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Generalized Maxwell Equations and the Gauge Mixing Mechanism of Mass Generation

Antonio AURILIA* and Yasushi TAKAHASHI

Theoretical Physics Institute, University of Alberta
Edmonton, Alberta, T6G 2J1
and
**Istituto Nazionale di Fisica Nucleare*
Sezione di Trieste

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The content of this paper consists of three main steps: (1) the construction of antisymmetric tensor field representations of massive particles, (2) the transition to the *massless* "Maxwell theory" endowed with a generalized abelian gauge invariance, (3) a *mixing mechanism* of the massless gauge fields in step (2) to generate the massive tensor field representations of step (1). The gauge mixing of the massless fields exhibits the phenomenon of "spin jumping" from the massless representation to the massive one. This spin discontinuity is strictly a consequence of the gauge invariance of the Maxwell theory and is discussed in connection with the established inequivalence of different field representations sharing the same number of degrees of freedom. A full understanding of this paper involves some properties of relativistic strings and bags. As a byproduct of this work we recover the results that the "bag constant" of hadrondynamics and the gravitational cosmological constant can be reformulated as the "Maxwell theory" of the antisymmetric tensor field $A_{\mu\nu\rho}$.

§ 1. Introduction and summary

There is growing evidence that totally antisymmetric tensor gauge fields play an important role in many branches of contemporary physics. They appear in a natural way in the study of dual resonance models,[1] in the relativistic theory of strings and membranes[2)~5] and the associated models of quark confinement,[3),4),6] in quantum gravity[7),8] as well as in various formulations of extended gravity[9] and supergravity.[10),11] Moreover, in the effective lagrangian approach to QCD, the $U(1)$ problem and the question of quark confinement were discussed recently in terms of an antisymmetric tensor gauge field $A_{\mu\nu\rho}(X)$ to which no degrees of freedom are associated in 3+1 dimensions.[6),12] This same field $A_{\mu\nu\rho}(X)$, when coupled to the Einstein tensor provides an unusual realization of the cosmological term.[8),11),13),14] Finally, some effective rules of mode counting for skew-symmetric tensor fields in covariant gauges have been derived by several authors.[15),16] A systematic method for the construction of arbitrarily mixed higher rank field representations of the inhomogeneous Lorentz group is based on the use of covariant spin projection operators.[17] However, at present we have in mind a specific dynamical framework (Maxwell theory) involving pure *tensor* field representations for which the property of antisymmetry is postulated at the outset. A unified discussion of the various aspects of this paper starts therefore with the geometric form of Maxwell's equations

$$dF=0\,, \tag{1·1}$$

$$d^*F=(-1)^{r*}J\,, \tag{1·2}$$

where F represents a differential form of degree $r+1$ ($r=0, 1, 2, 3$) in Minkowski space, $^*F$ is the dual field strength, a differential form of degree $4-r-1$ and $^*J$ is a $(4-r)$-form acting as source of the Maxwell field. Note that for $r=0$ Maxwell's equations reduce to d'Alambert's equation for a massless spin-0 field in presence of a scalar source. The standard electromagnetic theory corresponds to $r=1$. Then, it seems natural to us to inquire about the physical significance of the Maxwell fields of higher rank. It is a virtue of the geometric approach that the formal structure of the Maxwell equations is independent of the value of the parameter r. Thus, if spacetime is simply connected, Eq. (1·1) implies, at least locally

$$F=dA\,, \tag{1·3}$$

where A represents a potential r-form. Thus, Maxwell's equations are invariant under the group of generalized gauge transformations

$$A\to A+d\Lambda\,, \tag{1·4}$$

where Λ is an arbitrary $(r-1)$ differential form. Note that for $r=0$ the gauge transformation is by definition

$$A\to A+\text{constant} \tag{1·5}$$

and A is a scalar potential.

For $r=2$ and $r=3$ the geometric formulation leads to two new theories for the antisymmetric tensor potentials $A_{\mu\nu}(x)$ and $A_{\mu\nu\rho}(x)$ which only recently have received some careful consideration. In § 2 we discuss the *massive* field equations and their quantization. When a mass term is present we find that the field $A_{\mu\nu}(x)$ represents 3 degrees of freedom and satisfies a gauge invariant Proca-type equation discussed long ago by Kemmer[18] and also by Takahashi and Palmer.[19] The field $A_{\mu\nu\rho}(x)$ describes instead a massive spin-0 particle[17] in a representation which is dual to the Proca field.

In § 3 we discuss the generalized Maxwell equations and the associated gauge invariance. As expected on the basis of the electromagnetic case, the transition from the massive to the massless case does not preserve the number of degrees of freedom. In addition, higher rank tensors exhibit a discontinuity in the *spin content*. This is the phenomenon of "spin jumping" recently discussed by Deser, Townsend and Siegel.[20] Thus, in the massless case the field $A_{\mu\nu}(x)$ possesses *one* radiative degree of freedom satisfying the Ogievetskyii-Polubarinov equation[21] while $A_{\mu\nu\rho}(x)$ does not propagate physical quanta. Thus the free Maxwell equation for $r=2$ reduce to d'Alambert's equation ($r=0$) while the *free* Maxwell equation for $r=3$ is pure gauge. However, it has long been known that in the presence of interactions different field representations sharing the same number of degrees of freedom are not necessarily equivalent.[17),22)] Thus, although the tensor potential $A_{\mu\nu\rho}(x)$ does not radiate at all, it is nevertheless capable of interaction and definitely contributes through the energy momentum tensor to the background energy density of a physical system. We give two explicit examples of this phenomenon. First, we show that in the bag formulation of hadrondynamics the "bag constant" B, or vacuum tension, can be reformulated as the gauge theory of $A_{\mu\nu\rho}(x)$; the bag constant acts as a "cosmological constant" in the interior of the bag. More generally, in Appendix A we show that in a spacetime manifold of arbitrary dimensions n the totally antisymmetric tensor field $A_{\mu_1\cdots\mu_{n-1}}$ is just a gauge representation of the cosmological constant.

The inequivalence of different field representations with the same number of degrees of freedom is exploited in § 4 to give a dynamical interpretation of the "spin jumping" phenomenon. We show how the gauge degrees of freedom which are frozen in the free field case can be excited in the presence of interactions, the net result being the appearance of a mass term in the physical spectrum accompanied by a discontinuity in the spin content. For the fields $A_{\mu\nu}(x)$ and $A_{\mu\nu\rho}(x)$ this mechanism of mass generation is the same as the Schwinger mechanism in two spacetime dimensions. Indeed, the Schwinger model is perhaps the simplest field theory where the phenomenon of spin jumping is clearly exhibited. In § 4 we also offer an interpretation of the appearance of a mass term and its relationship to gauge invariance in terms of the classical interaction between strings and membranes with boundaries. Finally we devote Appendix B to discuss the quantum commutation relations of the Maxwell field strengths in the general case.

