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GENERALIZED KNOPP IDENTITIES FOR HOMOGENEOUS HARDY SUMS AND COCHRANE-HARDY SUMS

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Abstract. Let q, h, a, b be integers with $q > 0$. The classical and the homogeneous Dedekind sums are defined by

$$s(h,q)=\sum_{j=1}^{q}\Big(\Big(\frac{j}{q}\Big)\Big)\Big(\Big(\frac{hj}{q}\Big)\Big),\quad s(a,b,q)=\sum_{j=1}^{q}\Big(\Big(\frac{aj}{q}\Big)\Big)\Big(\Big(\frac{bj}{q}\Big)\Big),$$

respectively, where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

The Knopp identities for the classical and the homogeneous Dedekind sum were the following:

$$\sum_{d\mid n}\sum_{r=1}^{d} s\Big(\frac{n}{d}a+rq, dq\Big)=\sigma(n)s(a,q),$$

$$\sum_{d\mid n}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s\Big(\frac{n}{d}a+r_1q, \frac{n}{d}b+r_2q, dq\Big)=n\sigma(n)s(a,b,q),$$

where $\sigma(n)=\sum_{d\mid n} d$.

In this paper generalized homogeneous Hardy sums and Cochrane-Hardy sums are defined, and their arithmetic properties are studied. Generalized Knopp identities for homogeneous Hardy sums and Cochrane-Hardy sums are given.

Keywords: Dedekind sum, Cochrane sum, Knopp identity

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1. Introduction

For a positive integer q and an arbitrary integer h, the classical Dedekind sum is defined by

$$s(h,q)=\sum_{j=1}^{q}\Big(\Big(\frac{j}{q}\Big)\Big)\Big(\Big(\frac{hj}{q}\Big)\Big),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

The sum $s(h,q)$ plays an important role in the transformation theory of the Dedekind η function (see [9] and Chapter 3 of [1] for details).

In [6], Knopp derived the following arithmetical identity

$$\sum_{d\mid n}\sum_{r=1}^{d} s\Big(\frac{n}{d}a+rq,dq\Big)=\sigma(n)s(a,q),$$

where $\sigma(n)=\sum_{d\mid n} d$. Many authors have given elementary proofs for the Knopp identity, for example, Goldberg [4], Parson [7] and Zheng [15].

According to [5], the homogeneous Dedekind sum is defined by

$$s(a,b,q)=\sum_{j=1}^{q}\Big(\Big(\frac{aj}{q}\Big)\Big)\Big(\Big(\frac{bj}{q}\Big)\Big).$$

Zheng [16] extended the Knopp identity to homogeneous Dedekind sums as follows:

$$\sum_{d\mid n}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s\Big(\frac{n}{d}a+r_1q,\frac{n}{d}b+r_2q,dq\Big)=n\sigma(n)s(a,b,q).$$

Moreover, let $B_r(x)$ be the Bernoulli polynomials given by

$$\frac{z\mathrm{e}^{xz}}{\mathrm{e}^{z}-1}=\sum_{r=0}^{\infty}B_r(x)\frac{z^r}{r!}\quad(|z|<2\pi)$$

and

$$P_r(x)=\begin{cases} B_r(x-[x]), & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

Clearly $P_1(x)=((x))$. He defined the generalized homogeneous Dedekind sum by

$$s_{\alpha,\beta}(a,b,q)=\sum_{j=1}^{q}P_\alpha\Big(\frac{aj}{q}\Big)P_\beta\Big(\frac{bj}{q}\Big),$$

and proved the generalized Knopp identity for $s_{\alpha,\beta}(a,b,q)$ as follows:

$$(1.1)\quad \sum_{d|n} d^{\alpha+\beta-2}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s_{\alpha,\beta}\Big(\frac{n}{d}a+r_1q,\frac{n}{d}b+r_2q,dq\Big)=n\sigma_{\alpha+\beta-1}(n)s_{\alpha,\beta}(a,b,q),$$

where $\sigma_r(n)=\sum_{d|n} d^r$.

