

GENERALIZED INVEX SETS AND PREINVEX FUNCTIONS ON RIEMANNIAN MANIFOLDS

R. P. Agarwal*, I. Ahmad, Akhlag Iqbal and Shahid Ali

Abstract. In this paper, a geodesic α -invex subset of a Riemannian manifold is introduced. Geodesic α -invex and α -preinvex functions on a geodesic α -invex set with respect to particular maps are also defined. Further, we study the relationships between geodesic α -invex and α -preinvex functions on Riemannian manifolds. Some results of a non smooth geodesic α -preinvex function are also discussed using proximal subdifferentiation. At the end, mean value inequality and the mean value theorem in α -invexity analysis are extended to Cartan-Hadamard manifolds. Our results extend and generalize the known results in the literature.

1. INTRODUCTION

The notion of convexity plays an important and significant role in the theory of optimization. Since, convexity is often not enjoyed by real problems, various approaches have been proposed by several researchers in order to extend the validity of results to the larger classes of optimization. An important and significant generalization of convexity is invexity, which was introduced by Hanson [7] in 1981. Hanson's initial results inspired a great deal of subsequent work which has greatly expanded the roles and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Jeyakumar [8] and Noor [12, 13] have studied the properties of preinvex and α -preinvex functions, respectively, and their roles in optimization and mathematical programming.

It has been found that few properties of convex functions are needed for establishing the results on Riemannian manifolds. Rapcsak [15] and Udriste [16] proposed a generalization which differs from the others in the use of a Riemannian manifold.

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*Corresponding author.

In their setting, the linear space is replaced by Riemannian manifold and the line segment by a geodesic. Pini [14] introduced the notion of invex function on Riemannian manifold, while Mititelu [11] investigated its generalization. Ferrara and Mititelu [4] developed Mond-Weir type duality for vector programming problems on a differentiable manifold. Barani et al. [3] introduced the concepts of geodesic invex set and geodesic preinvex functions on Riemannian manifolds with respect to the particular maps. Recently, Li et al. [10] studied the weak sharp minima for constrained optimization problems on Riemannian manifolds and their characterizations. The methods involve appropriate tools of variational analysis and generalized differentiation on Riemannian and Hadamard manifolds.

In this paper, we define the concepts of geodesic α -invex set and geodesic α -preinvex function on Riemannian manifold. Using suitable conditions, some relations between geodesic α -invex set and geodesic α -preinvex function are established. We prove the existence condition for global minima of these functions by relaxing the smoothness condition on geodesic α -preinvex functions and considering lower semi-continuity. At the end, we prove the mean value theorem for differentiable functions on α -invex subsets of Riemannian manifolds which extends the results of Antczak [1] and Azagra et al. [2].

2. PRELIMINARIES

First, we recall some basic definitions and known results about Riemannian manifolds. For the standard material on differential geometry, we refer to [9].

Let M be a C^∞ smooth manifold modelled on a Hilbert space H , either finite dimensional or infinite dimensional, endowed with a Riemannian metric $\langle \cdot, \cdot \rangle_p$ on the tangent space $T_p M \cong H$. The corresponding norm is denoted by $\| \cdot \|_p$ and the length of a piecewise C^1 curve $\gamma : [a, b] \rightarrow M$ is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)} dt.$$

For any two points $p, q \in M$, we define

$$d(p, q) := \inf\{L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve joining } p \text{ to } q\}.$$

Then, d is a distance which induces the original topology on M . We know that on every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection, denoted by $\nabla_X Y$, for any vector fields $X, Y \in M$. We also recall that a geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ , that is, γ satisfies the equation $\nabla_{d\gamma(t)/dt} d\gamma(t)/dt = 0$. Any path γ joining p and q in M such that $L(\gamma) = d(p, q)$ is a geodesic and is called a minimal geodesic. The existence theorem for ordinary differential equation implies that for every $v \in TM$,

