SIAM J. OPTIM.
Vol. 17, No. 1, pp. 243–258

GENERALIZED LEVITIN–POLYAK WELL-POSEDNESS IN CONSTRAINED OPTIMIZATION*

X. X. HUANG† AND X. Q. YANG‡

Abstract. In this paper, we consider Levitin–Polyak-type well-posedness for a general constrained optimization problem. We introduce generalized Levitin–Polyak well-posedness and strongly generalized Levitin–Polyak well-posedness. Necessary and sufficient conditions for these types of well-posedness are given. Relations among these types of well-posedness are investigated. Finally, we consider convergence of a class of penalty methods and a class of augmented Lagrangian methods under the assumption of strongly generalized Levitin–Polyak well-posedness.

Key words. constrained optimization, generalized minimizing sequence, generalized Levitin–Polyak well-posedness, penalty-type methods

DOI. 10.1137/040614943

1. Introduction.

The study of well-posedness originates from Tykhonov [26] in dealing with unconstrained optimization problems. Its extension to the constrained case was developed by Levitin and Polyak [18]. Since then, various notions of well-posedness have been defined and extensively studied (see, e.g., [22, 6, 24, 28, 29, 9, 24, 30]). It is worth noting that recent research on well-posedness has been extended to vector optimization problems (see, e.g., [3, 20, 21, 12, 13, 7]).

Let (X, d_1) and (Y, d_2) be two metric spaces, and let $X_1 \subset X$ and $K \subset Y$ be two nonempty and closed sets. Consider the following constrained optimization problem:

$$\text{(P)} \qquad \begin{array}{l} \min f(x) \\ \text{s.t. } x \in X_1, \quad g(x) \in K, \end{array}$$

where $f : X \to R^1$ is a lower semicontinuous function and $g : X \to Y$ is a continuous function. Denote by X_0 the set of feasible solutions of (P), i.e.,

$$X_0 = \{x \in X_1 : g(x) \in K\}.$$

Denote by $\bar{X}$ and $\bar{v}$ the optimal solution set and the optimal value of (P), respectively. Throughout the paper, we always assume that $X_0 \neq \emptyset$ and $\bar{v} > -\infty$.

Let (Z, d) be a metric space and $Z_1 \subset Z$. We denote by $d_{Z_1}(z) = \inf\{d(z, z') : z' \in Z_1\}$ the distance from the point z to the set Z_1.

Levitin–Polyak (LP) well-posedness of (P) in the usual sense (when the optimal set of (P) is not necessarily a singleton) says that, for any sequence $\{x_n\} \subset X_1$ satisfying (i) $d_{X_0}(x_n) \to 0$ and (ii) $f(x_n) \to \bar{v}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \to \bar{x}$.

*Received by the editors September 14, 2004; accepted for publication (in revised form) December 22, 2005; published electronically May 12, 2006. This work was supported by the Research Grants Council of Hong Kong (BQ-654), the National Science Foundation of China, and a small grant from Fudan University, China.

http://www.siam.org/journals/siopt/17-1/61494.html

†School of Management, Fudan University, Shanghai 200433, China, and Department of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China (xxhuang@fudan.edu.cn).

‡Department of Applied Mathematics, Hong Kong Polytechnic University, Kowloon, Hong Kong (mayangxq@polyu.edu.uk).

It should be noted that many optimization algorithms, such as penalty-type methods, e.g., penalty function methods and augmented Lagrangian methods, terminate when the constraint is approximately satisfied; i.e., $d_K(g(\bar{x})) \leq \epsilon$ for some $\epsilon > 0$ sufficiently small, and $\bar{x}$ is taken as an approximate solution of (P). These methods may generate sequences $\{x_n\} \subset X_1$ that satisfy $d_K(g(x_n)) \to 0$, not necessarily $d_{X_0}(x_n) \to 0$, as shown in the following simple example.

Example 1.1. Let $\alpha > 0$. Let $X = R^1$, $X_1 = R^1_+$, $K = R^1_-$, and

$$f(x) = \begin{cases} -x^\alpha & \text{if } x \in [0,1]; \\ -1/x^\alpha & \text{if } x \geq 1, \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x \in [0,1]; \\ 1/x^2 & \text{if } x \geq 1. \end{cases}$$

Consider the following penalty problem:

$$(PP_\alpha(n)) \quad \min_{x \in X_1} f(x) + n\left[\max\{0, g(x)\}\right]^\alpha, \quad n \in N.$$

It is easily verified that $x_n = 2^{1/\alpha} n^{1/\alpha}$ is the unique global solution to $(PP_\alpha(n))$ for each $n \in N$. Note that $X_0 = \{0\}$. It follows that we have $d_K(g(x_n)) = 1/(2^{2/\alpha} n^{2/\alpha}) \to 0$, while $d_{X_0}(x_n) = 2^{1/\alpha} n^{1/\alpha} \to +\infty$.

Thus, it is useful to consider sequences that satisfy $d_K(g(x_n)) \to 0$ instead of $d_{X_0}(x_n) \to 0$ as $n \to \infty$ in order to study convergence of penalty-type methods.

The sequence $\{x_n\}$ satisfying (i) and (ii) above is called an LP minimizing sequence. In what follows, we introduce two more types of generalized LP well-posedness.

