



# Generalized Kenmotsu Manifolds

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**Abstract.** In 1972, K. Kenmotsu studied a class of almost contact Riemannian manifolds which later are called a Kenmotsu manifold. In this paper, we study Kenmotsu manifolds with  $(2n + s)$ -dimensional  $s$ -contact metric manifold that we call generalized Kenmotsu manifolds. Necessary and sufficient condition is given for a  $s$ -contact metric manifold to be a generalized Kenmotsu manifold. We show that a generalized Kenmotsu manifold is a locally warped product space. In addition, we study some curvature properties of generalized Kenmotsu manifolds. Moreover, we obtain that the  $\varphi$ -sectional curvature of any semi-symmetric and projective semi-symmetric  $(2n + s)$ -dimensional generalized Kenmotsu manifold is  $-s$ .

**Keywords.** Kenmotsu manifolds; Metric  $f$ -manifolds; Generalized Kenmotsu manifolds; Semi-symmetric; Ricci semi-symmetric; Projective semi-symmetric

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## 1. Introduction

In [22], K. Yano introduced the notion of a  $f$ -structure on a differentiable manifold  $M$ , i.e., a tensor field  $f$  of type  $(1, 1)$  and rank  $2n$  satisfying  $f^3 + f = 0$  as a generalization of both (almost) contact (for  $s = 1$ ) and (almost) complex structures (for  $s = 0$ ).  $TM$  splits into two complementary subbundles  $\mathcal{L} = \text{Im } \varphi$  and  $\mathcal{M} = \ker \varphi$ . The existence of which is equivalent to a reduction of the structural group of the tangent bundle to  $U(n) \times O(s)$  [4]. H. Nakagawa in [16, 17] introduced the notion of globally framed  $f$ -manifolds (called  $f$ -manifolds), later developed and studied by Goldberg and Yano [10, 11, 12]. A wide class of globally frame  $f$ -manifolds was introduced in [4]

by Blair according to the following definition. A metric  $f$ -structure is said to be a  $K$ -structure if the fundamental 2-form  $\Phi$ , defined usually as  $\Phi(X, Y) = g(X, \varphi Y)$  for any vector fields  $X$  and  $Y$  on  $M$ , is closed and the normality condition holds, that is,  $[\varphi, \varphi] + 2 \sum_{i=1}^s d\eta^i \otimes \xi_i = 0$  where  $[\varphi, \varphi]$  denotes the Nijenhuis torsion of  $\varphi$ . Some authors studied  $f$ -structure [5, 7, 23]. The Riemannian connection  $\nabla$  of a metric  $f$ -manifold satisfies the following formula [6],

$$2g((\nabla_X \varphi)Y, Z) = 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g(N^1(Y, Z), \varphi X) + \sum_{i=1}^s \{N^2(Y, Z)\eta^i(X) + 2d\eta^i(\varphi Y, X)\eta^i(Z) - 2d\eta^i(\varphi Z, X)\eta^i(Y)\}, \quad (1.1)$$

where the tensor fields  $N^1$  and  $N^2$  are defined by  $N^1 = [\varphi, \varphi] + 2 \sum_{i=1}^s d\eta^i \otimes \xi_i$ ,  $N^2(X, Y) = (L_{\varphi X} \eta^i)(Y) - (L_{\varphi Y} \eta^i)(X)$  respectively, which is by a simple computation can be rewritten as:  $N^2(X, Y) = 2d\eta^i(\varphi X, Y) - 2d\eta^i(\varphi Y, X)$ .

Let  $M$  be a  $(2n + 1)$  dimensional differentiable manifold.  $M$  is called an almost contact metric manifold if  $\varphi$  is  $(1, 1)$  type tensor field,  $\xi$  is vector field,  $\eta$  is 1-form and  $g$  is a compatible Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1 \quad (1.2)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (1.3)$$

for all  $X, Y \in \Gamma(TM)$ .

In addition, we have

$$\eta(X) = g(X, \xi), \quad \varphi(\xi) = 0, \quad \eta(\varphi) = 0 \quad (1.4)$$

for all  $X, Y \in \Gamma(TM)$  [6].

To study manifolds with negative curvature, Bishop and O'Neill introduced the notion of warped product as a generalization of Riemannian product [3]. In 1960's and 1970's, when almost contact manifolds were studied as an odd dimensional counterpart of almost complex manifolds, the warped product was used to make examples of almost contact manifolds [20]. In addition, S. Tanno classified the connected  $(2n + 1)$  dimensional almost contact manifold  $M$  whose automorphism group has maximum dimension  $(n + 1)^2$  in [20]. For such a manifold, the sectional curvature of plane sections containing  $\xi$  is a constant, say  $c$ . Then there are three cases.

- (i)  $c > 0$ ,  $M$  is homogeneous Sasakian manifold of constant holomorphic sectional curvature.
- (ii)  $c = 0$ ,  $M$  is the global Riemannian product of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature.
- (iii)  $c < 0$ ,  $M$  is warped product space  $R \times_f C^n$ .

Kenmotsu obtained some tensorial equations to characterize manifolds of the third case.

In 1972, Kenmotsu abstracted the differential geometric properties of the third case. In [15], Kenmotsu studied a class of almost contact Riemannian manifold which satisfy the following two conditions,

$$(\nabla_X \varphi)Y = -\eta(Y)\varphi X - g(X, \varphi Y)\xi \quad (1.5)$$

$$\nabla_X \xi = X - \eta(X)\xi$$

He showed normal an almost contact Riemannian manifold with (1.5) but neither Sasakian nor quasi Sasakian. He was to characterize warped product space  $L \times_f CE^n$  by an almost contact Riemannian manifold with (1.5). Moreover, he show that every point of an almost contact Riemannian manifold with (1.5) has a neighborhood which is a warped product  $(-\epsilon, \epsilon) \times_f V$  where  $f(t) = ce^t$  and  $V$  is Kähler.

In 1981 [13], Janssens and Vanhecke, an almost contact metric manifold satisfying the condition (1.5) is called a Kenmotsu manifold. After this definition, some authors studied Kenmotsu manifold [8, 14, 19, 18].

