Generalized Mehler semigroups and applications

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Abstract. We construct and study generalized Mehler semigroups $(p_t)_{t\geq 0}$ and their associated Markov processes **M**. The construction methods for $(p_t)_{t\geq 0}$ are based on some new purely functional analytic results implying, in particular, that any strongly continuous semigroup on a Hilbert space H can be extended to some larger Hilbert space E, with the embedding $H \subset E$ being Hilbert-Schmidt. The same analytic extension results are applied to construct strong solutions to stochastic differential equations of type $dX_t = CdW_t + AX_t dt$ (with possibly unbounded linear operators A and C on H) on a suitably chosen larger space E. For Gaussian generalized Mehler semigroups $(p_t)_{t\geq 0}$ with corresponding Markov process **M**, the associated (non-symmetric) Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ are explicitly calculated and a necessary and sufficient condition for path regularity of **M** in terms of $(\mathcal{E}, D(\mathcal{E}))$ is proved. Then, using Dirichlet form methods it is shown that **M** weakly solves the above stochastic differential equation if the state space E is chosen appropriately. Finally, we discuss the differences between these two methods yielding strong resp. weak solutions.

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Contents.

0. Introduction.

The two main objectives of this paper are the study and construction of *generalized Mehler semigroups* $(p_t)_{t\geq 0}$, and the solution of stochastic differential equations of type

$$dX_t = AX_t dt + CdW_t, \tag{0.1}$$

on infinite dimensional spaces. Here A and C are (in general) unbounded linear operators on a separable Hilbert space H, W_t is a cylindrical Wiener process in H, and we always assume that A generates a strongly continuous semigroup on H. As will be seen, and is essentially known, solutions of (0.1) are connected with a special class of generalized Mehler semigroups, namely Gaussian ones.

Generalized Mehler semigroups $(p_t)_{t\geq 0}$ on a separable Banach space E are defined for bounded, Borel measurable functions $f : E \to I\!R$, by the formula

$$p_t f(x) = \int f(T_t x - y)\mu_t(dy) = (\mu_t * f)(T_t x) \quad t \geq 0. \tag{0.2}$$

Here $(T_t)_{t\geq 0}$ is a strongly continuous semigroup on E and μ_t, $t \geq 0$, are probability measures such that

$$\mu_{t+s} = \left(\mu_t \circ T_s^{-1}\right) * \mu_s \quad \text{ for all } s, t \geq 0. \tag{0.3}$$

Then $(p_t)_{t\geq 0}$ is always a Feller semigroup (i.e., $p_t(C_b(E)) \subset C_b(E)$ for all $t \geq 0$). In Section 2 below they are treated in detail. In particular, construction methods are described (cf. Lemma 2.6) and a class of *non-Gaussian* examples is given (cf. Example 2.7).

The construction methods for $(p_t)_{t\geq 0}$ are based on some purely functional analytic results on extensions of strongly continuous semigroups presented in Section 1. More precisely, we show that for a strongly continuous semigroup $(T_t)_{t\geq 0}$ on a Hilbert space H there exists a larger separable Hilbert space E such that the embedding $H \subset E$ is Hilbert-Schmidt, and $(T_t)_{t\geq 0}$ extends (uniquely) to a strongly continuous semigroup on E (see Corollary 1.4). Various extensions of this result (e.g. that E can be chosen in common for finitely many commuting semigroups), which are particularly important in connection with constructing solutions for (0.1), are also discussed (cf. Theorems 1.6, 1.8). These functional analytic results are of their own interest.

In Section 3 we consider the Markov processes on E whose transition functions are given by generalized Mehler semigroups. They always exist by the classical Kolmogorov theorem, but might (even in the Gaussian case) have no sample path regularity whatsoever (cf. Remark 3.1 below). However, generalizing a result in [BR 95], we show that if the Markov process started with one probability measure on E has continuous (resp. cadlag resp. right continuous) sample paths, then it has the same sample path regularity when started with any other probability measure on E (cf. Theorem 3.2). In Section 4 we study the most general type of Gaussian Mehler semigroup on a Hilbert space E. It turns out that in the Gaussian case, (0.3) implies that $(t, x) \to p_t f(x)$ is continuous on $(0, \infty) \times E$ for all $f \in C_b(E)$, in particular,

$$\lim_{t\to 0} p_t f(x) = f(x) \quad \text{ for all } x \in E.$$

A way to obtain examples for $(p_t)_{t\geq 0}$ is described in Remark 4.5 (ii). We also consider invariant measures for $(p_t)_{t\geq 0}$ in this case. However, the corresponding results are more or less well-known (cf. [DPZ 92]), but they read a little differently in our framework.