Definition 1.1. (P) *is called LP well-posedness in the generalized sense if, for any sequence* $\{x_n\} \subset X_1$ *satisfying* (i) $d_K(g(x_n)) \to 0$ *and* (ii) $f(x_n) \to \bar{v}$*, there exist a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *and some* $\bar{x} \in \bar{X}$ *such that* $x_{n_k} \to \bar{x}$*. The sequence* $\{x_n\}$ *is called a generalized LP minimizing sequence.*

Definition 1.2. (P) *is called LP well-posedness in the strongly generalized sense if, for any sequence* $\{x_n\} \subset X_1$ *satisfying* (i) $d_K(g(x_n)) \to 0$ *and* (ii) $\limsup_{n \to +\infty} f(x_n) \leq \bar{v}$*, there exist a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *and some* $\bar{x} \in \bar{X}$ *such that* $x_{n_k} \to \bar{x}$*. The sequence* $\{x_n\}$ *is called a weakly generalized LP minimizing sequence.*

Remark 1.1. (i) The study of well-posedness for optimization problems with explicit constraints dates back to [17] when the abstract set X_1 does not appear. In [17], it was assumed that X is a Banach space and Y is a Banach space ordered by a closed and convex cone with some special properties; see [17] for details. What is worth emphasizing is that [17] studied only the case when (P) is a convex program. However, it is well known that penalty-type methods such as penalization methods and augmented Lagrangian methods are mostly developed for constrained nonconvex optimization problems. This is the main motivation of this paper.

(ii) The LP well-posedness in the strongly generalized sense defined above was called well-posedness in the strongly generalized sense in [17], while a weakly generalized LP minimizing sequence in the above definition is called a generalized minimizing sequence in [17].

(iii) It is obvious that LP well-posedness in the strongly generalized sense implies LP well-posedness in the generalized sense because a generalized LP minimizing sequence is a weakly generalized LP minimizing sequence.

(iv) If there exists some $\delta_0 > 0$ such that g is uniformly continuous on the set

$$\{x \in X_1 : d_{X_0}(x) \leq \delta_0\},$$

then it is not difficult to see that LP well-posedness in the generalized sense implies LP well-posedness.

(v) Any one type of (generalized) LP well-posedness defined above implies that the optimal set $\bar{X}$ of (P) is nonempty and compact.

The paper is organized as follows. In section 2, we investigate characterizations and criteria for the three types of (generalized) LP well-posednesses. In section 3, we establish relations among the three types of (generalized) LP well-posednesses. In section 4, we obtain convergence of a class of penalty methods and a class of augmented Lagrangian methods under the assumption of strongly generalized LP well-posedness.

2. Necessary and sufficient conditions for three types of (generalized) LP well-posedness. In this section, we present some criteria and characterizations for the three types of (generalized) LP well-posedness defined in section 1.

Consider the following statement:

$$(1)\quad \begin{array}{l}[\bar{X} \neq \emptyset \text{ and, for any LP minimizing sequence (resp.,}\\ \text{generalized LP minimizing sequence, weakly generalized LP minimizing sequence)}\\ \{x_n\}, \text{ we have } d_{\bar{X}}(x_n) \to 0].\end{array}$$

The proof of the following proposition is elementary and thus omitted.

PROPOSITION 2.1. *If* (P) *is LP well-posed (resp., LP well-posed in the generalized sense and LP well-posed in the strongly generalized sense), then* (1) *holds. Conversely, if* (1) *holds and* $\bar{X}$ *is compact, then* (P) *is LP well-posed (resp., LP well-posed in the generalized sense and LP well-posed in the strongly generalized sense).*

Consider a real-valued function $c = c(t, s)$ defined for $t, s \geq 0$ sufficiently small, such that

$$(2)\quad c(t,s) \geq 0 \quad \forall t, s, \quad c(0,0) = 0,$$

$$(3)\quad s_k \to 0, t_k \geq 0, c(t_k, s_k) \to 0 \text{ imply } t_k \to 0.$$

THEOREM 2.1. *If* (P) *is LP well-posed, then there exists a function c satisfying* (2) *and* (3) *such that*

$$(4)\quad |f(x) - \bar{v}| \geq c(d_{\bar{X}}(x), d_{X_0}(x)) \quad \forall x \in X_1.$$

Conversely, suppose that $\bar{X}$ *is nonempty and compact, and* (4) *holds for some c satisfying* (2) *and* (3). *Then* (P) *is LP well-posed.*

Proof. Define

$$c(t,s) = \inf\{|f(x) - \bar{v}| : x \in X_1, d_{\bar{X}}(x) = t, d_{X_0}(x) = s\}.$$

It is obvious that $c(0,0) = 0$. Moreover, if $s_n \to 0$, $t_n \geq 0$ and $c(t_n, s_n) \to 0$, then there exists a sequence $\{x_n\} \subset X_1$ with

$$(5)\quad d_{\bar{X}}(x_n) = t_n,$$

$$(6)\quad d_{X_0}(x_n) = s_n$$

such that

$$(7)\quad |f(x_n) - \bar{v}| \to 0.$$

Note that $s_n \to 0$. Equations (6) and (7) jointly imply that $\{x_n\}$ is an LP minimizing sequence. By Proposition 2.1, we have $t_n \to 0$. This completes the proof of the first half of the theorem. Conversely, let $\{x_n\}$ be an LP minimizing sequence. Then, by (4), we have

$$|f(x_n) - \bar{v}| \geq c(d_{\bar{X}}(x_n), d_{X_0}(x_n)) \quad \forall x \in X_1. \tag{8}$$

Let

$$t_n = d_{\bar{X}}(x_n), \quad s_n = d_{X_0}(x_n).$$

Then $s_n \to 0$. In addition, $|f(x_n) - \bar{v}| \to 0$. These facts together with (8) as well as the properties of the function c imply that $t_n \to 0$. By Proposition 2.1, we see that (P) is LP well-posed. ∎

