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Generalized Inversions of Hadamard and Tensor Products for Matrices

Saburou Saitoh

Institute of Reproducing Kernels, Kiryu, Japan
Email: saburou.saitoh@gmail.com

Abstract

We shall give natural generalized solutions of Hadamard and tensor products equations for matrices by the concept of the Tikhonov regularization combined with the theory of reproducing kernels.

1. Introduction

In this paper we shall consider the matrix equation

$$AB = C, \tag{1}$$

however, here we shall consider the product

$$AB \tag{2}$$

in the two senses; that is, the Hadamard product

$$AB = A * B \tag{3}$$

and the tensor (Kronecker) product

$$AB = A \otimes B. \tag{4}$$

Then, for the Hadamard product, for some same type matrices, the product may be considered as the matrix C made by each matrix elements-wise products, meanwhile, for the tensor product $A \otimes B$, there is no any restriction on the matrices types. In this paper, we shall consider natural inversions A for given C and B

by the idea of the Tikhonov regularization and the Moore-Penrose generalized inverses from the viewpoint of the theory of reproducing kernels.

2. Needed Backgrounds

We shall need the basic theory of reproducing kernels in connection with matrices. We shall fix the minimum requests for our purpose.

2.1. Reproducing Kernels and Positive Definite Hermitian Matrices

Let E be an arbitrary abstract (non-void) set. Let $\mathcal{F}(E)$ denote the set of all complex-valued functions on E. A reproducing kernel Hilbert spaces (RKHS for short) on the set E is a Hilbert space $\mathcal{H} \subset \mathcal{F}(E)$ endowed with a function $K : E \times E \to \mathcal{H}$, which is called the reproducing kernel and which satisfies **the reproducing property**. Namely we have

$$K_p \equiv K(\cdot, p) \in \mathcal{H} \text{ for all } p \in E \tag{5}$$

and

$$f(p) = \langle f, K_p \rangle_{\mathcal{H}} \tag{6}$$

for all $p \in E$ and for all $f \in \mathcal{H}$. We denote by $H_K(E)$ (or H_K) the reproducing kernel Hilbert space $\mathcal{H}$ whose corresponding reproducing function is K.

A complex-valued function $K : E \times E \to \mathbb{C}$ is called a **positive definite quadratic form function** on the set E, or shortly, **positive definite function**, if, for an arbitrary function $X : E \to \mathbb{C}$ and for any finite subset F of E, one has

$$\sum_{p,q \in F} \overline{X(p)} X(q) K(p,q) \geq 0. \tag{7}$$

By a fundamental theorem, we know that, for any positive definite quadratic form function K on E, there exists a unique reproducing kernel Hilbert space on $E \times E$ with reproducing kernel K. So, in a sense, the correspondence between the reproducing kernel K and the reproducing kernel Hilbert space $H_K(E)$ is one to one.

A simple example of positive definite quadratic form function is a positive definite Hermitian matrix and we obtain the fundamental relation:

Example 2.1 *Let* $E = \{p_1, p_2, \cdots, p_n\}$ *be a set consisting of* n *distinct points. Let* $A_n \equiv \{a_{ij}\}_{i,j=1}^n$ *be a strictly positive* $n \times n$ *Hermitian matrix. Let* $A_n^{-1} \equiv \{b_{ij}\}_{i,j=1}^n$ *denote the inverse of* A_n. *Then the space* $\mathcal{F}(E)$ *of the complex valued functions on* E, *endowed with the inner product*

$$\langle f, g \rangle_{H_{\overline{A}_n}(E)} \equiv \sum_{i,j=1}^n f(p_i) b_{ij} \overline{g(p_j)} = \overline{(g(p_1), g(p_2), \cdots, g(p_n))} A_n^{-1} \begin{pmatrix} f(p_1) \\ f(p_2) \\ \vdots \\ f(p_n) \end{pmatrix},$$

is a reproducing kernel Hilbert (complex Euclidean) space with reproducing kernel K *defined by* $K(p_i, p_j) = a_{ji}$ *for all* $i, j = 1, \cdots, n$.