We emphasize that throughout this paper our viewpoint with respect to both the construction of $(p_t)_{t\geq 0}$ and the solution of (0.1) is the following: we think of the quantities involved, i.e., operators, processes, and measures, as being given on a separable Hilbert space H as operators, cylindrical processes, and cylindrical measures. Our task is to construct a larger state space E such that all quantities have natural extensions to E so that (0.2) makes sense and (in the strongest possible sense) also (0.1).

Equations (0.1) and their non-linear perturbations have been studied extensively (see [Ro 90], [DPZ 92], and the references therein; for the linear case see also [MS 92]). It turns out that because the linear operators in our infinite dimensional setting are unbounded and are only defined on subdomains of the Hilbert space, a substantial part of the work is "to give the best possible sense" to the linear equation (0.1). Then a variety of well-understood (but nevertheless sophisticated) methods lead to solutions for the non-linear ones. However, in most of the literature the authors try to find solutions in an a priori chosen state space E, which if this space is too small, might require restrictive conditions on the operators A, C. Our approach is to first construct an appropriate sufficiently big state space E on which (0.1) can be solved (uniquely) in the strongest possible sense, then try to solve the perturbed non-linear equation and (simultaneously) try to determine a smaller natural sub-manifold of E which carries the process.

Step 1 of this programme is carried out in Section 5 of this paper (cf. Theorem 5.1) on the basis of the analytic extension results in Section 1. We emphasize that we do not assume the diffusion operator C to be bounded on H, but only need that it can be represented as the composition of a bounded operator and the generator of a strongly continuous semigroup on H, that commutes with the semigroup generated by A. As is well-known (and in this case particularly easy to see), the transition probabilities of the *strong* solutions of (0.1) on the enlarged state space E are then given by Gaussian Mehler semigroups. They are determined explicitly in Proposition 5.3.

Sections 6 and 7 are devoted to a different approach to (0.1), namely to construct weak solutions via Dirichlet forms using the method developed in [AR 91]. In contrast to [AR 91] we treat here only linear cases, but these are much more general, in particular non-symmetric. Moreover, the solutions are constructed on a state space E, so that their transition probabilities are given by Gaussian Mehler semigroups. In particular, they are Feller.

The starting point in Section 6 is a general Gaussian Mehler semigroup $(p_t)_{t\geq 0}$ which has an invariant measure μ on one of the enlarged spaces E constructed in Section 1. We first calculate the $L^2(E;\mu)$-generator L of $(p_t)_{t\geq 0}$ and the corresponding positive definite bilinear form $\mathcal{E}$. Assuming that the sector condition is fulfilled (cf. Condition 6.2) we prove (Theorem 6.3) that its closure $(\mathcal{E}, D(\mathcal{E}))$ is a local (non-symmetric) Dirichlet form on $L^2(E;\mu)$ (cf. [MR 92]). We characterize when $(\mathcal{E}, D(\mathcal{E}))$ is symmetric (cf. Remark 6.4). We point out that the symmetry of $(\mathcal{E}, D(\mathcal{E}))$ depends on symmetry properties of the involved operators on H and *not* of their extensions to the state space E.

In Section 7 we prove that $(\mathcal{E}, D(\mathcal{E}))$ is *quasi-regular* if and only if the Markov process

$\mathbf{M}$ associated with $(p_t)_{t\geq 0}$ (constructed in Section 3) has cadlag resp. continuous sample paths. We also prove a sufficient condition for $(\mathcal{E}, D(\mathcal{E}))$ to be quasi-regular and identify an enlarging space E constructed in Section 1 for which this is always the case. Subsequently, we prove that $\mathbf{M}$ weakly solves (0.1) on this particular state space E. We emphasize that the diffusion coefficient C may not be an operator of the type as in Section 5 and that this solution of (0.1) may not be a strong one (see the final discussion in Remark 7.11 about the connection with the results in Section 5).

The idea of extending equations to larger spaces E is quite standard. In many concrete situations, say where A is a differential operator on L^2, an extension may be obvious with some Sobolev space of negative order for E. Also if E is not required to be a Banach or Hilbert space; or if A is self-adjoint on H with discrete spectrum the extension results in Section 1 are entirely trivial. However, in more general situations, even for self-adjoint A, the existence of suitable extensions on Banach or even Hilbert spaces as above is not trivial. To the best of our knowledge [R 88a,b] seem to be the first papers in this direction. In [BR 95] a complete solution to the case with an arbitrary self-adjoint A was presented. In fact [BR 95] together with [S 93] were the starting point of this work. A substantial part of the results of this paper are extensions of results in [BR 95], [S 93].

Finally, we would like to note one technical point which might appear odd to the reader. Since we mostly work with two different Hilbert spaces H and E with $H \subset E$, to avoid confusion we identify neither H nor E with its dual.