there exists an open interval $J(v)$ containing 0 and exactly one geodesic $\gamma_v : J(v) \rightarrow M$ with $d\gamma_v(0)/dt = v$. This implies that there is an open neighborhood \overline{TM} of the submanifold M of TM such that for every $v \in \overline{TM}$, the geodesic $\gamma_v(t)$ is defined for $|t| < 2$. The exponential mapping $\exp : \overline{TM} \rightarrow M$ is then defined as $\exp(v) = J_v(1)$ and the restriction of \exp to a fiber T_pM in \overline{TM} is denoted by \exp_p for every $p \in M$. We use parallel transport of vectors along geodesic. Recall that for a given curve $\gamma : I \rightarrow M$, a number $t_0 \in I$ and a vector $v_0 \in T_{\gamma(t_0)}M$, there exists exactly one parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = v_0$. Moreover, the mapping defined by $v_0 \mapsto V(t)$ is linear isometry between the tangent spaces $T_{\gamma(t_0)}M$ and $T_{\gamma(t)}M$, for each $t \in I$. We denote this mapping by $P_{t_0,\gamma}^t$ and we call it the parallel translation from $T_{\gamma(t_0)}M$ to $T_{\gamma(t)}M$ along the curve γ .

If f is a differentiable map from the manifold M to the manifold N , then df_x , denotes the differential of f at x . We also recall that a simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Cartan-Hadamard manifold.

3. GEODESIC α -INVEX SETS AND α -INVEX FUNCTIONS

In this section, we define geodesic α -invex sets and α -invex functions by introducing a bifunction $\alpha : M \times M \rightarrow R \setminus \{0\}$.

Definition 3.1. Let M be a Riemannian manifold and $\eta : M \times M \rightarrow TM$ be a function, and $\alpha : M \times M \rightarrow R \setminus \{0\}$ be a bifunction such that for every $x, y \in M$, $\alpha(x, y)\eta(x, y) \in T_yM$. A non-empty subset S of M is said to be a geodesic α -invex set with respect to η and α if for every $x, y \in S$, there exists a unique geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\gamma_{x,y}(0) = y, \gamma'_{x,y}(0) = \alpha(x, y)\eta(x, y), \gamma_{x,y}(t) \in S, \text{ for all } t \in [0, 1].$$

Remark 3.1. If $\alpha(x, y) = 1$, then above definition reduces to that of geodesic invex set defined in [3].

We know that a subset S of a Riemannian manifold is called geodesic convex if for any two points $x, y \in S$ there exists exactly one geodesic of length $d(x, y)$ which belongs entirely to S (see [9, 16]).

Remark 3.2. If we consider M to be a Cartan-Hadamard manifold (either finite dimensional or infinite dimensional), then on M there exists a natural map η playing the role of $x - y$ in the Euclidean space R^n , for every $x, y \in R^n$. Indeed, we can define the function η as

$$(*) \quad \alpha(x, y)\eta(x, y) := \gamma'_{x,y}(0), \text{ for all } x, y \in M.$$

where $\gamma_{x,y}$ is the unique minimal geodesic joining y to x (see [9, p.253]) as follows:

$$\gamma_{x,y}(t) := \exp_y(t\alpha(x, y) \exp_y^{-1} x), \text{ for all } t \in [0, 1].$$

Therefore, every geodesic convex set $S \subseteq M$ is a geodesic α -invex set with respect to η and α defined in (*). Note that the converse is not true in general.

Example 3.1. Let M be a Cartan-Hadamard manifold and $x_0, y_0 \in M$, $x_0 \neq y_0$. Let $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$ for some $0 < r_1, r_2 < \frac{1}{2}d(x_0, y_0)$, where $B(x, r) = \{y \in M : d(x, y) < r\}$ is an open ball with centre x and radius r . We define

$$S := B(x_0, r_1) \cup B(y_0, r_2).$$

Then, S is not a geodesic convex set because every geodesic curve passing through x_0 and y_0 does not completely lie in S . Now we define the function $\eta : M \times M \rightarrow M$ such that

$$\eta(x, y) := \begin{cases} \exp_y^{-1} x; & x, y \in B(x_0, r_1) \text{ or } x, y \in B(y_0, r_2), \\ 0_y; & \text{otherwise.} \end{cases}$$

For every $x, y \in M$, consider a bifunction $\alpha : M \times M \rightarrow R \setminus \{0\}$ and $\gamma : [0, 1] \rightarrow M$ defined by

$$\gamma_{x,y}(t) := \exp_y(t\alpha(x, y)\eta(x, y)), \text{ for all } t \in [0, 1].$$