Indeed, the validity of (5) follows by a straightforward calculation. To prove (6) we observe that

$$\langle f, K(\cdot, p_k) \rangle_{H_{K_{\overline{A}_n}}(E)} = \sum_{i,j=1}^n f(p_i) b_{ij} \overline{K(p_j, p_k)} = \sum_{i,j=1}^n f(p_i) b_{ij} a_{jk} = f(p_k)$$

for all $k = 1, \cdots, n$ (note that $\overline{a_{kj}} = a_{jk}$). Thus $H_{\overline{A}_n}(E)$ coincides with the reproducing kernel Hilbert space $H_K(E)$. In particular the norm induced by the product $\langle \cdot, \cdot \rangle_{H_{\overline{A}_n}(E)}$ coincides with the norm of $H_K(E)$.

We can thus combine the two theories of positive definite Hermitian matrices and of reproducing kernels. See [1]-[9] for various applications and numerical problems.

2.2. Tensor Products of Reproducing Kernel Hilbert Spaces and Restrictions

For any two positive definite quadratic form functions $K_1(p_1, q_1)$ and $K_2(p_2, q_2)$ on $E_1 \times E_1$ and $E_2 \times E_2$, respectively, the usual product $K_1(p_1, q_1) K_2(p_2, q_2)$ is again a positive definite quadratic form function on $E_1 \times E_1 \times E_2 \times E_2$. Then, it is the reproducing kernel for the tensor product $H_{K_1}(E_1) \otimes H_{K_2}(E_2)$ for their reproducing kernel Hilbert spaces.

We would like to recall that for any two positive definite quadratic form functions $K_1(p, q)$ and $K_2(p, q)$ on $E \times E$, the usual product $K(p, q) = K_1(p, q) K_2(p, q)$ is again a positive definite quadratic form function on E. Consequently, the reproducing kernel Hilbert space H_K which admits the kernel $K(p, q)$ can be seen as the restriction of the tensor product $H_{K_1}(E) \otimes H_{K_2}(E)$ to the diagonal set.

Proposition 2.1 (see, [10] [11]) *Let us consider* $\left\{ f_j^{(1)} \right\}_j$ *and* $\left\{ f_j^{(2)} \right\}_j$ *to be complete orthonormal systems in* $H_{K_1}(E)$ *and* $H_{K_2}(E)$, *respectively. Then, the reproducing kernel Hilbert space* H_K *is comprised of all functions on* E *which are represented by*

$$f(p) = \sum_{i,j} \alpha_{i,j} f_i^{(1)}(p) f_j^{(2)}(p) \quad \text{on } E, \quad \sum_{i,j} \left| \alpha_{i,j} \right|^2 < \infty, \tag{8}$$

in the sense of absolutely convergence on E, *and its norm in* H_K *is given by*

$$\|f\|_{H_K}^2 = \min \sum_{i,j} \left| \alpha_{i,j} \right|^2$$

where $\left\{ \alpha_{i,j} \right\}$ *are considered to satisfy* (8). *Moreover, the norm inequality*

$$\|f_1 f_2\|_{H_{K_1 K_2}(E)} \le \|f_1\|_{H_{K_1}(E)} \|f_2\|_{H_{K_2}(E)}$$

holds true.

2.3. Generalized Inverses and the Tikhonov Regularization

Let L be any bounded linear operator from a reproducing kernel Hilbert space H_K into a Hilbert space $\mathcal{H}$. Then, the following problem is a classical and fundamental problem (known as the best approximate mean square norm problem): For any member $\mathbf{d}$ of $\mathcal{H}$, we would like to find

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}}.$$

It is clear that we are considering operator equations, generalized solutions and corresponding generalized inverses within the framework of $f \in H_K$ and $\mathbf{d} \in \mathcal{H}$, having in mind the equation

$$Lf = \mathbf{d}. \tag{9}$$

However, this problem has a complicated structure, specially in the infinite dimensional Hilbert spaces case, leading in fact to the consideration of generalized inverses (in the Moore-Penrose sense). Anyway, the problem turns to be well-posed within the reproducing kernels theory framework in the following proposition:

Proposition 2.2 *For any member* $\mathbf{d}$ *of* $\mathcal{H}$, *there exists a function* $\tilde{f}$ *in* H_K *satisfying*

