TOKYO J. MATH.
VOL. 31, NO. 1, 2008

Generalized (κ, μ)-contact Metric Manifolds with $\xi\mu = 0$

Themis KOUFOGIORGOS[1] and Charalambos TSICHLIAS

University of Ioannina[1]

(Communicated by M. Guest)

Abstract. This paper analytically describes the local geometry of a generalized (κ, μ)-manifold $M(\eta, \xi, \phi, g)$ with $\kappa < 1$ which satisfies the condition "the function μ is constant along the integral curves of the characteristic vector field ξ". This class of manifolds is especially rich, since it is possible to construct in R^3 two families of such manifolds, for any smooth function κ ($\kappa < 1$) of one variable. Every family is determined by two arbitrary functions of one variable.

1. Introduction

The class of 3-dimensional generalized (κ, μ)-contact metric manifolds, which we study in this paper, is important because it contains several interesting classes of Riemannian manifolds, such as Sasakian, η-Einstein and (κ, μ)-contact metric manifolds. In what follows in this section we refer to these classes of manifolds as well as to our motivation to study generalized (κ, μ)-contact metric manifolds which satisfy the condition $\xi\mu = 0$.

In [2] Blair, Koufogiorgos and Papantoniou studied for the first time the class of $(2m+1)$-dimensional contact metric manifolds $M(\eta, \xi, \phi, g)$ for which the vector field ξ belongs to the (κ, μ)-nullity distribution, for some real numbers κ and μ ($\kappa \leq 1$). The curvature tensor R of the above class of manifolds satisfies the condition

$$R(X, Y)\xi = (\kappa I + \mu h)[\eta(Y)X - \eta(X)Y] \tag{$*$}$$

for all vector fields $X, Y \in \mathcal{X}(M)$, where I is the identity and h denotes, up to a scaling factor, the Lie derivative of the structure tensor ϕ in the direction of ξ. For convenience, we will call such a contact metric manifold a "(κ, μ)-manifold". The special case $\kappa = 1$ characterizes the well known class of Sasakian manifolds, while the case $\mu = 0$ characterizes the class of η-Einstein manifolds. Within contact geometry, (κ, μ)-manifolds received attention mainly because the unit tangent sphere bundle of a Riemannian manifold of constant curvature belongs to this class. A (κ, μ)-manifold with $\kappa < 1$, is locally homogeneous and its local geometry is now completely known (see [2], [3], [4]). In particular, a 3-dimensional (κ, μ)-manifold with $\kappa < 1$, is locally isometric to one of the Lie groups $SU(2)$, $SO(3)$, $SL(2, R)$, $O(1, 2)$, $E(2)$, $E(1, 1)$ equipped with a left invariant metric (see [2] for more details).

Received July 24, 2006; revised November 16, 2007

In [5] the authors of the present paper gave an answer to the following question: Do contact metric manifolds exist satisfying the condition (*), with κ, μ non-constant smooth functions ? The answer is affirmative only for the 3-dimensional case. So in [5] a new class of 3-dimensional contact metric manifolds was introduced. A manifold of this class will be referred to as "a generalized (κ, μ)-manifold". We note that in contrast to (κ, μ)-manifolds the generalized (κ, μ)-manifolds are not locally homogeneous. Within contact geometry, a generalized (κ, μ)-manifold, with $\kappa < 1$, $M(\eta, \xi, \phi, g)$ is characterized by the fact that the vector field ξ defines almost everywhere in M a harmonic map from M into its unit tangent sphere bundle T_1M equipped with the Sasakian metric [7]. In [6] the generalized (κ, μ)-manifolds, which satisfy the assumption $\| \operatorname{grad} \kappa \| = c$ (constant $\neq 0$) have been studied. These manifolds satisfy the condition $\xi\mu = 0$ as well. On the other hand it is well known [5, examples 1, 2] that there exist generalized (κ, μ)-manifolds with $\xi\mu = 0$ and non-constant $\| \operatorname{grad} \kappa \|$. This has been our motivation for studying generalized (κ, μ)-manifolds with $\xi\mu = 0$. We would like to emphasize that, as will be shown in this paper, the class of generalized (κ, μ)-manifolds with $\xi\mu = 0$ is much more interesting than the class of generalized (κ, μ)-manifolds with $\| \operatorname{grad} \kappa \| =$ constant. For example, in the latter class the scalar curvature is a non-constant negative function, while the first class includes manifolds in which the scalar curvature can have any sign or be constant.

The paper is organized as follows. Section 2 contains necessary details about contact metric manifolds. In section 3, we give some results concerning generalized (κ, μ)-manifolds. In the last section we locally classify and construct any generalized (κ, μ)-manifold with $\xi\mu = 0$. All manifolds are assumed to be connected.

2. Preliminaries

In this section we collect some basic facts about contact metric manifolds. We refer the reader to [1] for a more detailed treatment. A differentiable $(2m+1)$- dimensional manifold M is called a **contact manifold** if it carries a global differential 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on M. The form η is usually called the **contact form** of M. It is well known that a contact manifold admits an almost contact metric structure (η, ξ, ϕ, g), i.e. a global vector field ξ, which is called the **characteristic vector field**, a $(1, 1)$-tensor field ϕ and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi\,, \quad \eta(\xi) = 1\,, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\,, \tag{2.1}$$

for all vector fields $X, Y \in \mathcal{X}(M)$. Moreover, (η, ξ, ϕ, g) can be chosen such that

$$d\eta(X, Y) = g(X, \phi Y)\,, \quad X, Y \in \mathcal{X}(M) \tag{2.2}$$

and we then call the structure a **contact metric structure**. A manifold M carrying such a structure is said to be a **contact metric manifold** and it is denoted by $M(\eta, \xi, \phi, g)$. As a consequence of the above relations we have $\eta(\xi) = 1$, $\phi\xi = 0$, $\eta \circ \phi = 0$ and $d\eta(\xi, X) = 0$. If ∇ denotes the Riemannian connection of $M(\eta, \xi, \phi, g)$, then following [1], we define the $(1,1)$-tensor fields h and l by $h = (1/2)(\mathcal{L}_\xi\phi)$ and $l = R(.,\xi)\xi$, where $\mathcal{L}_\xi$ is the Lie differentiation in the direction of ξ and R is the curvature tensor, which is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,, \tag{2.3}$$

for all vector fields $X, Y, Z \in \mathcal{X}(M)$. The tensor fields h, l are self adjoint and satisfy $h\xi = 0$, $l\xi = 0$, $\operatorname{Tr} h = \operatorname{Tr} h\phi = 0$, $\phi h + h\phi = 0$. Since h anti-commutes with ϕ, if $X \neq 0$ is an eigenvector of h corresponding to the eigenvalue λ, then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. Therefore, on any contact metric manifold $M(\eta,\xi,\phi,g)$ the following formulas are valid $\nabla\xi = -\phi - \phi h$ (and so $\nabla_\xi\xi = 0$), $\nabla_\xi h = \phi - \phi l - \phi h^2$, $\nabla_\xi\phi = 0$ and $\phi l\phi - l = 2(\phi^2 + h^2)$. A contact metric structure (η,ξ,ϕ,g) on M gives rise to an almost complex structure on the product $M \times R$. If this structure is integrable, then the contact metric manifold $M(\eta,\xi,\phi,g)$ is said to be Sasakian. Equivalently, a contact metric manifold $M(\eta,\xi,\phi,g)$ is Sasakian if and only if $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$, for all $X, Y \in \mathcal{X}(M)$.