In [2], Berndt studied the following Hardy sums:

$$S(h,k)=\sum_{j=1}^{k-1}(-1)^{j+1+[hj/k]},\qquad s_1(h,k)=\sum_{j=1}^{k}(-1)^{[hj/k]}\Big(\Big(\frac{j}{k}\Big)\Big),$$

$$s_2(h,k)=\sum_{j=1}^{k}(-1)^{j}\Big(\Big(\frac{j}{k}\Big)\Big)\Big(\Big(\frac{hj}{k}\Big)\Big),\quad s_3(h,k)=\sum_{j=1}^{k}(-1)^{j}\Big(\Big(\frac{hj}{k}\Big)\Big),$$

$$s_4(h,k)=\sum_{j=1}^{k-1}(-1)^{[hj/k]},\qquad s_5(h,k)=\sum_{j=1}^{k}(-1)^{j+[hj/k]}\Big(\Big(\frac{j}{k}\Big)\Big),$$

which are related to the classical Dedekind sums, and obtained some arithmetic properties (see [3]). Sitaramachandrarao [10] and Pettet [8] expressed the Hardy sums in terms of the Dedekind sums as follows:

$$\begin{aligned}
S(h,k)&=8s(h,2k)+8s(2h,k)-20s(h,k), &&\text{if } h+k \text{ is odd;}\\
s_1(h,k)&=2s(h,k)-4s(h,2k), &&\text{if } h \text{ is even;}\\
s_2(h,k)&=-s(h,k)+2s(2h,k), &&\text{if } k \text{ is even;}\\
s_3(h,k)&=2s(h,k)-4s(2h,k), &&\text{if } k \text{ is odd;}\\
s_4(h,k)&=-4s(h,k)+8s(h,2k), &&\text{if } h \text{ is odd;}\\
s_5(h,k)&=-10s(h,k)+4s(2h,k)+4s(h,2k), &&\text{if } h+k \text{ is even.}
\end{aligned}$$

Naturally, one might ask whether there is an analogous class of Hardy sums for the generalized homogeneous Dedekind sums? If yes, then what results can be expected?

We define the generalized homogeneous Hardy sums as follows:

$$\begin{aligned}
s_{\beta}^{(1)}(a,b,q)&=\sum_{j=1}^{q}(-1)^{[aj/q]}P_\beta\Big(\frac{bj}{q}\Big), &&\beta\equiv 1 \bmod 2;\\
s_{\alpha,\beta}^{(2)}(a,b,q)&=\sum_{j=1}^{q}(-1)^{j}P_\alpha\Big(\frac{aj}{q}\Big)P_\beta\Big(\frac{bj}{q}\Big), &&\alpha\equiv \beta \bmod 2;\\
s_{\beta}^{(5)}(a,b,q)&=\sum_{j=1}^{q}(-1)^{j+[aj/q]}P_\beta\Big(\frac{bj}{q}\Big), &&\beta\equiv 1 \bmod 2.
\end{aligned}$$

We will express the generalized homogeneous Hardy sums in terms of the generalized homogeneous Dedekind sums in Section 2, and give the generalized Knopp identities for the homogeneous Hardy sums in Section 3. Next we study the properties of the generalized homogeneous Cochrane-Hardy sums. Finally in Section 5, the generalized Knopp identities for the generalized homogeneous Cochrane-Hardy sums are given.

2. The generalized homogeneous Hardy sums

In this section, we shall express the generalized homogeneous Hardy sums in terms of the generalized homogeneous Dedekind sums. First we need the following lemmas.

Lemma 2.1. *If m, a and q are integers with $m \geqslant 0$ and $q > 0$, then for any real x, we have*

$$\sum_{r=1}^{q} P_m\Big(x + \frac{ar}{q}\Big) = (a,q)^m q^{1-m} P_m\Big(\frac{qx}{(a,q)}\Big).$$

Proof. This is Lemma 3 of [16]. □

Lemma 2.2. *For any non-negative integers α and β, and any positive integer k, we have*

(i) $s_{\alpha,\beta}(ak,b,qk) = (b,k)^{\beta} k^{1-\beta} s_{\alpha,\beta}(a,b/(b,k),q)$;
(ii) $s_{\alpha,\beta}(a,bk,qk) = (a,k)^{\alpha} k^{1-\alpha} s_{\alpha,\beta}(a/(a,k),b,q)$;
(iii) $s_{\alpha,\beta}(ak,bk,qk) = k s_{\alpha,\beta}(a,b,q)$.

