r \subset \Theta H_r^2 \cap K_r$. The proof of the proposition is complete.

Completion of the proof of Theorem 1. Let T be an operator on K that commutes with S. We want to show that T lies in the weak closure of the set of polynomials in S. In other words, we want to show that if $g_1, \ldots, g_r, h_1, \ldots, h_r$ are arbitrary functions in K, then there is a polynomial p such that

$$|(Tg_k, h_k) - (p(S)g_k, h_k)| < 1, \qquad k = 1, \ldots, r. \tag{7}$$

From this we form the function $G = g_1 x_1 + \cdots + g_r x_r$ in K_r. By Lemma 2.2 and Proposition 2.4, the operator T_r on K_r leaves invariant every invariant subspace of S_r. Hence if M is the invariant subspace of S_r generated by G, then TG lies in M. Since M is spanned by the set of functions $p(S_r)G$ with p a polynomial, there is some polynomial p such that $\|T_r G - p(S_r)G\|_2 < \min (1/\|h_k\|_2)$, and this p obviously satisfies (7). The proof of Theorem 1 is complete.

3. **The Carathéodory interpolation problem.** The interpolation problem of Carathéodory asks: given n complex constants $c_0, c_1, \ldots, c_n$, can one find a function analytic and with nonnegative real part in the unit disk whose power series begins with the terms $c_0 + c_1 z + \cdots + c_n z^n$? The condition for interpolation can be expressed as follows.

The Carathéodory problem has a solution if and only if the matrix

$$\begin{bmatrix} c_0+\bar{c}_0 & \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \\ c_1 & c_0+\bar{c}_0 & \bar{c}_1 & \cdots & \bar{c}_{n-1} \\ c_2 & c_1 & c_0+\bar{c}_0 & \cdots & \bar{c}_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0+\bar{c}_0 \end{bmatrix}$$

is nonnegative definite.

Carathéodory originally expressed the interpolation condition in different terms [1], [2]; the above formulation was pointed out by Toeplitz [38]. The Carathéodory problem has been studied by many authors from many different points of view; see Carathéodory and Fejer [3], Fischer [9], Frobenius [10], Gronwall [12], F. Riesz [27, pp. 58–61], [28], [29], Schur [34], [35], and Szegö [11, pp. 56–60].

To see how the Carathéodory problem fits into the present context, we consider the case where $\psi(z)=z^{n+1}$. In this case the subspace K is $(n+1)$-dimensional, and it has an orthonormal basis consisting of the functions $e_k(z)=z^k$, $k=0,\ldots,n$. The operator S is the shift with respect to this basis: $Se_k=e_{k+1}$ for $k<n$ and $Se_n=0$. It is easy to see that an operator on K commutes with S if and only if its matrix with respect to the basis $e_0,\ldots,e_n$ has the form

$$\begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix}. \tag{8}$$

Further, a function in H^∞ interpolates the operator on K having the matrix (8) if and only if its power series begins with the terms $c_0+c_1z+\cdots+c_nz^n$. Therefore, on the basis of Proposition 2.1, we can draw the following conclusion: *In order for there to exist a function in H^∞ of norm less than or equal to* 1 *whose power series begins with the terms $c_0+c_1z+\cdots+c_nz^n$, it is necessary and sufficient that the matrix* (8) *have norm less than or equal to* 1 *as an operator on an $(n+1)$-dimensional Hilbert space.* This result, although it concerns interpolation by bounded functions rather than by functions with nonnegative real parts, is equivalent to Carathéodory's theorem. To see this, one has only to apply a suitable transformation from the disk to the right half-plane. We omit the details and state the final conclusion: *In order for there to exist a function analytic and with nonnegative real part in the unit disk whose power series begins with the terms $c_0+c_1z+\cdots+c_nz^n$, it is necessary and sufficient that the matrix* (8) *have a nonnegative real part as an operator on an $(n+1)$-dimensional Hilbert space.* This obviously coincides with the Carathéodory-Toeplitz interpolation condition.

Another aspect of the Carathéodory problem is to study the class of functions that perform a given interpolation. For example, in the context of interpolation by bounded functions one might ask the following question: given $c_0, c_1,\ldots, c_n$, what is the nature of the function of minimum H^∞-norm whose power series begins with the terms $c_0+c_1z+\cdots+c_nz^n$? This problem will be handled with the techniques of the present paper in §5.

4. **The Nevanlinna-Pick interpolation problem.** An interpolation problem related to that of Carathéodory was first studied independently by Nevanlinna [22] and Pick [24]. It asks: given n distinct points $z_1,\ldots,z_n$ in the unit disk and n

complex numbers $w_1, \ldots, w_n$, can one find a function analytic and with nonnegative real part in the unit disk that takes at $z_1, \ldots, z_n$ the respective values $w_1, \ldots, w_n$? Nevanlinna and Pick used quite different techniques in studying this problem, and they found quite different interpolation conditions. The condition of Pick is the one of interest here:

The Nevanlinna-Pick problem has a solution if and only if the matrix

$$\begin{bmatrix} \frac{w_1+\bar{w}_1}{1-z_1\bar{z}_1} & \frac{w_1+\bar{w}_2}{1-z_1\bar{z}_2} & \cdots & \frac{w_1+\bar{w}_n}{1-z_1\bar{z}_n} \\ \frac{w_2+\bar{w}_1}{1-z_2\bar{z}_1} & \frac{w_2+\bar{w}_2}{1-z_2\bar{z}_2} & \cdots & \frac{w_2+\bar{w}_n}{1-z_2\bar{z}_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{w_n+\bar{w}_1}{1-z_n\bar{z}_1} & \frac{w_n+\bar{w}_2}{1-z_n\bar{z}_2} & \cdots & \frac{w_n+\bar{w}_n}{1-z_n\bar{z}_n} \end{bmatrix} \tag{9}$$

is nonnegative definite.

The Nevanlinna-Pick problem has been studied further by Denjoy [4], Nevanlinna [23], Pick [25], [26], Sz.-Nagy and Korányi [37], and Walsh [39, Chapter X].

To put the Nevanlinna-Pick problem into the context of the present paper, we consider the case where ψ is the finite Blaschke product having simple zeros at $z_1, \ldots, z_n$. For this case the subspace K is n-dimensional, and it is spanned by the functions $g_k(z) = 1/(1-\bar{z}_k z)$, $k=1, \ldots, n$. The function g_k is the kernel function for the functional on H^2 of evaluation at z_k (in other words, $(g, g_k) = g(z_k)$ for all g in H^2).