1. Functional-analytic preliminaries: Extensions of semigroups.

For a Banach space E, let $\mathcal{L}(E)$ denote the set of all bounded linear operators on E with the corresponding norm $\|\cdot\|_{\mathcal{L}(E)}$. Also, we let $\mathcal{B}(E)$ denote the σ-algebra of Borel sets in E.

Let H be a separable Hilbert space. A set $Q \subset H$ is called a *(nondegenerate) Hilbert-Schmidt ellipsoid*, if there exists a Hilbert-Schmidt operator T on H with dense range such that $Q = T(U_H)$, where U_H is the unit ball in H, i.e., $U_H = \{h \in H \mid \|h\|_H \leq 1\}$.

Remark 1.1. (i) We note that the operator T above can be assumed to be injective, self-adjoint, and nonnegative. Indeed, to get an injective Hilbert-Schmidt operator T_1 with $T_1(U_H) = T(U_H)$ we define $\widetilde{T}$ to be the restriction of T to $\widetilde{H} := (\ker T)^{\perp}$. Clearly, $\widetilde{T}$ is an injective Hilbert-Schmidt operator with the same range as T. It follows that $\widetilde{H}$ is infinite dimensional, and since H is separable there exists a unitary map $\Lambda\colon H \to \tilde{H}$. We then can take $T_1 := \widetilde{T}\Lambda$. Assuming that T is already injective, let $T = T_0 J$ be the polar decomposition of T, that is, J is a partial isometry, and T_0 is self-adjoint and nonnegative. Since T has dense range, it follows that J is onto and so $J(U_H) = U_H$ and T_0 is injective. Thus $T_0(U_H) = T_0(J(U_H)) = T(U_H)$ where T_0 is injective, self-adjoint, and non-negative.

(ii) Clearly, $Q \subset H$ is a Hilbert-Schmidt ellipsoid if and only if $Q = T(U_{H_1})$ for some Hilbert space H_1 and a Hilbert-Schmidt operator $T : H_1 \to H$ with dense range.

(iii) Note that, of course, every Hilbert-Schmidt ellipsoid is compact in H.

(iv) If T is injective, then the space $H_0 = T(H)$ possesses a natural Hilbert space structure given by

$$(x,y)_{H_0} = (T^{-1}x, T^{-1}y)_H, \quad x, y \in H_0. \tag{1.1}$$

The set $Q := T(U_H)$ is the unit ball in $(H_0, \|\cdot\|_{H_0})$ and $\|\cdot\|_H \leq a\|\cdot\|_{H_0}$, with $a = \|T\|_{\mathcal{L}(H)}$.

Lemma 1.2. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup in a separable Hilbert space H, such that $\|T_t\|_{\mathcal{L}(H)} < 1$ for some $t \geq 0$. Then there exists a Hilbert-Schmidt ellipsoid Q so that $T_t(Q) \subseteq Q$ for all $t \geq 0$. Moreover, if $T_t^1, \ldots, T_t^m$ are commuting semigroups, satisfying the condition above, then Q can be chosen common for all of them.*

Proof. We note that the assumption $\|T_t\|_{\mathcal{L}(H)} < 1$ for some $t \geq 0$, implies that there exist positive constants b, c so that $\|T_t\|_{\mathcal{L}(H)} \leq be^{-ct}$ for all $t \geq 0$.

Let H_0 be a Hilbert space in H corresponding to any nondegenerate Hilbert-Schmidt ellipsoid, as in Remark 1.1 (iv), and a the corresponding constant. Let $X := L^2([0,\infty) \to H_0; ds)$, be the L^2-space, with respect to Lebesgue measure, of H_0-valued functions on $[0,\infty)$. Define an operator $T\colon X \to H$ by $Tx := \int_0^\infty T_s x(s)\, ds$. This integral is well-defined since $s \to T_s x(s)$ is weakly measurable, and hence strongly measurable, and for every $x \in X$

$$\int_0^\infty \|T_s x(s)\|_H \, ds \leq ba \int_0^\infty \|x(s)\|_{H_0} \, e^{-cs} \, ds \leq ba \Big(\int_0^\infty e^{-2cs} \, ds \Big)^{1/2} \|x\|_X. \tag{1.2}$$

In particular, the operator T is bounded.

We claim that T is a Hilbert-Schmidt operator with dense range, and that $T_t(T(U_X)) \subseteq T(U_X)$ for every $t \geq 0$. If we can prove these statements, then the result follows by letting $Q = T(U_X)$.

First we prove that T is a Hilbert-Schmidt operator. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis in H_0. For m of the form $m = 2^q$ consider the orthonormal system in X defined by

$$e_{i,k}^m(s) = \begin{cases} \sqrt{m}\, e_k, & \text{if } \frac{i}{m} \leq s \leq \frac{i+1}{m}, \\ 0, & \text{otherwise.} \end{cases} \quad i, k \in \mathbb{N} \tag{1.3}$$

Denote by X_m the closed subspace generated by $(e_{i,k}^m)_{i,k\in\mathbb{N}}$.