Proof. This is Lemma 4 of [16]. □

From Lemma 2.1 and Lemma 2.2 we can get the following:

Lemma 2.3. *For any prime p, we have*

$$\sum_{r=0}^{p-1} s_{\alpha,\beta}(a+rq,b,pq) = p^{1-\alpha} s_{\alpha,\beta}(ap,b,pq) - p^{1-\alpha} s_{\alpha,\beta}(ap,b,q) + p s_{\alpha,\beta}(a,b,q).$$

For the special case $p = 2$, from Lemma 2.2 and Lemma 2.3 we immediately get

$$\begin{aligned} s_{\alpha,\beta}(a+q,b,2q) = {} & 2^{2-\alpha-\beta}(b,2)^{\beta} s_{\alpha,\beta}(a,b/(b,2),q) - 2^{1-\alpha} s_{\alpha,\beta}(2a,b,q) \\ & + 2 s_{\alpha,\beta}(a,b,q) - s_{\alpha,\beta}(a,b,2q). \end{aligned}$$

Proof. By Lemma 2.1 we easily get

$$\begin{aligned}
\sum_{r=0}^{p-1} s_{\alpha,\beta}(a+rq,b,pq) &= \sum_{r=0}^{p-1}\sum_{j=1}^{pq} P_\alpha\Big(\frac{(a+rq)j}{pq}\Big)P_\beta\Big(\frac{bj}{pq}\Big)\\
&= \sum_{j=1}^{pq} P_\beta\Big(\frac{bj}{pq}\Big)\sum_{r=0}^{p-1} P_\alpha\Big(\frac{aj}{pq}+\frac{rj}{p}\Big)\\
&= p^{1-\alpha}\sum_{j=1}^{pq}(j,p)^\alpha P_\alpha\Big(\frac{aj}{(j,p)q}\Big)P_\beta\Big(\frac{bj}{pq}\Big)\\
&= p^{1-\alpha}\sum_{\substack{j=1\\(j,p)=1}}^{pq} P_\alpha\Big(\frac{aj}{q}\Big)P_\beta\Big(\frac{bj}{pq}\Big) + p\sum_{\substack{j=1\\(j,p)=p}}^{pq} P_\alpha\Big(\frac{aj}{pq}\Big)P_\beta\Big(\frac{bj}{pq}\Big)\\
&= p^{1-\alpha}\Big[\sum_{j=1}^{pq} P_\alpha\Big(\frac{aj}{q}\Big)P_\beta\Big(\frac{bj}{pq}\Big) - \sum_{j=1}^{q} P_\alpha\Big(\frac{apj}{q}\Big)P_\beta\Big(\frac{bj}{q}\Big)\Big]\\
&\quad + p\sum_{j=1}^{q} P_\alpha\Big(\frac{aj}{q}\Big)P_\beta\Big(\frac{bj}{q}\Big)\\
&= p^{1-\alpha}s_{\alpha,\beta}(ap,b,pq) - p^{1-\alpha}s_{\alpha,\beta}(ap,b,q) + ps_{\alpha,\beta}(a,b,q).
\end{aligned}$$

This proves Lemma 2.3. □

Now we shall prove the following:

Theorem 2.1. *Let $(ab,q)=1$. Then we have*

$$s_\beta^{(1)}(a,b,q) = 2s_{1,\beta}(a,b,q) - 2^{\beta+1}(b,2)^{-\beta}s_{1,\beta}(a,b(b,2),2q), \quad \text{if } a \text{ is even;} \tag{2.1}$$

$$s_{\alpha,\beta}^{(2)}(a,b,q) = 2^\beta(b,2)^{-\beta}s_{\alpha,\beta}(2a,b(b,2),q) - s_{\alpha,\beta}(a,b,q), \quad \text{if } q \text{ is even;} \tag{2.2}$$

$$\begin{aligned}
s_\beta^{(5)}(a,b,q) = &-2s_{1,\beta}(a,b,q) - 2^{\beta+2}(b,2)^{-\beta}s_{1,\beta}(a,b(b,2),q)\\
&+ 2^{\beta+1}(b,2)^{-\beta}s_{1,\beta}(2a,b(b,2),q)\\
&+ 2^{\beta+1}(b,2)^{-\beta}s_{1,\beta}(a,b(b,2),2q),\\
&\text{if } a+q \text{ is even.}
\end{aligned} \tag{2.3}$$

Moreover, each one of

$$s_\beta^{(1)}(a,b,q) \quad (a \text{ odd}), \quad s_{\alpha,\beta}^{(2)}(a,b,q) \quad (q \text{ odd}), \quad s_\beta^{(5)}(a,b,q) \quad (a+q \text{ odd}) \tag{2.4}$$

is zero.

