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## *Generalized Linear Quadratic Control*

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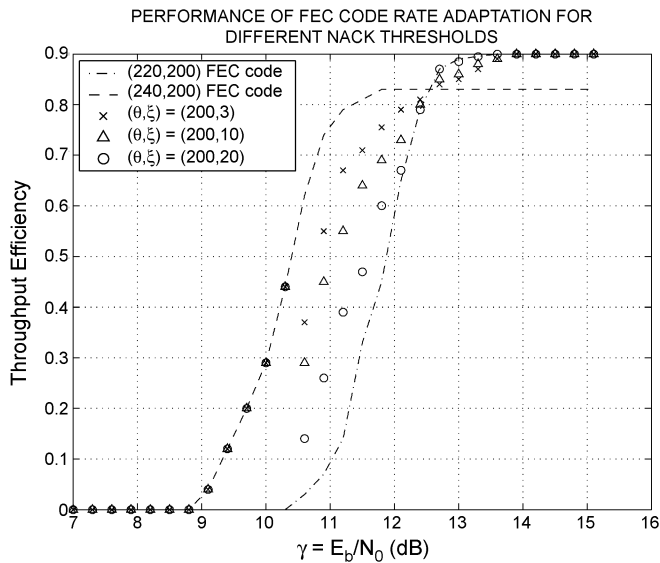


Fig. 3. FEC code rate adaptation: throughput for different NACK thresholds.

that the system does not unnecessarily switch to low-rate code, and it operates with the (220,200) one. The curve corresponding to  $\theta = 2000$ ,  $\xi = 3$  yields the closest approximation to the envelope. We observe that ACK thresholds are much larger than NACK thresholds. Small NACK thresholds (a few NACKs) imply that, in case of link quality deterioration the system must respond fast and decrease rate so that no throughput loss is incurred. The order of ACK thresholds (few to several hundreds) implies a conservative policy when link conditions improve, since rate increase is decided after a large number of ACKs.

#### IV. CONCLUSION

We studied the problem of optimal transmission rate control in a two-state model of a wireless link with two control actions, in which link state information is partially available to the controller. For time-invariant, yet unknown link state, we showed that the optimal policy, in the sense of maximizing link throughput has a threshold structure. We discussed how the analysis can provide valuable insights and guidelines for designing practical threshold based rate adaptation policies under general link conditions. There exist several directions for future study. Different variants of the problem require different assumptions. For instance, adoption of a Poisson packet arrival assumption necessitates the introduction of a random, exponential distributed slot duration. A discounted version of the problem would give more emphasis to the tradeoff between probing the state and reaping the benefits of transmission. For the corresponding problem with more states and control actions, the threshold structure for the optimal policy does not hold. The analogy with the multisequential probability ratio test [13] could be investigated so as to derive close-to-optimal policies. Finally, the problem with time-varying link state, an instance of a restless multi-armed bandit, warrants further investigation.

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## Generalized Linear Quadratic Control

Ather Gattami

**Abstract**—We consider the problem of stochastic finite- and infinite-horizon linear quadratic control under power constraints. The calculations of the optimal control law can be done off-line as in the classical linear quadratic Gaussian control theory using dynamic programming, which turns out to be a special case of the new theory developed in this technical note. A numerical example is solved using the new methods.

**Index Terms**—Linear quadratic control.

#### I. INTRODUCTION

In this technical note we consider the problem of linear quadratic control with power constraints. Power constraints are very common in control problems. For instance, we often have some limitations on the control signal, which we can express as  $\mathbf{E}u^T u \leq \gamma$ . Also, Gaussian channel capacity limitation can be modeled through power constraints ([7]). There has been much work on control with power constraints, see [2], [4], [6], [10]. Rantzer [5] showed how to use power constraints for distributed state feedback control. What is common to previously published papers is that they solve the stationary state-feedback infinite-horizon case using convex optimization. Output-feedback is only discussed in [6], where the quadratic (power) constraints are restricted to be positive semi-definite. In [1], [9], a suboptimal solution to linear

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The author is with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139 USA. He is also with the Royal Institute of Technology, Stockholm 100 44, Sweden (e-mail: ather@mit.edu; gattami@kth.se).

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quadratic control problems with *linear* constraints on the control signal and state. The piecewise linear controller is obtained by solving a sequence of finite-horizon problems, where each problem is posed as a static quadratic program. The aim of this technical note is to give a complete *optimal* solution to the *non-stationary* and *finite-horizon* problem of linear systems, including *time-varying*, with power constraints. The solution is obtained using dynamic programming. A solution of the stationary infinite-horizon linear quadratic control problem is derived from the finite-horizon results. Also, the *output-feedback* problem with *non-convex* quadratic constraints is solved.