Hence,

$$\gamma_{x,y}(0) = y, \quad \gamma'_{x,y}(0) = \alpha(x, y)\eta(x, y).$$

Now, we show that S is a geodesic α -invex set with respect to η and α . Let $x, y \in B(x_0, r_1)$, since $B(x_0, r_1)$ is geodesic convex (see [9, p.259]), it follows that,

$$\gamma_{x,y}(t) := \exp_y(t\alpha(x, y)\exp_y^{-1} x) \in B(x_0, r_1) \subset S, \text{ for all } t \in [0, 1].$$

Similarly, for $x, y \in B(y_0, r_2)$, we have

$$\gamma_{x,y}(t) \in S, \text{ for all } t \in [0, 1].$$

If $x \in B(x_0, r_1)$ and $y \in B(y_0, r_2)$ or $x \in B(y_0, r_2)$ and $y \in B(x_0, r_1)$ then, we have

$$\gamma_{x,y}(t) := \exp_y(t\alpha(x, y)0_y) = y \in S, \text{ for all } t \in [0, 1].$$

Hence, S is a geodesic α -invex set with respect to η and α .

Let S be a geodesic convex subset of a finite dimensional Cartan-Hadamard manifold M and $x \in M$. Then, there exists a unique point $p_s(x) \in S$ such that for each $y \in S$, $d(x, p_s(x)) \leq d(x, y)$. The point $p_s(x)$ is called the projection of x onto S (see [6, p. 262]).

Example 3.2. Let S_1 and S_2 be two nonempty closed geodesic convex subsets of a finite dimensional Cartan-Hadamard manifold M such that $S_1 \cap S_2 = \emptyset$. Let us define

$S := S_1 \cup S_2$ and $\eta : M \times M \rightarrow M$ by

$$\eta(x, y) := \begin{cases} \exp_y^{-1}(p_{S_1}(x)); & y \in S_1, x \in M, \\ \exp_y^{-1}(p_{S_2}(x)); & y \in S_2, x \in M, \\ 0_y; & \text{otherwise.} \end{cases}$$

Now, for every $x, y \in S$ we define

$$\gamma_{x,y}(t) := \exp_y(t\alpha(x, y)\eta(x, y)), \text{ for all } t \in [0, 1],$$

we can see easily

$$\gamma_{x,y}(0) = y, \gamma'_{x,y}(0) = \alpha(x, y)\eta(x, y), \gamma_{x,y}(t) \in S, \text{ for all } t \in [0, 1].$$

Hence, S is a geodesic α -invex set with respect to η and α .

Now, we introduce α -invexity and geodesic α -preinvexity on geodesic α -invex subset of a Riemannian manifold.

Definition 3.2. Let M be a Riemannian manifold and S be an open subset of M which is geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$. Let f be a real differentiable function on S . Then, f is said to be α -invex with respect to η and α on S if

$$f(x) - f(y) \geq df_y(\alpha(x, y)\eta(x, y)) \text{ for all } x, y \in S.$$

Definition 3.3. Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$. The function $f : S \rightarrow R$ is said to be geodesic α -preinvex if for every $x, y \in S$, we have

$$f(\gamma_{x,y}(t)) \leq tf(x) + (1 - t)f(y), \text{ for all } t \in [0, 1],$$

where $\gamma_{x,y}$ is the unique geodesic defined in Definition 3.1. If the above inequality is strict, then f is called strictly geodesic α -preinvex function.

Proposition 3.1. Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$. Suppose that $f : S \rightarrow R$ is a geodesic α -preinvex function, then every lower section of f defined by

$$S_\lambda := \{x \in S : f(x) \leq \lambda\}, \lambda \in R,$$

is a geodesic α -invex set with respect to η and α .

Proof. Let $x, y \in S_\lambda$. Since S is a geodesic α -invex set with respect to η and α , there exists exactly one geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\gamma_{x,y}(0) = y, \gamma'_{x,y}(0) = \alpha(x, y)\eta(x, y), \gamma_{x,y}(t) \in S, \text{ for all } t \in [0, 1].$$

By the geodesic α -preinvexity of f , we have for all $t \in [0, 1]$

$$\begin{aligned} f(\gamma_{x,y}(t)) &\leq tf(x) + (1-t)f(y) \\ &\leq t\lambda + (1-t)\lambda = \lambda. \end{aligned}$$

Therefore, $\gamma_{x,y}(t) \in S_\lambda$ for all $t \in [0, 1]$, and the result is proved.