P r o o f. By (5.8) of [10] we know that

$$(-1)^{[x]} = 2((x)) - 4\Big(\Big(\frac{x}{2}\Big)\Big), \quad \text{if } x \text{ is not an integer.} \tag{2.5}$$

Thus for any even number a, by Lemma 2.2 we have

$$\begin{aligned} s_{\beta}^{(1)}(a,b,q) &= \sum_{j=1}^{q}(-1)^{[aj/q]}P_{\beta}\Big(\frac{bj}{q}\Big) = 2\sum_{j=1}^{q-1}\Big(\Big(\frac{aj}{q}\Big)\Big)P_{\beta}\Big(\frac{bj}{q}\Big) - 4\sum_{j=1}^{q-1}\Big(\Big(\frac{aj}{2q}\Big)\Big)P_{\beta}\Big(\frac{bj}{q}\Big) \\ &= 2s_{1,\beta}(a,b,q) - 4s_{1,\beta}\Big(\frac{a}{2},b,q\Big) \\ &= 2s_{1,\beta}(a,b,q) - 2^{\beta+1}(b,2)^{-\beta}s_{1,\beta}(a,b(b,2),2q). \end{aligned}$$

This proves (2.1).

If q is even, then by Lemma 2.2 we also have

$$\begin{aligned} s_{\alpha,\beta}^{(2)}(a,b,q) &= \sum_{j=1}^{q}(-1)^{j}P_{\alpha}\Big(\frac{aj}{q}\Big)P_{\beta}\Big(\frac{bj}{q}\Big) \\ &= 2\sum_{\substack{j=1\\2|j}}^{q}P_{\alpha}\Big(\frac{aj}{q}\Big)P_{\beta}\Big(\frac{bj}{q}\Big) - \sum_{j=1}^{q}P_{\alpha}\Big(\frac{aj}{q}\Big)P_{\beta}\Big(\frac{bj}{q}\Big) \\ &= 2\sum_{j=1}^{q/2}P_{\alpha}\Big(\frac{2aj}{q}\Big)P_{\beta}\Big(\frac{2bj}{q}\Big) - s_{\alpha,\beta}(a,b,q) \\ &= 2s_{\alpha,\beta}\Big(a,b,\frac{q}{2}\Big) - s_{\alpha,\beta}(a,b,q) \\ &= 2^{\beta}(b,2)^{-\beta}s_{\alpha,\beta}(2a,b(b,2),q) - s_{\alpha,\beta}(a,b,q). \end{aligned}$$

This proves (2.2).

If $a+q$ is even, from Lemma 2.2 and Lemma 2.3 we get

$$\begin{aligned} s_{\beta}^{(5)}(a,b,q) &= \sum_{j=1}^{q}(-1)^{j+[aj/q]}P_{\beta}\Big(\frac{bj}{q}\Big) = \sum_{j=1}^{q}(-1)^{[j(a+q)/q]}P_{\beta}\Big(\frac{bj}{q}\Big) \\ &= \sum_{j=1}^{q-1}\Big[2\Big(\Big(\frac{j(a+q)}{q}\Big)\Big) - 4\Big(\Big(\frac{j(a+q)}{2q}\Big)\Big)\Big]P_{\beta}\Big(\frac{bj}{q}\Big) \\ &= 2s_{1,\beta}(a,b,q) - 4s_{1,\beta}\Big(\frac{a+q}{2},b,q\Big) \\ &= 2s_{1,\beta}(a,b,q) - 2^{\beta+1}(b,2)^{-\beta}s_{1,\beta}(a+q,b(b,2),2q) \\ &= -2s_{1,\beta}(a,b,q) - 2^{\beta+2}(b,2)^{-\beta}s_{1,\beta}(a,b(b,2),q) \\ &\quad + 2^{\beta+1}(b,2)^{-\beta}s_{1,\beta}(2a,b(b,2),q) + 2^{\beta+1}(b,2)^{-\beta}s_{1,\beta}(a,b(b,2),2q). \end{aligned}$$

This proves (2.3).