The outline of the technical note is as follows. We first start with notation used in this technical note. In Section III, we introduce a novel approach for solving the state-feedback linear quadratic control problem. Relations to the classical approach are discussed in Section IV. The new approach is then used in Section V to give the main result of the technical note, the finite horizon state-feedback linear quadratic control with *power constraints*. We show how a solution to the constrained infinite-horizon control problem can be derived in Section VI. The constrained output-feedback control problem is solved in Section VII. A numerical example is given in Section VIII.

#### NOTATION

We write  $Q \succ 0$  ( $Q \succeq 0$ ) to denote that  $Q$  is positive definite (semi-definite).  $Q \succ P$  ( $Q \succeq P$ ) means that  $Q - P \succ 0$  ( $Q - P \succeq 0$ ).  $\text{Tr } A$  denotes the trace of the square matrix  $A$ .  $\mathbb{S}^n$  denotes the set of  $n \times n$  symmetric matrices.

## II. OPTIMAL STATE FEEDBACK LINEAR QUADRATIC CONTROL THROUGH DUALITY

In this section we will derive a state-feedback solution to the classical linear quadratic control problem using duality. This method will be used to solve the problem of linear quadratic optimal control with power constraints. Consider the linear quadratic stochastic control problem

$$\begin{aligned} \min_{\mu_k} \quad & \mathbf{E} x^T(N) Q_{xx} x(N) \\ & + \sum_{k=0}^{N-1} \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\ \text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) + w(k) \\ & \mathbf{E} w(k) x^T(l) = 0, \quad \forall l \leq k \\ & \mathbf{E} w(k) w^T(k) = V_{ww}(k) \\ & u(k) = \mu_k(x(0), \dots, x(k)) \end{aligned} \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , and  $Q \succ 0$  which is partitioned according to the dimensions of  $x$  and  $u$  as

$$Q = \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{xu}^T & Q_{uu} \end{bmatrix}.$$

Without loss of generality, we assume that

$$\mathbf{E} x(0) = \mathbf{E} w(k) = 0, \quad \mathbf{E} x(0) x^T(0) = V_{ww}(k) = I$$

for all  $k \geq 0$ . The quadratic cost in (1) can be written as

$$\begin{aligned} & \mathbf{E} \left\{ \text{Tr} Q_{xx} x(N) x^T(N) \right\} \\ & + \sum_{k=0}^{N-1} \mathbf{E} \left\{ \text{Tr} Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \right\} \\ & = \text{Tr} Q_{xx} \{ \mathbf{E} x(N) x^T(N) \} \\ & + \sum_{k=0}^{N-1} \text{Tr} Q \left\{ \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \right\} \end{aligned}$$

$$= \text{Tr} Q_{xx} V_{xx}(N) + \sum_{k=0}^{N-1} \text{Tr} Q V(k),$$

where

$$V(k) = \begin{bmatrix} V_{xx}(k) & V_{xu}(k) \\ V_{xu}^T(k) & V_{uu}(k) \end{bmatrix} = \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T.$$

Let  $F = [I \ 0]$ , where the blocks of  $F$  are dimensioned such that  $V_{xx}(k) = FV(k)F^T$ . The system dynamics implicate the following recursive equation for the covariance matrices  $V(k)$

$$\begin{aligned} FV(k+1)F^T &= V_{xx}(k+1) = \mathbf{E} x(k+1) x^T(k+1) \\ &= \mathbf{E} (Ax(k) + Bu(k) + w(k)) \\ &\quad \times (Ax(k) + Bu(k) + w(k))^T \\ &= \mathbf{E} \left\{ \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \right. \\ &\quad \left. \times [A \ B]^T + w(k) w^T(k) \right\} \\ &= [A \ B] V(k) [A \ B]^T + I. \end{aligned}$$

The initial condition  $V_{xx}(0) = \mathbf{E} x(0) x^T(0) = I$  can be written as  $FV(0)F^T = I$ . We conclude:

*Proposition 1:* The linear quadratic problem (1) is equivalent to the covariance selection problem

$$\begin{aligned} \min_{V(0), \dots, V(N) \succeq 0} \quad & \text{Tr} Q_{xx} V_{xx}(N) + \sum_{k=0}^{N-1} \text{Tr} Q V(k) \\ \text{subject to} \quad & FV(0)F^T = I \\ & [A \ B] V(k) [A \ B]^T + I \\ & = FV(k+1)F^T \\ & k = 0, \dots, N-1. \end{aligned} \quad (2)$$