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = \|L\tilde{f} - \mathbf{d}\|_{\mathcal{H}} \tag{10}$$

if and only if, for the reproducing kernel Hilbert space H_k, *admitting the kernel defined by* $k(p, q) = \left(L^* L K(\cdot, q), L^* L K(\cdot, p) \right)_{H_K}$, *we have*

$$L^* \mathbf{d} \in H_k. \tag{11}$$

Furthermore, when there exists a function $\tilde{f}$ *satisfying* (10), *there exists a uniquely determined function that minimizes the norms in* H_K *among the functions satisfying the equality, and its function* $f_{\mathbf{d}}$ *is*

represented as follows:

$$f_{\mathbf{d}}(p) = \left(L^* \mathbf{d}, L^* L K(\cdot, p) \right)_{H_k} \quad \text{on } E. \tag{12}$$

From this proposition, we see that the problem is considered in a very good way by the theory of reproducing kernels. Namely, the existence, uniqueness and representation of the solutions for this problem are well-formulated. In particular, note that the adjoint operator is represented in a useful way (which will be very important in our framework later on). The extremal function $f_{\mathbf{d}}$ is the Moore-Penrose generalized inverse $L^{\dagger}\mathbf{d}$ of the equation $Lf = \mathbf{d}$. The criteria (11) is involved and so the Moore-Penrose generalized inverse $f_{\mathbf{d}}$ is—in general—not good, specially when the data contain error or noises in some practical cases. To overcome this difficulty, we will need the idea of Tikhonov regularization.

Proposition 2.3 *Let $L : H_K \to \mathcal{H}$ be a bounded linear operator defined from any reproducing kernel Hilbert space H_K into any Hilbert space $\mathcal{H}$, and define the inner product*

$$\langle f_1, f_2 \rangle_{H_{K_\lambda}} = \lambda \langle f_1, f_2 \rangle_{H_K} + \langle L f_1, L f_2 \rangle_{\mathcal{H}}$$

for $f_1, f_2 \in H_K$. Then $\left(H_K, \langle \cdot, \cdot \rangle_{H_{K_\lambda}} \right)$ is a reproducing kernel Hilbert space whose reproducing kernel is given by

$$K_\lambda(p, q) = \left[\left(\lambda + L^* L \right)^{-1} K_q \right](p).$$

Here, $K_\lambda(p, q)$ is the uniquely determined solution $\tilde{K}_\lambda(p, q)$ of the functional equation

$$\tilde{K}_\lambda(p, q) + \frac{1}{\lambda} \left(L \tilde{K}_q, L K_p \right)_{\mathcal{H}} = \frac{1}{\lambda} K(p, q), \tag{13}$$

(that is corresponding to the Fredholm integral equation of the second kind for many concrete cases). Moreover, we are using

$$\tilde{K}_q = \tilde{K}_\lambda(\cdot, q) \in H_K \quad \text{for } q \in E, \qquad K_p = K(\cdot, p) \quad \text{for } p \in E.$$

Proposition 2.4 *The Tikhonov functional*

$$H_K \ni f \mapsto \left\{ \lambda \| f \|_{H_K}^2 + \| L f - \mathbf{d} \|_{\mathcal{H}}^2 \right\} \in \mathbb{R}$$

attains the minimum and the minimum is attained only by the function $f_{\mathbf{d}, \lambda} \in H_K$ such that

$$\left(f_{\mathbf{d}, \lambda} \right)(p) = \left\langle \mathbf{d}, L K_\lambda(\cdot, p) \right\rangle_{\mathcal{H}}. \tag{14}$$

Furthermore, $\left(f_{\mathbf{d}, \lambda} \right)(p)$ satisfies

$$\left| \left(f_{\mathbf{d}, \lambda} \right)(p) \right| \leq \sqrt{\frac{K(p, p)}{2\lambda}} \| \mathbf{d} \|_{\mathcal{H}}. \tag{15}$$

For up-to-date versions of the Tikhonov regularization by using the theory of reproducing kernels, see [12] [13]. Furthermore, various applications and numerical problems, see, for example, [14] [15].