The paper is organized as follows: after a preliminary basic notions of  $s$ -contact metric manifolds theory, in Section 2, we introduced generalized almost Kenmotsu manifolds and generalized Kenmotsu manifolds. Necessary and sufficient condition is given for a  $s$ -contact metric manifold to be a generalized Kenmotsu manifold. The warped product  $L^s \times_f V^{2n}$  provides an example. In section 3, some curvature properties are given. In section 4, we study Ricci curvature tensor. In section 5, we study semi-symmetric properties of generalized Kenmotsu manifolds. We show that the  $\varphi$ -sectional curvature of any semi-symmetric and projective semi-symmetric  $(2n + s)$ -dimensional generalized Kenmotsu manifold is  $-s$ .

## 2. Preliminaries

In [11], a  $(2n + s)$ -dimensional differentiable manifold  $M$  is called metric  $f$ -manifold if there exist an  $(1, 1)$  type tensor field  $\varphi$ ,  $s$ -differentiable vector fields  $\xi_1, \dots, \xi_s$  (called structure vector fields),  $s$ -differentiable 1-forms  $\eta^1, \dots, \eta^s$  and a Riemannian metric  $g$  on  $M$  such that

$$\varphi^2 = -I + \sum_{i=1}^s \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta_{ij} \quad (2.1)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^s \eta^i(X)\eta^i(Y), \quad (2.2)$$

for any  $X, Y \in \Gamma(TM)$ ,  $i, j \in \{1, \dots, s\}$ . In addition, we have

$$\eta^i(X) = g(X, \xi_i), \quad g(X, \varphi Y) = -g(\varphi X, Y), \quad \varphi \xi_i = 0, \quad \eta^i \circ \varphi = 0, \quad i \in \{1, \dots, s\}. \quad (2.3)$$

Then, a 2-form  $\Phi$  is defined by  $\Phi(X, Y) = g(X, \varphi Y)$ , for any  $X, Y \in \Gamma(TM)$ , called the fundamental 2-form.

In what follows, we denote by  $\mathcal{M}$  the distribution spanned by the structure vector fields  $\xi_1, \dots, \xi_s$  and by  $\mathcal{L}$  its orthogonal complementary distribution. Then,  $TM = \mathcal{L} \oplus \mathcal{M}$ . If  $X \in \mathcal{M}$  we have  $\varphi X = 0$  and if  $X \in \mathcal{L}$  we have  $\eta^i(X) = 0$ , for any  $i \in \{1, \dots, s\}$ , that is,  $\varphi^2 X = -X$ .

In a metric  $f$ -manifold, special local orthonormal basis of vector fields can be considered. Let  $U$  be a coordinate neighborhood and  $E_1$  be a unit vector field on  $U$  orthogonal to the structure vector fields. Then, from (2.1), (2.2) and (2.3),  $\varphi E_1$  is also a unit vector field on  $U$  orthogonal to  $E_1$  and to the structure vector fields. Next, if it is possible, let  $E_2$  be a unit vector field on  $U$  orthogonal to  $E_1$ ,  $\varphi E_1$  and to the structure vector fields and so on. The local orthonormal basis

$$\{E_1, \dots, E_n, \varphi E_1, \dots, \varphi E_n, \xi_1, \dots, \xi_s\},$$

so obtained is called an  $f$ -basis. Moreover, a metric  $f$ -manifold is normal if

$$[\varphi, \varphi] + 2 \sum_{i=1}^s d\eta^i \otimes \xi_i = 0,$$

where  $[\varphi, \varphi]$  is denoting the Nijenhuis tensor field associated to  $\varphi$ .

In [21], let  $M$  be a  $(2n + s)$ -dimensional metric  $f$ -manifold. If there exists 2-form  $\Phi$  such that  $\eta^1 \wedge \dots \wedge \eta^s \wedge \Phi^n \neq 0$  on  $M$  then  $M$  is called an almost  $s$ -contact metric manifold. A normal almost  $s$ -contact metric manifold is called an  $s$ -contact metric manifold.

### 3. Generalized Kenmotsu Manifolds

As it is known in Kenmotsu manifold  $\dim \ker \varphi = 1$ , since  $\ker \varphi = sp\{\xi\}$ . The case  $\dim \ker \varphi > 1$  is an open question. Firstly in 2003, L. Bhatt and K. K. Dube introduced Kenmotsu  $s$ -structure, that is, an almost  $s$ -contact metric manifold  $M$  is called a Kenmotsu  $s$ -manifold if

$$(\nabla_X \varphi)Y = g(\varphi X, Y) \sum_{i=1}^s \xi_i - \varphi X \sum_{i=1}^s \eta^i(Y)$$

for any  $X, Y \in \Gamma(TM)$  [1]. We will give their definition as a theorem in this paper.

Afterwards in 2006, M. Falcitelli and A.M. Pastore introduced Kenmotsu  $f \cdot pk$ -manifold. In [9], a metric  $f \cdot pk$ -manifold  $M$  of dimension  $2n + s$ ,  $s \geq 1$ , with  $f \cdot pk$ -structure which is a metric  $f$ -structure with parallelizable kernel  $(\varphi, \xi_i, \eta^i, g)$  is said to be a Kenmotsu  $f \cdot pk$ -manifold if it is normal, the 1-forms  $\eta^i$  are closed and  $d\Phi = 2\eta^1 \wedge \Phi$ . They assume that  $d\Phi = 2\eta^i \wedge \Phi$  for all  $i = 1, 2, \dots, s$  in the definition of Kenmotsu  $f \cdot pk$ -manifold. So, they remark that, since the 1-forms  $\eta^i$  are linearly independent and  $\eta^i \wedge \Phi = \eta^j \wedge \Phi$  implies  $\eta^i = \eta^j$ , then the condition on  $d\Phi$  can be satisfied by a unique  $\eta^i$ , and they can assume that  $d\Phi = 2\eta^1 \wedge \Phi$ . It is clear that authors were equated 1-forms  $\eta^1, \dots, \eta^s$ , which dual of  $\xi_1, \dots, \xi_s$ . Thus, they studied unique 1-form  $\eta^1$ .

In this paper, all  $\eta^1, \dots, \eta^s$  1-forms are unequated at the definition of generalized Kenmotsu manifolds.

**Definition 3.1.** Let  $M$  be an almost  $s$ -contact metric manifold of dimension  $(2n + s)$ ,  $s \geq 1$ , with  $(\varphi, \xi_i, \eta^i, g)$ .  $M$  is said to be a generalized almost Kenmotsu manifold if for all  $1 \leq i \leq s$ , 1-forms  $\eta^i$  are closed and  $d\Phi = 2 \sum_{i=1}^s \eta^i \wedge \Phi$ . A normal generalized almost Kenmotsu manifold  $M$  is called a generalized Kenmotsu manifold.