In particular, it is convex in  $V(0), \dots, V(N)$ .  $\blacksquare$

The dual problem of (2) is given by

$$\begin{aligned} \max_{S(0), \dots, S(N)} \quad & \min_{V(0), \dots, V(N) \succeq 0} J(V(0), \dots, V(N), S(0), \dots, S(N)) \\ J(V(0), \dots, V(N), S(0), \dots, S(N)) &= \text{Tr} Q_{xx} V_{xx}(N) \\ & + \sum_{k=0}^{N-1} \text{Tr} Q V(k) + \text{Tr} \left\{ S(0) (I - FV(0)F^T) \right\} \\ & + \sum_{k=0}^{N-1} \text{Tr} \left\{ S(k+1) \left( [A \ B] V(k) [A \ B]^T \right. \right. \\ & \quad \left. \left. + I - FV(k+1)F^T \right) \right\} \\ & = \text{Tr} Q_{xx} V_{xx}(N) + \sum_{k=0}^{N-1} \text{Tr} Q V(k) + \text{Tr} S(0) \\ & \quad - \text{Tr} \{ S(0) FV(0)F^T \} + \sum_{k=0}^{N-1} \text{Tr} S(k+1) \\ & \quad + \sum_{k=0}^{N-1} \text{Tr} \left\{ S(k+1) [A \ B] V(k) [A \ B]^T \right\} \\ & \quad - \sum_{k=0}^{N-1} \text{Tr} \{ S(k+1) FV(k+1)F^T \} \\ & = \text{Tr} Q_{xx} V_{xx}(N) - \text{Tr} \{ S(N) FV(N)F^T \} + \text{Tr} S(0) \\ & \quad + \sum_{k=0}^{N-1} \text{Tr} S(k+1) + \sum_{k=0}^{N-1} \text{Tr} Q V(k) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=0}^{N-1} \text{Tr} \left\{ S(k+1) \begin{bmatrix} A & B \end{bmatrix} V(k) \begin{bmatrix} A & B \end{bmatrix}^T \right\} \\
 & - \sum_{k=0}^{N-1} \text{Tr} \{ S(k) F V(k) F^T \}
 \end{aligned} \quad (3)$$

and  $S(0), \dots, S(N) \in \mathbb{S}^n$  are the Lagrange multipliers. Thus, the dual problem can be written as

$$\begin{aligned}
 \max_{S(0), \dots, S(N)} \min_{V(0), \dots, V(N) \succeq 0} & \text{Tr} S(0) + H(N) \\
 & + \sum_{k=0}^{N-1} \{ H(k) + \text{Tr} S(k+1) \}
 \end{aligned} \quad (4)$$

where

$$\begin{aligned}
 H(N) & = \text{Tr} \{ Q_{xx} V_{xx}(N) - S(N) V_{xx}(N) \} \\
 & = \text{Tr} \{ [Q_{xx} - S(N)] V_{xx}(N) \}
 \end{aligned}$$

and

$$\begin{aligned}
 H(k) & = \text{Tr} \left\{ Q V(k) + S(k+1) \begin{bmatrix} A & B \end{bmatrix} V(k) \begin{bmatrix} A & B \end{bmatrix}^T \right. \\
 & \quad \left. - S(k) F V(k) F^T \right\} \\
 & = \text{Tr} \left\{ [Q + \begin{bmatrix} A & B \end{bmatrix}^T S(k+1) \begin{bmatrix} A & B \end{bmatrix} \right. \\
 & \quad \left. - F^T S(k) F] V(k) \right\}
 \end{aligned} \quad (5)$$

for  $k = 0, \dots, N-1$ . Here,  $H(k)$  plays the role of the *Hamiltonian* of the system. The duality gap between (2) and (3) is zero, since Slater's condition is satisfied for the primal (and dual) problem (see [3] for a reference on Slater's condition). Now for the optimal selection of the dual variables  $S(k)$ , we must have  $Q_{xx} - S(N) \succeq 0$ , and

$$Q + \begin{bmatrix} A & B \end{bmatrix}^T S(k+1) \begin{bmatrix} A & B \end{bmatrix} - F^T S(k) F \succeq 0 \quad (6)$$

because otherwise, the value of the cost function in (3) becomes infinite.

To maximize the cost in (3),  $S(N)$  is chosen equal to  $Q_{xx}$ ,  $S(N) = Q_{xx}$ , and  $S(k)$  is chosen to maximize  $\text{Tr} S(k)$  subject to the constraint (6).

Now

$$\begin{aligned}
 F^T S(k) F & = \begin{bmatrix} I \\ 0 \end{bmatrix} S(k) \begin{bmatrix} I & 0 \end{bmatrix} = \begin{bmatrix} S(k) & 0 \\ 0 & 0 \end{bmatrix}, \\
 & \text{and} \\
 Q + \begin{bmatrix} A & B \end{bmatrix}^T S(k+1) \begin{bmatrix} A & B \end{bmatrix} \\
 & = \begin{bmatrix} A^T S(k+1) A + Q_{xx} & A^T S(k+1) B + Q_{xu} \\ B^T S(k+1) A + Q_{xu}^T & B^T S(k+1) B + Q_{uu} \end{bmatrix}.
 \end{aligned}$$