By a **generalized** (κ,μ)-**manifold** we mean a 3-dimensional contact metric manifold such that

$$R(X,Y)\xi = (\kappa I + \mu h)[\eta(Y)X - \eta(X)Y]\,, \tag{2.4}$$

for all $X, Y \in \mathcal{X}(M)$, where κ, μ are smooth non-constant real functions on M. In the special case, where κ, μ are constant, then $M(\eta,\xi,\phi,g)$ is called a (κ,μ)-**manifold**. We note that $h = 0$ and $\kappa = 1$ on any Sasakian manifold.

Let M be a $(2m+1)$-dimensional contact metric manifold. By a D_a-**homothetic deformation** [8], we mean a change of structure tensors of the form

$$\bar{\eta} = a\eta\,, \quad \bar{\xi} = (1/a)\xi\,, \quad \bar{\phi} = \phi\,, \quad \bar{g} = ag + a(a-1)\eta\otimes\eta\,, \tag{2.5}$$

where a is a positive number. It is well known that $M(\bar{\eta},\bar{\xi},\bar{\phi},\bar{g})$ is also a contact metric manifold. The tensor h and the curvature tensor R transform in the following manner ([2]):

$$\bar{h} = (1/a)h \tag{2.6}$$

and

$$\begin{aligned} a\bar{R}(X,Y)\bar{\xi} = R(X,Y)\xi &+ (a-1)^2(\eta(Y)X - \eta(X)Y) \\ &- (a-1)\{(\nabla_X\phi)Y - (\nabla_Y\phi)X + \eta(X)(Y + hY) \\ &- \eta(Y)(X + hX)\}\,, \end{aligned} \tag{2.7}$$

for any $X, Y \in \mathcal{X}(M)$. Additionally, it is well known [9, pp 446–447], that any 3-dimensional contact metric manifold $M(\eta,\xi,\phi,g)$ satisfies

$$(\nabla_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{2.8}$$

for any $X, Y \in \mathcal{X}(M)$. Substituting (2.8) in (2.7) and using (2.6), (2.7), we see that if $M(\eta,\xi,\phi,g)$ is a generalized (κ,μ)-manifold, then $M(\bar{\eta},\bar{\xi},\bar{\phi},\bar{g})$ is also a generalized

$(\bar{\kappa}, \bar{\mu})$-manifold (see [5]) with

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2(a-1)}{a}. \tag{2.9}$$

Finally, we mention that on any Riemannian manifold (M, g), the metric g and the Riemannian connection ∇ are related by the formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \end{aligned} \tag{2.10}$$

for all $X, Y, Z \in \mathcal{X}(M)$.

3. Generalized (κ, μ)-manifolds

This section contains some basic results concerning generalized (κ, μ)-manifolds.

LEMMA 3.1. *On any generalized (κ, μ)-manifold $M(\eta, \xi, \phi, g)$ the following formulas are valid*

$$h^2 = (\kappa - 1)\phi^2, \quad \kappa = \frac{\operatorname{Tr} l}{2} \leq 1, \tag{3.1}$$

$$\xi\kappa = 0, \tag{3.2}$$

$$h \operatorname{grad} \mu = \operatorname{grad} \kappa, \tag{3.3}$$

$$Q\xi = 2\kappa\xi, \tag{3.4}$$

where Q is the Ricci operator ($QX = \sum_{i=1}^{3} R(X, E_i)E_i$, where $\{E_i\}$, $i = 1, 2, 3$, is an orthonormal frame and $X \in \mathcal{X}(M)$).

PROOF. For the proof of Lemma see [6].

LEMMA 3.2. *Let $M(\eta, \xi, \phi, g)$ be a generalized (κ, μ)-manifold. Then, for any point $P \in M$, with $\kappa(P) < 1$ there exist a neighbourhood U of P and an h-frame on U, i.e. orthonormal vector fields $\xi, X, \phi X$, defined on U, such that*

$$hX = \lambda X, \quad h\phi X = -\lambda\phi X, \quad h\xi = 0, \quad \lambda = \sqrt{1-\kappa} \tag{3.5}$$

at any point $q \in U$. Moreover, putting $A = X\lambda$ and $B = \phi X\lambda$, the following formulas are valid on U:

$$\nabla_X \xi = -(\lambda + 1)\phi X, \quad \nabla_{\phi X} \xi = (1 - \lambda)X, \tag{3.6}$$

$$\nabla_\xi X = -\frac{\mu}{2}\phi X, \quad \nabla_\xi \phi X = \frac{\mu}{2}X, \tag{3.7}$$

$$\nabla_X X = \frac{B}{2\lambda}\phi X, \quad \nabla_{\phi X}\phi X = \frac{A}{2\lambda}X, \tag{3.8}$$

$$\nabla_{\phi X} X = -\frac{A}{2\lambda}\phi X + (\lambda - 1)\xi, \quad \nabla_X \phi X = -\frac{B}{2\lambda}X + (\lambda + 1)\xi, \tag{3.9}$$

$$[\xi, X] = \left(1 + \lambda - \frac{\mu}{2}\right)\phi X\,, \qquad [\xi, \phi X] = \left(\lambda - 1 + \frac{\mu}{2}\right)X\,, \tag{3.10}$$

$$[X, \phi X] = -\frac{B}{2\lambda}X + \frac{A}{2\lambda}\phi X + 2\xi\,, \tag{3.11}$$

$$X\mu = -2X\lambda = -2A\,, \tag{3.12}$$

$$\phi X\mu = 2\phi X\lambda = 2B\,, \tag{3.13}$$

$$\xi A = \left(1 + \lambda - \frac{\mu}{2}\right)B\,, \tag{3.14}$$

$$\xi B = \left(\lambda - 1 + \frac{\mu}{2}\right)A\,, \tag{3.15}$$

$$[\xi, \phi\, \mathrm{grad}\,\lambda] = 0\,, \tag{3.16}$$

$$(\phi\, \mathrm{grad}\,\lambda)\mu = 4AB\,, \tag{3.17}$$

$$XB = \phi XA = \frac{1}{2}\left\{\xi\mu + \frac{1}{4\lambda}(\phi\, \mathrm{grad}\,\lambda)\mu\right\} = \frac{1}{2}\left(\xi\mu + \frac{1}{\lambda}AB\right), \tag{3.18}$$

$$\Delta\lambda = XA + \phi XB - \frac{1}{2\lambda}(A^2 + B^2)\,, \tag{3.19}$$

$$\xi XA = 2\left(1 + \lambda - \frac{\mu}{2}\right)XB + 2AB\,, \tag{3.20}$$

$$\xi\phi XB = 2\left(\lambda - 1 + \frac{\mu}{2}\right)XB + 2AB\,, \tag{3.21}$$

$$\xi\|\,\mathrm{grad}\,\lambda\|^2 = \xi(A^2 + B^2) = 4\lambda AB\,, \tag{3.22}$$

$$\xi\Delta\lambda = 2\lambda\xi\mu + 4AB\,, \tag{3.23}$$

where $\Delta\lambda$ is the Laplacian of λ, ($\Delta\lambda = \mathrm{div}\,\mathrm{grad}\,\lambda$).