It is a little easier to work with the operator S^* than with S. The functions $g_1, \ldots, g_n$ are eigenvectors of S^* with the respective eigenvalues $\bar{z}_1, \ldots, \bar{z}_n$. Hence an operator T on K commutes with S if and only if $g_1, \ldots, g_n$ are eigenvectors of T^*. Further, if T is the operator on K defined by

$$T^* g_k = \bar{w}_k g_k, \qquad k = 1, \ldots, n, \tag{10}$$

then a function ϕ in H^∞ interpolates T if and only if $\phi(z_k) = w_k$, $k=1, \ldots, n$. The following conclusion therefore follows immediately from Proposition 2.1: *In order for there to exist a function in H^∞ of norm less than or equal to* 1 *that takes at $z_1, \ldots, z_n$ the respective values $w_1, \ldots, w_n$, it is necessary and sufficient that the operator T on K defined by* (10) *have norm less than or equal to* 1. As in the case of the Carathéodory problem, we can transform this, by means of a map from the disk onto the right half-plane, into a result about interpolation by functions with nonnegative real parts, namely: *In order for there to exist a function analytic and with nonnegative real part in the unit disk that takes at $z_1, \ldots, z_n$ the respective values $w_1, \ldots, w_n$, it is necessary and sufficient that the operator T on K defined by* (10) *have a nonnegative real part.*

If T is the operator on K defined by (10) and $g=a_1g_1+\cdots+a_ng_n$ is an arbitrary function in K, then

$$\begin{aligned}2\,\mathrm{Re}\,(Tg,g) &= (T^*g,g)+(g,T^*g)\\ &= \sum_{j,k=1}^{n} a_j\bar{a}_k\bar{w}_j(g_j,g_k)+\sum_{j,k=1}^{n} a_j\bar{a}_k w_k(g_j,g_k)\\ &= \sum_{j,k=1}^{n}\frac{a_j\bar{a}_k(\bar{w}_j+w_k)}{1-\bar{z}_jz_k}.\end{aligned}$$

Thus T will have a nonnegative real part if and only if the matrix (9) is nonnegative definite. This shows that the interpolation condition just stated is equivalent to the one given by Pick.

5. **Uniqueness.** In this section we consider the question of when the function ϕ of Theorem 1 is uniquely determined by the operator T. For the Nevanlinna-Pick problem corresponding to an infinite sequence of points in the disk, a necessary and sufficient condition for a given interpolation to have a unique solution has been given by Denjoy [4]. This in principle has a bearing on our problem in the case where ψ is an infinite Blaschke product. However, the condition of Denjoy is very implicit, and it does not have any obvious reformulation in terms of the associated operator T. What one would hope for is some simple condition on T equivalent to the uniqueness of ϕ, although there is no *a priori* reason for expecting such a condition to exist. Nevertheless, it is very easy to obtain a simple sufficient condition for uniqueness, which applies, in particular, whenever T is completely continuous.

By a *maximal vector* for an operator T we shall mean a unit vector whose image under T has norm equal to $\|T\|$.

PROPOSITION 5.1. *Let T be an operator on K of unit norm that commutes with S, and assume that T has a maximal vector. Then there is a unique function ϕ in H^∞ such that $\|\phi\|_\infty=1$ and $\phi(S)=T$. This function ϕ is an inner function, and it is the quotient of two functions in K.*

Proof. Take any ϕ in H^∞ such that $\|\phi\|_\infty=1$ and $\phi(S)=T$. Let g be a maximal vector for T. Then

$$1=\|Tg\|_2=\|P\phi g\|_2\leqq\|\phi g\|_2\leqq\|g\|_2=1.$$

It follows that $P\phi g=\phi g$, and that the modulus of ϕ cannot be less than 1 on a set of positive measure. In other words, $\phi=Tg/g$ and ϕ is inner. This proves the proposition.

The following corollary applies in particular to the Carathéodory and Nevanlinna-Pick problems.

COROLLARY. *Assume ψ is a finite Blaschke product, and let T be a nonzero operator on K that commutes with S. Then there is a unique ϕ in H^∞ of norm $\|T\|$ that interpolates T. This ϕ is a rational function having constant modulus on the unit circle, and it has fewer zeros than does ψ.*

Proof. Under the present hypotheses the subspace K is finite dimensional, and so T has a maximal vector. We may thus conclude by the preceding proposition that ϕ is unique, that it has constant modulus on the unit circle, and that it is the quotient of two functions in K. Now the nonzero functions in K are easily seen to be rational functions whose numerators have degrees less than the degree of the numerator of ψ. Hence ϕ is a rational function whose numerator has a degree less than that of the numerator of ψ (i.e., ϕ has fewer zeros than ψ).

Thus when ψ is a finite Blaschke product, each operator in $H^\infty(S)$ has a unique interpolating function of minimum norm. However, Walsh [39, pp. 292–293] has proved in the context of the Nevanlinna-Pick problem that this is no longer the case when ψ is an infinite Blaschke product. A trivial modification of Walsh's reasoning will enable us to handle the case where ψ has a singular factor as well.

PROPOSITION 5.2. *Assume ψ is not a finite Blaschke product. Then there is an operator T in $H^\infty(S)$ of unit norm which is interpolated by two distinct functions in H^∞ of unit norm.*

Proof. Since ψ is not a finite Blaschke product it has a singularity on the unit circle, which, without loss of generality, we may assume occurs at $z=1$. Let Ω be the open set $\{z : |z|+\frac{1}{2}|1-z|<1\}$. It is not hard to show that Ω is a Jordan domain contained in the unit disk D, and that the point $z=1$ lies on the boundary of Ω (see [39] for the details). Let ϕ_1 be a conformal map of D onto Ω, say with $\phi_1(1)=1$. We then have

$$|\phi_1(z)|+\tfrac{1}{2}|1-\phi_1(z)| < 1, \qquad z \in D. \tag{11}$$

Let $T=\phi_1(S)$. Then since $\|\phi_1\|_\infty=1$, we have $\|T\|\leqq 1$. On the other hand, the point $z=1$ is in the spectrum of T. For otherwise, the operator $(T-1)^{-1}$ would commute with S and so would have the form $\phi(S)$ for some ϕ in H^∞. The function $\phi(\phi_1-1)$ would then interpolate the identity operator on K, so we would have

$$\phi(\phi_1-1) = 1+\psi h$$

for some h in H^∞. But the latter is absurd, because $\phi_1(z)\to 1$ as $z\to 1$ and 0 is a cluster value of ψ at $z=1$. This proves the assertion that $z=1$ belongs to the spectrum of T, and we may conclude that $\|T\|=1$.