THEOREM 2.2. *If* (P) *is LP well-posed in the generalized sense, then there exists a function* c *satisfying* (2) *and* (3) *such that*

$$|f(x) - \bar{v}| \geq c(d_{\bar{X}}(x), d_K(g(x))) \quad \forall x \in X_1. \tag{9}$$

Conversely, suppose that $\bar{X}$ *is nonempty and compact, and* (9) *holds for some* c *satisfying* (2) *and* (3). *Then* (P) *is LP well-posed in the generalized sense.*

Proof. The proof is almost the same as that of Theorem 2.1. The only difference lies in the proof of the first part of Theorem 2.1. Here we define

$$c(t,s) = \inf\{|f(x) - \bar{v}| : x \in X_1, d_{\bar{X}}(x) = t, d_K(g(x)) = s\}. \qquad ∎$$

Next we give a necessary and sufficient condition in the form of Furi and Vignoli [10] to characterize the LP well-posedness in the strongly generalized sense.

Let

$$\Omega(\epsilon) = \{x \in X_1 : f(x) \leq \bar{v} + \epsilon, d_K(g(x)) \leq \epsilon\}.$$

Let (X, d_1) be a complete metric space. Recall that the Kuratowski measure of noncompactness for a subset A of X is defined as

$$\alpha(A) = \inf\left\{\epsilon > 0 : A \subset \bigcup_{1 \leq i \leq n} C_i, \text{ for some } C_i, diam(C_i) \leq \epsilon\right\},$$

where $diam(C_i)$ is the diameter of C_i defined by

$$diam(C_i) = \sup\{d_1(x_1, x_2) : x_1, x_2 \in C_i\}.$$

The next theorem can be proved analogously to [17, Theorem 5.5].

THEOREM 2.3. *Let* (X, d_1) *be a complete metric space and* f *be bounded below on* X_0. *Then* (P) *is LP well-posed in the strongly generalized sense if and only if*

$$\alpha(\Omega(\epsilon))) \to 0 \text{ as } \epsilon \to 0.$$

DEFINITION 2.1. *Let* Z *be a topological space and* $Z_1 \subset Z$ *be nonempty. Suppose that* $h : Z \to R^1 \cup \{+\infty\}$ *is an extended real-valued function.* h *is said to be level-compact on* Z_1 *if, for any* $s \in R^1$, *the subset* $\{z \in Z_1 : h(z) \leq s\}$ *is compact.*

For any $\delta \geq 0$, define

$$X_1(\delta) = \{x \in X_1 : d_K(g(x)) \leq \delta\}. \tag{10}$$

The following proposition gives sufficient conditions that guarantee LP well-posedness in the strongly generalized sense.

PROPOSITION 2.2. *Let one of the following conditions hold.*

(i) *There exists $\delta_0 > 0$ such that $X_1(\delta_0)$ is compact.*

(ii) *f is level-compact on X_1.*

(iii) *X is a finite dimensional normed space and*

$$\lim_{x \in X_1, \|x\| \to +\infty} \max\{f(x), d_K(g(x))\} = +\infty. \tag{11}$$

(iv) *There exists $\delta_0 > 0$ such that f is level-compact on $X_1(\delta_0)$.*

Then (P) *is LP well-posed in the strongly generalized sense.*

Proof. Let $\{x_n\} \subset X_1$ be a weakly generalized LP minimizing sequence. Then

$$\limsup_{n \to +\infty} f(x_n) \leq \bar{v}, \tag{12}$$

$$d_K(g(x_n)) \to 0. \tag{13}$$

The proof of (i) is elementary. It is obvious that condition (ii) implies (iv). Now we show that (iii) implies (iv). Indeed, we need only to show that for any $s \in R^1$ and any $\delta > 0$, the set

$$A = \{x \in X_1(\delta) : f(x) \leq s\}$$

is bounded since X is a finite dimensional space. Suppose to the contrary that there exist $\delta > 0$, $s > 0$, and $\{x'_n\} \subset X_1(\delta)$ such that

$$\|x'_n\| \to +\infty \text{ and } f(x'_n) \leq s.$$

By $\{x'_n\} \subset X_1(\delta)$, we have $\{x'_n\} \subset X_1$ and

$$d_K(g(x'_n)) \leq \delta.$$

As a result,

$$\max\{f(x'_n), d_K(g(x'_n))\} \leq \max\{s, \delta\},$$

contradicting (11).

Thus, we need only to prove that if (iv) holds, then (P) is LP well-posed in the strongly generalized sense. By (13), it is apparent that we can assume without loss of generality that $\{x_n\} \subset X_1(\delta_0)$. By (12), we can assume without loss of generality that

$$\{x_n\} \subset \{x \in X_1 : f(x) \leq \bar{v} + 1\}.$$

By the level-compactness of f on $X_1(\delta_0)$, we deduce that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in X_1$ such that $x_{n_k} \to \bar{x}$. It is obvious from (13) that $\bar{x} \in X_0$. Furthermore, from (12), we deduce that $f(\bar{x}) \leq \bar{v}$. So we have $f(\bar{x}) = \bar{v}$. That is, $\bar{x} \in \bar{X}$. Hence, (P) is LP well-posed in the strongly generalized sense. ∎