PROOF. For the proofs of (3.5)–(3.11) see [5], [6]. The proofs of (3.12), (3.13) are immediate consequences of (3.3), (3.5) and the symmetry of h. In order to prove (3.14) we calculate, using (3.2) and (3.10),

$$\xi A = \xi X\lambda = [\xi, X]\lambda + X\xi\lambda = \left(1 + \lambda - \frac{\mu}{2}\right)\phi X\lambda = \left(1 + \lambda - \frac{\mu}{2}\right)B\,.$$

The relation (3.15) is proved similarly. Using (3.2) and the first of (2.1) we have

$$\mathrm{grad}\,\lambda = AX + B\phi X\,, \quad \phi\,\mathrm{grad}\,\lambda = A\phi X - BX\,.$$

From the last relation, (3.10), (3.14) and (3.15) we obtain

$$\begin{aligned}[\xi, \phi\,\mathrm{grad}\,\lambda] &= [\xi, A\phi X - BX]\\ &= (\xi A)\phi X + A[\xi, \phi X] - (\xi B)X - B[\xi, X] = 0\,.\end{aligned}$$

In order to prove (3.17) we use (3.12) and (3.13) and we obtain

$$(\phi \operatorname{grad} \lambda)\mu = (A\phi X - BX)\mu = A\phi X\mu - BX\mu = 4AB\,.$$

Letting the vector field $[X, \phi X]$, given by (3.10), act on the function λ and by using (3.2), we obtain

$$X(\phi X\lambda) - \phi X(X\lambda) = -\frac{B}{2\lambda}X\lambda + \frac{A}{2\lambda}\phi X\lambda + 2\xi\lambda$$

or,

$$XB - \phi XA = -\frac{AB}{2\lambda} + \frac{AB}{2\lambda} = 0\,.$$

Similarly, from the action of vector field $[X, \phi X]$ on the function μ and the use of the last relation, (3.12), (3.13) and (3.17) we obtain

$$XB = \frac{1}{2}\left(\xi\mu + \frac{1}{\lambda}AB\right) = \frac{1}{2}\left\{\xi\mu + \frac{1}{4\lambda}(\phi \operatorname{grad} \lambda)\mu\right\}.$$

Using the definition of the Laplacian and the relations (3.2), (3.8), (3.18) we obtain

$$\begin{aligned}\Delta\lambda &= XX\lambda + \phi X\phi X\lambda + \xi\xi\lambda - (\nabla_X X)\lambda - (\nabla_{\phi X}\phi X)\lambda - (\nabla_\xi \xi)\lambda\\ &= XA + \phi XB - \frac{1}{2\lambda}(A^2 + B^2)\,.\end{aligned}$$

For the proofs of (3.21), (3.22), using (3.10), (3.12)–(3.15), (3.18), we calculate

$$\begin{aligned}\xi XA &= [\xi, X]A + X\xi A = \left(1 + \lambda - \frac{\mu}{2}\right)\phi XA + X\left\{\left(1 + \lambda - \frac{\mu}{2}\right)B\right\}\\ &= \left(1 + \lambda - \frac{\mu}{2}\right)XB + \left(1 + \lambda - \frac{\mu}{2}\right)XB + B\left\{X\lambda - X(\frac{\mu}{2})\right\}\\ &= 2\left(1 + \lambda - \frac{\mu}{2}\right)XB + 2AB\,,\end{aligned}$$

$$\begin{aligned}\xi\phi XB &= [\xi, \phi X]B + \phi X\xi B = \left(\lambda - 1 + \frac{\mu}{2}\right)XB + \phi X\left\{\left(\lambda - 1 + \frac{\mu}{2}\right)A\right\}\\ &= \left(\lambda - 1 + \frac{\mu}{2}\right)XB + \left(\lambda - 1 + \frac{\mu}{2}\right)\phi XA + A\left\{\phi X\lambda + \phi X\left(\frac{\mu}{2}\right)\right\}\\ &= 2\left(\lambda - 1 + \frac{\mu}{2}\right)XB + 2AB\,.\end{aligned}$$

The relation (3.22) is an immediate consequence of (3.14) and (3.15). Differentiating (3.19) with respect to ξ and using (3.20)–(3.22), (3.2) and (3.18), then (3.23) follows, and thus the proof of Lemma is completed.

LEMMA 3.3. *On any generalized (κ,μ)-manifold $M(\eta,\xi,\phi,g)$ with $\kappa < 1$, the scalar curvature $S = \operatorname{Tr} Q$ is given by*

$$S = \frac{1}{\lambda}\Delta\lambda - \frac{1}{\lambda^2}\|\operatorname{grad}\lambda\|^2 + 2(\kappa-\mu)\,, \quad \lambda = \sqrt{1-\kappa}\,. \tag{3.24}$$

PROOF. Using (2.3), (3.6)–(3.9), we calculate

$$\begin{aligned}
R(X,\phi X)\phi X &= \nabla_X\nabla_{\phi X}\phi X - \nabla_{\phi X}\nabla_X\phi X - \nabla_{[X,\phi X]}\phi X\\
&= \nabla_X\Big(\frac{A}{2\lambda}X\Big) - \nabla_{\phi X}\Big(-\frac{B}{2\lambda}X + (1+\lambda)\xi\Big) - \nabla_{-\frac{B}{2\lambda}X+\frac{A}{2\lambda}\phi X+2\xi}\phi X\\
&= X\Big(\frac{A}{2\lambda}\Big)X + \frac{A}{2\lambda}\nabla_X X + \phi X\Big(\frac{B}{2\lambda}\Big)X + \frac{B}{2\lambda}\nabla_{\phi X}X\\
&\quad -(\phi X\lambda)\xi - (1+\lambda)\nabla_{\phi X}\xi + \frac{B}{2\lambda}\nabla_X\phi X - \frac{A}{2\lambda}\nabla_{\phi X}\phi X - 2\nabla_\xi\phi X\\
&= \frac{\lambda XA - A^2}{2\lambda^2}X + \frac{AB}{4\lambda^2}\phi X + \frac{\lambda\phi XB - B^2}{2\lambda^2}X\\
&\quad + \frac{B}{2\lambda}\Big(-\frac{A}{2\lambda}\phi X + (\lambda-1)\xi\Big) - B\xi - (1+\lambda)(1-\lambda)X\\
&\quad + \frac{B}{2\lambda}\Big(-\frac{B}{2\lambda}X + (1+\lambda)\xi\Big) - \frac{A^2}{4\lambda^2}X - \mu X\\
&= \Big\{\frac{1}{2\lambda}(XA + \phi XB) - \frac{1}{2\lambda^2}(A^2+B^2) - (1-\lambda^2) - \frac{1}{4\lambda^2}(A^2+B^2) - \mu\Big\}X\\
&= \Big\{\frac{1}{2\lambda}\Big(XA + \phi XB - \frac{1}{2\lambda}(A^2+B^2)\Big) - \frac{1}{2\lambda^2}(A^2+B^2) - \kappa - \mu\Big\}X\,.
\end{aligned}$$

Combining this and (3.19) we obtain

$$R(X,\phi X)\phi X = \Big\{\frac{1}{2\lambda}\Delta\lambda - \frac{1}{2\lambda^2}(A^2+B^2) - \kappa - \mu\Big\}X$$

and thus

$$g(R(X,\phi X)\phi X, X) = \frac{1}{2\lambda}\Delta\lambda - \frac{1}{2\lambda^2}(A^2+B^2) - \kappa - \mu\,.$$

The relation (3.24) is an immediate consequence of (3.5), (3.4) and $S = \operatorname{Tr} Q = g(QX,X) + g(Q\phi X,\phi X) + g(Q\xi,\xi)$.

4. Generalized (κ,μ)-manifolds with $\xi\mu = 0$

In the following Theorem, the generalized (κ,μ)-manifolds with $\kappa < 1$ that satisfy the condition $\xi\mu = 0$, are locally described.