Now let

$$\phi_2 = \phi_1+\tfrac{1}{2}\psi(1-\phi_1).$$

By (11) we have $\|\phi_2\|_\infty=1$, and obviously $\phi_2(S)=T$. The proof is complete.

The following question, related to that of uniqueness, seems worth mentioning.

Suppose T is an operator on K that commutes with S and has norm less than or equal to 1. *Can one describe the family of functions in H^∞ that interpolate T and have norms less than or equal to* 1?

In other words, if a coset in $H^\infty/\psi H^\infty$ has norm less than or equal to 1, can one describe the family of functions in this coset whose norms are less than or equal to 1? Nevanlinna [22], [23] has obtained beautiful results for the case where ψ is a Blaschke product (see also Pick [26]). It seems probable that Nevanlinna's results extend to the case where ψ has a singular factor, even though his proofs do not.

In particular, one would like to know the following: if T is an operator on K that commutes with S and has norm less than or equal to 1, can T be interpolated by an inner function? In other words, does every coset in $H^\infty/\psi H^\infty$ of norm less than or equal to 1 contain an inner function? Nevanlinna [23, Satz 7] has shown that the answer is yes when ψ is a Blaschke product.

The latter question seems of interest for the theory of unitary dilations. An affirmative answer would mean that every contraction on K commuting with S can be dilated by a unitary operator on L^2 that commutes with U.

6. **Complete continuity.** In this section we derive a necessary and sufficient condition on ϕ for the complete continuity of $\phi(S)$. Let $\mathscr{C}$ denote the space of continuous complex valued functions on the unit circle.

THEOREM 2. *Let ϕ be a function in H^∞. Then $\phi(S)$ is completely continuous if and only if $\bar{\psi}\phi$ belongs to $H^\infty+\mathscr{C}$.*

Proof. Suppose first that $\bar{\psi}\phi$ is in $H^\infty+\mathscr{C}$. Then there is a ϕ_0 in the coset $\phi+\psi H^\infty$ such that $\bar{\psi}\phi_0$ is in $\mathscr{C}$. So we loose no generality in assuming that $\bar{\psi}\phi$ is itself in $\mathscr{C}$. We thus have $\phi=\psi w$ with w in $\mathscr{C}$.

Let $\{w_n\}$ be a sequence of trigonometric polynomials that converges uniformly to w. For each n let T_n be the projection onto K of the operator on L^2 of multiplication by ψw_n. We then obviously have $T_n\to\phi(S)$ in operator norm.

Now the subspace ψH^2 has an orthonormal basis consisting of the functions ψ, $U\psi$, $U^2\psi,\dots$. From this it is clear that the linear manifold $PU^k\psi H^2$ is finite dimensional for any integer k. Hence the linear manifolds $P\psi w_n K$ are all finite dimensional, and it follows that each operator T_n is of finite rank. Thus $\phi(S)$ is the norm limit of a sequence of operators of finite rank, and so $\phi(S)$ is completely continuous. This proves the first half of the theorem.

To prove the other half of the theorem, suppose $\phi(S)$ is completely continuous. Let A be the subspace of $\mathscr{C}$ consisting of those functions whose Fourier coefficients with negative indices vanish. By the F. and M. Riesz theorem, H_0^1 is the annihilator of A in $\mathscr{C}^*$, and so H_0^1 is the dual of $\mathscr{C}/A$. Also, L^∞/H^∞ is the dual of H_0^1, and hence is the bidual of $\mathscr{C}/A$. The canonical embedding of $\mathscr{C}/A$ into L^∞/H^∞ sends a coset of the form $w+A$ ($w \in \mathscr{C}$) onto the coset $w+H^\infty$. What we want to prove, therefore, is that the coset $\bar{\psi}\phi+H^\infty$ lies in the canonical image of $\mathscr{C}/A$. For this it will suffice to show that the functional induced by $\bar{\psi}\phi$ on H_0^1—call it λ—is continuous with respect to the weak-star topology of H_0^1 as the dual of $\mathscr{C}/A$. For the latter, in turn, it will be enough to show that the kernel of λ is weak-star sequentially closed [8, Theorem 1 on p. 426 and Theorem 7 on p. 429].

Suppose $\{f_n\}$ is a sequence in the kernel of λ which converges weak-star to the function f. By the factorization theorem used in the proof of Lemma 2.1, there are for each n functions f_{1n} and f_{2n} in H^2 and H_0^2 respectively such that $|f_{1n}|^2=|f_{2n}|^2 =|f_n|$ and $f_n=f_{1n}f_{2n}$. As the sequence $\{f_n\}$ is bounded in L^1-norm, both of the sequences $\{f_{1n}\}$ and $\{f_{2n}\}$ are bounded in L^2-norm. So by passing to a subsequence, we may assume that each of the sequences $\{f_{1n}\}$ and $\{f_{2n}\}$ converges weakly in L^2, say to f_1 and f_2 respectively. The sequence $\{f_{1n}f_{2n}\}$ is then bounded in L^1-norm and converges to f_1f_2 at each point of the open unit disk. This implies that $f_{1n}f_{2n}\to f_1f_2$ weak-star, and so we have $f_1f_2=f$.

Let

$$g_1 = Pf_1, \qquad g_{1n} = Pf_{1n},$$
$$g_2 = P\psi\bar{f}_2, \qquad g_{2n} = P\psi\bar{f}_{2n}.$$

By the reasoning used in the proof of Lemma 2.1,

$$(\phi(S)g_1, g_2) = \int \bar{\psi}\phi f\, dm = \lambda(f), \tag{12}$$

$$(\phi(S)g_{1n}, g_{2n}) = \int \bar{\psi}\phi f_n\, dm = \lambda(f_n) = 0, \qquad n = 1, 2, \ldots. \tag{12'}$$

Moreover, we obviously have $g_{1n}\to g_1$ and $g_{2n}\to g_2$ weakly in L^2. Since $\phi(S)$ is completely continuous, it follows that $\phi(S)g_{1n}\to\phi(S)g_1$ in L^2-norm. Consequently $(\phi(S)g_{1n}, g_{2n})\to(\phi(S)g_1, g_2)$. This together with (12) and (12′) implies that $\lambda(f)=0$, which is the desired conclusion. The proof of the theorem is complete.

On the basis of the identification of L^∞/H^∞ as the bidual of $\mathscr{C}/A$, it is a simple matter to show that the linear manifold $H^\infty+\mathscr{C}$ is closed in L^∞. This fact seems to have been hitherto overlooked. (See for example the question raised by Devinatz at the 1965 Lexington Symposium [Bull. Amer. Math. Soc. **71** (1965), Problem B on p. 855].) One consequence is that $H^\infty+\mathscr{C}$ is an algebra, and this algebra appears to have interesting properties. For instance, its maximal ideal space is obtained from the maximal ideal space of H^∞ by deleting the unit disk.