2.4. The Solution of the Tikhonov Functional Equation

For our purpose, we use a natural representation of the Tikhonov extremal function considered in Proposition 2.4 by using complete orthogonal systems in the Hilbert spaces H_K and $\mathcal{H}$. The original result is given by [16].

Let $\{\Phi_\mu\}_{\mu=0}^{\infty}$ and $\{\Psi_\nu\}_{\nu=0}^{\infty}$ be any fixed complete orthonormal systems of the Hilbert spaces $\mathcal{H}$ and H_K, respectively.

Therefore, from the form (13) and the representation (14) in Proposition 2.4 of the Tikhonov extremal function $f_{\mathbf{d}, \lambda}$, we shall assume the representation

$$L \tilde{K}_\lambda(\cdot, q) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} d_{\mu, \nu} \Phi_\mu \otimes \overline{\Psi_\nu(q)}, \tag{16}$$

in the sense of the tensor product $\mathcal{H} \otimes \overline{H_K}$.

In the expansion of the reproducing kernel $K(p,q)$,

$$K(p,q) = \sum_{n=0}^{\infty} \Psi_n(p) \overline{\Psi_n(q)}, \tag{17}$$

since $LK(\cdot, q)$ belongs to $\mathcal{H}$, we can set

$$L\Psi_n(\cdot) = \sum_{m=0}^{\infty} D_{n,m} \Phi_m \tag{18}$$

with the uniquely determined constants $\{D_{n,m}\}$ satisfying, for any fixed n,

$$\sum_{m=0}^{\infty} \left| D_{n,m} \right|^2 < \infty.$$

Note that the natural condition

$$\sum_{n=0}^{\infty} \sum_{\mu=0}^{\infty} \left| D_{n,\mu} \right|^2 < \infty \tag{19}$$

is equivalent to the circumstance of the operator L be a Hilbert-Schmidt operator from H_K into $\mathcal{H}$.

By setting

$$\sum_{n=0}^{\infty} \overline{D_{n,m}} D_{n,\mu} = A_{m,\mu},$$

we consider the infinite equations

$$\lambda d_{\mu,\nu} + \sum_{m=0}^{\infty} d_{m,\nu} A_{m,\mu} = D_{\nu,\mu}. \tag{20}$$

Then, we obtain:

Proposition 2.5 *Assume that the operator L is a Hilbert-Schmidt operator and the Equation* (20) *have the solutions $\{d_{\mu,\nu}\}$ satisfying*

$$\sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \left| d_{\mu,\nu} \right|^2 < \infty. \tag{21}$$

Then, for any $\mathbf{d} = \sum_{\mu=0}^{\infty} b_\mu \Phi_\mu \in \mathcal{H}$, the Tikhonov extremal function $(f_{\mathbf{b},\lambda})(p)$ in the sense of Proposition 2.4 is given by

$$(f_{\mathbf{d},\lambda})(p) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} b_\mu \overline{d_{\nu,\mu}} \Psi_\nu(p) \tag{22}$$

and the following fundamental estimates hold:

$$\left| (f_{\mathbf{d},\lambda})(p) \right| \le \left(\sum_{\mu=0}^{\infty} \left| b_\mu \right|^2 \right)^{1/2} \left(\sum_{\mu=0}^{\infty} \left| \sum_{\nu=0}^{\infty} \overline{d_{\nu,\mu}} \Psi_\nu(p) \right|^2 \right)^{1/2} \le \left(\sum_{\mu=0}^{\infty} \left| b_\mu \right|^2 \right)^{1/2} \left(\sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \left| d_{\nu,\mu} \right|^2 \right)^{1/2} K(p,p)^{1/2}$$

and

$$\left\| f_{\mathbf{d},\lambda} \right\|_{H_K}^2 = \sum_{\nu=0}^{\infty} \left| \sum_{\mu=0}^{\infty} b_\mu \overline{d_{\nu,\mu}} \right|^2. \tag{23}$$

Here, note that $d_{\nu,\mu}$ are depending on λ.

In particular, if H_K is finite dimensional, the infinite Equation (20) are solved explicitly and so, then, it is enough to check the convergence (21).

Of course, if both spaces H_K and $\mathcal{H}$ are finite dimensional, then we obtain the complete representation for the Tikhonov extremal function $(f_{\mathbf{d},\lambda})(p)$.