This is an increasing sequence of subspaces with union dense in X. So for any vector $x \in X$, the sequence $P_m x$ of the projections of x onto X_m converges to x; in particular, $TP_m x \to Tx$. Hence by Fatou's lemma it suffices to prove that the operators TP_m have uniformly bounded Hilbert-Schmidt norms. We may now estimate the Hilbert-Schmidt norm of TP_m directly:

$$
\begin{aligned}
\sum_{i,k\in \mathbb{N}} \|TP_m e^m_{i,k}\|_H^2 &= \sum_{i,k\in \mathbb{N}} \|Te^m_{i,k}\|_H^2 \\
&= \sum_{i,k\in \mathbb{N}} \left\| \int\limits_{i/m}^{(i+1)/m} \sqrt{m}\, T_s e_k \, ds \right\|_H^2 \\
&\le \sum_{i,k\in \mathbb{N}} m\, \|e_k\|_H^2 \left(\int\limits_{i/m}^{(i+1)/m} \|T_s\|_{\mathcal{L}(H)} \, ds \right)^2 \\
&\le \sum_{i,k\in \mathbb{N}} m\, \|e_k\|_H^2 (b/m) \left(\int\limits_{i/m}^{(i+1)/m} \|T_s\|_{\mathcal{L}(H)} \; , ds \right) \\
&\le \alpha^2 b \int_0^\infty \|T_s\|_{\mathcal{L}(H)} \, ds,
\end{aligned} \tag{1.4}
$$

where α is the Hilbert-Schmidt norm of the natural embedding $H_0 \to H$. The left hand side of this estimate is precisely the Hilbert-Schmidt norm of TP_m, as TP_m is zero on the orthogonal complement of X_m and the Hilbert-Schmidt norm does not depend upon the choice of an orthogonal basis. This proves that T is Hilbert-Schmidt.

For any $h \in H_0$, the sequence $T(n1_{[0,1/n]}h) = n\int_0^{1/n} T_s h \, ds$ converges to h in H-norm as $n \to \infty$. This shows that the range of T is dense in H_0, and hence also in H.

Finally, we see that $T_t(Tx) = T(x_t)$ where

$$
x_t(s) = \begin{cases} x(s-t); & \text{if } s \ge t, \\ 0; & \text{otherwise.} \end{cases} \tag{1.5}
$$

If $x \in U_X$, then $x_t \in U_X$ and so it follows that $T_t(T(U_X)) \subseteq T(U_X)$.

The case with finitely many commuting semigroups is handled similarly, with X being a space of H_0-valued functions on $[\,0,\infty)^m$ and the operator T given by

$$
Tx = \int_0^\infty \cdots \int_0^\infty T^1_{s_1} \dots T^m_{s_m} x(s_1,\dots,s_m)\, ds_1 \dots ds_m. \qquad \square \tag{1.6}
$$

Theorem 1.3. *Let $(T_t)_{t\ge 0}$ be a continuous semigroup on a separable Hilbert space H. Assume that $\|T_t\|_{\mathcal{L}(H)} < 1$, for some $t \ge 0$. Then H can be linearly and continuously embedded into a Hilbert space E such that H is dense in E, the embedding is Hilbert-Schmidt, and so that $(T_t)_{t\ge 0}$ admits an extension to a strongly continuous contraction semigroup $(T^E_t)_{t\ge 0}$ on E.*

Moreover, finitely many commuting semigroups satisfying the condition above can be extended simultaneously on such a space.

Remark. In the symmetric case, Bogachev and Röckner [BR 95] have proved this extension result where $(T_t)_{t\geq 0}$ was only assumed to be a strongly continuous contraction semigroup without the exponential decay of $\|T_t\|_{\mathcal{L}(H)}$.

Proof. Since $\|T_t^*\|_{\mathcal{L}(H^*)} = \|T_t\|_{\mathcal{L}(H)}$ we can consider the Hilbert-Schmidt ellipsoid Q obtained by applying Lemma 1.2 to the dual semigroup $(T_t^*)_{t\geq 0}$. For $x \in H$, define $p(x) := \sup_{q\in Q} {}_{H^*}\langle q, x\rangle_H$. Since Q is absolutely convex with dense linear span, it follows that p is a norm on H. Let E be the completion of H with respect to the norm p. If T^* is a self-adjoint Hilbert-Schmidt operator on H^* that maps the unit ball U_{H^*} onto Q, then for all $x \in H$,

$$p(x) = \sup_{q\in Q} {}_{H^*}\langle q, x\rangle_H = \sup_{u\in U_{H^*}} {}_{H^*}\langle T^*u, x\rangle_H = \sup_{u\in U_{H^*}} {}_{H^*}\langle u, Tx\rangle_H = \|Tx\|_H. \qquad (1.7)$$

It follows that E is a Hilbert space and that the natural embedding $H \subset E$ is Hilbert-Schmidt.