Now the matrix

$$S(k) = A^T S(k+1) A + Q_{xx} - R(k) \quad (7)$$

with

$$R(k) = L(k)^T \left( B^T S(k+1) B + Q_{uu} \right) L(k) \quad (8)$$

and  $L(k)$  such that

$$\left( B^T S(k+1) B + Q_{uu} \right) L(k) = B^T S(k+1) A + Q_{xu}^T \quad (9)$$

fulfills (6) or equivalently fulfills the inequality

$$Q + \begin{bmatrix} A^T S(k+1) A - S(k) & A^T S(k+1) B \\ B^T S(k+1) A & B^T S(k+1) B \end{bmatrix} \succeq 0 \quad (10)$$

and any other matrix  $P$  with  $\text{Tr} P > \text{Tr} S(k)$  violates the inequality in (10). Hence, the choice of  $S(k)$  given by (7) is optimal.

*Theorem 1:* The dual problem of (1) and (2) is given by

$$\begin{aligned}
 \max_{S(k) \in \mathbb{S}^n} & \sum_{k=0}^N \text{Tr} S(k) \\
 \text{subject to} & \quad Q + \begin{bmatrix} A^T S(k+1) A - S(k) & A^T S(k+1) B \\ B^T S(k+1) A & B^T S(k+1) B \end{bmatrix} \\
 & \succeq 0 \\
 & S(N+1) = 0 \text{ for } k = 0, \dots, N.
 \end{aligned} \quad (11)$$

The problem (11) can be solved dynamically by sequentially solving

$$\begin{aligned}
 \max_{S(k) \in \mathbb{S}^n} & \text{Tr} S(k) \\
 \text{subject to} & \quad Q + \begin{bmatrix} A^T S(k+1) A - S(k) & A^T S(k+1) B \\ B^T S(k+1) A & B^T S(k+1) B \end{bmatrix} \\
 & \succeq 0
 \end{aligned} \quad (12)$$

for  $k = 0, \dots, N$ , with  $S(N+1) = 0$ .  $\blacksquare$

With this optimal choice of the multipliers  $S(0), \dots, S(N)$ , the dual problem (3) becomes

$$\begin{aligned}
 \min_{V(0), \dots, V(N-1) \succeq 0} & \sum_{k=0}^{N-1} \text{Tr} Z(k) V(k) + \sum_{k=0}^N \text{Tr} S(k) \\
 & \text{where} \\
 Z(k) & = \begin{bmatrix} R(k) & A^T S(k+1) B + Q_{xu} \\ B^T S(k+1) A + Q_{xu}^T & B^T S(k+1) B + Q_{uu} \end{bmatrix}
 \end{aligned}$$

and  $R(k)$  is given by (8). The matrix  $Z(k)$  is of the form

$$Z = \begin{bmatrix} X Y^{-1} X^T & X \\ X^T & Y \end{bmatrix}$$

where  $X = A^T S(k+1) B + Q_{xu}$  and  $Y = B^T S(k+1) B + Q_{uu}$ . In general, for any given matrix  $V_{xx} \succeq 0$ , we can choose  $V$  as

$$V = \begin{bmatrix} V_{xx} & -V_{xx} X Y^{-1} \\ -Y^{-1} X^T V_{xx} & Y^{-1} X^T V_{xx} X Y^{-1} \end{bmatrix}$$

and we get  $\text{Tr} Z V = 0$ . Note that the matrix  $Y$  is invertible since  $Q_{uu} \succ 0$  and  $S(k+1) \succeq 0$ . The matrix  $V$  given by

$$V = \begin{bmatrix} I & -X Y^{-1} \\ -Y^{-1} X^T & Y^{-1} X^T X Y^{-1} \end{bmatrix}$$

is such that  $\text{Tr} Z V = 0$ . Therefore, we see that the minimizing covariances are given by

$$\begin{aligned}
 V_{xx}(0) & = I \\
 V_{ux}(k) & = -L(k) V_{xx}(k) \\
 V_{uu}(k) & = V_{ux}(k) V_{xx}^{-1}(k) V_{xu}(k) \\
 V_{xx}(k+1) & = \begin{bmatrix} A & B \end{bmatrix} V(k) \begin{bmatrix} A & B \end{bmatrix}^T + I
 \end{aligned} \quad (13)$$

and the optimal cost is given by  $\sum_{k=0}^N \text{Tr} S(k)$ . Now we have found the covariances, it is easy to see that the optimal control law

$$u(k) = V_{ux}(k) V_{xx}^{-1}(k) x(k) = -L(k) x(k) \quad (14)$$

where  $L(k)$  is, as before, given by (9).