7. **The Volterra operator.**

7.1. In this section we consider the case where ψ is the inner function $\exp[(z+1)/(z-1)]$. The operator S for this ψ is closely related to the Volterra operator, that is, to the operator V on $L^2[0, 1]$ defined by

$$(Vg)(x) = \int_0^x g(t)\, dt.$$

In fact, let W_1 be the isometry of L^2 onto $L^2(-\infty,\infty)$ defined by

$$(W_1 f)(x) = \frac{\pi^{1/2} f((x-i)/(x+i))}{x+i},$$

and let W be the Fourier-Plancherel transformation on $L^2(-\infty,\infty)$. Then WW_1 sends L^2 isometrically onto $L^2(-\infty,\infty)$ and sends K isometrically onto $L^2[0, 1]$ (regarded as the subspace of $L^2(-\infty,\infty)$ consisting of the functions that vanish outside of $[0, 1]$). Moreover, WW_1 transforms S into the operator $(1-V)(1+V)^{-1}$ (see [31]). Thus, an operator on K commutes with S if and only if its transform under WW_1 commutes with V. Theorem 1 therefore characterizes the commutant of V. If we let $H^\infty(-\infty,\infty)$ denote the space of boundary functions on the real line for functions bounded and analytic in the upper half-plane, we can re-express the content of Theorem 1 for the present special case as follows.

If T is an operator on $L^2[0, 1]$ that commutes with V, then there is a function ϕ in $H^\infty(-\infty,\infty)$ in terms of which T is given by

$$(Tg)(x) = (W\phi W^{-1}g)(x), \qquad 0 \leqq x \leqq 1. \tag{13}$$

Moreover, ϕ can be chosen such that $\|\phi\|_\infty = \|T\|$.

When an operator T on $L^2[0, 1]$ and a function ϕ in $H^\infty(-\infty,\infty)$ are related as in (13), we shall say that ϕ *interpolates* T. Notice that two functions in $H^\infty(-\infty,\infty)$ interpolate the same operator on $L^2[0, 1]$ if and only if their difference is divisible, in the algebra $H^\infty(-\infty,\infty)$, by the function e^{ix}.

Henceforth we shall identify the functions in $H^\infty(-\infty,\infty)$ with their natural analytic extensions into the upper half-plane.

7.2. The convolution operators on $L^2[0, 1]$ induced by measures on $[0, 1)$ obviously commute with V. However, not all operators in the commutant of V arise in this way. In fact, if T is the operator on $L^2[0, 1]$ of convolution with the measure μ on $[0, 1)$, then T is interpolated by the inverse Fourier transform of μ, say ϕ_0. Clearly $\lim_{y\to+\infty} \phi_0(iy) = \mu(\{0\})$. If ϕ is any function in $H^\infty(-\infty,\infty)$ that interpolates T, then $\phi-\phi_0$ is divisible by e^{iz}, and so $\lim_{y\to+\infty} \phi(iy) = \mu(\{0\})$ also. Therefore, to obtain an operator that commutes with V and is not the operator of convolution with a measure, it suffices to take an operator interpolated by a function in $H^\infty(-\infty,\infty)$ that does not approach a limit as $z\to\infty$ along the positive imaginary axis.

The object in mentioning the foregoing is to contrast the Volterra operator on $L^2[0, 1]$ with the Volterra operator on $L^1[0, 1]$. The latter operator has been studied by Dixmier [6], and he has found that the operators commuting with it are precisely the convolution operators induced by measures on $[0, 1)$.

7.3. It is well known that the operator V belongs to a one-parameter semigroup, and so, in particular, it has an nth root for every positive integer n [18, pp. 663 ff.]. As an application of some of the above results, I shall show that V has precisely n nth roots.

The operator V is interpolated by the inverse Fourier transform of Lebesgue measure on [0, 1], i.e., by the function

$$\phi(z) = (e^{iz}-1)/iz.$$

It follows from a lemma of deLeeuw and Rudin [21, Lemma 1.4] that the function $e^{iz}-1$ is outer (in $H^\infty(-\infty,\infty)$). Therefore ϕ is also outer, and this fact will be needed.

Let ϕ_1 be an nth root of ϕ (n a positive integer exceeding 1), and let T_1 be the operator on $L^2[0, 1]$ interpolated by ϕ_1. Obviously T_1 is an nth root of V. Let T_2 be any nth root of V. Then T_2 commutes with V and so it is interpolated by some function ϕ_2 in $H^\infty(-\infty,\infty)$. The functions ϕ_1^n and ϕ_2^n then both interpolate V, and so $\phi_2^n-\phi_1^n$ is divisible by e^{iz}.

Let ω be a primitive nth root of unity. Then we can write

$$\phi_2^n-\phi_1^n = \prod_{k=0}^{n-1} (\phi_2-\omega^k\phi_1).$$

For $k=0,\dots,n-1$ let a_k be the largest nonnegative number such that $\phi_2-\omega^k\phi_1$ is divisible by the inner function $\psi_k(z)=\exp(ia_kz)$. The sum of the a_k's is then at least 1. For each k there is a function f_k in $H^\infty(-\infty,\infty)$ such that $\phi_2-\omega^k\phi_1=\psi_kf_k$. Thus if j and k are distinct we have

$$\phi_1 = (\psi_jf_j-\psi_kf_k)/(\omega^k-\omega^j).$$

If both a_j and a_k were nonzero it would follow that ϕ_1 is divisible by a nonconstant inner function. But that is impossible because $\phi_1^n=\phi$ is outer. Hence at most one of the a_k's is nonzero. This particular a_k is then at least 1, so that $\phi_2-\omega^k\phi_1$ is divisible by e^{iz}. In other words, for some k we have $T_2=\omega^kT_1$, as desired.

7.4. The operator V is completely continuous. Therefore, by Proposition 5.1, the operator $V/\|V\|$ can be interpolated by an inner function ϕ_0. The proof of Proposition 5.1 tells us how to find ϕ_0. Namely, the first step is to find a maximal vector for V. This is equivalent to finding an eigenvector for the largest eigenvalue of V^*V. A simple calculation shows that the function

$$g(x) = \cos(\pi x/2), \qquad 0 \leqq x \leqq 1,$$

is such an eigenvector. The same calculation shows that $\|V\|=2/\pi$; moreover

$$(Vg)(x) = \frac{2}{\pi}\sin\frac{\pi x}{2}.$$

The function ϕ_0 is equal to H/G, where G is the inverse Fourier transform of g and H is the inverse Fourier transform of $\pi Vg/2$. A computation gives

$$(2\pi)^{1/2}G(z) = \frac{-\pi e^{iz}+2iz}{2(t^2-\pi^2/4)}, \qquad (2\pi)^{1/2}H(z) = \frac{-2ize^{iz}-\pi}{2(t^2-\pi^2/4)}.$$

Hence

$$\phi_0(z) = \frac{(2iz/\pi)e^{iz}+1}{e^{iz}-2iz/\pi}.$$

We shall prove the following properties of ϕ_0.