For each $t \geq 0$ we have $T_t^*Q \subset Q$, so for all $x \in H$

$$p(T_tx) = \sup_{q\in Q} {}_{H^*}\langle q, T_tx\rangle_H = \sup_{q\in Q} {}_{H^*}\langle T_t^*q, x\rangle_H \leq \sup_{q\in Q} {}_{H^*}\langle q, x\rangle_H = p(x), \qquad (1.8)$$

from which it follows that T_t can be extended to a linear operator T_t^E on E with $\|T_t^E\|_{\mathcal{L}(E)} \leq 1$. The strong continuity of $(T_t^E)_{t\geq 0}$ on E follows from the strong continuity of $(T_t)_{t\geq 0}$ on H, the density of H in E, and the fact that $\|T_t^E\|_{\mathcal{L}(E)} \leq 1$ for all $t \geq 0$. In the case of finitely many commuting semigroups, (1.8) holds for all of them by Lemma 1.2. □

Corollary 1.4. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a separable Hilbert space H. Then H can be linearly and continuously embedded into a Hilbert space E such that H is dense in E, the embedding is Hilbert-Schmidt, and so that $(T_t)_{t\geq 0}$ admits an extension to a strongly continuous semigroup $(T_t^E)_{t\geq 0}$ on E. If $T_t^1, \ldots, T_t^m$ are strongly continuous commuting semigroups on H, then the space E as above can be chosen common for all of them.*

Proof. Since T_t is strongly continuous on H, $\|T_t\|_{\mathcal{L}(H)} \leq be^{ct}$ for some $c > 0$. We can apply Theorem 1.3 to the semigroup $\widetilde{T}_t = e^{-2ct}T_t$. Now we can take its extension and multiply it by e^{2ct}. In the case of finitely many semigroups the proof is the same. □

Remark 1.5. In Corollary 1.4, we have $H \subset E$, and the semigroup $(T_t)_{t\geq 0}$ extends to a strongly continuous semigroup $(T_t^E)_{t\geq 0}$ on E. Letting $(A, D(A))$ resp. $(A^E, D(A^E))$ be the generator of $(T_t)_{t\geq 0}$ resp. $(T_t^E)_{t\geq 0}$ on E, it is not hard to show that

$$D(A) \subset D(A^E) \quad \text{and} \quad A^E = A \text{ on } D(A), \qquad (1.9)$$

since $\|\cdot\|_E \leq$ constant $\|\cdot\|_H$ on H. On the dual spaces, everything is reversed. That is, $E^* \subset H^*$ and if $(T_t^*)_{t\geq 0}$ and $((T_t^E)^*)_{t\geq 0}$ are the dual semigroups on H^* and E^* respectively, then $(T_t^E)^*$ is just the restriction of the operator T_t^* to the subspace E^*. In addition, the generators satisfy $D((A^E)^*) \subset D(A^*)$ and $(A^E)^* = A^*$ on $D((A^E)^*)$.

In Sections 5 and 7 (cf. Theorems 5.1, 5.2, and Proposition 7.5 below) we shall need the following stronger version of Corollary 1.4.

Theorem 1.6. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ on a separable Hilbert space H. Then H can be linearly and continuously embedded into a Hilbert space E such that H is dense in E, $(T_t)_{t\geq 0}$ admits an extension to a strongly continuous semigroup $(T_t^E)_{t\geq 0}$ on E, and so that H embeds into $D(A^E)$ (equipped with the graph norm) with a Hilbert-Schmidt map.*

Proof. For any fixed $t \in (0, \infty)$, there exists $c \in (0, \infty)$ so that for $\tilde{T}_t := e^{-ct} T_t$, we have $\|\tilde{T}_t\|_{\mathcal{L}(H)} =: \gamma < 1$. Then also $\|\tilde{T}_t^*\|_{\mathcal{L}(H^*)} = \gamma < 1$.