THEOREM 4.1. *Let $M(\eta, \xi, \phi, g)$ be a generalized (κ, μ)-manifold with $\kappa < 1$ and $\xi\mu = 0$. Then*

1) *At any point of M, precisely one of the following relations is valid: $\mu = 2(1 + \sqrt{1-\kappa})$, or $\mu = 2(1 - \sqrt{1-\kappa})$*

2) *At any point $P \in M$ there exists a chart $(U, (x, y, z))$ with $P \in U \subseteq M$, such that*

i) *the functions κ, μ depend only on the variable z*

ii) *if $\mu = 2(1 + \sqrt{1-\kappa})$,* (resp. $\mu = 2(1 - \sqrt{1-\kappa})$), *the tensor fields η, ξ, ϕ, g are given by the relations,*

$$\xi = \frac{\partial}{\partial x}, \quad \eta = dx - a dz \quad (\text{resp.} \quad \eta = dx - a dz)$$

$$g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1 + a^2 + b^2 \end{pmatrix} \quad \left(\text{resp.} \quad g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1 + a^2 + b^2 \end{pmatrix}\right)$$

$$\phi = \begin{pmatrix} 0 & a & -ab \\ 0 & b & -1 - b^2 \\ 0 & 1 & -b \end{pmatrix} \quad \left(\text{resp.} \quad \phi = \begin{pmatrix} 0 & -a & ab \\ 0 & -b & 1 + b^2 \\ 0 & -1 & b \end{pmatrix}\right)$$

with respect to the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, *where* $a = 2y + f(z)$ (resp. $a = -2y + f(z)$), $b = 2\lambda(z)x - \frac{\lambda'(z)}{2\lambda(z)}y + h(z)$, $\lambda = \lambda(z) = \sqrt{1 - \kappa(z)}$, $\lambda'(z) = \frac{d\lambda}{dz}$ *and $f(z), h(z)$ are arbitrary smooth functions of z.*

PROOF. Let $\{\xi, X, \phi X\}$ be an h-frame, such that

$$hX = \lambda X, \quad h\phi X = -\lambda\phi X, \quad \lambda = \sqrt{1-\kappa}$$

in an appropriate neighbourhood of an arbitrary point of M. Using the hypothesis $\xi\mu = 0$ and the relations (3.16), (3.17), (3.14), (3.15) of Lemma 3.2, we successively obtain

$$\begin{aligned}
&[\xi, \phi \operatorname{grad} \lambda]\mu = 0 \\
&\xi(\phi \operatorname{grad} \lambda)\mu - (\phi \operatorname{grad} \lambda)\xi\mu = 0 \\
&\xi(AB) = 0 \\
&A\xi B + B\xi A = 0 \\
&A^2\left(\lambda - 1 + \frac{\mu}{2}\right) + B^2\left(1 + \lambda - \frac{\mu}{2}\right) = 0\,.
\end{aligned}$$

Differentiating the last relation with respect to ξ and using the relations (3.2), $\xi\mu = 0$, (3.14), (3.15) we are led through simple calculations to

$$\left(1 + \lambda - \frac{\mu}{2}\right)\left(\lambda - 1 + \frac{\mu}{2}\right)AB = 0\,. \tag{4.1}$$

We put $F = \left(1+\lambda-\frac{\mu}{2}\right)\left(\lambda-1+\frac{\mu}{2}\right)$ and consider the set $N = \{P \in M | (\operatorname{grad}\lambda)(P) \neq 0\}$. We will prove that $F = 0$ at any point of N. Let $P \in N$ be such that $F(P) \neq 0$. From (4.1) we obtain $(AB)(P) = 0$. We distinguish the cases $\{A(P) = B(P) = 0\}$, $\{A(P) \neq 0, B(P) = 0\}$ and $\{A(P) = 0, B(P) \neq 0\}$. The first case is impossible, because the relations $A(P) = B(P) = 0$ and (3.2) lead to $(\operatorname{grad}\lambda)(P) = 0$. Let us suppose that $\{A(P) \neq 0, B(P) = 0\}$. Since the function F is continuous, we find that a neighbourhood $U \subseteq N$ exists, with $P \in U$ such that $F \neq 0$ at any point of U. Similarly, due to the fact that the function A is continuous on its domain, a neighbourhood V of P exists with $P \in V \subset U$, such that $A \neq 0$ at any point of V, and thus $B = 0$ on V. Differentiating $B = 0$ with respect to ξ and using (3.15) we obtain $A\left(1 + \lambda - \frac{\mu}{2}\right) = 0$. Therefore, $1 + \lambda - \frac{\mu}{2} = 0$ at any point of V and thus $F = 0$ on V, which is a contradiction. Similarly, by supposing that$\{A(P) = 0, B(P) \neq 0\}$ we are led to a contradiction. Therefore, $F = 0$ at any point of N. In what follows, we will work on the complement N^c of set N, in order to prove that $F = 0$ on M. If $N^c = \emptyset$, then $F = 0$ on M. If $N^c \neq \emptyset$, then $\operatorname{grad}\lambda = 0$ on N^c and thus the function λ is constant at any connected component of the interior $(N^c)^o$ of N^c. From the constancy of λ and the relations (3.12), (3.13), $\xi\mu = 0$, the function μ is also constant. As a result we find that F is constant on any connected component of $(N^c)^o$. Because M is connected and $F = 0$ on N and $F =$ constant on any connected component of $(N^c)^o$ we conclude that $F = 0$, or equivalently $\left(1 + \lambda - \frac{\mu}{2}\right)\left(\lambda - 1 + \frac{\mu}{2}\right) = 0$ at any point of M. In what follows, we consider the open and disjoint sets

$$C = \left\{P \in M \Big/ \left(1 + \lambda - \frac{\mu}{2}\right)(P) \neq 0\right\} \quad \text{and} \quad D = \left\{P \in M \Big/ \left(\lambda - 1 + \frac{\mu}{2}\right)(P) \neq 0\right\}.$$

We have $C \cup D = M$. In fact, if there was $P \in M$, with $P \notin C$ and $P \notin D$, then we would obtain $\lambda(P) = 0$, or equivalently $\kappa(P) = 1$, which is impossible by the assumption of the Theorem. Since M is connected we conclude that $\{C = M$ and $D = \emptyset\}$ or $\{C = \emptyset$ and $D = M\}$. Regarding the first case we obtain $1+\lambda-\frac{\mu}{2} = 0$, or equivalently $\mu = 2(1+\sqrt{1-\kappa})$ at any point of M. Similarly, regarding the second case we obtain $\mu = 2(1 - \sqrt{1-\kappa})$. Therefore, the proof of (1) is completed. Now, we will examine the cases $\mu = 2(1 + \sqrt{1-\kappa})$ and $\mu = 2(1 - \sqrt{1-\kappa})$ separately.

Case 1. $\mu = 2(1 + \sqrt{1-\kappa}) = 2(1 + \lambda)$.