Now, we construct an example of generalized Kenmotsu manifold.

**Example 3.2.** We consider  $(2n + s)$ -dimensional manifold

$$M = \left\{ (x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s) \in R^{2n+s} : \sum_{\alpha=1}^s z_\alpha \neq 0 \right\}.$$

We choose the vector fields

$$X_i = e^{-\sum_{\alpha=1}^s z_\alpha} \frac{\partial}{\partial x_i}, \quad Y_i = e^{-\sum_{\alpha=1}^s z_\alpha} \frac{\partial}{\partial y_i}, \quad \xi_\alpha = \frac{\partial}{\partial z_\alpha}, \quad 1 \leq i \leq n, \quad 1 \leq \alpha \leq s$$

which are linearly independent at each point of  $M$ . Let  $g$  be the Riemannian metric defined by

$$g = e^{2 \sum_{\alpha=1}^s z_\alpha} \left[ \sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i) \right] + \sum_{\alpha=1}^s \eta^\alpha \otimes \eta^\alpha.$$

Hence,  $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi_1, \dots, \xi_s\}$  is an orthonormal basis. Thus,  $\eta^\alpha$  be the 1-form defined by  $\eta^\alpha(X) = g(X, \xi_\alpha)$ ,  $\alpha = 1, \dots, s$  for any vector field  $X$  on  $TM$ . We defined the  $(1, 1)$ -tensor field  $\varphi$  as

$$\varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i, \quad \varphi(\xi_\alpha) = 0, \quad 1 \leq i \leq n, \quad 1 \leq \alpha \leq s.$$

The linearity property of  $\varphi$  and  $g$  yields that

$$\eta^\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad \varphi^2 X = -X + \sum_{\alpha=1}^s \eta^\alpha(X) \xi_\alpha,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^s \eta^\alpha(X) \eta^\alpha(Y),$$

for any vector fields  $X, Y$  on  $M$ . Therefore,  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  defines a metric  $f$ -manifold. We have  $\Phi(X_i, Y_i) = -1$  and others are zero. Therefore, the essential non-zero component of  $\Phi$  is

$$\Phi\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) = g\left(\frac{\partial}{\partial x_i}, \varphi \frac{\partial}{\partial y_i}\right) = -e^{2 \sum_{\alpha=1}^s z_\alpha}$$

and hence, we have

$$\Phi = -e^{2 \sum_{\alpha=1}^s z_\alpha} \sum_{i=1}^n dx_i \wedge dy_i.$$

Therefore, we get  $\eta^1 \wedge \dots \wedge \eta^s \wedge \Phi^n \neq 0$  on  $M$ . Thus  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  is almost  $s$ -contact manifold. Consequently, the exterior derivative  $d\Phi$  is given by

$$d\Phi = 2 \sum_{\alpha=1}^s dz_\alpha \wedge (-e^{2 \sum_{\alpha=1}^s z_\alpha}) \sum_{i=1}^n dx_i \wedge dy_i.$$

Therefore,  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  is a generalized almost Kenmotsu manifold. It can be seen that  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  is normal. So, it is a generalized Kenmotsu manifold. Moreover, we get

$$[X_i, \xi_\alpha] = X_i, \quad [Y_i, \xi_\alpha] = Y_i,$$

$$\begin{aligned}[X_i, X_j] &= 0, \quad [X_i, Y_i] = 0, \quad [X_i, Y_j] = 0 \\ [Y_i, Y_j] &= 0, \quad 1 \leq i, j \leq n, \quad 1 \leq \alpha \leq s.\end{aligned}$$

The Riemannian connection  $\nabla$  of the metric  $g$  is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y).\end{aligned}$$

Using the Koszul's formula, we obtain

$$\begin{aligned}\nabla_{X_i} X_i &= \sum_{\alpha=1}^s \xi_\alpha, \quad \nabla_{Y_i} Y_i = \sum_{\alpha=1}^s \xi_\alpha, \\ \nabla_{X_i} X_j &= \nabla_{Y_i} Y_j = \nabla_{X_i} Y_i = \nabla_{X_i} Y_j = 0 \\ \nabla_{X_i} \xi_\alpha &= X_i, \quad \nabla_{Y_i} \xi_\alpha = Y_i, \quad 1 \leq i, j \leq n, \quad 1 \leq \alpha \leq s.\end{aligned}$$

We construct an example of generalized Kenmotsu manifold of dimension 7.

**Example 3.3.** Let  $n = 2$  and  $s = 3$ . The vector fields

$$\begin{aligned}e_1 &= f_1(z_1, z_2, z_3) \frac{\partial}{\partial x_1} + f_2(z_1, z_2, z_3) \frac{\partial}{\partial y_1}, \\ e_2 &= -f_2(z_1, z_2, z_3) \frac{\partial}{\partial x_1} + f_1(z_1, z_2, z_3) \frac{\partial}{\partial y_1}, \\ e_3 &= f_1(z_1, z_2, z_3) \frac{\partial}{\partial x_2} + f_2(z_1, z_2, z_3) \frac{\partial}{\partial y_2}, \\ e_4 &= -f_2(z_1, z_2, z_3) \frac{\partial}{\partial x_2} + f_1(z_1, z_2, z_3) \frac{\partial}{\partial y_2}, \\ e_5 &= \frac{\partial}{\partial z_1}, e_6 = \frac{\partial}{\partial z_2}, e_7 = \frac{\partial}{\partial z_3}\end{aligned}$$

where  $f_1$  and  $f_2$  are given by

$$\begin{aligned}f_1(z_1, z_2, z_3) &= c_2 e^{-(z_1+z_2+z_3)} \cos(z_1+z_2+z_3) - c_1 e^{-(z_1+z_2+z_3)} \sin(z_1+z_2+z_3), \\ f_2(z_1, z_2, z_3) &= c_1 e^{-(z_1+z_2+z_3)} \cos(z_1+z_2+z_3) + c_2 e^{-(z_1+z_2+z_3)} \sin(z_1+z_2+z_3)\end{aligned}$$

for nonzero constant  $c_1, c_2$ . It is obvious that  $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$  are linearly independent at each point of  $M$ . Let  $g$  be the Riemannian metric given by

$$g = \frac{1}{f_1^2 + f_2^2} \sum_{i=1}^2 (dx_i \otimes dx_i + dy_i \otimes dy_i) + dz_1 \otimes dz_1 + dz_2 \otimes dz_2 + dz_3 \otimes dz_3,$$

where  $\{x_1, y_1, x_2, y_2, z_1, z_2, z_3\}$  are standard coordinates in  $R^7$ . Let  $\eta^1, \eta^2$  and  $\eta^3$  be the 1-form defined by  $\eta^1(X) = g(X, e_5)$ ,  $\eta^2(X) = g(X, e_6)$  and  $\eta^3(X) = g(X, e_7)$ , respectively, for any vector field  $X$  on  $M$  and  $\varphi$  be the  $(1, 1)$ -tensor field defined by

$$\varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(e_3) = e_4, \quad \varphi(e_4) = -e_3,$$

$$\varphi(e_5 = \xi_1) = 0, \quad \varphi(e_6 = \xi_2) = 0, \quad \varphi(e_7 = \xi_3) = 0.$$