4. GEODESIC α -PREINVEXITY AND DIFFERENTIABILITY

In this section, we introduce property and the condition (say condition (C)) on the function $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$ which will be use in the subsequent analysis.

Definition 4.1. (Property (P)). Let M be a Riemannian manifold and $\gamma : [0, 1] \rightarrow M$ be a curve on M such that $\gamma_{x,y}(0) = y$ and $\gamma_{x,y}(1) = x$. Then, $\gamma_{x,y}$ is said to possess the property (P) with respect to $y, x \in M$ if

$$\gamma'_{x,y}(s)(t-s) = \alpha(\gamma_{x,y}(t), \gamma_{x,y}(s))\eta(\gamma_{x,y}(t), \gamma_{x,y}(s)), \quad \forall s, t \in [0, 1].$$

Remark 4.1. If $\alpha = 1$, then the above property reduces to the property defined by Pini [14].

Let M be a Riemannian manifold and $\gamma_{x,y}$ possessing the property (P) with respect to $y, x \in M$, then

$$\alpha(x, y)\eta(x, y) = \alpha(\gamma_{x,y}(1), \gamma_{x,y}(0))\eta(\gamma_{x,y}(1), \gamma_{x,y}(0)) = \gamma'_{x,y}(0).$$

the case when $\gamma_{x,y}$ is a geodesic, then

$$\begin{aligned} \alpha(\gamma_{x,y}(0), \gamma_{x,y}(s))\eta(\gamma_{x,y}(0), \gamma_{x,y}(s)) &= -s\gamma'_{x,y}(s) \\ &= -sP_{0,\gamma_{x,y}}^s[\gamma'_{x,y}(0)] \\ &= -sP_{0,\gamma_{x,y}}^s[\alpha(x, y)\eta(x, y)], \end{aligned}$$

or

$$P_{s,\gamma_{x,y}}^0[\alpha(y, \gamma_{x,y}(s))\eta(y, \gamma_{x,y}(s))] = -s\alpha(x, y)\eta(x, y)$$

and

$$\begin{aligned} \alpha(\gamma_{x,y}(1), \gamma_{x,y}(s))\eta(\gamma_{x,y}(1), \gamma_{x,y}(s)) &= (1-s)\gamma'_{x,y}(s) \\ &= (1-s)P_{0,\gamma_{x,y}}^s[\gamma'_{x,y}(0)] \\ &= (1-s)P_{0,\gamma_{x,y}}^s[\alpha(x, y)\eta(x, y)], \end{aligned}$$

or

$$P_{s,\gamma_{x,y}}^0[\alpha(x, \gamma_{x,y}(s))\eta(x, \gamma_{x,y}(s))] = (1-s)\alpha(x, y)\eta(x, y).$$

Hence

$$(C) \left. \begin{aligned} (C_1) P_{s,\gamma_{x,y}}^0[\alpha(y, \gamma_{x,y}(s))\eta(y, \gamma_{x,y}(s))] &= -s\alpha(x, y)\eta(x, y), \\ (C_2) P_{s,\gamma_{x,y}}^0[\alpha(x, \gamma_{x,y}(s))\eta(x, \gamma_{x,y}(s))] &= (1 - s)\alpha(x, y)\eta(x, y) \end{aligned} \right\}$$

for all $s \in [0, 1]$, which together called condition (C).

Theorem 4.1. *Let M be a Riemannian manifold and S be an open subset of M which is geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$. Let $f : S \rightarrow R$ be a differentiable and geodesic α -preinvex function on S . Then, f is an α -invex function on S .*

Proof. Since S is geodesic α -invex set with respect to η and α for all $x, y \in S$, there exists a unique geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\gamma_{x,y}(0) = y, \gamma'_{x,y}(0) = \alpha(x, y)\eta(x, y), \gamma_{x,y}(t) \in S, \text{ for all } t \in [0, 1].$$

But f is geodesic α -preinvex for $t \in (0, 1)$, we have

$$f(\gamma_{x,y}(t)) \leq tf(x) + (1 - t)f(y),$$

or

$$f(\gamma_{x,y}(t)) - f(y) \leq t(f(x) - f(y)).$$

On dividing by t ,

$$\frac{1}{t}[f(\gamma_{x,y}(t)) - f(y)] \leq f(x) - f(y).$$

Taking the limit as $t \rightarrow 0$, we have

$$df_{\gamma_{x,y}(0)}(\gamma'_{x,y}(0)) \leq f(x) - f(y).$$

Therefore,

$$df_y(\alpha(x, y)\eta(x, y)) \leq f(x) - f(y).$$

Hence, f is α -invex on S .