Consider the Hilbert space $D(A^*)$ equipped with the graph norm $\|h\|_{D(A^*)} = (\|h\|^2_{H^*} + \|A^*h\|^2_{H^*})^{1/2}$. One can easily check that $(T_t^*)_{t\geq 0}$ is a strongly continuous semigroup on $D(A^*)$. In addition, for the same value of t as above,

$$\begin{aligned}\|T_t^* h\|^2_{D(A^*)} &= \|T_t^* h\|^2_{H^*} + \|A^* T_t^* h\|^2_{H^*} = \|T_t^* h\|^2_{H^*} + \|T_t^* A^* h\|^2_{H^*} \\ &\leq e^{2ct}\gamma^2 \|h\|^2_{H^*} + e^{2ct}\gamma^2 \|A^* h\|^2_{H^*} = e^{2ct}\gamma^2 \|h\|^2_{D(A^*)}.\end{aligned} \tag{1.10}$$

Therefore we can apply Lemma 1.2 to $(\tilde{T}_t^*)_{t\geq 0}$ on $D(A^*)$ and let Q be a nondegenerate Hilbert-Schmidt ellipsoid in $D(A^*)$ such that $\tilde{T}_t^*(Q) \subseteq Q$ for every $t \geq 0$. Since $A^*\colon D(A^*) \to H^*$ is continuous it follows that $A^*(Q)$ is a (possibly degenerate) Hilbert-Schmidt ellipsoid in H^*.

Since Q is also a Hilbert-Schmidt ellipsoid in H^*, as in the proof of Theorem 1.3, we can now define E to be the completion of H with respect to the following Hilbertian norm,

$$\|x\|_E := \sup_{q\in Q} {}_{H^*}\langle q, x\rangle_H, \quad x \in H. \tag{1.11}$$

As in Corollary 1.4, the semigroup $(T_t)_{t\geq 0}$ extends to a strongly continuous semigroup $(T_t^E)_{t\geq 0}$ on E. Let $(A^E, D(A^E))$ be the generator of $(T_t^E)_{t\geq 0}$ on E. Using (1.9), we can evaluate the graph norm of A^E for $x \in D(A)$,

$$\begin{aligned}\|x\|^2_{D(A^E)} &= \|A^E x\|^2_E + \|x\|^2_E \\ &= \sup_{q\in Q} {}_{H^*}\langle q, Ax\rangle^2{}_H + \sup_{q\in Q} {}_{H^*}\langle q, x\rangle^2{}_H \\ &= \sup_{p\in A^*(Q)} {}_{H^*}\langle p, x\rangle^2{}_H + \sup_{q\in Q} {}_{H^*}\langle q, x\rangle^2{}_H.\end{aligned} \tag{1.12}$$

Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis for H sitting in $D(A)$. Since both Q and $A^*(Q)$ are Hilbert-Schmidt ellipsoids in H^*, from (1.12) and (1.7) we see that $\sum_{k=1}^{\infty} \|e_k\|^2_{D(A^E)} < \infty$. This shows that the injection $(D(A), \|\cdot\|_H) \to (D(A^E), \|\cdot\|_{D(A^E)})$ is Hilbert-Schmidt and hence extends to a unique Hilbert-Schmidt map from H into $D(A^E)$. Using (1.12) and the fact that Q is non-degenerate shows that this extension is one-to-one, i.e., it is an embedding. □

In Section 5 below we need the following stronger results, involving a second strongly continuous semigroup.

Theorem 1.7. *Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be two commuting strongly continuous semigroups on a separable Hilbert space H with generators $(A, D(A))$ and $(G, D(G))$. Then*

(i) $T_t(D(G)) \subset D(G)$ and T_t commutes with G on $D(G)$ for all $t \geq 0$.

(ii) The domain

$$D_{A,G} = \{h \in D(A) \cap D(G) \mid Ah \in D(G), Gh \in D(A)\} \tag{1.13}$$

is dense in H and $AG = GA$ on $D_{A,G}$. In addition, $D_{A,G}$ is complete with the norm $\|h\|_{A,G} := (\|h\|_H^2 + \|Ah\|_H^2 + \|Gh\|_H^2 + \|AGh\|_H^2 + \|GAh\|_H^2)^{1/2}$.

Proof. (i) Let $h \in D(G)$ and $t > 0$. Then the map $\tau \mapsto S_\tau h$ is differentiable and so is the map $\tau \mapsto T_t S_\tau h = S_\tau T_t h$. Hence $T_t h \in D(G)$ and $T_t G h = G T_t h$.