The function ϕ_0 is an infinite Blaschke product (for the upper half-plane) with simple zeros. Its zeros cluster only at ∞.

Proof. It is trivial to check that ϕ_0 is continuous on the real axis. Hence the only singular functions that can possibly divide it are the functions e^{iaz}, $a \geqq 0$. But obviously, if $a>0$ then $e^{-iaz}\phi_0(z)$ is unbounded on the positive imaginary axis. Hence ϕ_0 is not divisible by any nonconstant singular function, i.e., it is a Blaschke product. As ϕ_0 is continuous on the real axis, its zeros can cluster only at ∞.

To go further we need the following lemma.

LEMMA 7.1. *The equation*

$$e^{iz} = 2iz/\pi \tag{14}$$

has precisely two solutions on the real axis, none in the upper half-plane, and infinitely many in the lower half-plane.

Proof. It is trivial that the only solutions of (14) on the real axis are $z = \pm \pi/2$. If we can prove that there are no solutions in the upper half-plane, then it will follow by Picard's theorem (applied to the function e^{iz}/z) that there must be infinitely many solutions in the lower half-plane. Now the solutions of (14) are precisely the zeros of the denominator in the above expression for ϕ_0. Any such zero in the upper half-plane must also be a zero of the numerator (because ϕ_0 is analytic in the upper half-plane). But a simple computation shows that the only common zeros of the numerator and denominator are at $z = \pm\pi/2$. This proves the lemma.

We can now complete the discussion of ϕ_0. Obviously, if z is a solution of (14) then $-z$ is a zero of the numerator in the expression for ϕ_0. Hence, by the lemma, the numerator has infinitely many zeros in the upper half-plane. Moreover, the lemma implies that none of these is a zero of the denominator. Thus ϕ_0 itself has infinitely many zeros in the upper half-plane. A trivial computation shows that the numerator of ϕ_0 has no zeroes in common with its derivative, and thus the zeros of ϕ_0 are simple. All the asserted properties of ϕ_0 have now been proved.

Notice that the function $-e^{-iz}\phi_0$ is continuous on the one-point compactification of the real line and takes the value 1 at ∞. From this it is not hard to show that e^{iz} and $-\phi_0$ have the same cluster values at ∞. In particular, if $\{z_n\}$ is a sequence in the upper half-plane converging to ∞, then $\lim e^{iz_n}$ exists if and only if $\lim -\phi_0(z_n)$ exists, and when these limits exist they are equal. It seems remarkable that there is a Blaschke product that behaves in the same way as e^{iz} at ∞.

8. **The operator valued theorem.** We shall be concerned in what follows with functions on the unit circle C whose values are vectors in a separable Hilbert space,

and with functions on C whose values are bounded operators on a separable Hilbert space. Without mentioning it explicitly each time, we shall always assume that any such functions we consider are weakly measurable [18, §3.5]. An equality between two of these functions is to be interpreted as holding modulo null sets.

For $\mathscr{V}$ a separable Hilbert space, we let $L^2[\mathscr{V}]$ denote the space of $\mathscr{V}$-valued functions G on C satisfying

$$\int \|G(z)\|^2 \, dm(z) < \infty.$$

The space $L^2[\mathscr{V}]$ is a Hilbert space under the inner product

$$(G_1, G_2) = \int (G_1(z), G_2(z)) \, dm(z).$$

(The ambiguity in notation between the inner products in $\mathscr{V}$ and in $L^2[\mathscr{V}]$ will not cause confusion in practice.) A function G in $L^2[\mathscr{V}]$ is called *analytic* if the scalar function $(G(z), y)$ belongs to H^2 for each vector y in $\mathscr{V}$. The analytic functions in $L^2[\mathscr{V}]$ form a subspace, which we denote by $H^2[\mathscr{V}]$.

The space of bounded operators on $\mathscr{V}$ will be denoted by $\mathscr{B}(\mathscr{V})$. We let $L^\infty[\mathscr{B}(\mathscr{V})]$ denote the space of essentially bounded $\mathscr{B}(\mathscr{V})$-valued functions on C with the essential supremum norm. A function Φ in $L^\infty[\mathscr{B}(\mathscr{V})]$ is called *analytic* if the scalar function $(\Phi(z)x, y)$ is in H^∞ for each pair of vectors x and y in $\mathscr{V}$. The space of analytic functions in $L^\infty[\mathscr{B}(\mathscr{V})]$ will be denoted by $H^\infty[\mathscr{B}(\mathscr{V})]$.

We now suppose that $\mathscr{V}_1$ and $\mathscr{V}_2$ are two separable Hilbert spaces, and we consider the space $\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)$ of Hilbert-Schmidt operators of $\mathscr{V}_2$ into $\mathscr{V}_1$. The space $\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)$ is a separable Hilbert space under the inner product

$$(Q_1, Q_2) = \operatorname{tr}(Q_1 Q_2^*)$$

[33, pp. 29 ff.]. The space $L^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ provides the natural setting for generalizing Theorem 1. The functions in $L^\infty[\mathscr{B}(\mathscr{V}_1)]$ operate on $L^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ by means of multiplication from the left, and those in $L^\infty[\mathscr{B}(\mathscr{V}_2)]$ operate on $L^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ by means of multiplication from the right. The functions in $H^\infty[\mathscr{B}(\mathscr{V}_1)]$ and $H^\infty[\mathscr{B}(\mathscr{V}_2)]$ send $H^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ into itself.

As in the scalar case, we consider a nonconstant inner function ψ in H^∞, and we form the subspace

$$K = H^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)] \ominus \psi H^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)].$$

The subspace K is semi-invariant under the left multiplications by functions in $H^\infty[\mathscr{B}(\mathscr{V}_1)]$ and under the right multiplications by functions in $H^\infty[\mathscr{B}(\mathscr{V}_2)]$. For Φ in $H^\infty[\mathscr{B}(\mathscr{V}_1)]$ we let λ_Φ denote the projection onto K of the operator on $L^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ of left multiplication by Φ, and we let $\mathscr{L}$ denote the class of all such operators λ_Φ. Similarly, for Φ in $H^\infty[\mathscr{B}(\mathscr{V}_2)]$ we let ρ_Φ denote the projection onto K of the operator on $L^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ of right multiplication by Φ, and we let $\mathscr{R}$ denote the class of all such operators ρ_Φ. The classes $\mathscr{L}$ and $\mathscr{R}$ are algebras and they commute with each other.