Therefore, the essential non-zero component of  $\Phi$  is

$$\Phi\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) = -\frac{1}{f_1^2 + f_2^2} = -\frac{2e^{2(z_1+z_2+z_3)}}{c_1^2 + c_2^2}, \quad i = 1, 2$$

and hence

$$\Phi = -\frac{2e^{2(z_1+z_2+z_3)}}{c_1^2 + c_2^2} \sum_{i=1}^2 dx_i \wedge dy_i.$$

Thus, we have  $\eta^1 \wedge \dots \wedge \eta^s \wedge \Phi^n \neq 0$  on  $M$ . Consequently, the exterior derivative  $d\Phi$  is given by

$$d\Phi = -\frac{4e^{2(z_1+z_2+z_3)}}{c_1^2 + c_2^2} (dz_1 + dz_2 + dz_3) \wedge \sum_{i=1}^2 dx_i \wedge dy_i.$$

Since  $\eta^1 = dz_1$ ,  $\eta^2 = dz_2$  and  $\eta^3 = dz_3$ , we find

$$d\Phi = 2(\eta^1 + \eta^2 + \eta^3) \wedge \Phi.$$

In addition, Nijenhuis tensor of  $\varphi$  is equal to zero.

**Theorem 3.4.** *An almost  $s$ -contact metric manifold  $(M, \varphi, \xi_i, \eta^i, g)$  is a generalized Kenmotsu manifold if and only if*

$$(\nabla_X \varphi)Y = \sum_{i=1}^s \left\{ g(\varphi X, Y) \xi_i - \eta^i(Y) \varphi X \right\} \quad (3.1)$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ , where  $\nabla$  is Riemannian connection on  $M$ .

*Proof.* Let  $M$  be a generalized Kenmotsu manifold. From (1.1), (2.1), (2.2) and (2.3) for all  $X, Y \in \Gamma(TM)$ , we have

$$\begin{aligned} g((\nabla_X \varphi)Y, Z) &= 3 \left\{ \sum_{i=1}^s (\eta^i \wedge \Phi)(X, \varphi Y, \varphi Z) - \sum_{i=1}^s (\eta^i \wedge \Phi)(X, Y, Z) \right\} \\ &= \sum_{i=1}^s \left\{ -\eta^i(X)g(\varphi Y, \varphi^2 Z) + \eta^i(X)g(Y, \varphi Z) - \eta^i(Y)g(Z, \varphi X) - \eta^i(Z)g(X, \varphi Y) \right\} \\ &= \sum_{i=1}^s \left\{ -\eta^i(Y)g(Z, \varphi X) - \eta^i(Z)g(X, \varphi Y) \right\} \\ &= g \left( \sum_{i=1}^s \left\{ g(\varphi X, Y) \xi_i - \eta^i(Y) \varphi X \right\}, Z \right). \end{aligned}$$

Conversely, using (3.1) and (2.3), we get

$$\varphi \nabla_X \xi_j = \sum_{i=1}^s \left\{ g(\varphi X, \xi_j) \xi_i - \eta^i(\xi_j) \varphi X \right\}$$

hence, we get

$$\varphi^2 \nabla_X \xi_j = \varphi^2 X.$$

Therefore, we have

$$\nabla_X \xi_j = -\varphi^2 X.$$

On the other hand, we obtain

$$d\eta^i(X, Y) = \frac{1}{2} \{g(Y, -\varphi^2 X) - g(X, -\varphi^2 Y)\} = 0$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ . In addition, we know that

$$\begin{aligned} 3d\Phi(X, Y, Z) &= Xg(Y, \varphi Z) - Yg(X, \varphi Z) - Zg(X, \varphi Y) - g([X, Y], \varphi Z) \\ &\quad + g([X, Z], \varphi Y) - g([Y, Z], \varphi X) \\ &= g(Y, \nabla_X \varphi Z - \varphi \nabla_X Z) - g(X, \nabla_Y \varphi Z - \varphi \nabla_Y Z) + g(X, \nabla_Z \varphi Y - \varphi \nabla_Z Y). \end{aligned}$$

From the hypothesis, we have

$$\begin{aligned} 3d\Phi(X, Y, Z) &= \sum_{i=1}^s \{g(\varphi X, Z)g(Y, \xi_i) - \eta^i(Z)g(Y, \varphi X) - g(\varphi Y, Z)g(X, \xi_i) + \eta^i(Z)g(X, \varphi Y) \\ &\quad + g(\varphi Z, Y)g(X, \xi_i) - \eta^i(Y)g(X, \varphi Z)\} \\ &= 2 \sum_{i=1}^s \{\Phi(Z, X)\eta^i(Y) + \Phi(X, Y)\eta^i(Z) + \Phi(Y, Z)\eta^i(X)\} \end{aligned}$$

which leads to,

$$d\Phi = 2 \sum_{i=1}^s \eta^i \wedge \Phi.$$

Moreover, the Nijenhuis tensor of  $\varphi$  is obtained

$$\begin{aligned} N_\varphi(X, Y) &= \varphi \left( -\sum_{i=1}^s \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\} + \sum_{i=1}^s \{g(\varphi Y, X)\xi_i - \eta^i(X)\varphi Y\} \right) \\ &\quad + \sum_{i=1}^s \{g(\varphi^2 X, Y)\xi_i - \eta^i(Y)\varphi^2 X\} - \sum_{i=1}^s \{g(\varphi^2 Y, X)\xi_i - \eta^i(X)\varphi^2 Y\} \\ &= 0. \end{aligned}$$

Hence, we have

$$[\varphi, \varphi] + 2 \sum_{i=1}^s d\eta^i \otimes \xi_i = 0.$$

The proof is completed. □

**Corollary 3.5.** Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have

$$\nabla_X \xi_j = -\varphi^2 X \quad (3.2)$$

for all  $X \in \Gamma(TM)$ ,  $i, j \in \{1, 2, \dots, s\}$ .