(ii) If $h \in D_{A,G}$, then $AGh = GAh$. Indeed, for $t > 0$,

$$G\frac{T_t h - h}{t} = \frac{1}{t} G \int_0^t T_s A h \, ds = \frac{1}{t} \int_0^t T_s G A h \, ds, \tag{1.14}$$

which converges to GAh as t goes to zero. On the other hand, the derivative of $t \mapsto T_t G h$ at zero equals AGh. Furthermore, note that vectors of the form

$$z = \int_0^t \int_0^s T_r S_\tau h \, dr d\tau, \quad h \in H, \tag{1.15}$$

belong to $D_{A,G}$. Indeed, they are clearly in $D(A) \cap D(G)$ and (see [P 83; Theorem 2.4]),

$$\begin{aligned} A\Big(\int_0^t \int_0^s T_r S_\tau h \, dr d\tau\Big) &= A\Big(\int_0^s T_r \big(\int_0^t S_\tau h \, d\tau\big) \, dr\Big) \\ &= T_s\big(\int_0^t S_\tau h \, d\tau\big) - \int_0^t S_\tau h \, d\tau \\ &= \int_0^t S_\tau T_s h \, d\tau - \int_0^t S_\tau h \, d\tau, \end{aligned} \tag{1.16}$$

which belongs to the domain of G. Similarly, $Gz \in D(A)$. The collection of vectors given by (1.15) span a dense subspace in H, since if some h_0 is orthogonal to all of them, then differentiating the scalar product of h_0 with such a vector in s and in t at zero we get $(h_0, h)_H = 0$ for all $h \in H$. This implies that $h_0 = 0$.

Finally, to prove that $D_{A,G}$ is complete, assume that $(h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the norm above. Then the sequences $(h_n)_{n\in\mathbb{N}}$, $(Ah_n)_{n\in\mathbb{N}}$, $(Gh_n)_{n\in\mathbb{N}}$, $(GAh_n)_{n\in\mathbb{N}}$, $(AGh_n)_{n\in\mathbb{N}}$ are fundamental in H. Since both A and G are closed operators, this implies that there exists $h = \lim_{n\to\infty} h_n$ such that $h \in D(A) \cap D(G)$ and $Ah = \lim_{n\to\infty} Ah_n$, $Gh = \lim_{n\to\infty} Gh_n$. For the same reason, $Gh \in D(A)$, $AGh = \lim_{n\to\infty} AGh_n$, and $Ah \in D(G)$, $GAh = \lim_{n\to\infty} GAh_n$ and the proof is complete. □

Theorem 1.8. *Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be two commuting strongly continuous semigroups on a separable Hilbert space H with generators $(A, D(A))$ and $(G, D(G))$. Then H can be linearly and continuously embedded into a Hilbert space E such that H is dense in E, $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ admit extensions to strongly continuous commuting semigroups $(T_t^E)_{t\geq 0}$ and $(S_t^E)_{t\geq 0}$ on E, and so that H embeds into D_{A^E,G^E} (equipped with $\| \ \|_{A^E,G^E}$) with a Hilbert-Schmidt map.*

Proof. Using the last part of Lemma 1.2 and Lemma 1.7, the proof is done in exactly the same way as that of Theorem 1.6 replacing the role of $D(A)$ and $D(A^*)$ (with the graph norm) by $D_{A,G}$ resp. D_{A^*,G^*} (with $\| \ \|_{A,G}$ resp. $\| \ \|_{A^*,G^*}$). We leave the details to the reader. □

Remark 1.9. (i) Let $(T_t)_{t\geq 0}$ on H be as in Corollary 1.4 and let E_0 be a Hilbert space into which H is densely embedded by a Hilbert–Schmidt operator. Then the space E described in Corollary 1.4 can be chosen in such a way that E_0 is continuously embedded into E. Indeed, it suffices to take the initial ellipsoid Q_0 in the proof of Lemma 1.2 so that it is a Hilbert–Schmidt ellipsoid in the Hilbert space for which the polar of U_{E_0} is the unit ball.

(ii) Note that both Theorem 1.3 and Theorem 1.6 admit iterating (e.g., one can choose E in such a way that the initial space H is embedded into E with a nuclear operator).

2. Generalized Mehler semigroups.

Let E be a separable Banach space and $(T_t)_{t\geq 0}$ a strongly continuous semigroup on E. Given a family $(\mu_t)_{t\geq 0}$ of probability measures on $\mathcal{B}(E)$ we define for $f \in \mathcal{B}_b(E)$ (:= the set of all bounded $\mathcal{B}(E)$-measurable functions on E), $t \geq 0$

$$p_t f(x) := \int f(T_t x - y)\mu_t(dy) = (\mu_t * f)(T_t x), \quad x \in E. \tag{2.1}$$

Note that for each $t \geq 0$, p_t is *Feller* (i.e., p_t maps $C_b(E)$ into itself, where $C_b(E)$ is the space of bounded, continuous functions on E).