THEOREM 3. *The algebras $\mathscr{L}$ and $\mathscr{R}$ are commutants of each other. Further, if T is an operator in $\mathscr{L}$ then there is a function Φ in $H^\infty[\mathscr{B}(\mathscr{V}_1)]$ with $\|\Phi\|_\infty=\|T\|$ such that $\lambda_\Phi=T$. The analogous conclusion holds for $\mathscr{R}$.*

The proof of this theorem will be sketched in the next section. Some special cases should perhaps be commented on.

(a) Theorem 1 is precisely the special case of Theorem 3 in which

$$\dim \mathscr{V}_1 = \dim \mathscr{V}_2 = 1.$$

(b) In case $\dim \mathscr{V}_1=n_1<\infty$ and $\dim \mathscr{V}_2=n_2<\infty$, the space $L^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ can be naturally identified with the space of n_1-by-n_2 matrix valued functions on C having square-summable entries. The subspace K consists of all such matrix functions whose entries belong to $H^2 \ominus \psi H^2$ (the K of Theorem 1).

(c) If $\dim \mathscr{V}_2=1$, then $L^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ can be naturally identified with $L^2[\mathscr{V}_1]$. The subspace K can in this case be identified with a direct sum of copies of $H^2 \ominus \psi H^2$, the number of direct summands being equal to the dimension of $\mathscr{V}_1$. The algebra $\mathscr{R}$ is generated by a single operator, which is a direct sum of copies of the operator S of Theorem 1.

Theorem 3 of course enables one to prove operator valued generalizations of the interpolation theorems of Carathéodory and Pick. These, however, can be obtained more simply by other means. Sz.-Nagy and Korányi [37] have proved the operator valued Pick theorem by an ingenious elementary argument, and a modification of their reasoning yields the Carathéodory theorem as well. I do not know whether the method of Sz.-Nagy and Korányi can be applied to obtain any of the other results of this paper.

It is natural to ask whether there is a version of Theorem 3 that holds when the scalar inner function ψ is replaced by an operator inner function [15, p. 68]. This question may be difficult.

9. Sketch of the proof of Theorem 3.

9.1. The main ideas for the proof of Theorem 3 are already contained in the proof of Theorem 1. The proof that $\mathscr{R}$ is the commutant of $\mathscr{L}$, for example, involves two steps:

(a) the proof that $\mathscr{R}$ is weakly closed;

(b) the proof that every operator commuting with $\mathscr{L}$ lies in the weak closure of $\mathscr{R}$.

We describe step (b) first.

One first needs the following generalization of the theorem of Lax that was used in the proof of Theorem 1:

The subspaces of $H^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ invariant under right multiplication by all *functions in* $H^\infty[\mathscr{B}(\mathscr{V}_2)]$ *are precisely those of the form* $\Theta H^2[\mathscr{H}(\mathscr{V}_1, \mathscr{V}_2)]$ *with* Θ *a rigid function in* $H^\infty[\mathscr{B}(\mathscr{V}_1)]$.

This follows in trivial fashion from the theorem Halmos proves in [14] (see also [15, p. 61]). (The Halmos theorem is just the special case where $\dim \mathscr{V}_2 = 1$.) From it and the argument used to prove Proposition 2.4, one can characterize the invariant subspaces of $\mathscr{R}$:

The invariant subspaces of $\mathscr{R}$ *are precisely those of the form* $\lambda_\Theta K$ *with* Θ *a rigid function in* $H^\infty[\mathscr{B}(\mathscr{V}_1)]$.

In particular, therefore, an operator commuting with $\mathscr{L}$ leaves invariant every invariant subspace of $\mathscr{R}$. In order now to show that every operator commuting with $\mathscr{L}$ lies in the weak closure of $\mathscr{R}$, one can proceed just as in the proof of Theorem 1. The idea is to use the preceding result, but with $\mathscr{V}_1$ replaced by a direct sum of copies of itself.

The same argument of course shows that every operator commuting with $\mathscr{R}$ lies in the weak closure of $\mathscr{L}$.

9.2. To complete the proof of Theorem 3, one must show that $\mathscr{L}$ and $\mathscr{R}$ are weakly closed and that the interpolations in question can be carried out without increasing norms. For this, as is easily seen, one may assume without loss of generality that $\mathscr{V}_1 = \mathscr{V}_2$. We shall therefore suppose from now on that $\mathscr{V}_1$ and $\mathscr{V}_2$ are the same Hilbert space $\mathscr{V}$, and we shall write $\mathscr{H}(\mathscr{V})$ in place of $\mathscr{H}(\mathscr{V}, \mathscr{V})$. As in the scalar case, we have natural algebraic isomorphisms of $H^\infty[\mathscr{B}(\mathscr{V})]/\psi H^\infty[\mathscr{B}(\mathscr{V})]$ onto $\mathscr{R}$ and $\mathscr{L}$. The proof of Theorem 1 suggests that we should try to identify $H^\infty[\mathscr{B}(\mathscr{V})]/\psi H^\infty[\mathscr{B}(\mathscr{V})]$ as a dual space. This is precisely what we shall do.

Let $\mathscr{T}(\mathscr{V})$ be the space of trace class operators on $\mathscr{V}$ [33, pp. 36 ff.]. The trace norm of an operator Q in $\mathscr{T}(\mathscr{V})$ will be denoted by $\|Q\|_1$. With this norm, $\mathscr{T}(\mathscr{V})$ is a separable Banach space, and $\mathscr{B}(\mathscr{V})$ is its dual under the duality

$$\langle Q', Q\rangle = \operatorname{tr}(Q'Q), \qquad Q' \in \mathscr{B}(\mathscr{V}), \qquad Q \in \mathscr{T}(\mathscr{V}).$$

Let $L^1[\mathscr{T}(\mathscr{V})]$ be the space of $\mathscr{T}(\mathscr{V})$-valued functions F on C such that

$$\|F\|_1 = \int \|F(z)\|_1 \, dm(z) < \infty.$$

Under the indicated norm, $L^1[\mathscr{T}(\mathscr{V})]$ is a Banach space. Let $H^1[\mathscr{T}(\mathscr{V})]$ be the subspace of functions F in $L^1[\mathscr{T}(\mathscr{V})]$ with the property that for every pair of vectors x and y in $\mathscr{V}$, the scalar function $(F(z)x, y)$ is in H^1. Define $H^1_0[\mathscr{T}(\mathscr{V})]$ analogously. Then the following properties hold.