Let $P \in M$ and $\{\xi, X, \phi X\}$ be an h-frame on an appropriate neighborhood V of P. From the assumption $\mu = 2(1 + \lambda)$ and (3.12) we obtain $A = 0$ and thus the relations (3.10), (3.11) are

$$[\xi, X] = 0\,, \quad [\xi, \phi X] = 2\lambda X\,, \quad [X, \phi X] = -\frac{B}{2\lambda}X + 2\xi\,. \tag{4.2}$$

Because the linearly independent vector fields ξ, X satisfy the relation $[\xi, X] = 0$ on V, the distribution which is spanned by ξ and X is integrable and so for any point $q \in V$, there exists a chart $(U, (x, y, z))$ such that $P \in U \subset V$ and

$$\xi = \frac{\partial}{\partial x}, \quad X = \frac{\partial}{\partial y} \tag{4.3}$$

at any point of U. The vector field ϕX can be written on U as

$$\phi X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}, \tag{4.4}$$

where a, b, c are smooth functions defined on U. Since $\xi, X, \phi X$ are linearly independent, we have $c \neq 0$ at any point of U. By using (4.3), (3.2) and $X\lambda = A = 0$ we obtain

$$\frac{\partial \lambda}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \lambda}{\partial y} = 0\,.$$

From these relations we conclude that the function λ depends only on the variable z, i.e. $\lambda = \lambda(z)$, and thus from (4.4) we obtain

$$B = \phi X\lambda = c\frac{\partial \lambda}{\partial z}\,. \tag{4.5}$$

By using (4.2)–(4.4) we obtain

$$\begin{aligned} 2\lambda\frac{\partial}{\partial y} = 2\lambda X = [\xi, \phi X] &= \left[\frac{\partial}{\partial x}, a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right] \\ &= \frac{\partial a}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b}{\partial x}\frac{\partial}{\partial y} + \frac{\partial c}{\partial x}\frac{\partial}{\partial z}\,. \end{aligned}$$

Thus

$$\frac{\partial a}{\partial x} = 0\,, \quad \frac{\partial b}{\partial x} = 2\lambda\,, \quad \frac{\partial c}{\partial x} = 0\,. \tag{4.6}$$

Similarly, from (4.3), (4.4) and the third equation of (4.2) we obtain

$$\frac{\partial a}{\partial y} = 2\,, \quad \frac{\partial b}{\partial y} = -\frac{B}{2\lambda}\,, \quad \frac{\partial c}{\partial y} = 0\,. \tag{4.7}$$

From $\frac{\partial c}{\partial x} = \frac{\partial c}{\partial y} = 0$ it follows that $c = c(z)$ and because of the fact that $c \neq 0$, we can suppose that $c = 1$, through a reparametrization of the variable z. For the sake of simplicity we will continue to use the same coordinates (x, y, z), taking into account that $c = 1$ in the relations that we have occurred. From the solution of the system of the differential equations

$$\left\{\frac{\partial a}{\partial x} = 0, \frac{\partial a}{\partial y} = 2, \frac{\partial b}{\partial x} = 2\lambda, \frac{\partial b}{\partial y} = -\frac{B}{2\lambda}\right\} \tag{4.8}$$

where $B = \phi X\lambda = \frac{\partial \lambda}{\partial z} = \lambda'(z)$, we easily obtain

$$a = a(x, y, z) = 2y + f(z)$$

$$b = b(x, y, z) = 2\lambda(z)x - \frac{\lambda'(z)}{2\lambda(z)}y + h(z)\,,$$

where $f(z), h(z)$ are arbitrary smooth functions of z defined on U. In what follows, we will calculate the tensor fields g, η, ϕ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. For the components g_{ij} of the Riemannian metric g, we calculate, using (4.3), (4.4, with $c = 1$), (4.8)

$$\begin{aligned}
g_{11} &= g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g(\xi, \xi) = 1\,, \quad g_{22} = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = g(X, X) = 1\\
g_{12} &= g_{21} = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = g(\xi, X) = 0\,,\\
g_{13} &= g_{31} = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, \phi X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right)\\
&= g(\xi, \phi X) - ag_{11} - bg_{12} = -a\\
g_{23} &= g_{32} = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial y}, \phi X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right)\\
&= g(X, \phi X) - ag_{12} - bg_{22} = -b\\
1 &= g(\phi X, \phi X) = a^2 g_{11} + b^2 g_{22} + g_{33} + 2abg_{12} + 2ag_{13} + 2bg_{23}\\
&= a^2 + b^2 + g_{33} - 2a^2 - 2b^2 = g_{33} - a^2 - b^2\,,
\end{aligned}$$

from which we obtain $g_{33} = 1 + a^2 + b^2$. The components of the tensor field ϕ are immediate consequences of

$$\phi\left(\frac{\partial}{\partial x}\right) = \phi\xi = 0\,, \quad \phi\left(\frac{\partial}{\partial y}\right) = \phi X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$$\begin{aligned}
\phi\left(\frac{\partial}{\partial z}\right) &= \phi\left(\phi X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right) = \phi^2 X - a\phi\frac{\partial}{\partial x} - b\phi\frac{\partial}{\partial y}\\
&= -X - b\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\\
&= -\frac{\partial}{\partial y} - ab\frac{\partial}{\partial x} - b^2\frac{\partial}{\partial y} - b\frac{\partial}{\partial z}\\
&= -ab\frac{\partial}{\partial x} - (1 + b^2)\frac{\partial}{\partial y} - b\frac{\partial}{\partial z}\,.
\end{aligned}$$

The expression for the contact form η, immediately follows from

$$\eta\left(\frac{\partial}{\partial x}\right) = \eta(\xi) = 1\,, \quad \eta\left(\frac{\partial}{\partial y}\right) = \eta(X) = g(X, \xi) = 0$$

$$\eta\left(\frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial z}, \xi\right) = g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}\right) = g_{13} = -a$$

and thus the proof of the case 1 is completed.

Case 2. $\mu = 2(1 - \sqrt{1-\kappa}) = 2(1-\lambda)$.

We work as in case 1, considering an h-frame $\{\xi, X, \phi X\}$. Using the assumption $\mu = 2(1-\lambda)$ and (3.13) we obtain $B = 0$ and thus the relation (3.10) is written as

$$[\xi, X] = 2\lambda\phi X\,, \quad [\xi, \phi X] = 0\,, \quad [X, \phi X] = \frac{A}{2\lambda}\phi X + 2\xi\,.$$

From $[\xi, \phi X] = 0$ we conclude that around any point $P \in M$ there is a chart $(U, (x, y, z))$ such that

$$\xi = \frac{\partial}{\partial x}\,, \quad \phi X = \frac{\partial}{\partial y}$$

on U. We put

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\,,$$

where a, b, c are smooth functions defined on U. The continuation of the proof is similar to the proof of the case 1 and for this reason we omit it. This completes the proof of the Theorem.

In the next Theorem, generalized (κ, μ)-manifolds with $\kappa < 1$ and $\xi\mu = 0$ are locally constructed.

THEOREM 4.2. *Let $\kappa : I \subset R \to R$ be a smooth function defined on an open interval I, such that $\kappa(z) < 1$ for any $z \in I$. Then, we can construct two families of generalized (κ_i, μ_i)-manifolds $M(\eta_i, \xi_i, \phi_i, g_i)$, $i = 1, 2$, in the set $M = R^2 \times I \subset R^3$, so that, for any $P(x, y, z) \in M$, the following are valid:*

$$\kappa_1(P) = \kappa_2(P) = \kappa(z)\,, \quad \mu_1(P) = 2(1+\sqrt{1-\kappa(z)}) \quad \textit{and} \quad \mu_2(P) = 2(1-\sqrt{1-\kappa(z)})\,.$$

Each family is determined by two arbitrary smooth functions of one variable.

PROOF. We put $\lambda = \sqrt{1-\kappa} > 0$, $\lambda'(z) = \frac{\partial\lambda}{\partial z}$ and we consider on M the linearly independent vector fields

$$\xi_1 = \frac{\partial}{\partial x}\,, \quad X_1 = \frac{\partial}{\partial y} \quad \text{and}$$

$$Y_1 = (2y + f(z))\frac{\partial}{\partial x} + \left(2\lambda(z)x - \frac{\lambda'(z)}{2\lambda(z)}y + h(z)\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\,, \tag{4.9}$$

where $f(z), h(z)$ are arbitrary functions of z. We define the tensor fields η_1, ϕ_1, g_1 as follows: g_1 is the Riemannian metric on M, with respect to which the vector fields ξ_1, X_1, Y_1 are orthonormal; η_1 is the 1-form on M which is defined from $\eta_1(Z) = g_1(Z, \xi_1)$ for any $Z \in \mathcal{X}(M)$; ϕ_1 is the $(1,1)$-tensor field that is defined by the relations $\phi_1\xi_1 = 0$, $\phi_1 X_1 = Y_1$ and $\phi_1 Y_1 = -X_1$. Initially we will show that $M(\eta_1, \xi_1, \phi_1, g_1)$ is a contact metric manifold.