To prove (2.4), noting that

$$P_m(-x) = (-1)^m P_m(x), \tag{2.6}$$

we have from the definition of $s_\beta^{(1)}(a,b,q)$

$$\begin{aligned}
s_\beta^{(1)}(a,b,q) &= \sum_{j=1}^{q-1}(-1)^{[aj/q]}P_\beta\Big(\frac{bj}{q}\Big) = \sum_{k=1}^{q-1}(-1)^{[a(q-k)/q]}P_\beta\Big(\frac{b(q-k)}{q}\Big)\\
&= \sum_{k=1}^{q-1}(-1)^{a+[-ak/q]}P_\beta\Big(-\frac{bk}{q}\Big) = \sum_{k=1}^{q-1}(-1)^{a-[ak/q]-1}P_\beta\Big(-\frac{bk}{q}\Big)\\
&= (-1)^{a+\beta-1}\sum_{k=1}^{q-1}(-1)^{[ak/q]}P_\beta\Big(\frac{bk}{q}\Big) = (-1)^a\sum_{k=1}^{q-1}(-1)^{[ak/q]}P_\beta\Big(\frac{bk}{q}\Big)\\
&= (-1)^a s_\beta^{(1)}(a,b,q).
\end{aligned}$$

Thus $s_\beta^{(1)}(a,b,q)=0$ if a is odd. The proofs of the remaining assertions in (2.4) are similar. □

Remark. Taking $\alpha=\beta=b=1$ in Theorem 2.1, we immediately get (1.3), (1.4) and (1.7).

3. Generalized Knopp identities for the generalized homogeneous Hardy sums

In this section, we shall give the generalized Knopp identities for the generalized homogeneous Hardy sums.

Theorem 3.1. *Let n be a positive odd integer, and $(a,q)=1$. Then we have*

$$\sum_{d|n} d^{\beta-1}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s_\beta^{(1)}\Big(\frac{n}{d}a+2r_1q, \frac{n}{d}b+r_2q, dq\Big) = n\sigma_\beta(n)s_\beta^{(1)}(a,b,q), \quad \text{if } a \text{ is even;} \tag{3.1}$$

$$\sum_{d|n} d^{\alpha+\beta-2}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s_{\alpha,\beta}^{(2)}\Big(\frac{n}{d}a+r_1q, \frac{n}{d}b+r_2q, dq\Big) = n\sigma_{\alpha+\beta-1}(n)s_{\alpha,\beta}^{(2)}(a,b,q), \quad \text{if } q \text{ is even;} \tag{3.2}$$

$$\sum_{d|n} d^{\beta-1}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s_\beta^{(5)}\Big(\frac{n}{d}a+2r_1q, \frac{n}{d}b+r_2q, dq\Big) = n\sigma_\beta(n)s_\beta^{(5)}(a,b,q), \quad \text{if } a+q \text{ is even.} \tag{3.3}$$

Proof. We only prove (3.1). The other identities can be deduced similarly. By Theorem 2.1 and (1.1) we have

$$\begin{aligned}
&\sum_{d\mid n} d^{\beta-1}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s_\beta^{(1)}\Bigl(\frac{n}{d}a+2r_1q,\frac{n}{d}b+r_2q,dq\Bigr)\\
&\quad=\sum_{d\mid n} d^{\beta-1}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d}\Bigl[2s_{1,\beta}\Bigl(\frac{n}{d}a+2r_1q,\frac{n}{d}b+r_2q,dq\Bigr)\\
&\qquad-4s_{1,\beta}\Bigl(\frac{n}{d}\cdot\frac{a}{2}+r_1q,\frac{n}{d}b+r_2q,dq\Bigr)\Bigr]\\
&\quad=2\sum_{d\mid n} d^{\beta-1}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s_{1,\beta}\Bigl(\frac{n}{d}a+r_1q,\frac{n}{d}b+r_2q,dq\Bigr)\\
&\qquad-4\sum_{d\mid n} d^{\beta-1}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} s_{1,\beta}\Bigl(\frac{n}{d}\cdot\frac{a}{2}+r_1q,\frac{n}{d}b+r_2q,dq\Bigr)\\
&\quad=2n\sigma_\beta(n)s_{1,\beta}(a,b,q)-4n\sigma_\beta(n)s_{1,\beta}\Bigl(\frac{a}{2},b,q\Bigr)=n\sigma_\beta(n)s_\beta^{(1)}(a,b,q).
\end{aligned}$$

This completes the proof of Theorem 3.1. □

4. Generalized homogeneous Cochrane sums and Cochrane-Hardy sums

For a positive integer q and an arbitrary integer h, the Cochrane sums are defined by

$$c(h,q)=\sideset{}{'}\sum_{a=1}^{q}\Bigl(\Bigl(\frac{a}{q}\Bigr)\Bigr)\Bigl(\Bigl(\frac{\bar a h}{q}\Bigr)\Bigr),$$

where $\sideset{}{'}\sum_{a}$ denotes the summation over all a such that $(a,q)=1$, and $\bar a$ is defined by the equation $a\bar a\equiv 1 \bmod q$. Wenpeng Zhang and Yuan Yi [14] gave the following upper bound estimate

$$|c(h,q)|\ll\sqrt{q}\,d(q)\ln^2 q$$

and the mean value estimate

$$\sum_{h=1}^{p-1} c^2(h,p)=\frac{5}{144}p^2+O\Bigl(p\exp\Bigl(\frac{4\ln p}{\ln\ln p}\Bigr)\Bigr),$$

where $d(q)$ is the divisor function, and $\exp(y)=\mathrm{e}^y$.