(i) *The space* $L^\infty[\mathscr{B}(\mathscr{V})]$ *is the dual of* $L^1[\mathscr{T}(\mathscr{V})]$ *under the duality*

$$\langle \Phi, F\rangle = \int \operatorname{tr}(\Phi(z)F(z))\, dm(z), \qquad \Phi \in L^\infty[\mathscr{B}(\mathscr{V})], \qquad F \in L^1[\mathscr{T}(\mathscr{V})],$$

and $H^\infty[\mathscr{B}(\mathscr{V})]$ *is weak-star closed in* $L^\infty[\mathscr{B}(\mathscr{V})]$.

(ii) *The annihilator of* $H^\infty[\mathscr{B}(\mathscr{V})]$ *in* $L^1[\mathscr{T}(\mathscr{V})]$ *is* $H_0^1[\mathscr{T}(\mathscr{V})]$, *and consequently* $H^\infty[\mathscr{B}(\mathscr{V})]/\psi H^\infty[\mathscr{B}(\mathscr{V})]$ *is the dual of* $\bar{\psi}H_0^1[\mathscr{T}(\mathscr{V})]/H_0^1[\mathscr{T}(\mathscr{V})]$.

(iii) *If* F_1 *and* F_2 *are in* $H^2[\mathscr{H}(\mathscr{V})]$ *then* F_1F_2 *is in* $H^1[\mathscr{T}(\mathscr{V})]$.

(iv) *The orthogonal complement of* $H^2[\mathscr{H}(\mathscr{V})]$ *in* $L^2[\mathscr{H}(\mathscr{V})]$ *is* $H_0^{2*}[\mathscr{H}(\mathscr{V})]$ (*the space of pointwise adjoints of functions in* $H_0^2[\mathscr{H}(\mathscr{V})]$, *the latter space being defined in the obvious way*).

(v) *Each function* F *in* $H^1[\mathscr{T}(\mathscr{V})]$ *has a factorization* $F=F_1F_2$, *where* F_1 *and* F_2 *are in* $H^2[\mathscr{H}(\mathscr{V})]$ *and*

$$F_2^*F_2 = (F^*F)^{1/2}, \qquad F_1^*F_1 = F_2F_2^*.$$

For the case where $\mathscr{V}$ is finite dimensional, properties (i)–(iv) are very easy consequences of the scalar theorems that they generalize. When $\mathscr{V}$ is infinite dimensional their proofs are rather more difficult, but not so much so as to require their inclusion here. Property (v) is a generalization of the factorization theorem of F. Riesz used in the proof of Lemma 2.1; its proof will be given in the next section.

Properties (ii)–(v) are operator valued versions of the basic facts about H^1, H^2, and H^∞ that were used in §2.1. With their aid, the arguments of §2.1 can be lifted to the present context. The first step is to prove the obvious generalization of Lemma 2.1. This requires properties (iii)–(v). Then, using (ii), one can show that the natural isomorphisms of $H^\infty[\mathscr{B}(\mathscr{V})]/\psi H^\infty[\mathscr{B}(\mathscr{V})]$ onto $\mathscr{L}$ and $\mathscr{R}$ are norm preserving, and that $\mathscr{L}$ and $\mathscr{R}$ are weakly closed. In this way one completes the proof of Theorem 3.

10. **Factorization of analytic operator functions.**

10.1. We prove now the factorization theorem stated in the preceding section as property (v). In fact, we shall prove a somewhat more general result, as this can be done without any extra effort.

Let $\mathscr{V}$ be a separable Hilbert space. In what follows we shall refer to $\mathscr{V}$-valued functions on C as *vector functions* and to $\mathscr{B}(\mathscr{V})$-valued functions on C as *operator functions*. An operator function A on C is called *integrable* if

$$\int \|A(z)\|\, dm(z) < \infty.$$

Square-integrability is defined analogously. An integrable operator function A is called *analytic* if the scalar function $(A(z)x, y)$ is in H^1 for every pair of vectors x and y in $\mathscr{V}$.

The following is the factorization theorem.

THEOREM 4. *Let* A *be an analytic integrable operator function on* C. *Then* A *has a factorization* $A=A_1A_2$, *where* A_1 *and* A_2 *are analytic square-integrable operator functions such that* $A_2^*A_2=(A^*A)^{1/2}$ *and* $A_1^*A_1=A_2A_2^*$.

This has been proved by Devinatz [5, Theorem 7.1] under the additional conditions that A be invertible almost everywhere and that the operator function $\log A^*A$ be integrable. Earlier, Helson and Lowdenslager proved the same result for the case where $\mathscr{V}$ is finite dimensional [16, Theorem 10]. The improvement given by the theorem above is the elimination of assumptions on the invertibility of A. The latter is of course essential for the purposes of the present paper.

The proof of Theorem 4 is based largely on results and ideas of Helson and Lowdenslager. Certain preliminaries are necessary before the proof can be given.

10.2. An operator function is called *positive* if its values almost everywhere are nonnegative Hermitian operators. A positive integrable operator function R is called *factorable* if it can be written as $R=B^*B$ with B an analytic square-integrable operator function. The first part of the proof of Theorem 4 will be devoted to showing that if A is an analytic integrable operator function, then $(A^*A)^{1/2}$ is factorable. This will first be done for the case where A has finite rank almost everywhere by using a theorem of Helson and Lowdenslager, which is stated below as Lemma 10.1. The general case will then be settled by appealing to a theorem of Douglas (Lemma 10.2 below). The idea of basing the factorization theorem for analytic operator functions on the factorization theorem for positive operator functions is due to Helson and Lowdenslager.

For Q a Hermitian operator of finite rank on $\mathscr{V}$, we let $\Delta[Q]$ denote the determinant of the restriction of Q to its range.

LEMMA 10.1. *Let R be a positive integrable operator function such that the rank of $R(z)$ is finite for almost all z. Then R is factorable if and only if its range function is conjugate analytic and*

$$\int \log \Delta[R(z)]\, dm(z) > -\infty.$$

Originally, Helson and Lowdenslager proved this under the stronger condition that $\mathscr{V}$ itself be finite dimensional [17, Theorem 13]. However, the proof in Helson's book establishes the result stated above [15, pp. 120–123]. (The reader is referred to Helson's book also for the definition of a range function.)

The result of Douglas we need can be stated as follows [7, Theorem 4].