Now we consider the case when Y is a normed space and K is a closed and convex cone with nonempty interior $\text{int}K$. Arbitrarily fix an $e \in \text{int}K$. Let $t \geq 0$ and consider the following perturbed problem of (P):

$$(P_t) \qquad \min f(x)$$
$$\text{s.t. } x \in X_1, \quad g(x) \in K - te. \tag{14}$$

Let

$$X_2(t) = \{x \in X_1 : g(x) \in K - te\}. \tag{15}$$

PROPOSITION 2.3. *Let one of the following conditions hold.*

(i) *There exists $t_0 > 0$ such that $X_2(t_0)$ is compact.*

(ii) *f is level-compact on X_1.*

(iii) *X is a finite dimensional normed space and*

$$\lim_{x \in X_1, \|x\| \to +\infty} \max\{f(x), d_K(g(x))\} = +\infty.$$

(iv) *There exists $t_0 > 0$ such that f is level-compact on $X_2(t_0)$.*

Then (P) *is LP well-posed in the strongly generalized sense.*

Proof. The proof is similar to that of Proposition 2.2. ∎

Now we make the following assumption.

ASSUMPTION 2.1. *X is a finite dimensional normed space, Y is a normed space, $X_1 \subset X$ is a nonempty, closed, and convex set, $K \subset Y$ is a closed, and convex cone with nonempty interior* $\text{int}K$ *and $e \in \text{int}K$, f and g are continuous on X_1, f is a convex function on X_1, and g is K-concave on X_1 (namely, for any $x_1, x_2 \in X_1$ and any $\theta \in (0, 1)$, there holds that $g(\theta x_1 + (1-\theta)x_2) - \theta g(x_1) - (1-\theta)g(x_2) \in K$).*

It is obvious that under Assumption 2.1, (P) is a convex program.

The next lemma can be proved similarly to that of [16, Proposition 2.4].

LEMMA 2.1. *Let Assumption* 2.1 *hold. Then the following two statements are equivalent.*

(i) *The optimal set $\bar{X}$ of* (P) *is nonempty and compact.*

(ii) *For any $t \geq 0$, f is level-compact on the set $X_2(t)$.*

THEOREM 2.4. *Let Assumption* 2.1 *hold. Then* (P) *is LP well-posed in the strongly generalized sense if and only if the optimal set $\bar{X}$ of* (P) *is nonempty and compact.*

Proof. The sufficiency part follows directly from Lemma 2.1 and Proposition 2.3, while the necessity part is obvious by Remark 1.1. ∎

The next two lemmas will be used to derive Theorem 2.5.

LEMMA 2.2 (see [1]). *Let (Z, d) be a complete metric space and $h : Z \to R^1 \cup \{+\infty\}$ be lower semicontinuous and bounded below. Let $\epsilon > 0$. Suppose that $z_0 \in Z$ satisfies $h(z_0) \leq \inf\{h(z) : z \in Z\} + \epsilon$. Then there exists $z_\epsilon \in Z$ such that*

(i) $h(z_\epsilon) \leq h(z_0)$;

(ii) $d(z_\epsilon, z_0) \leq \sqrt{\epsilon}$;

(iii) $h(z_\epsilon) < h(z) + \sqrt{\epsilon} d(z, z_\epsilon) \ \forall z \in Z \backslash \{z_\epsilon\}$.

LEMMA 2.3. *Let Y be a normed space and $K \subset Y$ be a closed and convex cone with* $\text{int}K \neq \emptyset$ *and $e \in \text{int}K$. Suppose that $\{y_n\} \subset Y$. Then $d_K(y_n) \to 0$ if and only if there exists a sequence $\{t_n\} \subset R^1_+$ with $t_n \to 0$ such that $y_n \in K - t_n e$.*

Proof. For the necessity part, from $d_K(y_n) \to 0$, we have $\{u_n\} \subset K$ such that $\|y_n - u_n\| \to 0$. Let $y'_n = y_n - u_n$. Then $\|y'_n\| \to 0$. Let $t_n = \sqrt{\|y'_n\|}$. Then $\{t_n\} \subset R^1_+$,

$t_n \to 0$ and $y'_n/t_n \to 0$. Since $e \in \text{int}K$, it follows that $e + y'_n/t_n \in K$ when n is sufficiently large. Consequently, $y'_n \in K - t_n e$. Hence, $y_n = u_n + y'_n \in K - t_n e$.

For the sufficiency part, as $y_n \in K - t_n e$, we have $y_n + t_n e \in K$. Thus,

$$d_K(y_n) \le \|y_n - (y_n + t_n e)\| = t_n\|e\|.$$

Hence, $d_K(y_n) \to 0$. □

Suppose that K is a cone. We denote by K^* the positive polar cone of K, i.e.,

$$K^* = \{\mu \in Y^* : \mu(u) \ge 0 \ \forall u \in K\}.$$

THEOREM 2.5. *Assume that X is a Banach space, Y is a normed space, and $X_1 \subset X$ is nonempty, closed, and convex. $K \subset Y$ is a closed and convex cone with* $\text{int}K \neq \emptyset$ *and* $e \in \text{int}K$. *Suppose that $f : X \to R^1$ is convex and continuously differentiable on X_1 and $g : X \to Y$ is K-concave and continuously differentiable on X_1. Let Slater constraint qualification for* (P) *hold: there exists $x_0 \in X_1$ such that* $g(x_0) \in \text{int}K$. *Assume that the optimal set $\bar{X}$ of* (P) *is nonempty. Further assume that there exists a convergent subsequence of $\{x_n\}$ for any sequences $\{x_n\} \subset X_1$ and $\{\mu_n\} \subset K^*$ satisfying the following.*

(i) $\lim_{n\to+\infty} d_K(g(x_n)) = 0$.