§ 2. The massive case

A number of seemingly different relativistic wave equations discussed in the old and recent literature can be reinterpreted, with technical and conceptual advantage, as special cases of "Maxwell's equations" with the addition of a mass term

$$dF=0\,, \tag{2·1}$$

$$d^*F=m^{2*}A\,. \tag{2·2}$$

Thus for $r=0$ and $r=1$ Eqs. (2·1) and (2·2) reduce to the Klein-Gordon equation and to the Proca equation respectively.

When $r=2$ both A and $^*A$ are differential 2-forms describing a massive spin-1 particle in dual representations. In component notation the field strength F is given by (cf. Eq. (1·3))

$$F_{\mu\nu\rho}(x)=\partial_\mu A_{\nu\rho}(x)+\partial_\nu A_{\rho\mu}(x)+\partial_\rho A_{\mu\nu}(x) \tag{2·3}$$

and the equation for $A_{\mu\nu}(x)$ is[3),18),19)]

$$(\Box-m^2)A_{\mu\nu}(x)+\partial_\mu\partial_\rho A_{\nu\rho}(x)+\partial_\nu\partial_\rho A_{\rho\mu}(x)=0\,. \tag{2·4}$$

Equation (2·4) is equivalent to the set of equations

$$(\Box-m^2)A_{\mu\nu}(x)=0\,, \tag{2·5}$$

$$\partial_\mu A_{\mu\nu}(x)=0\,. \tag{2·6}$$

In this representation the components A_{4i} $(i=1,2,3)$ vanish in the rest frame while the three independent components of A_{ij} propagate freely. In terms of the dual potential $^*A_{\mu\nu}(x)\equiv(i/2)\varepsilon_{\mu\nu\lambda\rho}A_{\lambda\rho}$ the Maxwell equations (2·1) and (2·2) (or Eq. (2·4)) lead to the following field equation:

$$\partial_\mu\partial_\alpha{}^*A_{\alpha\lambda}(x)-\partial_\lambda\partial_\alpha{}^*A_{\alpha\mu}(x)-m^{2*}A_{\mu\lambda}(x)=0\,, \tag{2·7}$$

which is equivalent to the set of equations

$$(\Box-m^2)^*A_{\mu\nu}(x)=0\,, \tag{2·8}$$

$$\varepsilon_{\mu\lambda\alpha\beta}\partial_\lambda{}^*A_{\alpha\beta}(x)=0\,. \tag{2·9}$$

In this dual representation the components $^*A_{ij}$ vanish in the rest frame while the three independent components $^*A_{4i}$ propagate freely.

Finally, when $r=3$ the potential is a differential 3-form $A\equiv(1/3!)\times A_{\mu\nu\rho}(x)dx^\mu\Lambda dx^\nu\Lambda dx^\rho$ and the corresponding equation represents a massive spin-0 particle. Indeed, in this case the components of the field strength are by definition

$$F_{\mu\nu\rho\sigma}(x)\equiv\partial_\mu A_{\nu\rho\sigma}(x)-\partial_\nu A_{\rho\sigma\mu}(x)+\partial_\rho A_{\sigma\mu\nu}(x)-\partial_\sigma A_{\mu\nu\rho}(x) \tag{2·10}$$

and the equation for $A_{\mu\nu\rho}(x)$ becomes

$$(\Box-m^2)A_{\nu\rho\sigma}(x)-\partial_\mu\partial_\nu A_{\rho\sigma\mu}(x)+\partial_\mu\partial_\rho A_{\sigma\mu\nu}(x)-\partial_\mu\partial_\sigma A_{\mu\nu\rho}(x)=0\,. \tag{2·11}$$

Equation (2·11) is equivalent to the equations

$$(\Box-m^2)A_{\mu\nu\rho}(x)=0\,, \tag{2·12}$$

$$\partial_\mu A_{\mu\nu\rho}(x)=0\,, \tag{2·13}$$

which are easily seen to represent a single propagating degree of freedom. In terms of the dual potential

$$^{*}A_{\mu}(x)\equiv\frac{i}{3!}\varepsilon_{\mu\nu\lambda\rho}A_{\nu\lambda\rho}(x),\tag{2·14}$$

the Maxwell equations (or Eq. (2·11)) lead to the field equation

$$\partial_{\lambda}\partial_{\mu}{}^{*}A_{\mu}(x)=m^{2*}A_{\lambda}(x).\tag{2·15}$$

Thus, in the rest frame $^{*}A_{i}(x)=0$ ($i=1,2,3$) so that only the "scalar mode" of $^{*}A_{\lambda}(x)$ propagates. The relation between Eq. (2·15) and the other standard equations for massive spin-0 particles is immediate. Thus, Eq. (2·15) can be cast into the Duffin-Kemmer form

$$(\beta\cdot\partial+m)\,\Psi(x)=0\,,\tag{2·16}$$

where $\Psi\equiv({}^{*}A_{1},{}^{*}A_{2},{}^{*}A_{3},-i^{*}A_{4},-(1/m)\Phi)$ and the matrices β_{μ} satisfy the wellknown algebraic relation: $\beta_{\mu}\beta_{\lambda}\beta_{\nu}+\beta_{\nu}\beta_{\lambda}\beta_{\mu}=\beta_{\mu}\delta_{\lambda\nu}+\beta_{\nu}\delta_{\lambda\mu}$. Moreover, it is a consequence of Eq. (2·15) that the scalar component $\Phi\equiv\partial_{\mu}{}^{*}A_{\mu}$ satisfies the Klein-Gordon equation

$$(\Box-m^{2})\partial_{\mu}{}^{*}A_{\mu}=0\,,\tag{2·17}$$

which is gauge invariant in spite of the mass term. Note however that the introduction of a mass term in the Maxwell theory for $A_{\mu\nu\rho}$ (or $^{*}A_{\mu}$, cf. Eqs. (3·1) and (3·2)) leads to the original equation (2·15) or (2·11) and not to Eq. (2·17). The appearance of Eqs. (2·17), (2·4) and (2·11) as a consequence of a gauge mixing of Maxwell fields will be discussed in § 4.