From (4.9) we easily obtain

$$[\xi_1, X_1] = 0\,, \quad [\xi_1, Y_1] = 2\lambda(z)X_1\,, \quad [X_1, Y_1] = -\frac{\lambda'(z)}{2\lambda(z)}X_1 + 2\xi_1\,. \tag{4.10}$$

Because $(\eta_1 \wedge d\eta_1)(\xi_1, X_1, Y_1) \neq 0$ everywhere on M, we conclude that η_1 is a contact form. From the definitions of ϕ_1, g_1 and the relations (4.10) it is easy to see that the following relations are valid

$$\begin{aligned}&\phi_1^2 Z = -Z + \eta_1(Z)\xi_1\,, \quad g_1(\phi_1 Z, \phi_1 W) = g_1(Z, W) - \eta_1(Z)\eta_1(W)\,,\\ &d\eta_1(Z, W) = g_1(Z, \phi_1 W)\end{aligned}$$

for any $Z, W \in \mathcal{X}(M)$. Therefore, by (2.1) and (2.2), $M(\eta_1, \xi_1, \phi_1, g_1)$ is a contact metric manifold. Let ∇ be the Riemannian connection of g_1. Using the well known formula (see (2.10))

$$\begin{aligned}2g_1(\nabla_Z W, T) = Zg_1(W, T) + Wg_1(T, Z) - Tg_1(Z, W)\\ - g_1(Z, [W, T]) + g_1(W, [T, Z]) + g_1(T, [Z, W])\end{aligned}$$

for any $Z, W, T \in \mathcal{X}(M)$, as well as (4.10), $h\xi_1 = 0$ and $\nabla\xi = -\phi - \phi h$, by direct calculations we obtain the following:

$$\begin{aligned}&\nabla_{\xi_1}\xi_1 = 0\,, \quad \nabla_{\xi_1}X_1 = -(1 + \lambda(z))Y_1\,, \quad \nabla_{\xi_1}Y_1 = (1 + \lambda(z))X_1\,,\\ &\nabla_{X_1}\xi_1 = -(1 + \lambda(z))Y_1\,, \quad \nabla_{Y_1}\xi_1 = (1 - \lambda(z))X_1\,, \quad \nabla_{X_1}X_1 = \frac{\lambda'(z)}{2\lambda(z)}Y_1\,,\\ &\nabla_{Y_1}Y_1 = 0\,, \quad \nabla_{X_1}Y_1 = -\frac{\lambda'(z)}{2\lambda(z)}X_1 + (1 + \lambda(z))\xi_1\,, \quad \nabla_{Y_1}X_1 = (\lambda(z) - 1)\xi_1\,.\end{aligned}$$

Furthermore, by using $\nabla\xi_1 = -\phi_1 - \phi_1 h_1$, $h_1\phi_1 + \phi_1 h_1 = 0$ and the first of (2.1) we obtain

$$h_1\phi_1 X_1 = -\lambda(z)\phi_1 X_1 \quad \text{and} \quad h_1 X_1 = \lambda(z)X_1\,.$$

Defining the functions $\kappa_1, \mu_1 : M \to R$ by $\kappa_1(x, y, z) = \kappa(z)$, $\mu_1(x, y, z) = 2(1 + \sqrt{1 - \kappa(z)})$ we will show that $M(\eta_1, \xi_1, \phi_1, g_1)$ is a generalized (κ_1, μ_1)-manifold. Indeed, using (2.3) and the derivates of ξ_1, X_1, Y_1 that we have calculated, we find that

$$\begin{aligned}&R(\xi_1, \xi_1)\xi_1 = 0\,, \quad R(X_1, \xi_1)\xi_1 = \kappa_1 X_1 + \mu_1 h_1 X_1\,,\\ &R(Y_1, \xi_1)\xi_1 = \kappa_1 Y_1 + \mu_1 h_1 Y_1\,, \quad R(X_1, X_1)\xi_1 = 0\,,\\ &R(Y_1, Y_1)\xi_1 = 0\,, \quad R(X_1, Y_1)\xi_1 = 0\,.\end{aligned}$$

From the above, as well as from the linearity of R, we conclude that

$$R(Z, W)\xi_1 = (\kappa_1 I + \mu_1 h_1)(\eta_1(W)Z - \eta_1(Z)W)$$

for any $Z, W \in \mathcal{X}(M)$, i.e. $M(\eta_1, \xi_1, \phi_1, g_1)$ is a generalized (κ_1, μ_1)-manifold (with $\xi_1\mu_1 = 0$) and thus the construction of the first family is completed. The construction of the second family occurs, if we consider the vector fields

$$\xi_2 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y} \quad \text{and}$$

$$X_2 = (-2y + f(z))\frac{\partial}{\partial x} + \left(2\lambda(z)x - \frac{\lambda'(z)}{2\lambda(z)}y + h(z)\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \tag{4.11}$$

and define the tensor fields g_2, ϕ_2, η_2 as follows: g_2 is the Riemannian metric on M with respect to which the vector fields ξ_2, X_2, Y_2 are orthonormal. The $(1, 1)$-tensor field ϕ_2 is defined by $\phi_2\xi_2 = 0$, $\phi_2 X_2 = Y_2$ and $\phi_2 Y_2 = -X_2$. The 1-form η_2 is defined by $\eta_2(Z) = g_2(Z, \xi_2)$ for any $Z \in \mathcal{X}(M)$.

Next, we work similarly with the case 1 arriving at the conclusion that $M(\eta_2, \xi_2, \phi_2, g_2)$ is a generalized (κ_2, μ_2)-manifold, where $\kappa_2(x, y, z) = k(z)$ and $\mu_2(x, y, z) = 2(1 - \sqrt{1 - \kappa(z)})$. This completes the proof of the Theorem.

In the following Proposition some conditions equivalent to $\xi\mu = 0$ are obtained.

PROPOSITION 4.3. *Let $M(\eta, \xi, \phi, g)$ be a generalized (κ, μ)-manifold with $\kappa < 1$. Then the following conditions are equivalent*,

a) $\xi\mu = 0$
b) $\mu = 2(1 \pm \lambda), \lambda = \sqrt{1 - \kappa}$
c) $\xi\xi\mu = 0$
d) $\xi\Delta\lambda = 0$.

PROOF. Conditions (a),(b) are equivalent. This is a direct consequence of Theorem 4.1 and (3.2). In order to complete the proof of the Proposition, we consider around an arbitrary point of M an h-frame $\{\xi, X, \phi X\}$ such that $hX = \lambda X$, $h\phi X = -\lambda\phi X$ (see Lemma 3.2). By using (3.10), (3.2) and (3.12)–(3.15) we easily obtain

$$X\xi\mu = -4B\left(1 + \lambda - \frac{\mu}{2}\right) \tag{4.12}$$

$$\xi X\xi\mu = -4A\left(1 + \lambda - \frac{\mu}{2}\right)\left(\lambda - 1 + \frac{\mu}{2}\right) + 2B\xi\mu \tag{4.13}$$

$$[X, \xi]\xi\mu = -4A\left(1 + \lambda - \frac{\mu}{2}\right)\left(\lambda - 1 + \frac{\mu}{2}\right) \tag{4.14}$$

$$\phi X\xi\mu = 4A\left(\lambda - 1 + \frac{\mu}{2}\right) \tag{4.15}$$

$$\xi\phi X\xi\mu = 4B\left(1 + \lambda - \frac{\mu}{2}\right)\left(\lambda - 1 + \frac{\mu}{2}\right) + 2A\xi\mu \tag{4.16}$$

$$[\phi X, \xi]\xi\mu = 4B\left(1 + \lambda - \frac{\mu}{2}\right)\left(\lambda - 1 + \frac{\mu}{2}\right). \tag{4.17}$$

Now, we will prove that (c)⇒(a).