Theorem 4.2. *Let M be a Riemannian manifold and S be an open subset of M , which is geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$. Let $f : S \rightarrow R$ be a differentiable function and η satisfies the condition (C). Then f is geodesic α -preinvex on S if f is α -invex on S .*

Proof. We know that for geodesic α -invex set with respect to η and α for every $x, y \in S$, there exists a unique geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\gamma_{x,y}(0) = y, \gamma'_{x,y}(0) = \alpha(x, y)\eta(x, y), \gamma_{x,y}(t) \in S, \text{ for all } t \in [0, 1].$$

Fix $t \in [0, 1]$ and set $\bar{x} := \gamma_{x,y}(t)$, then by geodesic α -invexity of f on S , we have

$$(1) \quad f(x) - f(\bar{x}) \geq df_{\bar{x}}(\alpha(x, \bar{x})\eta(x, \bar{x})),$$

$$(2) \quad f(y) - f(\bar{x}) \geq df_{\bar{x}}(\alpha(y, \bar{x})\eta(y, \bar{x})).$$

On multiplying (1) by t and (2) by $(1 - t)$, respectively, and then adding we get

$$(3) \quad tf(x) + (1 - t)f(y) - f(\bar{x}) \geq df_{\bar{x}}[t\alpha(x, \bar{x})\eta(x, \bar{x}) + (1 - t)\alpha(y, \bar{x})\eta(y, \bar{x})].$$

By the condition (C), we have

$$(4) \quad \begin{aligned} t\alpha(x, \bar{x})\eta(x, \bar{x}) + (1 - t)\alpha(y, \bar{x})\eta(y, \bar{x}) &= t(1 - t)P_{0,\gamma}^t[\alpha(x, y)\eta(x, y)] \\ &\quad - (1 - t)tP_{0,\gamma}^t[\alpha(x, y)\eta(x, y)] \\ &= 0. \end{aligned}$$

This together with (3) implies

$$tf(x) + (1 - t)f(y) \geq f(\bar{x}).$$

Hence, f is geodesic α -preinvex on S .

5. GEODESIC α -PREINVEXITY AND SEMICONTINUITY

In this section, we discuss α -preinvexity on Riemannian manifold under proximal subdifferential of a lower semicontinuous function. First, we recall the definition of a proximal subdifferential of a function defined on a Riemannian manifold in [5].

Definition 5.1. Let M be a Riemannian manifold and $f : M \rightarrow (-\infty, \infty]$ be a lower semicontinuous function. A point $\zeta \in T_y M$ is said to be a proximal subgradient of f at $y \in \text{dom}(f)$, if there exist positive numbers δ and σ such that

$$f(x) \geq f(y) + \langle \zeta, \exp_y^{-1} x \rangle - \sigma d^2(x, y), \text{ for all } x \in B(y, \delta),$$

where $\text{dom}(f) := \{x \in M : f(x) < \infty\}$.

The set of all proximal subgradients of f at $y \in M$ is denoted by $\partial_p f(y)$ and is called the proximal subdifferential of f at y .

Theorem 5.1. Let M be a Riemannian manifold and S be an open subset of M , which is geodesic α -invex with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$. Let $f : S \rightarrow R$ be geodesic α -preinvex. If $\bar{x} \in S$ is a local minimum of the problem

$$(P) \quad \begin{aligned} &\text{Minimize } f(x) \\ &\text{subject to } x \in S, \end{aligned}$$

then \bar{x} is a global minimum of (P) .