Finally, the quantization of the equations discussed so far can be carried out in a straightforward manner according to the Takahashi-Umezawa method:[23] in the presence of a source term, write Eqs. (2·4) and (2·11) in the form

$$\Lambda_{\mu\nu,\sigma\rho}(\partial)A_{\sigma\rho}(x)=J_{\mu\nu}(x),\qquad(\text{for } r=2)\tag{2·18}$$

$$\Lambda_{\mu\nu\lambda,\sigma\rho\tau}(\partial)A_{\sigma\rho\tau}(x)=J_{\mu\nu\lambda}(x)\qquad(\text{for } r=3)\tag{2·19}$$

and assume that the currents $J_{\mu\nu}(x)$ and $J_{\mu\nu\lambda}(x)$ are totally antisymmetric and divergenceless. With this assumption the antisymmetry of the potentials $A_{\sigma\rho}$ and $A_{\sigma\rho\tau}$ need not be postulated but follows from Eqs. (2·18) and (2·19). Next, introduce the "Klein-Gordon divisors" such that

$$\Lambda_{\mu\nu,\sigma\rho}(\partial)d_{\sigma\rho,\xi\eta}(\partial)=(\Box-m^{2})\delta_{\mu\xi}\delta_{\nu\eta}\tag{2·20}$$

and

$$\Lambda_{\mu\nu\lambda,\sigma\rho\tau}(\partial)d_{\sigma\rho\tau,\xi\eta\zeta}(\partial)=(\Box-m^{2})\delta_{\mu\xi}\delta_{\nu\eta}\delta_{\lambda\zeta}\,.\tag{2·21}$$

Equations (2·18) and (2·20) were derived and discussed in Ref. 19). By comparison with Eq. (2·11), it is easy to obtain the following form of $\Lambda_{\mu\nu\lambda,\sigma\rho\tau}$:

$$\Lambda_{\mu\nu\lambda,\sigma\rho\tau}(\partial)=\frac{1}{3!}\varepsilon_{\mu\nu\lambda\alpha}\varepsilon_{\sigma\rho\tau\beta}\partial_{\alpha}\partial_{\beta}-m^{2}\delta_{\mu\sigma}\delta_{\nu\rho}\delta_{\lambda\tau}\tag{2·22}$$

and to show that the Klein-Gordon divisor

$$d_{\sigma\rho\tau,\xi\eta\zeta}(\partial)=\frac{1}{3!}\frac{1}{m^2}\varepsilon_{\sigma\rho\tau\alpha}\varepsilon_{\xi\eta\zeta\beta}\partial_\alpha\partial_\beta-\frac{1}{m^2}(\Box-m^2)\delta_{\sigma\xi}\delta_{\rho\eta}\delta_{\tau\zeta} \tag{2·23}$$

does satisfy the relation (2·21). Then, the covariant commutator of field operators in the interaction representation is given by

$$[A_{\sigma\rho\tau}(x),\ \bar{A}_{\xi\eta\zeta}(x')]=id_{\sigma\rho\tau,\xi\eta\zeta}(\partial)\Delta(x-x'), \tag{2·24}$$

where

$$\bar{A}_{\xi\eta\zeta}(x)=A_{\sigma\rho\tau}(x)g_{\sigma\xi}g_{\rho\eta}g_{\tau\zeta}\ . \tag{2·25}$$

Given the d-operator as in Eq. (2·23), the Green functions, the explicit c-number wave functions together with the closure and orthonormality conditions can be constructed according to the general method of Ref. 23).

The quantization of the antisymmetric tensor fields in the massless case is outlined in Appendix B.

§ 3. Generalized Maxwell equations

The various properties of the field equations discussed in the previous sections are based on the fact that the mass parameter is nonvanishing. As it is well known, in the massless limit the Proca theory does not preserve the number of degrees of freedom. The same is true for the generalized equations (2·1) and (2·2). The number of degrees of freedom which survive in the massless limit is easily determined from Maxwell's equations in vacua

$$dF=0\ , \tag{3·1}$$

$$d^*F=0\ . \tag{3·2}$$

For $r=0$ and $r=1$ these equations reduce to d'Alambert's equation and to the standard Maxwell theory respectively.

When $r=2$ $^*F$ is a 1-form and Eq. (3·2) implies $^*F=d\rho$ where ρ is a scalar function. Thus Eq. (3·1) reduces to $d^*d\rho=0$ which, in Minkowski space, reduces again to d'Alambert's equation $\Box\rho=0$.

Finally, when $r=3$ $^*F$ is a scalar function and Eq. (3·1) is an identity. Thus Eq. (3·2) gives

$$^*F=\text{constant}\ . \tag{3·3}$$

In flat spacetime and in the absence of interaction the constant background field (3·3) cannot be distinguished from the vacuum and can be set equal to zero. Thus, the free Maxwell theories labelled by $r=0$ and $r=2$ represent different field representations of the same single degree of freedom while for $r=3$ the Maxwell field strength vanishes. However, in the presence of an interaction this degeneracy is removed. For instance, Duff and van Nieuwenhuizen[8] have observed that the gravitational trace anomaly of $A_{\mu\nu}(x)$ and $A(x)$ are different and that the gravitational trace anomaly of $A_{\mu\nu\rho}(x)$ is nonvanishing. Thus naive equivalence arguments for different field representations describing the same number of degrees of freedom breakdown whenever global properties of the spacetime manifold are taken into consideration. From the point of view advocated in this paper, however, this result is a special effect due to the gravitational interaction. The inequivalence in question persists in the presence of interactions even in flat spacetime with trivial topology;[22] the source of the inequivalence is the gauge structure associated with different field representations leading to inequivalent couplings and therefore to different physical effects. This is already transparent at *classical* level. Thus, a relativistic closed string emits longitudinal waves which belong to the $r=2$ representation of Maxwell's equations (1·1) and (1·2). The propagation of such massless spin-0 field is described by the tensor potential $A_{\mu\nu}(x)$ with field strength given by Eq. (2·3). This special choice of representation is strictly a consequence of the gauge principle. Indeed, as the electromagnetic potential couples to the timelike tangent element of the world-line $x^{\mu}(\tau)$ of a charged particle

$$S_{\rm INT}^{\rm particle}=e_1\int_{\tau_1}^{\tau_2}d\tau\frac{\partial x_\mu(\tau)}{\partial\tau}A_\mu(x(\tau)), \tag{3·4}$$

the tensor gauge potential $A_{\mu\nu}(x)$ couples to the timelike tangent element of the world-sheet $x^{\mu}(\tau,\sigma)$ spanned by a relativistic string in spacetime

$$S_{\rm INT}^{\rm string}=\frac{e_2}{2!}\int_{\tau_1}^{\tau_2}\int_{\sigma_1}^{\sigma_2}d\tau d\sigma\frac{\partial x_\mu}{\partial\tau}\Lambda\frac{\partial x_\nu}{\partial\sigma}A_{\mu\nu}[x(\tau,\sigma)]. \tag{3·5}$$

Here e_2 represents the strength of the coupling and the coordinates τ, σ represent an arbitrary parametrization of the world sheet of the string. If the string is closed, which amounts to impose periodic boundary conditions on the σ-dependence, one easily verifies that the interaction (3·5) is invariant under the gauge transformation

$$\delta A_{\mu\nu}(x)=\partial_\mu\Lambda_\nu(x)-\partial_\nu\Lambda_\mu(x), \tag{3·6}$$

which corresponds to Eq. (1·4) for $r=2$. We are not aware of any manifestly covariant and gauge invariant coupling of the string element to a massless scalar field $A(x)$ which leads to the same physical solutions[24] as (3·5).