Differentiating $\xi\xi\mu = 0$ with respect to X we obtain $X\xi\xi\mu = 0$, or equivalently $[X, \xi]\xi\mu + \xi X\xi\mu = 0$ and so using (4.13), (4.14) we obtain

$$B\xi\mu = 4A\left(1 + \lambda - \frac{\mu}{2}\right)\left(\lambda - 1 + \frac{\mu}{2}\right). \tag{4.18}$$

Similarly, differentiating $\xi\xi\mu = 0$ with respect to ϕX and using (4.16), (4.17) we obtain

$$A\xi\mu = -4B\left(1 + \lambda - \frac{\mu}{2}\right)\left(\lambda - 1 + \frac{\mu}{2}\right). \tag{4.19}$$

For the functions A, B there are the following possible cases: $\{A = 0, B = 0\}$, $\{AB \neq 0\}$, $\{A \neq 0, B = 0\}$, $\{A = 0, B \neq 0\}$. The two first possibilities cannot occur. Indeed, the combination of $A = 0$, $B = 0$ with (3.2) leads to κ =constant which is impossible. Furthermore, if $AB \neq 0$, then, multiplying (4.18), (4.19) with B, A respectively and adding the relations that occur we are led to $(A^2 + B^2)\xi\mu = 0$, from which we obtain $\xi\mu = 0$ or equivalently $\mu = 2(1 \pm \lambda)$. If $\mu = 2(1 + \lambda)$, then $X\mu = 2X\lambda = 2A$. From this and (3.12) we obtain $A = 0$, which is impossible. Similarly, supposing that $\mu = 2(1 - \lambda)$ we obtain $B = 0$, which is also impossible. Therefore, the only possible cases are $\{A \neq 0, B = 0\}$ and $\{A = 0, B \neq 0\}$. If we assume that $\{A \neq 0, B = 0\}$, then (4.19) gives $\xi\mu = 0$. Similarly, from $\{A = 0, B \neq 0\}$ and (4.18) we obtain $\xi\mu = 0$ and this completes the proof of (c)⇒(a).

The case (a)⇒(c) is obvious. In what follows, we will prove that (d)⇔(a).

Let us suppose that (a) is valid, i.e. $\xi\mu = 0$. Then, as it has been proved earlier, we obtain $AB = 0$ and thus from (3.23) we obtain $\xi\Delta\lambda = 0$, i.e. the condition (d). Conversely, let us assume that $\xi\Delta\lambda = 0$. Then (3.23) gives

$$\xi\mu = -\frac{2}{\lambda}AB. \tag{4.20}$$

If $AB = 0$, then $\xi\mu = 0$. We will prove that the case $AB \neq 0$ is impossible. Let $AB \neq 0$, therefore $\xi\mu \neq 0$. Differentiating (4.20) with respect to X and using (4.12), (3.18), (4.20) we calculate

$$\begin{aligned}
-4B\left(1 + \lambda - \frac{\mu}{2}\right) &= \frac{2}{\lambda^2}(X\lambda)AB - \frac{2}{\lambda}\{(XA)B + A(XB)\} \\
&= \frac{2}{\lambda^2}A^2B - \frac{2B}{\lambda}XA - \frac{2A}{\lambda}\left(\frac{1}{2}\xi\mu + \frac{1}{2\lambda}AB\right) \\
&= -\frac{A}{\lambda}\xi\mu - \frac{2B}{\lambda}XA - \frac{A}{2\lambda}\xi\mu \\
&= -\frac{3A}{2\lambda}\xi\mu - \frac{2B}{\lambda}XA
\end{aligned}$$

and so

$$\frac{2B}{\lambda}XA = 4B\left(1+\lambda-\frac{\mu}{2}\right) - \frac{3A}{2\lambda}\xi\mu\,. \tag{4.21}$$

Similarly, differentiating (4.20) with respect to ϕX and using (4.15), (3.18), (4.20) we are led to

$$\frac{2A}{\lambda}\phi XB = -4A\left(\lambda-1+\frac{\mu}{2}\right) - \frac{3B}{2\lambda}\xi\mu\,. \tag{4.22}$$

Multiplying (4.21) with A and (4.22) with B and adding the resulting relations, we obtain

$$\frac{2AB}{\lambda}(XA+\phi XB) = 4AB(2-\mu) - \frac{3}{2\lambda}(A^2+B^2)\xi\mu\,.$$

Furthermore, by using (3.19) and (4.20), the last relation leads to

$$\frac{1}{\lambda}\Delta\lambda - \frac{A^2+B^2}{\lambda^2} - 2(2-\mu) = 0\,.$$

Differentiating the last relation with respect to ξ and using $\xi\Delta\lambda = 0$, (3.22), we easily obtain $\xi\mu = \frac{2}{\lambda}AB$. From this and (4.20) we obtain the contradiction $AB = 0$ and thus the proof of the Proposition is completed.

REMARK. Theorem 4.1 can be reformulated by replacing the condition $\xi\mu = 0$ with any one of the equivalent conditions of Proposition 4.3.

In [6] the generalized (κ,μ)-manifolds $M(\eta,\xi,\phi,g)$ with $\|\operatorname{grad}\kappa\| = \text{constant} \neq 0$ have been studied. These manifolds satisfy $\mu = 2(1\pm\sqrt{1-\kappa})$ (see [6], Lemma 3) and thus by (3.2), the condition $\xi\mu = 0$ as well. Moreover, it is obvious that the function κ satisfies $\kappa < 1$. Thus this class of manifolds is a special case of generalized (κ,μ)-manifolds with $\kappa < 1$ and $\xi\mu = 0$. In the process of proving Theorem 4.1 (see relation(4.8)) we have shown that for the case $\{A = 0,\, B \neq 0,\, \mu = 2(1+\lambda)\}$ we have

$$B = \frac{d\lambda}{dz} \quad \text{and so} \quad \phi XB = \frac{d^2\lambda}{dz^2}\,. \tag{4.23}$$

From $B = \frac{d\lambda}{dz}$, $\|\operatorname{grad}\kappa\| = c$ and $\lambda^2 = 1-\kappa$ we are easily led to $4\lambda^2\left(\frac{d\lambda}{dz}\right)^2 = c^2$ and from the solution of this we obtain $\kappa = \pm cz + d < 1$, (d =constant). Furthermore, (4.23), (3.19) and (3.24) tell us that the scalar curvature of M is given by

$$S = -\frac{5c^2}{8\lambda^4} - 2(\lambda+1)^2\,. \tag{4.24}$$

Similarly, regarding the case $\{A \neq 0,\, B = 0,\, \mu = 2(1-\lambda)\}$ we have

$$A = \frac{d\lambda}{dz}\,, \quad XA = \frac{d^2\lambda}{dz^2} \tag{4.25}$$

and, therefore, in this case $\kappa = \pm cz + d < 1$ ($d =$ constant) and

$$S = -\frac{5c^2}{8\lambda^4} - 2(\lambda - 1)^2\,. \tag{4.26}$$

From (4.24) and (4.26) we find that the scalar curvature is a strictly negative function. Furthermore, S is non-constant. Indeed, if we suppose that $S =$ constant, then (4.24) or (4.26) show that κ is constant, which is impossible by definition. Summarizing the above we obtain the following Proposition.