Proof. Let $\bar{x} \in S$ be a local minimum. Then, there exists a neighborhood $N_\epsilon(\bar{x})$ such that

$$(5) \quad f(\bar{x}) \leq f(x), \text{ for all } x \in S \cap N_\epsilon(\bar{x}).$$

If \bar{x} is not a global minimum of f , then there exists a point $x^* \in S$ such that

$$f(x^*) < f(\bar{x}).$$

As S is a geodesic α -invex set with respect to η and α , there exists a unique geodesic γ such that

$$\gamma(0) = \bar{x}, \gamma'(0) = \alpha(x^*, \bar{x})\eta(x^*, \bar{x}), \gamma(t) \in S, \text{ for all } t \in [0, 1].$$

If we choose $\epsilon > 0$ such that $d(\gamma(t), \bar{x}) < \epsilon$, then $\gamma(t) \in N_\epsilon(\bar{x})$. From the geodesic α -preinvexity of f , we have

$$f(\gamma(t)) \leq tf(x^*) + (1 - t)f(\bar{x}) < f(\bar{x}), \text{ for all } t \in (0, 1).$$

Therefore, for each $\gamma(t) \in S \cap N_\epsilon(\bar{x})$, $f(\gamma(t)) < f(\bar{x})$, which is a contradiction to (5).

Theorem 5.2. *Let M be a Cartan-Hadamard manifold and S be an open subset of M , which is geodesic α -invex with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$ with $\alpha(x, y)\eta(x, y) \neq 0$ for all $x \neq y$. Assume that $f : S \rightarrow (-\infty, \infty]$ is a lower semicontinuous geodesic α -preinvex function and $y \in \text{dom}(f)$, $\zeta \in \partial_p f(y)$. Then there exists a positive number δ such that*

$$(6) \quad f(x) \geq f(y) + \langle \zeta, \alpha(x, y)\eta(x, y) \rangle_y, \text{ for all } x \in S \cap B(y, \delta).$$

Proof. From the definition of $\partial_p f(y)$, there are positive numbers δ and σ such that

$$(7) \quad f(x) \geq f(y) + \langle \zeta, \exp_y^{-1} x \rangle_y - \sigma d^2(x, y), \text{ for all } x \in B(y, \delta).$$

Now, fix $x \in S \cap B(y, \delta)$. Since S is a geodesic α -invex set with respect to η and α , there exists a unique geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\gamma_{x,y}(0) = y, \gamma'_{x,y}(0) = \alpha(x, y)\eta(x, y), \gamma_{x,y}(t) \in S, \text{ for all } t \in [0, 1].$$

Given that M is a Cartan-Hadamard manifold, then $\gamma_{x,y}(t) = \exp_y(t\alpha(x, y)\eta(x, y))$ for each $t \in [0, 1]$ (see [9, p. 253]). If we choose $t_0 = \frac{\delta}{\|\alpha(x, y)\eta(x, y)\|_y}$, then $\exp_y(t\alpha(x, y)\eta(x, y)) \in S \cap B(y, \delta)$ for all $t \in (0, t_0)$.

From the geodesic α -preinvexity of f , we get

$$(8) \quad f(\exp_y(t\alpha(x, y)\eta(x, y))) \leq tf(x) + (1 - t)f(y) \text{ for all, } t \in (0, t_0).$$

Using (7) for each $t \in (0, t_0)$, we get

$$\begin{aligned}
 (9) \quad f(\exp_y(t\alpha(x, y)\eta(x, y))) &\geq f(y) + \langle \zeta, \exp_y^{-1} \exp_y(t\alpha(x, y)\eta(x, y)) \rangle_y \\
 &\quad - \sigma d^2(\exp_y(t\alpha(x, y)\eta(x, y)), y) \\
 &= f(y) + \langle \zeta, t\alpha(x, y)\eta(x, y) \rangle_y \\
 &\quad - \sigma d^2(\exp_y(t\alpha(x, y)\eta(x, y)), y).
 \end{aligned}$$

Since M is a Cartan-Hadamard manifold, for each $t \in (0, t_0)$, we have

$$d^2(\exp_y(t\alpha(x, y)\eta(x, y)), y) = \|t\alpha(x, y)\eta(x, y)\|_y^2 = t^2\|\alpha(x, y)\eta(x, y)\|_y^2.$$

Thus, from (8) and (9), we have

$$\begin{aligned}
 tf(x) + (1-t)f(y) &\geq f(\exp_y(t\alpha(x, y)\eta(x, y))) \\
 &\geq f(y) + \langle \zeta, t\alpha(x, y)\eta(x, y) \rangle_y - \sigma t^2\|\alpha(x, y)\eta(x, y)\|_y^2.
 \end{aligned}$$