**Theorem 2:** The optimal solution of the covariance selection problem (2) is given by the equations in (13). The corresponding optimal control law is given by (14). ■

**Remark 1:** We could have assigned covariance matrices other than the identity matrix for the initial value of the state  $x(0)$  and the disturbances  $w(k)$ , and the solution would be similar to the case treated. We could also have treated a time-varying system with time-varying quadratic cost functions. The only change is that we replace  $Q$  with  $Q(k)$ ,  $A$  with  $A(k)$ , etc.

### III. RELATIONS TO CLASSICAL LINEAR QUADRATIC CONTROL

The covariance selection method developed in the previous section is very closely related to the classical way of calculating the optimal state-feedback control law. Consider the dual variable  $S(k)$ . At each time-step  $k$ ,  $S(k)$  was chosen to be

$$S(k) = A^T S(k+1)A + Q_{xx} - L^T(k) \left( B^T S(k+1)B + Q_{uu} \right) L(k)$$

with  $L(k)$  such that  $(B^T S(k+1)B + Q_{uu})L(k) = B^T S(k+1)A + Q_{xu}^T$ . This value of  $S(k)$  is exactly the quadratic matrix for the cost to go function from time-step  $k$  to  $N$ , given by  $x^T(k)S(k)x(k)$  (see e.g., [8]).

Now we will take a closer look at the optimal cost. In the previous section, we obtained the cost  $\sum_{k=0}^N \text{Tr} S(k)$ . In general, when  $\mathbf{E}x(0)x^T(0) = V_{xx}(0)$  and  $\mathbf{E}w(k)w^T(k) = V_{ww}(k)$ , it turns out that the cost becomes

$$\text{Tr} S(0)V_{xx}(0) + \sum_{k=1}^N \text{Tr} S(k)V_{ww}(k).$$

For  $\mathbf{E}x(0)x^T(0) = I$  and  $\mathbf{E}w(k)w^T(k) = I$ , we get the cost obtained in the previous section. Since

$$\text{Tr} S(0)V_{xx}(0) = \text{Tr} \mathbf{E} S(0)x(0)x^T(0) = \mathbf{E}x^T(0)S(0)x(0)$$

and  $\text{Tr} S(k)V_{ww}(k) = \text{Tr} \mathbf{E} S(k+1)w(k)w^T(k) = \mathbf{E}w^T(k)S(k+1)w(k)$ , the optimal cost can be written as

$$\sum_{k=0}^N \text{Tr} S(k) = \mathbf{E}x^T(0)S(0)x(0) + \sum_{k=0}^{N-1} \mathbf{E}w^T(k)S(k+1)w(k).$$

We see that the cost  $\mathbf{E}x^T(0)S(0)x(0)$  is due to the initial value  $x(0)$ , and  $\sum_{k=0}^{N-1} \mathbf{E}w^T(k)S(k+1)w(k)$  is the cost caused by the disturbance  $\{w(k)\}_{k=0}^{N-1}$ .

Having realized that the cost can be expressed as a quadratic function of the uncertainty represented by  $x(0)$  and  $\{w(k)\}_{k=1}^{N-1}$ , the dual (maximin) problem can be seen as a *game* between the controller and *nature's* choice of uncertainty.

### IV. OPTIMAL STATE FEEDBACK CONTROL WITH POWER CONSTRAINTS

In this section we consider a linear quadratic problem given by (1), with additional constraints of the form

$$\mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q_i \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \leq \gamma_i(k)$$

or equivalently

$$\text{Tr} Q_i V(k) \leq \gamma_i(k) \quad (15)$$

for  $k = 0, \dots, N-1$ ,  $i = 1, \dots, M$ . Note that we do *not* make any other assumptions about  $Q_i$  except that it is symmetric,  $Q_i \in \mathbb{S}^{m+n}$ . Note also that the covariance constraints in (15) are linear in the elements of the covariance matrices  $V(k)$ , and hence convex.

The dual problem, including the covariance constraints above, becomes

$$\begin{aligned} \max_{S(k), \tau_i(k)} \min_{V(k) \succeq 0} H(N) + \sum_{k=0}^{N-1} \{H(k) + \text{Tr} S(k+1)\} \\ + \text{Tr} S(0) - \sum_{k=0}^{N-1} \sum_{i=1}^M \tau_i(k) \gamma_i(k) \end{aligned} \quad (16)$$

where  $\tau_i(k) \geq 0$  and  $H(k)$ , the Hamiltonian of the system, is given by

$$H(k) = \text{Tr} \left\{ (Q + [A \ B]^T S(k+1) [A \ B] - F^T S(k) F + \sum_{i=1}^M \tau_i(k) Q_i) V(k) \right\} \quad (17)$$

The dual problem (16) is finite if and only if

$$Q + [A \ B]^T S(k+1) [A \ B] - F^T S(k) F + \sum_{i=1}^M \tau_i(k) Q_i \succeq 0.$$