PROPOSITION 4.4. *Let $M(\eta,\xi,\phi,g)$ be a generalized (κ,μ)-manifold with $\|\operatorname{grad}\kappa\| = c$ (constant) $\neq 0$. Then*

a) $\xi\mu = 0$

b) *At any point $P \in M$, there exist a chart $(U,(x,y,z))$ with $P \in U \subseteq M$, such that $\kappa(x,y,z) = cz + d$, ($d =$ constant) and $\mu = 2(1 \pm \sqrt{1-\kappa})$.*

c) *The scalar curvature of M is a negative non-constant function.*

REMARK. 1. Since $c \neq 0$ in Proposition 4.4, doing an appropriate reparametrization of the chart $(U,(x,y,z))$ we can find a chart $(V,(x,y,z))$ such that $\kappa(x,y,z) = z$, and thus the conclusion (b) of Proposition 4.4 is identified with the corresponding result of Theorem 5 of [6].

2. If we apply a D_a-homothetic deformation on a generalized (κ,μ)-manifold $M(\eta,\xi,\phi,g)$, $(\kappa < 1)$, with $\xi\mu = 0$, then from (2.9) it follows that the new manifold $M(\bar\eta,\bar\xi,\bar\phi,\bar g)$ is a generalized $(\bar\kappa,\bar\mu)$-manifold $(\bar\kappa < 1)$ with $\bar\xi\bar\mu = 0$ as well.

As we have seen in Proposition 4.4, in a generalized (κ,μ)-manifold with $\|\operatorname{grad}\kappa\| = c \neq 0$ the scalar curvature S is a non-constant negative function. In examples 4.5 and 4.6, below, we construct generalized (κ,μ)-manifolds with constant scalar curvature S of any sign.

EXAMPLE 4.5. For any $c \in R$, we will construct a family of generalized (κ,μ)-manifolds with $S = c$. In order to reach this construction, we consider the function $F : R \to R$, $F(z) = 8\log z + 4z - 2(c+2)z^{-1} + d$, where $z > 0$ and $d \in R$. Since $\lim_{z\to+\infty} F(z) = +\infty$, there exist $b \in R$ and a neighborhood $V \subset R$ with $b \in V$, such that the function $g : V \to R$, $g(z) = z^{3/2}(F(z))^{1/2}$, is smooth and positive for any $z \in V$. Let us consider the function $f : V \subset R \to R$ defined by

$$f(z) = \int_b^z \frac{1}{g(y)}dy\,.$$

Since $f'(z) \neq 0$ for any $z \in V$, we find that $f(z)$ is invertible in V. We consider now the manifold $M = \{(x,y,z) \in R^3/z \in f(V)\}$ and the function $\lambda : M \to R$: $\lambda(x,y,z) = l(z) = f^{-1}(z)$. By applying Theorem 4.2 we find that $M(\eta,\xi,\phi,g)$ is a generalized (κ,μ)-manifold with $\kappa = 1 - \lambda^2$ and $\mu = 2(1 + \sqrt{1-\kappa})$. The tensor fields (η,ξ,ϕ,g) of M are defined by the vector fields $\xi, X, Y = \phi X$ of the relation (4.9):

$$\xi = \frac{\partial}{\partial x}, \quad X = \frac{\partial}{\partial y}, \quad \phi X = (2y + u(z))\frac{\partial}{\partial x} + (2\lambda(z)x - \frac{\lambda'(z)}{2\lambda(z)}y + h(z))\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where $u(z), h(z)$ are arbitrary functions of z. In order to find the scalar curvature S, we calculate

$$\begin{aligned}
&\lambda' = \frac{\partial \lambda}{\partial z} = l'(z) = \lambda^{3/2}(8\log\lambda + 4\lambda - (2c+4)\lambda^{-1} + d)^{1/2} \\
&\lambda''(z) = 12\lambda^2 \log\lambda + 8\lambda^3 + \frac{3d+8}{2}\lambda^2 - (4+2c)\lambda \\
&A = X\lambda = \frac{\partial\lambda}{\partial y}, \quad XA = 0 \\
&B = \phi X\lambda = \frac{\partial\lambda}{\partial z} = \lambda', \quad \phi XB = \lambda'' \\
&\|\operatorname{grad}\lambda\|^2 = A^2 + B^2 = (\lambda')^2 \\
&\kappa - \mu = -(\lambda+1)^2 .
\end{aligned}$$

By using these relations, as well as (3.19), (3.24) we calculate

$$\begin{aligned}
S &= \frac{1}{\lambda}\Delta\lambda - \frac{1}{\lambda^2}\|\operatorname{grad}\lambda\|^2 + 2(\kappa - \mu) \\
&= \frac{1}{\lambda}\left\{XA + \phi XB - \frac{1}{2\lambda}(A^2 + B^2)\right\} - \frac{1}{\lambda^2}(A^2+B^2) + 2(\kappa-\mu) \\
&= \frac{\lambda''}{\lambda} - \frac{3\lambda'^2}{2\lambda^2} - 2(1+\lambda)^2 \\
&= \frac{1}{\lambda}\left\{12\lambda^2\log\lambda + 8\lambda^3 + \frac{1}{2}(3d+8)\lambda^2 - (4+2c)\lambda\right\} \\
&\quad - \frac{3}{2\lambda^2}\lambda^3(8\log\lambda + 4\lambda - (2c+4)\lambda^{-1} + d) - 2(1+\lambda)^2 = c .
\end{aligned}$$

Consequently, $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-manifold with $S = c$. Since the tensor fields (η, ξ, ϕ, g) depend on the arbitrary functions $u(z)$ and $h(z)$, a family of generalized (κ, μ)-manifolds finally occurs with $S = c$.

EXAMPLE 4.6. Using Theorem 4.2 for the smooth function $\kappa(z) = 1 - \frac{1}{2z^2}$, $z > 0$, we obtain the generalized (κ, μ)-manifold $M(\eta, \xi, \phi, g)$, where $M = \{(x, y, z) \in R^3/z > 0\}$, $\kappa = 1 - \frac{1}{2z^2}$ and $\mu = 2\big(1 - \frac{1}{\sqrt{2}z}\big)$. Using (3.19), (3.24), $\lambda^2 = 1 - \kappa$, $\mu = 2(1-\lambda)$, we finally find that the scalar curvature S of M is given by

$$S = \frac{1}{\lambda}\frac{d^2\lambda}{dz^2} - \frac{3}{2\lambda^2}\left(\frac{d\lambda}{dz}\right)^2 - 2(1-\lambda)^2 = -2\left(1 - \frac{1}{\sqrt{2}z}\right)^2 + \frac{1}{2z^2} = -\frac{1}{2z^2}(4z^2 - 4\sqrt{2}z + 1) .$$

Thus, we easily conclude that S can be of any sign.

ACKNOWLEDGEMENT. The authors thank the referee for his useful corrections and suggestions.

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Present Addresses:
THEMIS KOUFOGIORGOS
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF IOANNINA,
IOANNINA 45110, GREECE.
e-mail: tkoufog@cc.uoi.gr

TSICHLIAS CHARALAMBOS
SOULIOU 175, PETROUPOLI
ATHENS 13231, GREECE.
e-mail: c_tsichlias@yahoo.com