Hence

$$t(f(x) - f(y)) \geq t\langle \zeta, \alpha(x, y)\eta(x, y) \rangle_y - \sigma t^2\|\alpha(x, y)\eta(x, y)\|_y^2.$$

Dividing by t we obtain

$$f(x) - f(y) \geq \langle \zeta, \alpha(x, y)\eta(x, y) \rangle_y - \sigma t\|\alpha(x, y)\eta(x, y)\|_y^2.$$

Taking limit as $t \rightarrow 0$, we obtain

$$f(x) - f(y) \geq \langle \zeta, \alpha(x, y)\eta(x, y) \rangle_y.$$

Note that $x \in S \cap B(y, \delta)$ is arbitrary. Then relation (6) holds for all $x \in S \cap B(y, \delta)$.

Corollary 5.1. *Let M be a Cartan-Hadamard manifold and S be an open subset of M which is geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$. Assume that $f : S \rightarrow R$ be a lower semicontinuous geodesic α -preinvex function. Let $y \in S$ and $0 \in \partial_p f(y)$. Then, y is a global minimum of f .*

6. MEAN VALUE INEQUALITY AND MEAN VALUE THEOREM

In this section, we introduce mean value inequality and mean value theorem for Cartan-Hadamard manifold which are the extension of the results proved by Antczak [1].

Definition 6.1. Let S be a nonempty subset of a Riemannian manifold M , which is geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$;

and x and u be two arbitrary points of S . Let $\gamma : [0, 1] \rightarrow M$ be the unique geodesic such that

$$\gamma(0) = u, \gamma'(0) = \alpha(x, u)\eta(x, u), \gamma(t) \in S, \text{ for all } t \in [0, 1].$$

A set P_{uv} is said to be a closed η -path joining the points u and $v := \gamma(1)$, if

$$P_{uv} := \{y : y = \gamma(t); t \in [0, 1]\}.$$

An open η -path joining the points u and v is a set of the form

$$P_{uv}^0 := \{y : y = \gamma(t); t \in (0, 1)\}.$$

If $u = v$ we set $P_{uv}^0 := \emptyset$.

Example 6.1. Suppose that M is a Cartan-Hadamard manifold and S is a geodesic α -invex set with respect to η and α defined in Ex 3.2. Let x and u be two arbitrary points of S and $\gamma(t) := \exp_u(t\alpha(x, u)\eta(x, u))$. Then, for $u \in S_1$, $x \in S_2$ and $\alpha(x, u) = 1$, we have $v := \gamma(1) = P_{S_1}(x)$, and P_{uv} is the unique geodesic with end points u and $P_{S_1}(x)$.

Theorem 6.1. (Mean value inequality). *Let M be a Cartan-Hadamard manifold and S be an open subset of M , which is geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$ such that $\alpha(a, b)\eta(a, b) \neq 0$ for all $a, b \in S, a \neq b$. Let $\gamma_{b,a}(t) = \exp_a(t\alpha(b, a)\eta(b, a))$ for every $a, b \in S, t \in [0, 1]$ and $c = \gamma_{b,a}(1)$. Then, a necessary and sufficient condition for a function $f : S \rightarrow R$ to be geodesic α -preinvex is that the inequality*

$$(10) \quad f(x) \leq f(a) + \frac{f(b) - f(a)}{\alpha(b, a)\langle \eta(b, a), \eta(b, a) \rangle_a} \langle \exp_a^{-1} x, \eta(b, a) \rangle_a$$

is true for all $x \in P_{ca}$.

Proof. Let $f : S \rightarrow R$ be a geodesic α -preinvex function, $a, b \in S$ and $x \in P_{ca}$. If $x = a$ or $x = c$ then (10) is true trivially. If $x \in P_{ca}^0$, then $x := \exp_a(t\alpha(b, a)\eta(b, a))$, for some $t \in (0, 1)$. From the geodesic α -invexity of S , we have $x \in S$ and

$$t = \frac{\langle \exp_a^{-1} x, \alpha(b, a)\eta(b, a) \rangle_a}{\langle \alpha(b, a)\eta(b, a), \alpha(b, a)\eta(b, a) \rangle_a} = \frac{\langle \exp_a^{-1} x, \eta(b, a) \rangle_a}{\alpha(b, a)\langle \eta(b, a), \eta(b, a) \rangle_a}.$$