The duality gap is zero, since Slater's condition is satisfied for the dual problem (16) (see [3]). Just like in the previous section, for every time-step  $k$  and for fixed values of  $\tau_i(k)$ , we solve

$$\begin{aligned} \max_{S(k) \in \mathbb{S}^n} \text{Tr} S(k) - \sum_{i=1}^M \tau_i(k) \gamma_i(k) \\ \text{subject to } Q(k) \\ + \begin{bmatrix} A^T S(k+1)A - S(k) & A^T S(k+1)B \\ B^T S(k+1)A & B^T S(k+1)B \end{bmatrix} \\ \succeq 0 \end{aligned} \quad (18)$$

where  $Q(k) = Q + \sum_{i=1}^M \tau_i(k) Q_i$ . Now for any fixed values of  $\tau_i(k)$ ,  $\text{Tr} S(k)$  is maximized for

$$S(k) = A^T S(k+1)A + Q_{xx}(k) - R(k) \quad (19)$$

with  $R(k) = L(k)^T (B^T S(k+1)B + Q_{uu}(k)) L(k) \succ 0$  (20)

and  $L(k)$  such that

$$\left( B^T S(k+1)B + Q_{uu}(k) \right) L(k) = B^T S(k+1)A + Q_{xu}^T(k) \quad (21)$$

and any other matrix  $P$  with  $\text{Tr} P > \text{Tr} S(k)$  violates the inequality in (18). Hence, the choice of  $S(k)$  given by (19) is the optimal that is obtained through the eigenvalue problem (18). With the optimal values of  $S(k)$  and  $\tau_i(k)$ , the dual problem (16) becomes

$$\begin{aligned} \min_{V(0), \dots, V(N-1) \succeq 0} \sum_{k=0}^{N-1} \text{Tr} Z(k) V(k) + \sum_{k=0}^N \text{Tr} S(k) \\ - \sum_{k=0}^{N-1} \sum_{i=1}^M \tau_i(k) \gamma_i(k) \\ \text{subject to } FV(0)F^T - I = 0 \end{aligned}$$

where the matrix  $Z(k)$  is given by

$$Z = \begin{bmatrix} XY^{-1}X^T & X \\ X^T & Y \end{bmatrix}$$

with  $X = A^T S(k+1)B + Q_{xu}(k)$  and  $Y = B^T S(k+1)B + Q_{uu}(k)$ . The optimal covariances  $V(k)$  are obtained just like in the previous section, by taking

$$\begin{aligned} V_{xx}(0) &= I \\ V_{ux}(k) &= -L(k)V_{xx}(k) \\ V_{uu}(k) &= V_{ux}(k)V_{xx}^{-1}(k)V_{xu}(k) \\ V_{xx}(k+1) &= [A \ B]V(k)[A \ B]^T + I \\ u(k) &= V_{ux}(k)V_{xx}^{-1}(k)x(k) = -L(k)x(k) \end{aligned} \quad (22)$$

where  $L(k)$  is the solution of (21). The problem above can be solved efficiently using primal-dual interior point methods (see [3, pp. 609]), where iteration is made with respect to the dual variables  $\tau_i(0), \dots, \tau_i(N-1)$ ,  $i = 1, \dots, M$ .

## V. INFINITE-HORIZON LINEAR QUADRATIC CONTROL PROBLEM WITH POWER CONSTRAINTS

Consider the infinite-horizon linear quadratic control problem

$$\begin{aligned} \min_{\mu_k} \quad & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q_0 \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\ \text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) + w(k) \\ & u(k) = \mu_k(x(0), \dots, x(k)) \\ & \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q_i \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \leq \gamma_i \\ & \text{for } i = 1, \dots, M. \end{aligned} \quad (23)$$

The solution to this problem is easily obtained using the results for the finite-horizon problem in the previous section. We have seen that the optimal choice of  $S(k)$  is given by (19). When the control system is stationary, we have  $\lim_{k \rightarrow \infty} S(k) = S$ . Thus, when  $k \rightarrow \infty$ , the convex optimization problem of the cost in (18) becomes

$$\begin{aligned} \max_{S \in \mathbb{S}^p, \tau_i \geq 0} \quad & \text{Tr} S - \sum_{i=1}^M \tau_i \gamma_i \\ \text{subject to} \quad & Q + \begin{bmatrix} A^T S A - S & A^T S B \\ B^T S A & B^T S B \end{bmatrix} \succeq 0 \\ & Q = Q_0 + \sum_{i=1}^M \tau_i Q_i. \end{aligned} \quad (24)$$

The optimal control law is then given by  $u(k) = -Lx(k)$ , where  $L$  is the solution to  $(B^T S B + Q_{uu})L = B^T S A + Q_{xu}^T$ .