**Lemma 3.6.** Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have

$$\begin{aligned} \text{(i)} \quad & \nabla_{\xi_j} \varphi = 0, \quad \nabla_{\xi_j} \xi_i = 0 \\ \text{(ii)} \quad & (L_{\xi_i} \varphi)X = 0, \quad (L_{\xi_i} \eta^j)X = 0 \\ \text{(iii)} \quad & (L_{\xi_i} g)(X, Y) = 2\{g(X, Y) - \sum_{i=1}^s \eta^i(X) \eta^i(Y)\} \end{aligned} \quad (3.3)$$

for all  $X \in \Gamma(TM)$ ,  $i, j \in \{1, 2, \dots, s\}$ .

**Theorem 3.7.** Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have

$$(\nabla_X \eta^i)Y = g(X, Y) - \sum_{j=1}^s \eta^j(X) \eta^j(Y) \quad (3.4)$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

*Proof.* Using (2.3) and (3.2), we get the desired result.  $\square$

We have the following the corollary for the case  $s = 1$ .

**Corollary 3.8.** An almost contact metric manifold  $(M^{2n+1}, \varphi, \xi, \eta, g)$  is a Kenmotsu manifold if and only if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ , where  $\nabla$  is Riemannian connection on  $M$  [15].

**Theorem 3.9.** Let  $F$  be a Kähler manifold,  $f(t) = ke^{\sum_{i=1}^s t_i}$  be a function on  $R^s$ , and  $k$  be a non-zero constant. Then the warped product space  $M = R^s \times_f F$  have a generalized Kenmotsu manifold.

*Proof.* Let  $(F, J, G)$  be a Kähler manifold and consider  $M = R^s \times_f F$ , with coordinates  $(t_1, \dots, t_s, x_1, \dots, x_{2n})$ . We define  $\varphi$  tensor field, 1-forms  $\eta^i$ , vector fields  $\xi_i$  and Riemannian metric tensor  $g$  on  $M$  as follows:

$$\begin{aligned} \varphi\left(\frac{\partial}{\partial t_i}, U\right) &= (0, JU), \\ \eta^j\left(\frac{\partial}{\partial t_i}, U\right) &= \delta_{ij}, \quad \xi_i = \left(\frac{\partial}{\partial t_i}, 0\right) \end{aligned}$$

$$g_f = \sum_{i=1}^s dt^i \otimes dt^i + f^2 \pi^*(G)$$

where  $f(t) = k e^{\sum_{i=1}^s t_i}$ ,  $U \in \Gamma(F)$ . Then  $(M, \varphi, \eta^i, \xi_i, g_f)$  defines  $s$ -contact metric manifold.

Now let us show that this manifold is a generalized Kenmotsu manifold.

It is clear that  $\eta^i$  are closed. Thus, we have

$$\Phi(X, Y) = g_f(X, \varphi Y) = f^2 \pi^*(G(X, JY)) \quad \text{or} \quad \Phi = f^2 \pi^*(\Psi)$$

where  $\Psi$  is fundamental 2-form of Kähler manifold. Hence, we get

$$d\Phi = 2c \sum_{i=1}^s e^{\sum_{i=1}^s t_i} dt^i \wedge \pi^*(\Psi) = 2 \sum_{i=1}^s dt^i \wedge \Phi.$$

Finally Nijenhuis tensor  $N_\varphi$  of  $M$  is vanish, since  $\eta^i$  are closed and  $N_J = 0$ . Then  $(M = R^s \times_f F, \varphi, \eta^i, \xi_i, g_f)$  is a generalized Kenmotsu manifold.  $\square$

**Example 3.10.**  $(R^2 \times_f V^4, g_f = \sum_{i=1}^2 dt^i \otimes dt^i + f^2 G)$  is warped product with coordinates  $(t_1, t_2, x_1, x_2, x_3, x_4)$ , where  $f^2 = k^2 e^{\sum_{i=1}^2 t_i}$ . Take an orthonormal frame field  $\{\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4\}$  of  $V^4$  and  $\{\bar{e}_5, \bar{e}_6\}$  of  $R^2$  such that  $\bar{E}_2 = J\bar{E}_1$ ,  $\bar{E}_4 = J\bar{E}_3$ . Then, we obtain a local orthonormal field  $\{E_1, E_2, E_3, E_4, E_5, E_6\}$  of  $R^2 \times_f V^4$  by

$$\begin{aligned} E_1 &= k e^{-\sum_{i=1}^2 t_i} \bar{E}_1, & E_2 &= k e^{-\sum_{i=1}^2 t_i} \bar{E}_2 \\ E_3 &= -k e^{-\sum_{i=1}^2 t_i} \bar{E}_3, & E_4 &= -k e^{-\sum_{i=1}^2 t_i} \bar{E}_4 \\ E_5 &= \xi_1, & E_6 &= \xi_2. \end{aligned}$$

Then  $R^2 \times_f V^4$  is a generalized Kenmotsu manifold.

**Theorem 3.11.** Let  $(M^{2n+s}, \varphi, \eta^i, \xi_i, g)$  be a generalized Kenmotsu manifold,  $V^{2n}$  and  $L^s$  are Kähler and a flat manifold with local coordinates  $(x_1, \dots, x_{2n})$  and  $(t_1, \dots, t_s)$  respectively. Then  $M$  a locally warped product  $L^s \times_f V^{2n}$  where  $f(t) = k e^{\sum_{i=1}^s t_i}$  and  $k$  a nonzero positive constant.

*Proof.* We know that  $TM = \mathcal{L} \oplus \mathcal{M}$ .  $\mathcal{L}$  is clearly integrable, since  $d\eta^i = 0$ . Then  $V$  integral manifold of  $\mathcal{L}$  is totally umbilical because  $\nabla_X \xi_i = X$ . On the other hand  $[\xi_i, \xi_j] = 0$  and  $\nabla_{\xi_i} \xi_j = 0$ ,  $\mathcal{M}$  is integrable and  $\mathcal{L}$  integral manifold is totally geodesic.