(ii) *There exists a subsequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} = 0 \ \forall k$ or* $\lim_{n\to+\infty} \mu_n(g(x_n))/\|\mu_n\| = 0$.

(iii) $\lim_{n\to+\infty} d_{(-N_{X_1}(x_n))}(\bigtriangledown f(x_n) - \mu_n(\bigtriangledown g(x_n))) = 0$, *where $N_{X_1}(x_n)$ is the normal cone of X_1 at x_n.*

Then, (P) is LP well-posed in the strongly generalized sense.

Proof. Suppose that $\bar{x} \in \bar{X}$. Since Slater constraint qualification holds, we have $\bar{\mu} \in K^*$ such that

$$f(\bar{x}) \le f(x) - \bar{\mu}(g(x)) \quad \forall x \in X_1 \tag{16}$$

and

$$\bar{\mu}(g(\bar{x})) = 0. \tag{17}$$

Let $\{x_n\} \subset X_1$ be a weakly generalized LP minimizing sequence for (P). Then, by Lemma 2.3,

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v} \tag{18}$$

and

$$g(x_n) \in K - t_n e \tag{19}$$

for some $\{t_n\} \subset R^1_+$ with $t_n \to 0$. From (16), we have

$$f(\bar{x}) \le f(x) - \bar{\mu}(g(x)) \quad \forall x \in X_2(t_n).$$

Note that

$$-\bar{\mu}(g(x)) \le t_n\bar{\mu}(e) \quad \forall x \in X_2(t_n).$$

Thus,

$$f(\bar{x}) \le f(x) + t_n\bar{\mu}(e) \quad \forall x \in X_2(t_n). \tag{20}$$

Hence,

$$\inf_{x\in X_2(t_n)} f(x) > -\infty. \tag{21}$$

The combination of (19) and (20) gives

$$f(\bar{x}) \le f(x_n) + t_n\bar{\mu}(e).$$

Consequently,

$$f(\bar{x}) \le \liminf_{n\to+\infty} f(x_n).$$

This together with (18) yields

$$\lim_{n\to+\infty} f(x_n) = f(\bar{x}). \tag{22}$$

This combined with (20) implies that there exists $\epsilon_n \to 0^+$ such that

$$f(x_n) \le f(x) + \epsilon_n \quad \forall x \in X_2(t_n).$$

Note that $X_2(t_n) \subset X$ is nonempty and closed. $(X_2(t_n), \|\cdot\|)$ can be seen as a complete (metric) subspace of X. Applying Lemma 2.2, we obtain

$$x'_n \in X_2(t_n) \tag{23}$$

such that

$$\|x_n - x'_n\| \le \sqrt{\epsilon_n} \tag{24}$$

and

$$f(x'_n) \le f(x) + \sqrt{\epsilon_n}\|x - x'_n\| \quad \forall x \in X_2(t_n). \tag{25}$$

Note that Slater constraint qualification also holds for the following constrained optimization problem:

$$(P_n) \qquad \begin{array}{l} \min\ f(x) + \sqrt{\epsilon_n}\|x - x'_n\| \\ \text{s.t. } x \in X_1, \quad g(x) \in K - t_n e, \end{array}$$

and by (25), x'_n is an optimal solution of (P_n). Hence, there exists $\mu_n \in K^*$ such that

$$0 \in \triangledown f(x'_n) - \mu_n(\triangledown g(x'_n)) + \sqrt{\epsilon_n}B^* + N_{X_1}(x'_n) \tag{26}$$

and

$$\mu_n(g(x'_n) + t_n e) = \mu_n(g(x'_n)) + t_n\mu_n(e) = 0, \tag{27}$$

where B^* is the closed unit ball of X^*. Equation (26) implies that

$$\lim_{n\to+\infty} d_{(-N_{X_1}(x'_n))}(\triangledown f(x'_n) - \mu_n(\triangledown g(x'_n))) = 0. \tag{28}$$

From (27), we see that if there does not exist a subsequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} = 0\ \forall k$, then

$$\lim_{n\to+\infty} \mu_n(g(x_n))/\|\mu_n\| = 0. \tag{29}$$

The combination of (24), (28), and (29) implies that $\{x'_n\}$ and $\{\mu_n\}$ satisfy conditions (i)–(iii) of the theorem. Thus, $\{x'_n\}$ has a subsequence $\{x'_{n_k}\}$ which converges to some $\bar{x}' \in X_0$. From (24), we deduce that $x_{n_k} \to \bar{x}' \in X_0$. This combined with (22) implies $\bar{x}' \in \bar{X}$. Hence, (P) is LP well-posed in the strongly generalized sense. ∎

Remark 2.1. Conditions (i)–(iii) of Theorem 2.5 can be seen as the well-known Palais–Smale condition (C) [1] in the case of constrained optimization.

3. Relations among three types of (generalized) LP well-posedness. Simple relationships among the three types of LP well-posedness were mentioned in Remark 1.1. Now we investigate further relationships among them.

The proof of next theorem is elementary and is omitted.