**Theorem 3:** The dual of the infinite-horizon control problem (23) is given by (24). The optimal value of (24) is equal to the optimal value of the primal problem (23). The optimal control law is given by  $u(k) = -Lx(k)$ , where  $L$  solves  $(B^T S B + Q_{uu})L = B^T S A + Q_{xu}^T$ . ■

**Remark 2:** The result above is very similar to previous results obtained for the continuous-time infinite-horizon control problem with power constraints (see [2]).

## VI. OPTIMAL OUTPUT FEEDBACK CONTROL

The problem of optimal output feedback control will be treated in this section. The solution will be observer-based using the optimal Kalman filter.

The optimization problem to be considered is given by

$$\begin{aligned} \min_{\mu_k} \quad & \mathbf{E} x^T(N) Q_{xx} x(N) \\ & + \sum_{k=0}^{N-1} \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\ \text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) + w(k) \end{aligned}$$

$$\begin{aligned} y(k) &= Cx(k) + v(k) \\ u(k) &= \mu_k(y(0), \dots, y(k)) \\ \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q_i \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} &\leq \gamma_i(k) \\ \text{for } i &= 1, \dots, M. \end{aligned} \quad (25)$$

We make the following assumptions:

$$\begin{aligned} \mathbf{E} x(0) &= \mathbf{E} w(k) = \mathbf{E} v(k) = 0, \\ \mathbf{E} x(0) x^T(0) &= I \\ \mathbf{E} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w(l) \\ v(l) \end{bmatrix}^T &= \delta(k-l) \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}. \end{aligned}$$

Consider the standard Kalman filter (see [8])

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + K(k)(y(k) - C\hat{x}(k)), \\ K(k) &= \left( AP(k)C^T + R_{12} \right) \left( CP(k)C^T + R_2 \right)^{-1}, \end{aligned} \quad (26)$$

$$\begin{aligned} P(k+1) &= AP(k)A^T + R_1 - K(k) \\ &\quad \times \left( CP(k)C^T + R_2 \right) K^T(k), \end{aligned} \quad (28)$$

$$P(0) = \mathbf{E} x(0) x(0)^T = I. \quad (29)$$

$P(k)$  is the covariance matrix of the error  $\tilde{x}(k) = x(k) - \hat{x}(k)$ . Now define the innovations

$$e(k) = y(k) - C\hat{x}(k) = C\tilde{x}(k) + v(k).$$

The covariance matrix of the innovations is given by

$$\begin{aligned} V_{ee}(k) &= \mathbf{E} e(k) e^T(k) \\ &= C \mathbf{E} [\tilde{x}(k) \tilde{x}^T(k)] C^T + \mathbf{E} v(k) v^T(k) \\ &= CP(k)C^T + I. \end{aligned} \quad (30)$$

Define

$$\hat{w}(k) = K(k)e(k). \quad (31)$$

Then

$$\begin{aligned} V_{\hat{w}\hat{w}}(k) &= \mathbf{E} \hat{w}(k) \hat{w}^T(k) \\ &= \mathbf{E} K(k) e(k) e^T(k) K^T(k) \\ &= K(k) V_{ee}(k) K^T(k). \end{aligned} \quad (32)$$

Since  $\tilde{x}(k)$  is the error obtained from the Kalman filter, we have that  $\mathbf{E} y(t) \tilde{x}^T(k) = 0$  for  $t \leq k$  and  $u(k) = \mu(y(0), \dots, y(k))$  implies that  $\mathbf{E} u(k) \tilde{x}^T(k) = 0$ . Hence

$$\begin{aligned} J(x, u) &= \mathbf{E} x^T(N) Q_{xx} x(N) + \sum_{k=0}^{N-1} \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\ &= \mathbf{E} \hat{x}^T(N) Q_{xx} \hat{x}(N) + \sum_{k=0}^{N-1} \mathbf{E} \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix} \\ &\quad + \sum_{k=0}^N \mathbf{E} \tilde{x}^T(k) Q_{xx} \tilde{x}(k). \end{aligned} \quad (33)$$

Therefore, minimization of  $J(x, u)$  is the same as minimizing

$$\mathbf{E} \hat{x}^T(N) Q_{xx} \hat{x}(N) + \sum_{k=0}^{N-1} \mathbf{E} \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix}$$

since nothing can be done about the sum

$$\sum_{k=0}^N \mathbf{E} \tilde{x}^T(k) Q_{xx} \tilde{x}(k) = \sum_{k=0}^N \text{Tr} Q_{xx} P(k)$$

which is a constant. We also have the inequality

$$\begin{aligned}\gamma_i(k) &\geq \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q_i \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\ &= \mathbf{E} \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix}^T Q_i \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix} + \mathbf{E} \tilde{x}^T(k) Q_{i_{xx}} \tilde{x}(k) \\ &= \mathbf{E} \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix}^T Q_i \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix} + \mathbf{Tr} Q_{i_{xx}} P(k).\end{aligned}$$