In [11], Wenpeng Zhang found that there are some relationships between $c(h,q)$ and the Kloosterman sum

$$K(m,n;q)=\sideset{}{'}\sum_{a=1}^{q}\mathrm{e}\Big(\frac{ma+n\bar{a}}{q}\Big),$$

where $\mathrm{e}(y)=\mathrm{e}^{2\pi\mathrm{i}y}$. For example, if q is a square-full number, he obtained

$$\sideset{}{'}\sum_{h=1}^{q}c(h,q)K(h,1;q)=-\frac{1}{2\pi^2}q\varphi(q)+O\Big(q\exp\Big(\frac{3\ln q}{\ln\ln q}\Big)\Big),$$

where $\varphi(q)$ is the Euler function. For general integer $q\geqslant 3$, he obtained in [12] the asymptotic formula

$$\sideset{}{'}\sum_{h=1}^{q}c(h,q)K(h,1;q)=-\frac{1}{2\pi^2}q\varphi(q)\prod_{p\|q}\Big(1-\frac{1}{p(p-1)}\Big)+O(q^{\frac{3}{2}+\varepsilon}),\tag{4.1}$$

where ε is any fixed positive number. Later he and the author [13] proved that the error term in (4.1) is $O(q^{1+\varepsilon})$.

We turn now to the study of the generalized homogeneous Cochrane sums and Cochrane-Hardy sums. First we give the following definitions:

$$\begin{aligned}
c_{\alpha,\beta}(a,b,q)&=\sideset{}{'}\sum_{j=1}^{q}P_\alpha\Big(\frac{aj}{q}\Big)P_\beta\Big(\frac{b\bar{j}}{q}\Big), && \alpha\equiv\beta \bmod 2;\\
c_{\beta}^{(1)}(a,b,q)&=\sideset{}{'}\sum_{j=1}^{q}(-1)^{[aj/q]}P_\beta\Big(\frac{b\bar{j}}{q}\Big), && \beta\equiv 1 \bmod 2;\\
c_{\alpha,\beta}^{(2)}(a,b,q)&=\sideset{}{'}\sum_{j=1}^{q}(-1)^{j}P_\alpha\Big(\frac{aj}{q}\Big)P_\beta\Big(\frac{b\bar{j}}{q}\Big), && \alpha\equiv\beta \bmod 2;\\
c_{\beta}^{(5)}(a,b,q)&=\sideset{}{'}\sum_{j=1}^{q}(-1)^{j+[aj/q]}P_\beta\Big(\frac{b\bar{j}}{q}\Big), && \beta\equiv 1 \bmod 2.
\end{aligned}$$

Now we express the generalized homogeneous Cochrane-Hardy sums in terms of generalized homogeneous Cochrane sums.

Theorem 4.1. *Let $(ab, q) = 1$. Then we have*

$$c^{(1)}_{\beta}(a,b,q) = 2c_{1,\beta}(a,b,q) - 4c_{1,\beta}\Big(\frac{a}{2}, b, q\Big), \qquad \text{if } a \text{ is even;} \tag{4.2}$$

$$c^{(2)}_{\alpha,\beta}(a,b,q) = -\,c_{\alpha,\beta}(a,b,q), \qquad \text{if } q \text{ is even;} \tag{4.3}$$

$$c^{(5)}_{\beta}(a,b,q) = 2c_{1,\beta}(a,b,q) - 4c_{1,\beta}\Big(\frac{a+q}{2}, b, q\Big), \qquad \text{if } a+q \text{ is even.} \tag{4.4}$$

Moreover, each one of

$$c^{(1)}_{\beta}(a,b,q) \quad (a \text{ odd}), \quad c^{(2)}_{\alpha,\beta}(a,b,q) \quad (q \text{ odd}), \quad c^{(5)}_{\beta}(a,b,q) \quad (a+q \text{ odd}) \tag{4.5}$$

is zero.