THEOREM 3.1. *Suppose that there exist* $\delta > 0$, $\alpha > 0$, *and* $c > 0$ *such that*

$$d_{X_0}(x) \leq c d_K^{\alpha}(g(x)) \quad \forall x \in X_1(\delta), \tag{30}$$

where $X_1(\delta)$ *is defined by* (10). *If* (P) *is LP well-posed, then* (P) *is LP well-posed in the generalized sense.*

Remark 3.1. Equation (30) is an error bound condition for the set X_0 in terms of the residual function

$$r(x) = d_K(g(x)) \quad \forall x \in X_1.$$

When $X = R^l, Y = R^m$, $X_1 = X$, and $X_0 \neq \emptyset$, by Theorem 5 of [23], (30) holds if and only if, for any $y \in R^m$ with $\|y\| \leq \delta$,

$$\Psi(y) \subset \Psi(0) + c\|y\|^{\alpha} B,$$

where

$$\Psi(y) = \{x \in R^l : g(x) \in K + y\}, \quad y \in R^m,$$

and B is the closed unit ball of Y. Sufficient conditions guaranteeing (30) were given in numerous papers on error bounds for systems of inequalities and metric regularity of set-valued maps (when (30) holds locally with $\alpha = 1$) in finite and infinite dimensional spaces (see, e.g., [5, 8, 18] and the references therein).

DEFINITION 3.1 (see [4]). *Let* W *be a topological space and* $F : W \to 2^X$ *be a set-valued map.* F *is said to be upper Hausdorff semicontinuous (u.H.c.) at* $w \in W$ *if, for any* $\epsilon > 0$, *there exists a neighborhood* U *of* w *such that* $F(U) \subset B(F(w), \epsilon)$, *where, for* $Z \subset X$ *and* $r > 0$,

$$B(Z, r) = \{x \in X : d_Z(x) \leq r\}.$$

DEFINITION 3.2 (see [1]). *Let* W *be a topological space and* $F : W \to 2^X$ *be a set-valued map.* F *is said to be upper semicontinuous (u.s.c.) in the Berge's sense at* $w \in W$ *if, for any neighborhood* Ω *of* $F(w)$, *there exists a neighborhood* U *of* w *such that* $F(U) \subset \Omega$.

It is obvious that the notion of u.s.c. (in Berge's sense) is stronger than u.H.c.

Clearly, $X_1(\delta)$ given by (10) can be seen as a set-valued map from R^1_+ to X. The next two theorems use conditions similar to those for the general stability results presented in section 3 of [4], where the uniform continuity of the objective function around the feasible set and the u.H.c. of the perturbation set-valued map were considered.

THEOREM 3.2. *Assume that the set-valued map* $X_1(\delta)$ *defined by* (10) *is u.H.c. at* $0 \in R^1_+$. *If* (P) *is LP well-posed, then* (P) *is LP well-posed in the generalized sense.*

Proof. Let $\{x_n\} \subset X_1$ be a generalized LP minimizing sequence. That is,

$$f(x_n) \to \bar{v}, \tag{31}$$

$$d_K(g(x_n)) \to 0. \tag{32}$$

Equation (32), together with the u.H.c. of $X_1(\delta)$ at 0, implies that $d_{X_0}(x_n) \to 0$. This fact combined with (31) implies that $\{x_n\}$ is an LP minimizing sequence. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \to \bar{x}$. Hence, (P) is LP well-posed in the generalized sense. ∎

THEOREM 3.3. *Assume that there exists $\epsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \epsilon_0)$ and the set-valued map $X_1(\delta)$ is u.H.c. at 0. If (P) is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

Proof. Let $\{x_n\}$ be a weakly generalized LP minimizing sequence. That is,

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v}, \tag{33}$$

$$d_K(g(x_n)) \to 0. \tag{34}$$

Note that $X_1(\delta)$ is u.H.c. at 0. This fact together with (34) implies that $d_{X_0}(x_n) \to 0$. Note that f is uniformly continuous on $B(X_0, \epsilon_0)$. It follows that

$$\liminf_{n\to+\infty} f(x_n) \ge \bar{v}. \tag{35}$$

The combination of (33) and (35) yields that

$$f(x_n) \to \bar{v}.$$

Hence, $\{x_n\}$ is an LP minimizing sequence. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \to \bar{x}$. So, (P) is LP well-posed in the strongly generalized sense. ∎

Let $\delta \ge 0$. Consider the perturbed problem of (P):

$$(P_\delta) \qquad \begin{array}{l} \min\ f(x) \\ \text{s.t. } x \in X_1, \quad d_K(g(x)) \le \delta. \end{array}$$

Denote by $v_1(\delta)$ the optimal value of (P_δ). Clearly, $v_1(0) = \bar{v}$.

THEOREM 3.4. *Consider problems* (P) *and* (P_δ). *Suppose that* (P) *is LP well-posed in the generalized sense and*

$$\liminf_{\delta\to 0^+} v_1(\delta) = \bar{v}. \tag{36}$$

Then (P) *is LP well-posed in the strongly generalized sense.*

Proof. Let $\{x_n\} \subset X_1$ be a weakly generalized LP minimizing sequence. Then

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v} \tag{37}$$

and

$$\lim_{n\to+\infty} d_K(g(x_n)) = 0.$$

Let $\delta_n = d_K(g(x_n))$. Then x_n is feasible for (P_{δ_n}). Thus,

$$v_1(\delta_n) \le f(x_n).$$

Passing to the lower limit, we get

$$\liminf_{n\to+\infty} v_1(\delta_n) \le \liminf_{n\to+\infty} f(x_n).$$

This together with (37) and (36) yields

$$\lim_{n\to+\infty} f(x_n) = \bar{v}.$$

It follows that $\{x_n\}$ is a generalized LP minimizing sequence. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \to \bar{x}$. So, (P) is LP well-posed in the strongly generalized sense. □

Remark 3.2. If the set-valued map $X_1(\delta)$ defined by (10) is u.s.c. at $0 \in R^1_+$, by Theorem 4.2.3 (1) of [2], (36) holds. In this case, the generalized LP well-posedness of (P) implies the strongly generalized LP well-posedness of (P).