Similarly the tensor gauge potential $A_{\mu\nu\rho}(x)$ couples to the timelike tangent element of the 3-dimensional world-hypersurface $x^{\mu}(\tau,\sigma,\lambda)$ spanned by a relativistic membrane in Minkowski space,

$$S_{\rm INT}^{\rm membrane}=\frac{e_3}{3!}\int_{\tau_1}^{\tau_2}\int_{\sigma_1}^{\sigma_2}\int_{\lambda_1}^{\lambda_2}d\tau d\sigma d\lambda\frac{\partial x_\mu}{\partial\tau}\Lambda\frac{\partial x_\nu}{\partial\sigma}\Lambda\frac{\partial x_\rho}{\partial\lambda}A_{\mu\nu\rho}[x(\tau,\sigma,\lambda)], \tag{3·7}$$

where e_3 represents again the strength of the coupling. Note that $[e_r]=M^{r-1}$ $(r=1,2,3)$. The interaction(3·7) is invariant under a general change of the coordinates (τ,σ,λ) on the world track of the membrane. Furthermore, if the surface is closed, Eq.(3·4) is also invariant under the gauge transformation

$$\delta A_{\mu\nu\rho}(x)=\partial_\mu\Lambda_{\nu\rho}(x)+\partial_\nu\Lambda_{\rho\mu}(x)+\partial_\rho\Lambda_{\mu\nu}(x), \qquad (\Lambda_{\alpha\beta}=-\Lambda_{\beta\alpha}) \tag{3·8}$$

which corresponds to Eq. (1·4) for $r=3$.

The case of the tensor gauge potential $A_{\mu\nu\rho}(x)$ illustrates particularly well the inequivalence between different field representations possessing the same number of degrees of freedom. Thus there are no degrees of freedom associated with $A_{\mu\nu\rho}(x)$ and one can plausibly argue that the corresponding *free* Maxwell theory is equivalent to an empty theory. However, our argument shows the potential danger of setting the constant (3·3) equal to zero *before* the interaction is switched on: as *gauge* field, $A_{\mu\nu\rho}(x)$ mediates the interaction between the surface elements of a relativistic closed membrane. The physical significance of the interaction (3·7) can be appreciated as follows. By analogy with classical electrodynamics the interaction terms (3·4), (3·5) and (3·7) can be rewritten in the following form

$$S_{\rm INT}^{\rm object}=\frac{e_r}{3!}\int J_{\mu_1\cdots\mu_r}(y)A_{\mu_1\cdots\mu_r}(y)d^4y, \qquad (r=1,2,3) \tag{3·9}$$

where $J_{\mu_1\cdots\mu_r}(y)$ is the r-vector density

$$J_{\mu_1\cdots\mu_r}(y)=\int\cdot\cdot\int d^rt\frac{\partial x_{\mu_1}}{\partial t^1}\Lambda\cdots\Lambda\frac{\partial x_{\mu_r}}{\partial t^r}\delta[y-x(t^1\cdots t^r)] \tag{3·10}$$

integrated over the local coordinates $\{t^1\cdots t^r\}$ which parametrize the world track of the object. For spatially closed systems we have seen that the action

$$S=-\frac{1}{2(r+1)!}\int F_{\mu_1\cdots\mu_{r+1}}(x)F_{\mu_1\cdots\mu_{r+1}}(x)d^4x-\text{Eq. (3·9)} \tag{3·11}$$

is invariant under the group of generalized gauge transformations and the currents (3·10) represent the corresponding conserved Noether currents. Variation of the action (3·11) with respect to the potential leads to the Maxwell equations (1·2). For $r=3$ these equations are

$$\partial_\mu F_{\mu\nu\rho\sigma}(x)=cJ_{\nu\rho\sigma}(x) \qquad (c\equiv e_3) \tag{3·12}$$

and the Bianchi identities (1·1) are automatically satisfied. In terms of the dual quantities $^*F(x)\equiv(i/4!)\varepsilon_{\mu\nu\lambda\rho}F_{\mu\nu\lambda\rho}(x)=-\partial_\mu{}^*A_\mu(x)$ and $^*J_\mu(x)=(i/3!)\times\varepsilon_{\mu\nu\lambda\rho}J_{\nu\lambda\rho}(x)$ Eq. (3·12) reduces to

$$\partial_\mu{}^*F(x)=-c^*J_\mu(x). \tag{3·13}$$

By construction, the dual current $^*J_\mu$ is directed along the (spacelike) normal to the world-track of the membrane. In the instantaneous rest frame of a point on the membrane $n_0=0$, n_i is the ordinary unit space normal and $^*J_\mu$ possesses a surface δ-type singularity. Thus, at each point on the membrane, Eq. (3·13) becomes

$$\partial_\mu {}^*F(x) = -c\eta_\mu \delta_\Sigma(x), \tag{3·14}$$

where $\delta_\Sigma(x)$ is the surface δ-function with the property

$$\int d^4x\, \delta_\Sigma(x) f(x) = \int (d^3x)_\Sigma f(x) \tag{3·15}$$

and the integral on the r. h. s. is restricted to the volume enclosed by the membrane. The solution of Eq. (3·14) is[13]

$$^*F = -c\theta_V(x), \tag{3·16}$$

where

$$\theta_V(x) = \begin{cases} 1 & \text{inside the surface}, \\ 0 & \text{outside the surface}, \end{cases} \tag{3·17}$$

is the volume step function discontinuous at the membrane points. The value of the field strength on the surface is taken to be the arithmetic mean of the values inside and outside

$$^*F|_{\text{surface}} \equiv -\frac{c}{2}. \tag{3·18}$$

The reader familiar with the bag formulation of hadron dynamics will recognize that Eq. (3·14) with the solution (3·16) is the basic ingredient used in that formulation to enforce the confinement of quarks and gluons to the interior of the bag.[25] It is interesting that the mechanism of confinement in the bag model can be reformulated as the generalized Maxwell equation (3·12) and can be derived from the action (3·11). According to this generalized Maxwell theory, the "bag constant", i. e., the volume energy density of the bag is simply the field self-energy density. Unlike electrodynamics, the total self-energy is *finite* and equal to $\frac{1}{2}c^2 V$ where V is the volume enclosed by the mebrane at a given time. Indeed the field contribution to the energy momentum tensor of the system(3·11) is

$$T_{\mu\nu\text{field}}(x) = \frac{1}{3!} F_{\mu\lambda\rho\sigma}(x) F_{\nu\lambda\rho\sigma}(x) - \frac{1}{2\cdot 4!} g_{\mu\nu} F_{\alpha\beta\sigma\rho}(x) F_{\alpha\beta\sigma\rho}(x), \tag{3·19}$$

which in terms of $^*F(x)$ is simply

$$T_{\mu\nu\text{field}}(x) = -\frac{1}{2} g_{\mu\nu} ({}^*F)^2 = -\frac{1}{2} c^2 g_{\mu\nu} \theta_V(x). \tag{3·20}$$