Meanwhile, when we apply the above complete orthonormal systems to the Moore-Penrose generalized

problem, we see that we cannot, in general, obtain any valuable information, because we cannot analyze the structure of the important reproducing kernel Hilbert space H_k in Proposition 2.2.

3. Matrices and Reproducing Kernels

We shall combine matrices and reproducing kernels. In order to simplify the notations we shall consider the numbers on the real field R.

First note that R^m is the reproducing kernel Hilbert space admitting the reproducing kernel that is represented by the unit matrix U_m of the $m \times m$ and then the inner product $(x, y)_{R^m}$ in R^m is given by

$$(x, y)_{R^m} = \sum_{j=1}^{m} x_j y_j$$

and when we take the simplest standard orthonormal system $\{e_j\}_{j=1}^{m}$, the reproducing property is given by

$$(x, e_j)_{R^m} = x_j, \quad j = 1, 2, \cdots, m$$

and

$$U_m = (e_1, e_2, \cdots, e_m).$$

We shall consider another space R^n $(n \geq 1)$, similarly and we shall consider the tensor product $R^m \otimes R^n$. Then, the metric in the tensor product is given by

$$(x, y)_{R^m \otimes R^m} = \sum_{j=1}^{m} \sum_{j'=1}^{n} x_{j,j'} y_{j,j'}$$

for

$$x_{j,j'} = x_j x_{j'}, \quad y_{j,j'} = y_j y_{j'}$$

in the mn dimensional space R^{mn} for

$$x = \sum_{j=1}^{m} \sum_{j'=1}^{n} x_j x_{j'} e_j \otimes e_{j'}$$

and

$$y = \sum_{j=1}^{m} \sum_{j'=1}^{n} y_j y_{j'} e_j \otimes e_{j'}.$$

This will mean that the $m \times n$ matrices formed by, for $a \in R^m$ and $b \in R^n$

$$ab^{\mathrm{T}} \tag{24}$$

form the tensor product of the two reproducing spaces R^m and R^n and the metric is introduced by the l^2 in the mn dimensional space R^{mn} and the reproducing kernel may be considered as in the case R^m.

By this idea, we can consider any matrices space of any type may be considered as reproducing kernel spaces and when we consider the metric, we shall consider it as in the above sense. As we stated in the introduction, since we are interested in the Hadamard product and tensor product of matrices, we will see that it is enough to consider the case of vector spaces by transformed to vectors.

4. Hadamard Product Inversions

We shall consider on R^n and for any members $a, b \in R^n$, in the Hadamard product

$$a * b = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ \\ a_n b_n \end{pmatrix},$$

we shall consider the Tikhonov inverse in the sense, for any positive λ and for any $c \in \boldsymbol{R}^n$

$$\min_{x \in \boldsymbol{R}^n}\left\{\lambda\|x\|^2 + \|x * b - c\|^2\right\}. \tag{25}$$

Then, we obtain

Theorem 4.1 *In the sense of* (25), *the general inversion* $a_j(c;\lambda)$ *is given by*

$$a_j(c;\lambda) = c_j \frac{b_j}{\lambda + (b_j)^2}.$$

Proof. Note that from the general theory of reproducing kernels, we obtain the inequality

$$\|a * b\| \le \|a\|\|b\|.$$

However, this is just the Cauchy-Schwart inequality and the theory gives the background of the inequality with the precise structure of the Hadamard product. So, the multiplication operator, for any fixed b

$$x \to x * b$$

is a bounded linear operator and by Proposition 2.5, directly we obtain the result.

Of course, there exists a uniquely determined the Moore-Penrose generalized inverse, all the cases, by taking the limit

$$\lim_{\lambda \to +0} a_j(c;\lambda). \tag{26}$$

In particular, in the sense of the Moore-Penrose generalized inverse, of course, we have, on $\boldsymbol{R}$,

$$\frac{100}{0} = 0, \quad \frac{0}{0} = 0. \tag{27}$$

Mathematicians will expect for our mathematics, there exists some realization of our mathematics in some real world. So, we wonder: does there exist some real examples supporting the above results.