We select  $J = \varphi|_D$  such that  $J^2 = -I$ ,  $G = g|_D$ . Then  $(V, J, G)$  is almost Hermitian manifold. Also Nijenhuis tensor  $N_J = N_\varphi = 0$  and using (3.1), we get  $(\nabla_X J)Y = 0$ . Then  $(V, J, G)$  is Kähler manifold.

Then  $M = L \times_f V$  is locally a warped product and the metric is

$$g_f = \sum_{i=1}^s dt^i \otimes dt^i + f^2 G.$$

It follows that

$$(L_{\xi_i} g_f)(X, Y) = \frac{2\xi_i(f)}{f} G(X, Y)$$

and using (3.3), we get

$$\xi_i(f) = f, \quad i = 1, \dots, s.$$

Thus, we have

$$\frac{\partial f(t_1, \dots, t_s)}{\partial t_i} = f(t_1, \dots, t_s), \quad i = 1, \dots, s.$$

Therefore, we obtained  $f(t_1, \dots, t_s) = k e^{\sum_{i=1}^s t_i}$  where  $k$  is nonzero constant.  $\square$

**Example 3.12.** Let's go back to the Example ???. Hence,  $(R^7, \varphi, \eta^i, \xi_i, g)$  is a generalized Kenmotsu manifold where  $i = 1, 2, 3$ . Take an orthonormal frame field

$$\left\{ \frac{\partial}{\partial z_1} = \xi_1, \frac{\partial}{\partial z_2} = \xi_2, \frac{\partial}{\partial z_3} = \xi_3 \right\}$$

of  $R^3$  and

$$\left\{ \frac{e^{\sum_{i=1}^3 z_i}}{c_1^2 + c_2^2} (f_1 e_1 - f_2 e_2), \frac{e^{\sum_{i=1}^3 z_i}}{c_1^2 + c_2^2} (f_2 e_1 + f_1 e_2), \frac{e^{\sum_{i=1}^3 z_i}}{c_1^2 + c_2^2} (f_1 e_3 - f_2 e_4), \frac{e^{\sum_{i=1}^3 z_i}}{c_1^2 + c_2^2} (f_2 e_3 + f_1 e_4) \right\}$$

of  $R^4$ . Then  $R^7 = R^3 \times R^4$  is product manifold, the structure by tensor  $\varphi$  and metric tensor  $g$ .  $R^4$  is the standard Kähler structure  $(J, G)$ . Here the Riemannian metric  $g$  is warped product metric

$$g_0 + c f^2 G$$

where  $g_0$  is the Euclidean metric of  $R^3$ ,  $f$  is the function defined on  $R^3$  by

$$f(z_1, z_2, z_3) = e^{\sum_{i=1}^3 z_i} \quad \text{and} \quad c = \frac{1}{c_1^2 + c_2^2}.$$

## 4. Some Curvature Properties

**Theorem 4.1.** Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have

$$R(X, Y)\xi_i = \sum_{j=1}^s \{\eta^j(Y)\varphi^2 X - \eta^j(X)\varphi^2 Y\} \quad (4.1)$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

*Proof.* Using (3.2) and (2.1), we get

$$\nabla_X \nabla_Y \xi_i = \nabla_X Y + \varphi^2 X \sum_{j=1}^s \eta^j(Y) - \sum_{j=1}^s \{\eta^j(\nabla_X Y) \xi_j + g(Y, -\varphi^2 X) \xi_j\}$$

and

$$\nabla_{[X,Y]} \xi_i = -\varphi^2 \nabla_X Y + \varphi^2 \nabla_Y X.$$

Then,

$$\begin{aligned} R(X, Y) \xi_i &= \nabla_X Y + \varphi^2 X \sum_{j=1}^s \eta^j(Y) - \sum_{j=1}^s \{\eta^j(\nabla_X Y) \xi_j + g(Y, -\varphi^2 X) \xi_j\} \\ &\quad - \nabla_Y X - \varphi^2 Y \sum_{j=1}^s \eta^j(X) + \sum_{j=1}^s \{\eta^j(\nabla_Y X) \xi_j + g(X, -\varphi^2 Y) \xi_j\} + \varphi^2 \nabla_X Y - \varphi^2 \nabla_Y X. \end{aligned}$$

From (2.1), we obtain the desired result.  $\square$

**Corollary 4.2.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have*

$$R(X, \xi_i)Y = \sum_{j=1}^s \{\eta^j(Y) \varphi^2 X - g(X, \varphi^2 Y) \xi_j\}, \quad (4.2)$$

$$R(X, \xi_j) \xi_i = \varphi^2 X, \quad R(\xi_k, \xi_j) \xi_i = 0 \quad (4.3)$$

for all  $X, Y \in \Gamma(TM)$ ,  $i, j, k \in \{1, 2, \dots, s\}$ .

**Corollary 4.3** ([15]). *Let  $M$  be a  $(2n + 1)$ -dimensional Kenmotsu manifold with structure  $(\varphi, \xi, \eta, g)$ . Then we have*

$$R(X, Y) \xi = \eta(Y)X - \eta(X)Y,$$

$$R(X, \xi)Y = g(X, Y) \xi - \eta(Y)X$$

for all  $X, Y \in \Gamma(TM)$ .

**Theorem 4.4.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have*

$$\begin{aligned} (\nabla_Z R)(X, Y, \xi_i) &= sg(Z, X)Y - sg(Z, Y)X - R(X, Y)Z \\ &\quad + s \sum_{h=1}^s \eta^h(Z) \{\eta^h(Y)X - \eta^h(X)Y\} + \sum_{l=1}^s \eta^l(Z) R(X, Y) \xi_l \end{aligned}$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

*Proof.* Using (3.2) and (4.1), we have

$$(\nabla_Z R)(X, Y, \xi_i) = \nabla_Z \left\{ \sum_{j=1}^s \{ \eta^j(X)Y - \eta^j(Y)X \} \right\} - \sum_{j=1}^s \{ \eta^j(\nabla_Z X)Y - \eta^j(Y)\nabla_Z X \} \\ - \sum_{j=1}^s \{ \eta^j(X)\nabla_Z Y - \eta^j(\nabla_Z Y)X \} - R(X, Y)\varphi^2 Z.$$

From (2.1), we get

$$(\nabla_Z R)(X, Y, \xi_i) = \sum_{j=1}^s \{ g(X, \nabla_Z \xi_j)Y - g(Y, \nabla_Z \xi_j)X \} - R(X, Y)Z + \sum_{k=1}^s \eta^k(Y)R(X, Y)\xi_k.$$