Thus the tensor $F_{\mu\nu\rho\sigma}(x)$ carries no momentum but contributes to the background energy density of the bag in the amount $\frac{1}{2}c^2$ per unit volume. Thus the physical effect of the field strength $F_{\mu\nu\rho\sigma}(x)$ is to "polarize" the vacuum inside the membrane as postulated in the bag model and the bag constant can be reformulated as the gauge theory of the antisymmetric tensor field $A_{\mu\nu\rho}(x)$.

§ 4. Gauge field mixing and mass generation

In the previous sections we have shown how the transition from the massive to the massless field equations for antisymmetric tensor gauge fields does not preserve the number of degrees of freedom. The actual number of such degrees of freedom for an antisymmetric tensor potential of rank r in spacetimes of arbitrary dimensions n has been known[4] for some time and is given by $(n-2)!/r!(n-2-r)!$. However, the exact mechanism by which the gauge invariance operates in these new gauge theories was clarified only recently through the work of several authors.[15),16)] For antisymmetric tensor fields of higher rank the gauge fixing terms are themselves gauge invariant and require additional sequences of lower rank gauge fixing terms and compensating Faddeev-Popov ghosts. When finally the number of degrees of freedom is correctly accounted for, the gauge theory of the rank 2 potential $A_{\mu\nu}(x)$ turns out to be complementary[21)] to the electromagnetic theory $(r=1)$: In the limit of zero rest mass, the transverse polarization states of the massive spin-1 particle described by the Kemmer-Takahashi-Palmer equation are eliminated, on account of the gauge invariance, so that only the longitudinal component survives. Conversely, When the longitudinal degree of freedom represented by the massless tensor $A_{\mu\nu}(x)$ is mixed in a gauge invariant manner with the electromagnetic field, the linear equations describing the coupled system are soluble and yield a massive spin-1 particle obeying the K-T-P equation. Thus the lagrangian system

$$\begin{aligned}\mathcal{L}=&\frac{1}{2\cdot3!}F_{\mu\nu\rho}F_{\mu\nu\rho}-\frac{1}{3!}F_{\mu\nu\rho}(\partial_\mu A_{\nu\rho}+\partial_\nu A_{\rho\mu}+\partial_\rho A_{\mu\nu})\\&+\frac{1}{2\cdot2!}F_{\mu\nu}F_{\mu\nu}-\frac{1}{2!}F_{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)\\&+\frac{ik}{3!}\varepsilon_{\mu\nu\rho\beta}A_\beta(\partial_\mu A_{\nu\rho}+\partial_\nu A_{\rho\mu}+\partial_\rho A_{\mu\nu})\end{aligned}\tag{4·1}$$

is invariant under the combined gauge transformations (cf. (1·4))

$$(r=1):\quad \delta A_\mu=\partial_\mu\Lambda,\qquad (r=2):\quad \delta A_{\mu\nu}=\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu\tag{4·2}$$

and leads to the field equations

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,\tag{4·3}$$

$$F_{\mu\nu\rho}=\partial_\mu A_{\nu\rho}+\partial_\nu A_{\rho\mu}+\partial_\rho A_{\mu\nu}\ , \tag{4·4}$$

$$\partial_\mu F_{\mu\nu}=-k^*F_\nu\ , \tag{4·5}$$

$$\partial_\mu(F_{\mu\nu\rho}-ik\varepsilon_{\mu\nu\rho\beta}A_\beta)=0\ . \tag{4·6}$$

These equations can be combined in an elementary way to yield the following equations:

$$\Box^*F_{\nu\rho}=k^{2*}F_{\nu\rho}\ , \tag{4·7}$$

$$\partial_\nu{}^*F_{\nu\rho}=0\ . \tag{4·8}$$

Note that the above equations are *gauge invariant* in spite of the mass term. When the gauge is fixed they reduce[19] to the standard Proca theory.

Similarly, when the massless tensor $A_{\mu\nu\rho}(x)$ which represents a static polarization effect with no degrees of freedom is mixed in a gauge invariant manner with a massless scalar field $A(x)(r=0)$, the linear equations describing the coupled system are soluble and the physical spectrum consists of a massive spin-0 particle.

Take the lagrangian density

$$\mathcal{L}=\frac{1}{2}F_\mu F_\mu-F_\mu\partial_\mu A+\frac{1}{2}\frac{1}{4!}F_{\mu\nu\lambda\rho}F_{\mu\nu\lambda\rho}$$

$$-\frac{1}{4!}F_{\mu\nu\lambda\rho}(\partial_\mu A_{\nu\lambda\rho}+3\text{ terms})-\frac{if}{3!}\varepsilon_{\mu\nu\lambda\rho}A_{\mu\nu\lambda}\partial_\rho A \tag{4·9}$$

which is invariant under the combined gauge transformations (cf. Eq. (1·4))

$$r=0:\quad \delta A=\text{constant}\ ;\qquad r=3:\quad \delta A_{\mu\nu\rho}=\partial_\mu\Lambda_{\nu\rho}+2\text{ terms}\ . \tag{4·10}$$

In this case, the field equations

$$F_\mu=\partial_\mu A\ ;\qquad F_{\mu\nu\rho\sigma}=\partial_\mu A_{\nu\rho\sigma}+3\text{ terms}\ , \tag{4·11}$$

$$\partial_\mu F_\mu=-\frac{if}{3!}\partial_\rho\varepsilon_{\mu\nu\lambda\rho}A_{\mu\nu\lambda}\ , \tag{4·12}$$

$$\partial_\mu F_{\mu\nu\lambda\rho}=if\varepsilon_{\nu\lambda\rho\mu}\partial_\mu A\ , \tag{4·13}$$

combine to give the *gauge invariant* Klein-Gordon equation (cf. Eq. (2·17))