Since the value of  $\mathbf{Tr} Q_{i_{xx}} P(k)$  is known, we can define the new constant  $\hat{\gamma}_i(k) = \gamma_i(k) - \mathbf{Tr} Q_{i_{xx}} P(k)$  to obtain the equivalent inequality

$$\mathbf{E} \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix}^T Q_i \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix} \leq \hat{\gamma}_i(k).$$

Thus, our output feedback problem is equivalent to the following problem:

$$\begin{aligned}\min_{\mu_k} \quad & \mathbf{E} \hat{x}^T(N) Q_{xx} \hat{x}(N) \\ & + \sum_{k=0}^{N-1} \mathbf{E} \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix} \\ \text{subject to} \quad & \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + \hat{w}(k) \\ & \mathbf{E} \hat{w}(k) \hat{x}^T(0) = 0 \\ & \mathbf{E} \hat{w}(k) \hat{w}^T(k) = V_{\hat{w}\hat{w}}(k) \\ & u(k) = \mu_k(\hat{x}(0), \dots, \hat{x}(k)) \\ & \mathbf{E} \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix}^T Q_i \begin{bmatrix} \hat{x}(k) \\ u(k) \end{bmatrix} \leq \hat{\gamma}_i(k) \\ & \text{for } i = 1, \dots, M.\end{aligned}\tag{34}$$

We see that we have transformed the output feedback problem to a state feedback problem, which can be solved as in the previous sections. We have obtained

*Theorem 4:* The optimal output feedback problem (25) is equivalent to the static feedback problem (34), where  $\hat{x}(k)$  is the optimal estimate of  $x(k)$  obtained from the Kalman filter given by (26) with  $P(k)$  as the covariance matrix of the estimation error  $x(k) - \hat{x}(k)$ . The covariance matrix  $V_{\hat{w}\hat{w}}(k)$  is calculated according to (30)–(32), and  $\hat{\gamma}_i(k) = \gamma_i(k) - \mathbf{Tr} Q_{i_{xx}} P(k)$ . ■

## VII. EXAMPLE

Consider the following scalar stochastic linear quadratic control problem:

$$\begin{aligned}\min_{\mu_k} \quad & \mathbf{E} x_2^2 + \sum_{k=0}^1 \mathbf{E} \{x_k^2 + u_k^2\} \\ \text{subject to} \quad & x_{k+1} = x_k + u_k + w_k \\ & x_0, w_0, w_1 \in \mathcal{N}(0, 1) \\ & \mathbf{E} u_0^2 \leq \frac{1}{10}, \quad \mathbf{E} u_1^2 \leq \frac{1}{4} \mathbf{E} x_1^2 \\ & u_k = \mu_k(x_0, \dots, x_k).\end{aligned}\tag{35}$$

Note first that

$$\begin{aligned}x_k^2 + u_k^2 &= \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \\ u_0^2 &= \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}.\end{aligned}$$

Also,  $\mathbf{E} u_1^2 \leq (1/4) \mathbf{E} x_1^2$  can be written as

$$0 \geq \mathbf{E} \{4u_1^2 - x_1^2\} = \mathbf{E} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix}.$$

We have now the weighting matrices  $Q$ ,  $Q_1(0)$ , and  $Q_1(1)$ . The optimal control law of the optimization (35) can be calculated efficiently using the methods presented in Section V. It is given by  $u_0 = -0.3162x_0$  and  $u_1 = -0.5x_1$ . The minimal cost is 4.3013, compared to the cost of the unconstrained problem (i. e. without the third constraint in (35)) which is 4.1. We can also check that the quadratic constraints are satisfied;  $\mathbf{E} u_0^2 = (1/10) \mathbf{E} x_0^2 = (1/10)$ , and  $\mathbf{E} u_1^2 = (1/4) \mathbf{E} x_1^2$ . Now consider the stationary problem

$$\begin{aligned}\min_{\mu_k} \quad & \mathbf{E} \{x_k^2 + u_k^2\} \\ \text{subject to} \quad & x_{k+1} = x_k + u_k + w_k \\ & w_k \in \mathcal{N}(0, 1) \\ & u_k^2 \leq \frac{1}{4} \mathbf{E} x_k^2 \\ & u_k = \mu_k(x_0, \dots, x_k).\end{aligned}\tag{36}$$

The solution to the problem above is obtained easily by posing it as a set of LMI's, as in equation (24). The minimal cost is 1.6666, which is obtained using  $u_k = 0.5x_k$ .

## VIII. CONCLUSION

We have considered the problem of stochastic linear quadratic control under power constraints. The calculations of the optimal control law can be done off-line as in the classical linear quadratic Gaussian control theory using dynamic programming.

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