LEMMA 10.2. *Let R be a positive integrable operator function. Suppose there is a sequence $\{R_n\}$ of positive operator functions such that $R_n(z) \leq R(z)$ for all n and almost all z, each R_n is factorable, and $R_n(z) \to R(z)$ strongly almost everywhere. Then R is factorable.*

A *vector polynomial* is a vector function G on C of the form

$$G(z) = x_0 + zx_1 + \cdots + z^n x_n,$$

where $x_0, \ldots, x_n$ are vectors in $\mathscr{V}$. For B an analytic square-integrable operator function, we let $M(B)$ denote the subspace of $H^2[\mathscr{V}]$ spanned by all the functions

BG with G a vector polynomial. The function B is called an *outer function* if the closure of the range of $B(z)$ is almost everywhere a fixed subspace $\mathscr{W}$ of $\mathscr{V}$, and if $M(B)=H^2[\mathscr{W}]$. Our final preliminary lemma is proved in Helson's book [15, p. 121].

LEMMA 10.3. *If the positive integrable operator function R is factorable, then it has a factorization of the form $R=B^*B$ with B an outer function.*

10.3. We can now begin the proof of Theorem 4.

LEMMA 10.4. *Let A be an analytic integrable operator function such that $A(z)$ has finite rank for almost all z. Then $(A^*A)^{1/2}$ is factorable.*

Proof. Consider first the case where A is bounded. In this case, if $R=(A^*A)^{1/2}$, then R^2 is factorable, and so it satisfies the necessary and sufficient conditions of Lemma 10.1. From this it is immediate that R satisfies these conditions also, and therefore R is factorable.

Now consider the general case. By a classical theorem of Szegö [19, p. 53], there is an outer function h in H^∞ such that

$$\begin{aligned} |h(z)|^2 &= 1 && \text{where } \|A(z)\| \leqq 1, \\ &= 1/\|A(z)\| && \text{elsewhere.} \end{aligned}$$

The operator function h^2A is then bounded and analytic, and so $|h|^2(A^*A)^{1/2}$ is factorable by what we just proved. Therefore, there is an analytic operator function B_0 such that $|h|^2(A^*A)^{1/2}=B_0^*B_0$. Setting $B=h^{-1}B_0$, we have $(A^*A)^{1/2}=B^*B$, and it only remains to show that B is analytic. Now if x and y are vectors in $\mathscr{V}$, then the scalar function $(B(z)x, y)$ is in L^2, and it is equal to the H^∞ function $(B_0(z)x, y)$ divided by the outer function h. This implies by a well-known property of outer functions that $(B(z)x, y)$ is in H^2 [19, p. 75], and so B is analytic, as desired. The proof of the lemma is complete.

LEMMA 10.5. *Assume $\mathscr{V}$ is infinite dimensional, and let A be an analytic integrable operator function. Then $(A^*A)^{1/2}$ is factorable.*

Proof. Choose an orthonormal basis for $\mathscr{V}$, and for each positive integer n let P_n be the orthogonal projection onto the subspace spanned by the first n basis vectors. Each operator function P_nA is then integrable and analytic. Therefore $(A^*P_nA)^{1/2}$ is factorable for each n, by Lemma 10.4. Now $A(z)^*P_nA(z)\leqq A(z)^*A(z)$ for all z, and the inequality is preserved upon taking square roots [13, p. 168]. Further, $A(z)^*P_nA(z)\to A(z)^*A(z)$ strongly for all z, and this is easily seen to imply that $(A(z)^*P_nA(z))^{1/2}\to(A(z)^*A(z))^{1/2}$ strongly. The factorability of $(A^*A)^{1/2}$ therefore follows by Lemma 10.2.

Completion of the proof of Theorem 4. From this point the proof follows the lines laid out in [16] and [5]. Minor modifications are necessitated by the possible noninvertibility of our function.

Let A be an analytic integrable operator function, and let $R=(A^*A)^{1/2}$. By Lemmas 10.3–10.5 we have a factorization $R=A_2^*A_2$ with A_2 an outer analytic square-integrable operator function. Also, by the polar decomposition theorem, we have a factorization $A=JR$, where $J(z)$ is for each z a partial isometry whose initial space equals the closure of the range of $R(z)$ and whose range equals the closure of the range of $A(z)$. (A standard argument shows that J is measurable.) If we set $A_1=JA_2^*$, then we have $A=A_1A_2$. It remains to show that A_1 is analytic and that $A_1^*A_1=A_2A_2^*$.

The latter fact will be proved first. We have

$$A_1^*A_1 = A_2J^*JA_2^*. \tag{15}$$

For any z, the operator $J(z)^*J(z)$ is the orthogonal projection in $\mathscr{V}$ onto the range of $J(z)^*$. The range of $J(z)^*$ is the initial space of $J(z)$, which equals the closure of the range of $R(z)$. But the equality $R=A_2^*A_2$ shows that the ranges of $R(z)$ and $A_2(z)^*$ have almost everywhere the same closures. Hence $J(z)^*J(z)$ is almost everywhere the projection onto the closure of the range of $A_2(z)^*$, and we have $J^*JA_2^*=A_2^*$. This together with (15) yields the desired equality $A_1^*A_1=A_2A_2^*$.

To prove that A_1 is analytic, we note that because A_2 is outer, the closure of the range of $A_2(z)$ is almost everywhere a fixed subspace $\mathscr{W}$ of $\mathscr{V}$. The equality $A_1=JA_2^*$ shows that the null space of $A_1(z)$ is equal almost everywhere to $\mathscr{W}^\perp$. Hence it will be enough to show that the scalar function $(A_1(z)x, y)$ is in H^1 for all x in $\mathscr{W}$ and y in $\mathscr{V}$. Let such an x and such a y be given, and let x be regarded as a constant function in $H^2[\mathscr{W}]$. Because $M(A_2)=H^2[\mathscr{W}]$, there is a sequence $\{G_n\}$ of vector polynomials such that $A_2G_n \to x$ in the norm of $L^2[\mathscr{V}]$. A simple application of Schwarz's inequality gives

$$\int |(A_1(z)x, y)-(A(z)G_n(z), y)|\, dm(z) \leq \|y\|\left(\int \|A_1(z)\|^2\, dm(z)\right)^{1/2}\left(\int \|x-A_2(z)G_n(z)\|^2\, dm(z)\right)^{1/2}.$$

It follows that $(A(z)G_n(z), y) \to (A_1(z)x, y)$ in L^1. But each of the functions $(A(z)G_n(z), y)$ is in H^1 because A is analytic, and consequently $(A_1(z)x, y)$ is in H^1. The proof of the theorem is complete.

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