$$(\Box-f^2)\partial_\mu{}^*A_\mu=0\ . \tag{4·14}$$

The simplicity of the lagrangian system (4·9) is somewhat deceptive. It was shown in Ref. 12) how the gauge field mixing leading to Eq. (4·14) can be interpreted as a Higgs phenomenon. In covariant gauges, the structure of the Hilbert space associated with the system (4·9) is rather complex: consistent with the non propagating character of $A_{\mu\nu\rho}(x)$, there exist 32 unphysical modes which eliminate each other by a norm cancellation mechanism leaving a single massive spin-0 particle in the physical spectrum. In Ref. 12) it was also observed how the mechanism of mass generation in the lagrangian (4·9) is precisely the same as the Schwinger mechanism in two-dimensional spacetime. Indeed, in two dimensions the lagrangian (4·9) reduces to the bosonized version of the Schwinger model

$$\mathcal{L}(2\,\mathrm{dim.}) = -\frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\frac{1}{2!}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2!}F_{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - fi\varepsilon_{\mu\nu}A_\nu\partial_\mu\rho \tag{4·15}$$

and one easily verifies that the lagrangian system (4·15) leads again to Eq. (4·14) in two spacetime dimensions. In this connection we wish to note here that in two dimensions *both* lagrangian systems (4·1) and (4·9) reduce to the Schwinger lagrangian (4·15). Hence, both gauge field mixings represented by (4·1) and (4·9) are legitimate representations of the Schwinger mechanism in 4 dimensions. Of course this simply reflects the lack of transversality in a two-dimensional world. This degeneracy is removed in 4-dimensions and the physical spectrum of (4·1) consists of a massive spin-1 particle obeying Eqs. (4·7) and (4·8) while the physical spectrum of (4·9) consists of a massive spin-0 particle obeying Eq. (4·14).

In the remaining part of this section we wish to discuss two alternative but instructive derivations of Eqs. (2·4) and (2·11) by a *different gauge mixing* of antisymmetric tensor fields. In so doing we shall illustrate an interesting relationship between those equations and the classical couplings to relativistic strings and membranes discussed in the previous section. In § 3 we have observed that the massless field $A_{\mu\nu}(x)$ mediates the string-string interaction via the exchange of spin-0 quanta when the strings are closed. For strings with open ends this is no longer true as was first pointed out by Kalb and Ramond.[2)] For strings with open ends the response of the action (3·5) to the variation (3·6) is

$$\delta_1 S_{\mathrm{INT}}^{\mathrm{string}} \propto e_2\int_{\tau_1}^{\tau_2} d\tau \frac{\partial x_\mu}{\partial\tau}\Lambda_\mu\Big|_{\sigma_1}^{\sigma_2} \tag{4·16}$$

so that there is a breaking of gauge invariance due to the end point contribution. In order to compensate this "leakage" of symmetry from the end points one proceeds as follows, i) introduce a new coupling

$$S_{\mathrm{INT}}^{\mathrm{string}} \to \bar{S}_{\mathrm{INT}} = S_{\mathrm{INT}}^{\mathrm{string}} + e_1\int d\tau \int d\sigma \frac{\partial x_\mu}{\partial\tau}\Lambda\frac{\partial x_\nu}{\partial\sigma}F_{\mu\nu}\,, \tag{4·17}$$

where $F_{\mu\nu}$ is the field strength tensor associated with a vector potential A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{4·18}$$

ii) introduce the new skew-symmetric potential

$$H_{\mu\nu} \equiv \frac{e_2}{e_1} A_{\mu\nu} + F_{\mu\nu} \tag{4·19}$$

so that the new interaction becomes

$$\bar{S}_{\text{INT}} = e_1 \int d\tau \int d\sigma \frac{\partial x_\mu}{\partial \tau} \Lambda \frac{\partial x_\nu}{\partial \sigma} H_{\mu\nu} , \tag{4·20}$$

while the original field strength is form invariant

$$F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + 2 \text{ terms} = \frac{e_1}{e_2} (\partial_\mu H_{\nu\rho} + \partial_\nu H_{\rho\mu} + \partial_\rho H_{\mu\nu}) \tag{4·21}$$

as a consequence of the definition (4·19).

The group of transformations

$$\delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu , \tag{4·22}$$

$$\delta A_\mu = \partial_\mu \Lambda - \frac{e^2}{e_1} \Lambda_\mu , \tag{4·23}$$

leaves $F_{\mu\nu\rho}$, $H_{\mu\nu}$ and $\bar{S}_{\text{INT}}$ unchanged. *The transformation law (4·22), (4·23) also determines the structure of the free field contribution to the action.* The free lagrangian density is

$$\mathcal{L} = \frac{1}{2 \cdot 3!} F_{\mu\nu\rho} F_{\mu\nu\rho} - \frac{1}{3!} F_{\mu\nu\rho} (\partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu})$$

$$+ \frac{1}{2 \cdot 2!} H_{\mu\nu} H_{\mu\nu} - \frac{1}{2!} H_{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu + k A_{\mu\nu}), \tag{4·24}$$

where $k \equiv \frac{e_2}{e_1}$ has the dimensions of a mass. The field equation derived from (4·24) is

$$\partial_\mu F_{\mu\nu\rho}(x) = k H_{\nu\rho}(x), \tag{4·25}$$

which is the equation for a massive spin-1 discussed previously (cf. (2·4)). Thus, while the interaction between closed strings is mediated by massless spin-0 quanta, the full interaction between open strings, including boundary effects, requires a massive spin-1 field as the mediating field. *It is interesting that the mass arises as a gauge mixing of different antisymmetric tensor fields invariant under the wider group of transformations (4·22), (4·23).*

The gauge field mixing mechanism displayed by the lagrangian (4·24) is quite general. Thus one readily checks that the mass term in the standard Proca equation arises from the gauge mixed lagrangian density

$$\mathcal{L} = \frac{1}{2 \cdot 2!} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2!} F_{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} H_\mu H_\nu - H_\mu (\partial_\mu A + k A_\mu) \tag{4·26}$$

invariant under the group of transformations

$$\delta H_\mu=\delta F_{\mu\nu}=0, \qquad \delta A_\mu=\partial_\mu \Lambda, \qquad \delta A=\text{const}-k\Lambda . \tag{4·27}$$

The mixing of the (A_μ, A) fields according to (4·26) results in the massive Proca equation

$$\partial_\mu F_{\mu\nu}=kH_\nu . \tag{4·28}$$

The general pattern emerging from the above discussion is quite suggestive: there exists a one to one correspondence between geometric structures (points, strings, membranes) and antisymmetric tensor gauge fields of Maxwell type; the form of their mutual interaction (cf. Eq. (3·9)) is independent of the rank of the tensor and, for closed objects, is invariant under the gauge transformation (1·4); the agents of the interaction are respectively A_μ with 2 degrees of freedom, $A_{\mu\nu}$ with 1 degree of freedom and $A_{\mu\nu\rho}$ with no degrees of freedom, for open strings there is a breaking of symmetry due to boundary effects and the restoration of the symmetry by extra couplings of the boundaries to gauge fields of lower rank generates a gauge invariant mass term by the mechanism of gauge field mixing.