Since f is geodesic α -preinvex on S , it follows that

$$\begin{aligned} f(x) &= f(\exp_a(t\alpha(b, a)\eta(b, a))) \\ &\leq tf(b) + (1 - t)f(a) \\ &= f(a) + t[f(b) - f(a)] \\ &= f(a) + \frac{f(b) - f(a)}{\alpha(b, a)\langle \eta(b, a), \eta(b, a) \rangle_a} \langle \exp_a^{-1} x, \eta(b, a) \rangle_a. \end{aligned}$$

For sufficiency suppose that the mean value inequality (10) is true. Let $a, b \in S$ and $x := \exp_a(t\alpha(b, a)\eta(b, a))$, for some $t \in [0, 1]$. Then $x \in S$, and we have

$$\begin{aligned} f(x) &= f(\exp_a(t\alpha(b, a)\eta(b, a))) \\ &\leq f(a) + \frac{f(b) - f(a)}{\alpha(b, a)\langle \eta(b, a), \eta(b, a) \rangle_a} \langle \exp_a^{-1}(\exp_a(t\alpha(b, a)\eta(b, a))), \eta(b, a) \rangle_a \\ &= f(a) + \frac{f(b) - f(a)}{\alpha(b, a)\langle \eta(b, a), \eta(b, a) \rangle_a} t\alpha(b, a)\langle \eta(b, a), \eta(b, a) \rangle_a \\ &= f(a) + t[f(b) - f(a)] \\ &= tf(b) + (1 - t)f(a), \end{aligned}$$

which shows that f is a geodesic α -preinvex function on S .

Theorem 6.2. (Mean value theorem). *Let M be a Cartan-Hadamard manifold and S be an open subset of M , which is nonempty, open geodesic α -invex set with respect to $\eta : M \times M \rightarrow TM$ and $\alpha : M \times M \rightarrow R \setminus \{0\}$. Let $f : S \rightarrow R$ be differentiable on S . Then, for every $a, b \in S$ there exists $c \in P_{ab}^0$ such that*

$$f(\exp_a(\alpha(b, a)\eta(b, a))) - f(a) = df_c[(d \exp_a)_u(\alpha(b, a)\eta(b, a))],$$

where $u := t_0\alpha(b, a)\eta(b, a)$, $t_0 \in (0, 1)$ and $(d \exp_a)_u : T_u(T_aM) \cong T_aM \rightarrow T_cM$ is differential of \exp_a at u .

Proof. Let us define the function $g : [0, 1] \rightarrow R$ as follows

$$(11) \quad g(t) := f(\exp_a(t\alpha(b, a)\eta(b, a))) - f(a) - t[f(\exp_a(\alpha(b, a)\eta(b, a))) - f(a)].$$

Since $g(1) = g(0) = 0$, using Rolle's theorem, it follows that there exists $t_0 \in (0, 1)$ such that $g'(t_0) = 0$. Let $c := \exp_a(t_0\alpha(b, a)\eta(b, a))$, then from (11) we have

$$0 = g'(t_0) = df_c[(d \exp_a)_u(\alpha(b, a)\eta(b, a))] - f(\exp_a(\alpha(b, a)\eta(b, a))) + f(a).$$

or

$$f(\exp_a(\alpha(b, a)\eta(b, a))) - f(a) = df_c[(d \exp_a)_u(\alpha(b, a)\eta(b, a))].$$

Since $t_0 \in (0, 1)$, it follows from the definition that $c \in P_{ab}^0$, and the proof is complete.

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R. P. Agarwal
Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
P. O. Box 728, Dhahran 31261
Saudi Arabia
Permanant Address:
Department of Mathematics,
Texas A&M University - Kingsville
700 University Blvd.
Kingsville, TX 78363-8202
U.S.A.
E-mail: agarwal@tamuk.edu

I. Ahmad
Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
P. O. Box 728, Dhahran 31261
Saudi Arabia
Permanant Address:
Department of Mathematics
Aligarh Muslim University
Aligarh-202 002
India
E-mail: drizhar@kfupm.edu.sa

Akhlag Iqbal
Department of Mathematics
BITS Pilani Hyderabad Campus
Hyderabad, AP-500078
India
E-mail: akhlag@bits-hyderabad.ac.in

Shahid Ali
Department of Mathematics
Aligarh Muslim University
Aligarh-202 002
India
E-mail: shahid_rrs@yahoo.co.in