Now let Y be a normed space and $y \in Y$. Consider the following perturbed problem of (P):

$$(P_y) \qquad \min f(x)$$
$$\text{s.t. } x \in X_1, \quad g(x) \in K + y.$$

Denote by

$$X_3(y) = \{x \in X_1 : g(x) \in K + y\} \tag{38}$$

the feasible set of (P_y) and $v_3(y)$ the optimal value of (P_y). Here we note that if $X_3(y) = \emptyset$, we set $v_3(y) = +\infty$. It is obvious that $X_3(y)$ can be seen as a set-valued map from Y to X. Corresponding to Theorems 3.2–3.4, respectively, we have the following theorems.

THEOREM 3.5. *Assume that Y is a normed space and that the set-valued map $X_3(y)$ is u.H.c. at $0 \in Y$. If* (P) *is LP well-posed, then* (P) *is LP well-posed in the generalized sense.*

THEOREM 3.6. *Assume that Y is a normed space and that there exists $\epsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \epsilon_0)$ and the set-valued map $X_3(y)$ is u.H.c. at $0 \in Y$. If* (P) *is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

THEOREM 3.7. *Assume that Y is a normed space. Consider problems* (P) *and* (P_y). *Suppose that* (P) *is LP well-posed in the generalized sense and*

$$\liminf_{y\to 0} v_3(y) = \bar{v}. \tag{39}$$

Then (P) *is LP well-posed in the strongly generalized sense.*

Similar to Remark 3.2, when the set-valued map X_3 is u.s.c. at $0 \in Y$, then (39) holds. Thus, the generalized LP well-posedness of (P) implies its strongly generalized LP well-posedness.

In the special case when K is a closed and convex cone with nonempty interior int$K$, arbitrarily fix an $e \in \text{int}K$. It is obvious that $X_2(t)$ defined by (15) can be seen as a set-valued map from R^1_+ to X. Denote by $v_2(t)$ the optimal value of (P_t).

THEOREM 3.8. *Assume that K is a closed and convex cone with nonempty interior* int$K$ *and that the set-valued map $X_2(t)$ is u.H.c. at $0 \in R^1_+$. If* (P) *is LP well-posed, then* (P) *is LP well-posed in the generalized sense.*

THEOREM 3.9. *Assume that K is a closed and convex cone with nonempty interior* int$K$ *and that there exists $\epsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \epsilon_0)$ and the set-valued map $X_2(t)$ is u.H.c. at $0 \in R^1_+$. If* (P) *is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

THEOREM 3.10. *Assume that K is a closed and convex cone with nonempty interior* $\text{int}K$. *Consider problems* (P) *and* (P_t). *Suppose that* (P) *is LP well-posed in the generalized sense and*

$$\liminf_{t\to 0^+} v_2(t) = \bar{v}. \tag{40}$$

Then (P) *is LP well-posed in the strongly generalized sense.*

Again, as noted in Remark 3.2, when the set-valued map X_2 is u.s.c. at $0 \in R^1_+$, then (39) holds. Thus, the generalized LP well-posedness of (P) implies its strongly generalized LP well-posedness.

4. Applications to penalty-type methods. In this section, we consider the convergence of a class of penalty methods and a class of augmented Lagrangian methods under the assumption of strongly generalized LP well-posedness of (P).

4.1. Penalty methods. Let $\alpha > 0$. Consider the following penalty problem:

$$(PP_\alpha(r)) \quad \min_{x\in X_1} f(x) + r d_K^\alpha(g(x)), \quad r > 0.$$

Denote by $v_4(r)$ the optimal value of $(PP_\alpha(r))$. It is clear that

$$v_4(r) \le \bar{v} \quad \forall r > 0. \tag{41}$$

Remark 4.1. When $\alpha \in (0,1)$, $X = R^l$, $Y = R^m$, $K = R^{m_1}_- \times \{0_{m-m_1}\}$, where $m \ge m_1$ and 0_{m-m_1} is the origin of the space R^{m-m_1}, this class of penalty functions was applied to the study of mathematical programs with equilibrium constraints [19]. Necessary and sufficient conditions for the exact penalization of this class of penalty functions were derived in [14]. This class of penalty methods was also applied to mathematical programs with complementarity constraints [27] and nonlinear semidefinite programs [15]. An important advantage of this class of penalty methods is that it requires weaker conditions to guarantee its exact penalization property than the usual l_1 penalty function method (see [19]).

THEOREM 4.1. *Let* $0 < r_n \to +\infty$. *Consider problems* (P) *and* $(PP_\alpha(r_n))$. *Assume that there exist* $\bar{r} > 0$ *and* $m_0 \in R^1$ *such that*

$$f(x) + \bar{r} d_K^\alpha(g(x)) \ge m_0 \quad \forall x \in X_1. \tag{42}$$

Let $0 < \epsilon_n \to 0$. Suppose that each $x_n \in X_1$ satisfies

$$f(x_n) + r_n d_K^\alpha(g(x_n)) \le v_4(r_n) + \epsilon_n. \tag{43}$$

Further assume that (P) is LP well-posed in the strongly generalized sense. Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \to \bar{x}$.