From (2.1) and (3.2) the proof is completed.  $\square$

**Corollary 4.5.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold. Then we have*

$$(\nabla_Z R)(X, Y, \xi_i) = sg(Z, X)Y - sg(Z, Y)X - R(X, Y)Z, \quad Z \in \mathcal{L} \\ (\nabla_{\xi_j} R)(X, Y, \xi_i) = 0$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

**Corollary 4.6** ([15]). *Let  $M$  be a  $(2n + 1)$ -dimensional Kenmotsu manifold with structure  $(\varphi, \xi, \eta, g)$ . Then we have*

$$(\nabla_Z R)(X, Y, \xi) = g(Z, X)Y - g(Z, Y)X - R(X, Y)Z, \quad Z \in \mathcal{L}.$$

**Corollary 4.7.** *Let  $(M, \varphi, \xi_i, \eta^i, g)$  be a  $(2n + s)$ -dimensional locally-symmetric generalized Kenmotsu manifold. Then we have*

$$R(X, Y)Z = s\{g(Z, X)Y - g(Z, Y)X\}.$$

**Corollary 4.8** ([15]). *Let  $M$  be a  $(2n + 1)$ -dimensional Kenmotsu manifold with structure  $(\varphi, \xi, \eta, g)$ . If  $M$  is a locally symmetric then we have*

$$R(X, Y)Z = g(Z, X)Y - g(Z, Y)X.$$

**Corollary 4.9.** *The  $\varphi$ -sectional curvature of any locally symmetric generalized Kenmotsu manifold  $(M, \varphi, \xi_i, \eta^i, g)$  is equal to  $-s$ .*

In the case  $s = 1$ , we obtain that the  $\varphi$ -sectional curvature of any locally symmetric Kenmotsu manifold  $(M, \varphi, \xi, \eta, g)$  is equal to  $-1$  [15].

**Theorem 4.10.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have*

$$R(X, Y)\varphi Z - \varphi R(X, Y)Z = g(Y, Z)\varphi X - g(X, Z)\varphi Y - g(Y, \varphi Z)X + g(X, \varphi Z)Y, \\ R(\varphi X, \varphi Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y + g(Y, \varphi Z)\varphi X - g(X, \varphi Z)\varphi Y$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

## 5. Ricci Curvature Tensor

**Theorem 5.1.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have*

$$S(X, \xi_i) = -2n \sum_{j=1}^s \eta^j(X) \quad (5.1)$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

*Proof.* If  $\{E_1, E_2, \dots, E_{2n+s}\}$  are local orthonormal vector fields, then  $S(X, Y) = \sum_{k=1}^{2n+s} g(R(E_k, X)Y, E_k)$  defines a global tensor field  $S$  of type  $(0, 2)$ . Then, we obtain

$$\begin{aligned} S(X, \xi_i) &= \sum_{k=1}^{2n} g\left(\sum_{j=1}^s \{\eta^j(X)\varphi^2 E_k - \eta^j(E_k)\varphi^2 X\}, E_k\right) + \sum_{k=1}^s g(-\varphi^2 X, \xi_k) \\ &= \sum_{j=1}^s \eta^j(X) \sum_{k=1}^{2n} g(\varphi^2 E_k, E_k). \end{aligned} \quad \square$$

In the case  $s = 1$  we have  $S(X, \xi) = -2n\eta(X)$  in [15].

**Corollary 5.2.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have*

$$S(\xi_k, \xi_i) = -2n \quad (5.2)$$

for all  $X, Y \in \Gamma(TM)$   $i, k \in \{1, 2, \dots, s\}$ .

**Theorem 5.3.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have*

$$S(\varphi X, \varphi Y) = S(X, Y) + 2n \sum_{i=1}^s \eta^i(X)\eta^i(Y) \quad (5.3)$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

*Proof.* We can put

$$X = X_0 + \sum_{i=1}^s \eta^i(X)\xi_i \quad \text{and} \quad Y = Y_0 + \sum_{i=1}^s \eta^i(Y)\xi_i$$

where  $X_0, Y_0 \in \mathcal{L}$ . Then from (5.1) and (5.2) we have,

$$\begin{aligned} S(X, Y) &= S(X_0, Y_0) + \sum_{i=1}^s \eta^i(Y)\eta^i(X_0) + \sum_{i=1}^s \eta^i(X)\eta^i(Y_0) + \sum_{i=1}^s \eta^i(X)\eta^i(Y)S(\xi_i, \xi_i) \\ &= S(X_0, Y_0) - 2n \sum_{i=1}^s \eta^i(X)\eta^i(Y). \end{aligned}$$

Since  $\varphi X, \varphi Y \in \mathcal{L}$  we get  $S(X_0, Y_0) = S(\varphi X, \varphi Y)$  which implies the desired result.  $\square$

Considering  $s = 1$  in [14], we deduce

$$S(\varphi X, \varphi Y) = S(X, Y) + 2n\eta(X)\eta(Y).$$

**Theorem 5.4.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have*

$$\begin{aligned} (\nabla_{\varphi X} S)(\varphi Y, \varphi Z) &= (\nabla_{\varphi X} S)(Y, Z) - \sum_{i=1}^s \eta^i(Y) \{S(X, \varphi Z) + 2ng(X, \varphi Z)\} \\ &\quad - \sum_{i=1}^s \eta^i(Z) \{S(X, \varphi Y) + 2ng(X, \varphi Y)\} \end{aligned}$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

*Proof.* Using (3.1), we get

$$\begin{aligned} (\nabla_{\varphi X} S)(\varphi Y, \varphi Z) &= \nabla_{\varphi X} S(Y, Z) + 2n \sum_{i=1}^s \{\eta^i(Y) \nabla_{\varphi X} \eta^i(Z) + \eta^i(Z) \nabla_{\varphi X} \eta^i(Y)\} \\ &\quad + \sum_{i=1}^s \{-S(g(\varphi^2 X, Y) \xi_i - \eta^i(Y) \varphi^2 X, \varphi Z) - S(\nabla_{\varphi X} Y, Z) - 2n\eta^i(\nabla_{\varphi X} Y) \eta^i(Z) \\ &\quad - S(\varphi Y, g(\varphi^2 X, Z) \xi_i - \eta^i(Z) \varphi^2 X) - S(Y, \nabla_{\varphi X} Z) - 2n\eta^i(Y) \eta^i(\nabla_{\varphi X} Z)\}. \end{aligned}$$