With little change the above discussion applies to open membranes with boundaries represented by closed strings. Thus the first step is to change the original interaction term (3·7) into

$$\bar{S}_{\text{INT}}=e_2\int d\tau\int d\sigma\int d\lambda\frac{\partial x_\mu}{\partial\tau}\Lambda\frac{\partial x_\nu}{\partial\sigma}\Lambda\frac{\partial x_\rho}{\partial\lambda}H_{\mu\nu\rho} , \tag{4·29}$$

where we have introduced

$$H_{\mu\nu\rho}=\frac{e_3}{e_2}A_{\mu\nu\rho}+F_{\mu\nu\rho} \tag{4·30}$$

and

$$F_{\mu\nu\rho}=\partial_\mu A_{\nu\rho}+2\text{ terms} \tag{4·31}$$

is the field strength associated with the boundary (closed string). The original field strength is form invariant

$$F_{\mu\nu\rho\sigma}=\partial_\mu A_{\nu\rho\sigma}+3\text{ terms }=\frac{e_2}{e_3}(\partial_\mu H_{\nu\rho\sigma}+3\text{ terms}) \tag{4·32}$$

as consequence of the definition (4·30). The tensor $F_{\mu\nu\rho\sigma}$, $H_{\mu\nu\rho}$ and $\bar{S}_{\text{INT}}$ are now invariant under the group of transformations

$$\delta A_{\mu\nu\rho}=\partial_\mu\Lambda_{\nu\rho}+\partial_\nu\Lambda_{\rho\mu}+\partial_\rho\Lambda_{\mu\nu}, \qquad (\Lambda_{\alpha\beta}=-\Lambda_{\beta\alpha}) \tag{4·33}$$

$$\delta A_{\mu\nu}=\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu-\frac{e^3}{e_2}\Lambda_{\mu\nu} . \tag{4·34}$$

In this case, the free field lagrangian which reflects the symmetry under (4·33), (4·34) requires the following mixing of $A_{\mu\nu\rho}$ and $A_{\mu\nu}$

$$\mathcal{L}=\frac{1}{2}\frac{1}{4!}F_{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}-\frac{1}{4!}F_{\mu\nu\lambda\rho}(\partial_\mu A_{\nu\lambda\rho}+3\text{ terms})$$

$$+\frac{1}{2}\frac{1}{3!}H_{\mu\nu\rho}H_{\mu\nu\rho}-\frac{1}{3!}H_{\mu\nu\rho}(\partial_\mu A_{\nu\rho}+2\text{ terms}+kA_{\mu\nu\rho}),\quad(4\cdot35)$$

where $k\equiv\dfrac{e^3}{e_2}$.

The gauge invariant field equation resulting from (4·35) is

$$\partial_\mu F_{\mu\nu\rho\sigma}(x)=kH_{\mu\nu\rho\sigma}\qquad(4\cdot36)$$

with $F_{\mu\nu\rho\sigma}$ given by Eq. (4·32). Equation (4·36) was discussed in § 2 (cf., Eq. (2·11)) and describes a massive spin-0 particle which, as we have just seen, mediates the full interaction between open membranes including the boundary effects due to closed strings.

Appendix A

If M represents the spacetime manifold of dimension n endowed with a general lorentzian metric g,*) the action for our generalized Maxwell field $F=dA$ defined as an $r+1$ differential form on M is

$$S^{\text{field}}=-\frac{1}{2}\int\|F\|^2\mu_g\qquad(\text{A}\cdot1)$$

with norm given by

$$\|F\|^2=\frac{1}{(r+1)!}F_{\mu_1\cdots\mu_{r+1}}F^{\mu_1\cdots\mu_{r+1}}.\quad(r=0,1,2,n-1)\qquad(\text{A}\cdot2)$$

Minimizing the action with respect to A leads to the Maxwell equations in M which for $n=4$ were discussed in the text

$$d^*F=0\,,\qquad(\text{A}\cdot3)$$

$$dF=0\,.\qquad(\text{A}\cdot4)$$

Furthermore, from the action we can read off the general form of the energy momentum tensor

$$T^{\mu\nu}=2\frac{\delta S^{\text{field}}}{\delta g_{\mu\nu}}=\left\{\frac{1}{r!}F^{\mu}{}_{\lambda_1\cdots\lambda_r}F^{\nu\lambda_1\cdots\lambda_r}-\frac{1}{2}g^{\mu\nu}\|F\|^2\right\}\sqrt{-g}\,.\qquad(\text{A}\cdot5)$$

*) Note that throughout the text we have used the Pauli metric and the conventions of Ref. 23). However, in Appendix A the metric has signature $(-+++)$ and the conventions are those of C. W. Misner, K. S. Thorne and J. A. Wheeler in *Gravitation*, (W. H. Freeman and Company, San Francisco).

The tensor $T^{\mu\nu}$ is symmetric by construction and conserved, $\nabla_\mu T^{\mu\nu}=0$ by virtue of Maxwell's equations.

In particular, if F is a differential n-form on $M(r=3)$ then (A·5) gives for any n

$$T^{\mu\nu}=-\frac{1}{2}g^{\mu\nu}({}^*F)^2\sqrt{-g}\ . \tag{A·6}$$

Moreover, for any n (A·3) and (A·4) imply ${}^*F=$const (cf., Eq. (3·3)) and we see that (A·6) amounts to a cosmological term on the r. h. s. of Einstein equation. Thus we have the equivalence

$$\left.\begin{array}{l} G=8\pi T^{\text{field}} \\ dF=0 \\ d^*F=0 \end{array}\right\} \longleftrightarrow G+g\Lambda=0\ , \tag{A·7}$$

that is, the cosmological term can be reformulated as the Maxwell gauge theory of $A_{\mu_1\cdots\mu_{n-1}}$. If $n=2$ this "cosmological field" is represented by $F=\frac{1}{2}F_{\mu\nu}dx^\mu\Lambda dx^\nu$ with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ which happens to coincide with the usual form of the electromagnetic Maxwell field in 4 dimensions. In 2 spacetime dimensions the "cosmological field" can be identified with the constant background field which gives rise to the θ parameter in the Schwinger model.[6),11)]

Appendix B

——Field Commutator Algebra——

Our quantization procedure involves the Maxwell field strength directly. The basic assumption is the Heisenberg equation of motion

$$-i\partial_\mu F_{\lambda_1\cdots\lambda_{r+1}}=[F_{\lambda_1\cdots\lambda_{r+1}},\ P_\mu], \tag{B·1}$$

where

$$P_\nu=\int d\sigma_\mu T_{\mu\nu} \tag{B·2}$$

and $T^{\mu\nu}$ is given by (A·5) (in flat spacetime with $n=4$). We emphasize that the Heisenberg equation involves explicitly the time derivative *and* the space derivative in contrast with the usual formulation. In the present case the field commutator algebra is so constructed that the Heisenberg equation (B·1) implies the Maxwell equation (A·3) *and* the Bianchi identity (A·4). These equations, in turn, guarantee the conservation law

$$\partial_\mu T_{\mu\nu}=0\ . \tag{B·3}$$

In the case of the "cosmological field" ($r=3$) there are no degrees of freedom to be quantized (however, see Kimura, Ref. 16)).