In particular, Sin-Ei,Takahashi ([17]) established a simple and natural interpretation (27) by analyzing any extensions of fractions and by showing the complete characterization for such property (27). His result will show that our mathematics says that the results (27) should be accepted as natural ones. However, the results will be curious and so, the above question will be vital still as a very important problem.

For simple direct proofs and several physical meanings, see [18].

5. Inversions of Tensor Products of Matrices

Before a general situation, we shall show the simplest case as a prototype example in order to see the basic concept and problem for the inversion. We shall consider for $\boldsymbol{R}^2 \ni a$ and $\boldsymbol{R}^3 \ni b$ as follows:

$$a \otimes b = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{28}$$

$$\otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{29}$$

We can write the tensor product as follows:

$$\begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \end{pmatrix} \tag{30}$$

or with its transpose. However, as we stated, we shall consider it as follows:

$$\left(a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_2 b_2, a_2 b_3\right)^{\mathrm{T}} \in \boldsymbol{R}^6. \tag{31}$$

Now we are interested in the matrix equation, for given c, b

$$a \otimes b = c; \tag{32}$$

that will mean the equations:

$$a_i b_j = c_{ij}, \quad i = 1, 2; \, j = 1, 2, 3. \tag{33}$$

We are considering arbitrary m, n positive integers, and so we see that these equations will have important problems; the existence and representation problems. By the concept and theory in Proposition 2.5, we will be able to obtain the reasonable solutions.

For any members $b \in \mathbf{R}^n$ and $c \in \mathbf{R}^{mn}$, we shall consider the Tikhonov inverse in the sense, for any positive λ

$$\min_{x \in \mathbf{R}^m} \left\{ \lambda \|x\|_{\mathbf{R}^m}^2 + \|x \otimes b - c\|_{\mathbf{R}^{mn}}^2 \right\}. \tag{34}$$

Note that from the general theory of reproducing kernels, we obtain the equality

$$\|x \otimes b\|_{\mathbf{R}^{mn}} = \|x\|_{\mathbf{R}^m} \|b\|_{\mathbf{R}^n}.$$

That is, for any fixed $b \in \mathbf{R}^n$, the operator

$$x \in \mathbf{R}^m \to x \otimes b \in \mathbf{R}^{mn}$$

is a bounded linear operator into $\mathbf{R}^{mn}$. So, we can apply Proposition 2.5.

The Equation (16) corresponds to

$$L\tilde{K}_\lambda \left(\mu\mu', j \right) = \sum_{\mu=1, \mu'=1}^{n} \sum_{\nu=1}^{n} d_{\mu\mu', \nu} \boldsymbol{e}_\mu \otimes \boldsymbol{e}_{\mu'} \otimes \boldsymbol{e}_\nu \left(j \right). \tag{35}$$

The representation (18) corresponds to

$$L\boldsymbol{e}_\nu \left(\cdot \right) = \sum_{\mu=1, \mu'=1}^{n} D_{\nu, \mu\mu'} \boldsymbol{e}_\mu \otimes \boldsymbol{e}_{\mu'} \tag{36}$$

with the uniquely determined constants $\left\{ D_{\nu, \mu\mu'} \right\}$, for any fixed ν. We set

$$\sum_{\nu=1}^{n} D_{\nu, \mu\mu'} D_{\nu, \mu_* \mu_*'} = A_{\mu\mu', \mu_* \mu_*'}.$$

We consider the equations as in (20)

$$\lambda d_{\mu\mu', \nu} + \sum_{\mu_* \mu_*'=1}^{n} d_{\mu_* \mu_*', \nu} A_{\mu_* \mu_*', \mu\mu'} = D_{\nu, \mu\mu'}. \tag{37}$$

Then, we obtain

Theorem 5.1 *In the sense of (34), the general inversion* $a_j(c; \lambda)$ *is given, with the uniquely determined solutions* $d_{\mu\mu', \nu}$ *of the Equation (37)*

$$a_j(c; \lambda) = \sum_{\mu=1, \mu'=1}^{n} \sum_{\nu=1}^{m} c_\nu d_{\mu\mu', \nu} e_\nu(j),$$

and the corresponding results in Proposition 2.5 are valid.

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