From (2.1), (5.1) and (5.2) we have

$$\begin{aligned} (\nabla_{\varphi X} S)(\varphi Y, \varphi Z) &= (\nabla_{\varphi X} S)(Y, Z) + \sum_{i=1}^s \{2n\eta^i(Y)(\nabla_{\varphi X} \eta^i(Z) + 2n\eta^i(Z)(\nabla_{\varphi X} \eta^i(Y) \\ &\quad - \eta^i(Y)S(X, \varphi Z) - \eta^i(Z)S(\varphi Y, X)\}. \end{aligned}$$

□

**Corollary 5.5.** *Let  $M$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold with structure  $(\varphi, \xi_i, \eta^i, g)$ . Then we have*

$$\begin{aligned} (\nabla_X S)(\varphi Y, \varphi Z) &= (\nabla_X S)(Y, Z) + 2n \sum_{i=1}^s \{g(X, Y) \eta^i(Z) + g(X, Z) \eta^i(Y)\} \\ &\quad + \sum_{i=1}^s \{\eta^i(Y) S(X, Z) + \eta^i(Z) S(X, Y)\} \end{aligned}$$

for all  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

**Definition 5.6.** The Ricci tensor  $S$  of a  $(2n + s)$ -dimensional generalized Kenmotsu manifold  $M$  is called  $\eta$ -parallel, if it satisfies

$$(\nabla_X S)(\varphi Y, \varphi Z) = 0$$

for all vector fields  $X, Y$  and  $Z$  on  $M$ .

**Theorem 5.7.** Let  $(M, \varphi, \xi_i, \eta^i, g)$  be a  $(2n + s)$ -dimensional generalized Kenmotsu manifold.  $M$  is  $\eta$ -parallel if and only if

$$(\nabla_X S)(Y, Z) = -2n \sum_{i=1}^s \{g(X, Y)\eta^i(Z) + g(X, Z)\eta^i(Y)\} - \sum_{i=1}^s \{\eta^i(Y)S(X, Z) + \eta^i(Z)S(X, Y)\}$$

for all  $X, Y, Z \in \Gamma(TM)$ ,  $i \in \{1, 2, \dots, s\}$ .

**Corollary 5.8** ([15]). Let  $(M, \varphi, \xi, \eta, g)$  be a  $(2n + 1)$ -dimensional Kenmotsu manifold.  $M$  has  $\eta$ -parallel if and only if

$$(\nabla_X S)(Y, Z) = -2n\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y)\} - \eta(Y)S(X, Z) - \eta(Z)S(X, Y)$$

for all  $X, Y, Z \in \Gamma(TM)$ .

## 6. Semi-Symmetric Properties of Generalized Kenmotsu Manifolds

**Theorem 6.1.** The  $\varphi$ -sectional curvature of any semi-symmetric  $(2n + s)$ -dimensional generalized Kenmotsu manifold  $(M, \varphi, \xi_i, \eta^i, g)$  is equal to  $-s$ .

*Proof.* Let  $X$  be a unit vector field. Since  $(M, \varphi, \xi_i, \eta^i, g)$  is semi-symmetric, then

$$(R \cdot R)(X, \xi_i, X, \varphi X, \varphi X, \xi_i) = 0,$$

for any  $i, j \in \{1, 2, \dots, s\}$ . Expanding this formula from (2.2) and taking into account (4.2), we get

$$R(X, \varphi X, \varphi X, X) = -s,$$

which completes the proof.  $\square$

Observe that, in the case  $s = 1$ , by using the Theorem 6.1 we obtain that a semi-symmetric Kenmotsu manifolds is constant curvature equal to  $-1$  [2].

**Theorem 6.2.** Let  $(M, \varphi, \xi_i, \eta^i, g)$  be a  $(2n + s)$ -dimensional Ricci semi-symmetric generalized Kenmotsu manifold. Then its Ricci tensor field  $S$  respect the Riemannian connection satisfies

$$S(X, Y) = -2n \left\{ sg(\varphi X, \varphi Y) + \sum_{i,j=1}^s \eta^i(X)\eta^j(Y) \right\} \quad (6.1)$$

for any  $X, Y \in \Gamma(TM)$ .

*Proof.* Since  $(M, \varphi, \xi_i, \eta^i, g)$  is Ricci semi-symmetric, then

$$S(R(X, \xi_i)\xi_j, Y) + S(\xi_j, R(X, \xi_i)Y) = 0,$$

for any  $X, Y \in \Gamma(TM)$  and  $i, j \in \{1, 2, \dots, s\}$ . Now, from (4.2), (4.3) and (5.1) we get the desired result.  $\square$

In this case  $s = 1$  we have following the corollary.

**Corollary 6.3.** *Any Ricci semi-symmetric  $(2n+1)$ -dimensional Kenmotsu manifold is an Einstein manifold.*

*Proof.* Considering  $s = 1$  in (6.1), we deduce

$$S(X, Y) = -2ng(X, Y)$$

for any  $X, Y \in \Gamma(TM)$ . □

For the Weyl projective curvature tensor field  $P$ , the weyl projective curvature tensor  $P$  of a  $(2n + s)$ -dimensional generalized Kenmotsu manifold  $M$  is given by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n + s - 1} \{S(Y, Z)X - S(X, Z)Y\}$$

where  $R$  is curvature tensor and  $S$  is the ricci curvature tensor of  $M$ , we have the following theorem.

**Theorem 6.4.** *The  $\varphi$ -sectional curvature of any projectively semi-symmetric generalized Kenmotsu manifold  $(M, \varphi, \xi_i, \eta^i, g)$  is equal to  $-s$ .*

*Proof.* Let  $X$  be a unit vector field. Then, from (2.1) and taking into account (4.2) and (5.1) we have

$$(R.P)(X, \xi_i, X, \varphi X, \varphi X, \xi_j) = (R.R)(X, \xi_i, X, \varphi X, \varphi X, \xi_j).$$

This completes the proof from the Theorem 6.2. □

**Corollary 6.5.** *Let  $(M, \varphi, \xi, \eta, g)$  be a  $(2n + 1)$ -dimensional Kenmotsu manifold. The  $\varphi$ -sectional curvature of any projectively semi-symmetric Kenmotsu manifold if and only if  $M$  is an Einstein manifold.*

## 7. Conclusion

We introduced the new concept which is called almost Kenmotsu manifolds and generalized Kenmotsu manifolds, and some examples of generalized Kenmotsu manifold are given. In addition, the presented theorems extend and improve the corresponding results which given in the literature.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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