P r o o f. For any even number a, by (2.5) we have

$$\begin{aligned} c^{(1)}_{\beta}(a,b,q) &= \sum_{j=1}^{q}{}'(-1)^{[aj/q]} P_{\beta}\Big(\frac{b\bar{j}}{q}\Big) \\ &= 2\sum_{j=1}^{q}{}'\Big(\Big(\frac{aj}{q}\Big)\Big) P_{\beta}\Big(\frac{b\bar{j}}{q}\Big) - 4\sum_{j=1}^{q}{}'\Big(\Big(\frac{aj}{2q}\Big)\Big) P_{\beta}\Big(\frac{b\bar{j}}{q}\Big) \\ &= 2c_{1,\beta}(a,b,q) - 4c_{1,\beta}\Big(\frac{a}{2}, b, q\Big). \end{aligned}$$

This proves (4.2).

If q is even, then $(j, q) = 1$ only if j is odd. Therefore

$$\begin{aligned} c^{(2)}_{\alpha,\beta}(a,b,q) &= \sum_{j=1}^{q}{}'(-1)^{j} P_{\alpha}\Big(\frac{aj}{q}\Big) P_{\beta}\Big(\frac{b\bar{j}}{q}\Big) \\ &= -\sum_{j=1}^{q}{}' P_{\alpha}\Big(\frac{aj}{q}\Big) P_{\beta}\Big(\frac{b\bar{j}}{q}\Big) = -c_{\alpha,\beta}(a,b,q). \end{aligned}$$

This proves (4.3).

If $a + q$ is even, by (2.5) we get

$$\begin{aligned} c^{(5)}_{\beta}(a,b,q) &= \sum_{j=1}^{q}{}'(-1)^{j+[aj/q]} P_{\beta}\Big(\frac{b\bar{j}}{q}\Big) = \sum_{j=1}^{q}{}'(-1)^{[j(a+q)/q]} P_{\beta}\Big(\frac{b\bar{j}}{q}\Big) \\ &= \sum_{j=1}^{q}{}'\Big[2\Big(\Big(\frac{j(a+q)}{q}\Big)\Big) - 4\Big(\Big(\frac{j(a+q)}{2q}\Big)\Big)\Big] P_{\beta}\Big(\frac{b\bar{j}}{q}\Big) \\ &= 2c_{1,\beta}(a,b,q) - 4c_{1,\beta}\Big(\frac{a+q}{2}, b, q\Big). \end{aligned}$$

This proves (4.4).

To prove (4.5), from (2.6) and the definition of $c_\beta^{(1)}(a,b,q)$ we have

$$\begin{aligned} c_\beta^{(1)}(a,b,q) &= {\sum_{j=1}^{q}}'(-1)^{[aj/q]}P_\beta\Big(\frac{b\bar{j}}{q}\Big) = {\sum_{j=1}^{q}}'(-1)^{[a(q-j)/q]}P_\beta\Big(\frac{b\overline{q-j}}{q}\Big) \\ &= {\sum_{j=1}^{q}}'(-1)^{a-[aj/q]-1}P_\beta\Big(-\frac{b\bar{j}}{q}\Big) = (-1)^{a+\beta-1}{\sum_{j=1}^{q}}'(-1)^{[aj/q]}P_\beta\Big(\frac{b\bar{j}}{q}\Big) \\ &= (-1)^a c_\beta^{(1)}(a,b,q). \end{aligned}$$

Thus $c_\beta^{(1)}(a,b,q)=0$ if a is odd. The proofs of the remaining assertions in (4.5) are similar. □

5. Generalized Knopp identities for the generalized homogeneous Cochrane sums and Cochrane-Hardy sums

First we prove the generalized Knopp identity for the generalized homogeneous Cochrane sums as follows:

Theorem 5.1. *Let $(abn,q)=1$. Then we have*

$$\sum_{d\mid n} d^{\alpha+\beta-2}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} c_{\alpha,\beta}(a+r_1q,b+r_2q,dq) = nc_{\alpha,\beta}(a,b,q).$$