Proof. From (41) and (43), we have

$$f(x_n) \le \bar{v} + \epsilon_n.$$

Thus,

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v}. \tag{44}$$

Moreover, from (41)–(43), we deduce that

$$f(x_n) + \bar{r} d_K^\alpha(g(x_n)) + (r_n - \bar{r}) d_K^\alpha(g(x_n)) \le \bar{v} + \epsilon_n.$$

Thus,

$$m_0 + (r_n - \bar{r}) d_K^{\alpha}(g(x_n)) \leq \bar{v} + \epsilon_n,$$

implying

$$d_K(g(x_n)) \leq \left[\frac{\bar{v} + \epsilon_n - m_0}{r_n - \bar{r}} \right]^{1/\alpha}.$$

Passing to the limit, we get

$$\lim_{n \to +\infty} d_K(g(x_n)) = 0. \tag{45}$$

It follows from (44) and (45) that $\{x_n\}$ is a weakly generalized LP minimizing sequence. Hence, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \to \bar{x}$. ☐

4.2. Augmented Lagrangian methods. Let (X, d_1) be a metric space, let $Y = R^m$, and let $K \subset Y$ be a nonempty, closed, and convex set. Let $\sigma : R^m \to R^1 \cup \{+\infty\}$ be an augmenting function; namely, it is a lower semicontinuous, convex function satisfying

$$\min_{y \in R^m} \sigma(y) = 0 \text{ and } \sigma \text{ attains its unique minimum at } y = 0.$$

Following Example 11.46 in [25], we define the dualizing parametrization function by setting $X = X_1$ and $\theta = \delta_K$:

$$\bar{f}(x, u) = f(x) + \delta_{X_1}(x) + \delta_K(g(x) + u),$$

where δ_A is the indicator function of a subset A of a space Z, i.e.,

$$\delta_A(a) = \begin{cases} 0 & \text{if } a \in A, \\ +\infty & \text{if } a \in Z \backslash A. \end{cases}$$

Constructing the augmented Lagrangian as in Definition 11.55 of [25], we obtain the augmented Lagrangian:

$$\bar{l}(x, y, r) = \inf_{u \in R^m} \left\{ \bar{f}(x, u) + r\sigma(u) - \langle y, u \rangle \right\}, x \in X, y \in R^m, r > 0.$$

The augmented Lagrangian problem is

$$(ALP(y, r)) \quad \min_{x \in X} \bar{l}(x, y, r), \quad y \in R^m, r > 0.$$

Denote by $v_5(y, r)$ the optimal value of $(ALP(y, r))$.

We have the following result.

THEOREM 4.2. *Let $\{y_n\} \subset R^m$ be bounded and $0 < r_n \to +\infty$. Consider* (P) *and $(ALP(y_n, r_n))$. Assume that there exist $(\bar{y}, \bar{r}) \in R^m \times (0, +\infty)$ and $m_0 \in R^1$ such that*

$$\bar{l}(x, \bar{y}, \bar{r}) \geq m_0 \quad \forall x \in X. \tag{46}$$

Let $0 < \epsilon_n \to 0$. Suppose that each x_n satisfies

$$\bar{l}(x_n, y_n, r_n) \leq v_5(y_n, r_n) + \epsilon_n, \tag{47}$$

$v_5(y_n, r_n) > -\infty$ $\forall n$, and (P) *is LP well-posed in the strongly generalized sense. Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \to \bar{x}$.*

Proof. By the definition of $\bar{l}(x, y, r)$, it is easy to see that

$$\bar{l}(x, y, r) = f(x) \quad \forall x \in X_0.$$

It follows that

$$v_5(y, r) \le \bar{v} \quad \forall y \in R^m, r > 0.$$

Thus,

$$v_5(y_n, r_n) \le \bar{v} \quad \forall n. \tag{48}$$

By the definition of $\bar{l}(x_n, y_n, r_n)$ and (47), $\{x_n\} \subset X_1$ and there exists $\{u_n\} \subset R^m$ satisfying

$$g(x_n) + u_n \in K \quad \forall n \tag{49}$$

such that

$$f(x_n) + r_n\sigma(u_n) - \langle y_n, u_n \rangle \le v_5(y_n, r_n) + 2\epsilon_n. \tag{50}$$

This combined with (46) and (48) implies that

$$(r_n - \bar{r})\sigma(u_n) - \langle y_n - \bar{y}, u_n \rangle \le \bar{v} + 2\epsilon_n - m_0. \tag{51}$$

We assert that $\{u_n\}$ is bounded. Otherwise, we assume without loss of generality that $\|u_n\| \to +\infty$. Since the lower semicontinuous and convex function σ has a unique minimum, by Proposition 3.2.5 in IV of [11] and Corollary 3.27 of [25], $\liminf_{n\to+\infty} \sigma(u_n)/\|u_n\| > 0$. As $\{y_n\}$ is bounded, (51) cannot hold. So, $\{u_n\}$ should be bounded. Assume without loss of generality that $u_n \to u_0$. We deduce from (51) that

$$\sigma(u_0) \le \liminf_{n\to+\infty} \sigma(u_n) = 0.$$

It follows that $u_0 = 0$. We deduce from (48) and (50) that

$$f(x_n) - \langle y_n, u_n \rangle \le \bar{v} + 2\epsilon_n.$$

Passing to the limit, we get

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v}.$$

From (49) and the fact that $u_n \to 0$, we obtain

$$\lim_{n\to+\infty} d_K(g(x_n)) = 0.$$

Thus, $\{x_n\}$ is a weakly generalized LP minimizing sequence. Hence, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \to \bar{x}$. ∎

Acknowledgment. The authors would like to thank the two anonymous referees for their detailed and constructive comments which have helped to improve some results and the presentation of the paper.

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