B1) *The case* $r=0$

The symmetric energy momentum is

$$T_{\mu\nu}=F_\mu F_\nu-\frac{1}{2}\delta_{\mu\nu}F_\lambda F_\lambda\,. \tag{B·4}$$

The assumed form of the commutators is, at equal time

$$[F_0(x),\,F_0(x')]=0\,, \tag{B·5}$$

$$[F_0(x),\,F_i(x')]=-i\partial_i\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{B·6}$$

$$[F_i(x),\,F_j(x')=0\,. \tag{B·7}$$

Then

$$i\partial_t F_0(x)=[F_0(x),\,H]=-i\partial_i F_i(x), \tag{B·8}$$

$$i\partial_t F_i(x)=[F_i(x),\,H]=-i\partial_i F_0(x), \tag{B·9}$$

$$-i\partial_i F_0(x)=[F_0(x),\,P_i]=-i\partial_i F_0(x), \tag{B·10}$$

$$-i\partial_i F_j(x)=[F_j(x),\,P_i]=-i\partial_j F_i(x). \tag{B·11}$$

Equations (B·8)~(B·11) imply

$$\partial_\mu F^\mu=0\,, \tag{B·12}$$

$$\partial_\mu F_\nu-\partial F_\mu=0\,, \tag{B·13}$$

which correspond to (A·3) and (A·4) for $r=0$.

B2) *The case* $r=1$

The symmetry energy momentum tensor is

$$T_{\mu\nu}=F_{\mu\lambda}F_{\nu\lambda}-\frac{1}{2!}\frac{1}{2}\delta_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho} \tag{B·14}$$

and the equal time commutation relations are

$$[F_{0i}(x),\,F_{0j}(x')]=0\,, \tag{B·15}$$

$$[F_{0i}(x),\,F_{jk}(x')]=-i(\delta_{ik}\partial_j-\delta_{ij}\partial_k)\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{B·16}$$

$$[F_{ij}(x),\,F_{kl}(x')]=0\,. \tag{B·17}$$

Hence

$$\mathrm{i}\partial_t F_{0i}(x)=[F_{0i}(x),\,H]=-i\partial_k F_{ki}(x), \tag{B·18}$$

$$i\partial_t F_{ij}(x)=[F_{ij}(x),\,H]=-i\{\partial_j F_{0i}(x)-\partial_i F_{0j}(x)\}, \tag{B·19}$$

$$-i\partial_i F_{0j}(x)=[F_{0j}(x),\,P_i]=-i\{\partial_i F_{0j}(x)-\delta_{ij}\partial_k F_{0k}(x)\}, \tag{B·20}$$

$$-i\partial_i F_{jk}(x)=[F_{jk}(x),\ P_i]=-i\{\partial_j F_{ik}(x)-\partial_k F_{ij}(x)\}. \tag{B·21}$$

The above equations can be assembled to reproduce the standard covariant form of Maxwell equations.

B3) *The case* $r=2$

In this case the Bianchi identities are

$$\partial_\mu F_{\mu\nu\rho}-\partial_\nu F_{\lambda\rho\mu}+\partial_\lambda F_{\rho\mu\nu}-\partial_\rho F_{\mu\nu\lambda}=0\ . \tag{B·22}$$

Hence

$$F_{\mu\nu\lambda}=\partial_\mu A_{\nu\lambda}+\partial_\nu A_{\lambda\mu}+\partial_\nu A_{\mu\nu} \tag{B·23}$$

and the field equations

$$\partial_\mu F_{\mu\nu\lambda}=0 \tag{B·24}$$

imply

$$\Box A_{\nu\lambda}+\partial_\nu\partial_\nu A_{\lambda\mu}+\partial_\mu\partial_\lambda A_{\mu\nu}=0\ . \tag{B·25}$$

The energy momentum tensor is

$$T_{\mu\nu}=\frac{1}{2!}F_{\mu\lambda\rho}F_{\nu\lambda\rho}-\frac{1}{3!}\frac{1}{2}\delta_{\mu\nu}F_{\lambda\rho\tau}F_{\lambda\rho\tau} \tag{B·26}$$

and the commutation relations for the field components are at equal time

$$[F_{0ij}(x),\ F_{0kl}(x')]=0\ , \tag{B·27}$$

$$[F_{0ij}(x),\ F_{klm}(x')]=i\{(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\partial_k$$
$$+(\delta_{im}\delta_{jk}-\delta_{ik}\delta_{jm})\partial_l+(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\partial_m\}\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{B·28}$$

$$[\mathrm{F}_{ijk}(x),\ F_{lmn}(x')]=0\ . \tag{B·29}$$

From the above equations and the Heisenberg equations of motion we derive

$$i\partial_t F_{0ij}(x)=[F_{0ij}(x),\ H]=-i\partial_k F_{kij}(x), \tag{B·30}$$

$$i\partial_t F_{ijk}(x)=[F_{ijk}(x),\ H]$$
$$=-i\partial_{jk0}(x)+i\partial_j F_{k0i}(x)-i\partial_k F_{0ijk}(x)$$
$$=-i\{\partial_i F_{0ijk}(x)+\partial_j F_{0ki}(x)+\partial_k F_{0ij}(x)\}, \tag{B·31}$$

$$-i\partial_k F_{0ij}(x)=[F_{0ij}(x),\ P_k]$$
$$=i\partial_k F_{0ij}(x)+i\delta_{ik}\partial_l F_{0lj}(x)-i\delta_{ik}\partial_l F_{0li}(x). \tag{B·32}$$

This equation implies

$$\partial_l F_{lj0}(x)=0\ , \tag{B·33}$$

$$-i\partial_l F_{ijk}(x)=[F_{ijk}(x),\ P_l]$$
$$=-i\{\partial_i F_{jkl}(x)-\partial_j F_{kli}(x)+\partial_k F_{lij}(x)\}. \qquad (\text{B}\cdot 34)$$

Equations (B·30)~(B·34) reproduce the covariant form of the Bianchi identities (B·24) and the field equations (B·26) and therefore imply the conservation of the energy momentum tensor (B·28).

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