Proof. By Lemma 2.1 we get

$$\begin{aligned} &\sum_{d\mid n} d^{\alpha+\beta-2}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} c_{\alpha,\beta}(a+r_1q,b+r_2q,dq) \\ &= \sum_{d\mid n} d^{\alpha+\beta-2}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d}{\sum_{r=1}^{dq}}' P_\alpha\Big(\frac{(a+r_1q)r}{dq}\Big)P_\beta\Big(\frac{(b+r_2q)\bar{r}}{dq}\Big) \\ &= \sum_{d\mid n} d^{\alpha+\beta-2}{\sum_{r=1}^{dq}}'\sum_{r_1=1}^{d} P_\alpha\Big(\frac{ar}{dq}+\frac{r_1r}{d}\Big)\sum_{r_2=1}^{d} P_\beta\Big(\frac{b\bar{r}}{dq}+\frac{r_2\bar{r}}{d}\Big) \\ &= \sum_{d\mid n}{\sum_{r=1}^{dq}}' P_\alpha\Big(\frac{ar}{q}\Big)P_\beta\Big(\frac{b\bar{r}}{q}\Big). \end{aligned} \tag{5.1}$$

Since $(n,q)=1$, we have $(d,q)=1$. Let $r=xd+yq$ with $x=1,\dots,q$, $(x,q)=1$, $y=1,\dots,d$, $(y,d)=1$. Then

$$\bar{r}=\bar{x}\bar{d}^2d+\bar{y}\bar{q}^2q,$$

where

$$d\bar{d} \equiv 1 \bmod q, \quad q\bar{q} \equiv 1 \bmod d, \quad x\bar{x} \equiv 1 \bmod q, \quad y\bar{y} \equiv 1 \bmod d.$$

Therefore

$$\begin{aligned}
(5.2) \qquad & \sum_{d|n} \sum_{r=1}^{dq}{}' P_\alpha\Big(\frac{ar}{q}\Big) P_\beta\Big(\frac{b\bar{r}}{q}\Big) \\
&= \sum_{d|n} \sum_{x=1}^{q}{}' \sum_{y=1}^{d}{}' P_\alpha\Big(\frac{a(xd+yq)}{q}\Big) P_\beta\Big(\frac{b(\bar{x}\bar{d}^2 d + \bar{y}\bar{q}^2 q)}{q}\Big) \\
&= \sum_{d|n} \sum_{x=1}^{q}{}' \sum_{y=1}^{d}{}' P_\alpha\Big(\frac{axd}{q}\Big) P_\beta\Big(\frac{b\bar{x}\bar{d}}{q}\Big) \\
&= \sum_{d|n} \varphi(d) \sum_{x=1}^{q}{}' P_\alpha\Big(\frac{axd}{q}\Big) P_\beta\Big(\frac{b\bar{x}\bar{d}}{q}\Big).
\end{aligned}$$

As x runs over a reduced residue system modulo q, xd also runs over a reduced residue system modulo q. Thus we have

$$\begin{aligned}
(5.3) \qquad \sum_{d|n} \varphi(d) \sum_{x=1}^{q}{}' P_\alpha\Big(\frac{axd}{q}\Big) P_\beta\Big(\frac{b\bar{x}\bar{d}}{q}\Big) &= \sum_{d|n} \varphi(d) \sum_{x=1}^{q}{}' P_\alpha\Big(\frac{ax}{q}\Big) P_\beta\Big(\frac{b\bar{x}}{q}\Big) \\
&= n c_{\alpha,\beta}(a,b,q).
\end{aligned}$$

Combining (5.1)–(5.3), we immediately get Theorem 5.1. □

Using the same methods and Theorem 4.1 we also get the following generalized Knopp identities for the generalized homogeneous Cochrane-Hardy sums.

Theorem 5.2. *Let n be a positive odd integer with $(abn, q) = 1$. Then we have*

$$\sum_{d|n} d^{\beta-1} \sum_{r_1=1}^{d} \sum_{r_2=1}^{d} c_\beta^{(1)}(a+2r_1q, b+r_2q, dq) = n c_\beta^{(1)}(a,b,q), \qquad \text{if } a \text{ is even;}$$

$$\sum_{d|n} d^{\alpha+\beta-2} \sum_{r_1=1}^{d} \sum_{r_2=1}^{d} c_{\alpha,\beta}^{(2)}(a+r_1q, b+r_2q, dq) = n c_{\alpha,\beta}^{(2)}(a,b,q), \quad \text{if } q \text{ is even;}$$

$$\sum_{d|n} d^{\beta-1} \sum_{r_1=1}^{d} \sum_{r_2=1}^{d} c_\beta^{(5)}(a+2r_1q, b+r_2q, dq) = n c_\beta^{(5)}(a,b,q), \qquad \text{if } a+q \text{ is even.}$$

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