Lemma 2.1. *Assume that the map $t \mapsto \mu_t$ on $[0, \infty)$ is continuous in the weak topology. Then $(t, x) \mapsto p_t f(x)$ is continuous on $[0, \infty) \times E$ for all $f \in C_b(E)$.*

Proof. Fix $\epsilon > 0$, and suppose that $(t_n, x_n) \to (t, x)$ in $[0, \infty) \times E$. By the Prohorov theorem, there is a compact K with

$$\mu_s(E\backslash K) < \epsilon \ \text{ for all } s \in \{t, t_1, \ldots, t_n, \ldots\}. \tag{2.2}$$

By the strong continuity of the semigroup $(T_t)_{t\geq 0}$, the set $S = \{T_t x\} \cup \{T_{t_n} x_n | \ n \in I\!N\}$ is compact. Hence $S - K$ is compact, and since f is uniformly continuous on compacts, there exists $N \in I\!N$ such that for any $n > N$ and any $y \in K$

$$|f(T_{t_n} x_n - y) - f(T_t x - y)| \leq \epsilon. \tag{2.3}$$

Set $C = \|f\|_\infty$, and note that by increasing this N if necessary, we also have that for any $n > N$

$$\left| \int_K f(T_t x - y)\, \mu_t(dy) - \int_K f(T_t x - y)\, \mu_{t_n}(dy) \right| \le (2C+1)\epsilon, \tag{2.4}$$

since by the weak continuity $\int_E f(T_t x - y)\, \mu_{t_n}(dy) \to \int_E f(T_t x - y)\, \mu_t(dy)$ as $n \to \infty$. Finally we get for all $n > N$

$$\begin{aligned} &\left| \int_E f(T_t x - y)\, \mu_t(dy) - \int_E f(T_{t_n} x_n - y)\, \mu_{t_n}(dy) \right| \\ &\le \left| \int_K f(T_t x - y)\, \mu_t(dy) - \int_K f(T_{t_n} x_n - y)\, \mu_{t_n}(dy) \right| + 2C\epsilon \\ &\le \left| \int_K f(T_t x - y)\, \mu_t(dy) - \int_K f(T_t x - y)\, \mu_{t_n}(dy) \right| \\ &\quad + \int_K \big| f(T_{t_n} x_n - y) - f(T_t x - y) \big|\, \mu_{t_n}(dy) + 2C\epsilon \\ &\le (2C+1)\epsilon + \epsilon + 2C\epsilon = (4C+2)\epsilon. \end{aligned} \tag{2.5}$$

Since ϵ is arbitrary, this proves the result. □

Now we turn to the question when $(p_t)_{t\ge0}$ in (2.1) defines a semigroup. To this end, for a probability measure ν on $\mathcal{B}(E)$, (or just a cylindrical probability measure on E), we denote its Fourier transform by $\tilde{\nu}$, i.e.,

$$\tilde{\nu}(l) := \int_E e^{il}\, d\nu, \quad l \in E^*, \tag{2.6}$$

where E^* denotes the topological dual of E.

The following gives a characterization of the semigroup property of $(p_t)_{t\ge0}$.

Proposition 2.2. *$(p_t)_{t\ge0}$, as in (2.1), is a semigroup on $\mathcal{B}_b(E)$ if and only if for all $t, s \ge 0$*

$$\mu_{t+s} = \left(\mu_t \circ T_s^{-1}\right) * \mu_s, \tag{2.7}$$

where $\mu_t \circ T_s^{-1}$ is the image measure of μ_t under T_s. (2.7) is equivalent to

$$\widetilde{\mu}_{t+s}(l) = \tilde{\mu}_s(l)\, \tilde{\mu}_t(T_s^* l) \quad \textit{for all } l \in E^*. \tag{2.8}$$

Proof. Let $f \in \mathcal{B}_b(E)$ and $t,\ s \ge 0$. Then for all $x \in E$:

$$\begin{aligned} p_t(p_s f)(x) &= (\mu_t * p_s f)(T_t x) \\ &= (\mu_t * (\mu_s * f)(T_s \cdot))(T_t x) \\ &= (((\mu_t \circ T_s^{-1}) * \mu_s) * f)(T_{t+s} x), \end{aligned} \tag{2.9}$$

while

$$p_{t+s} f(x) = (\mu_{t+s} * f)(T_{t+s} x). \tag{2.10}$$

Comparing these two expressions the assertion follows (where for the converse one applies both to functions $f = e^{il},\ l \in E^*$). $\square$

Remark 2.3. Note that the validity of (2.8) for $t = s = 0$ implies that $\tilde{\mu}_0 = \tilde{\mu}_0^2$, hence by continuity, and since $\tilde{\mu}_0$ is equal to 1 at 0, we conclude that $\tilde{\mu}_0 \equiv 1$.

Definition 2.4. If $(T_t)_{t\geq 0}$, $(\mu_t)_{t\geq 0}$ satisfy (2.7), we call $(p_t)_{t\geq 0}$ defined in (2.1) a *generalized Mehler semigroup.*

In section 4 we shall see that the classical Gaussian Mehler semigroups (cf. e.g. [BR 95]) are special cases. Now we want to present a general method to construct examples of generalized Mehler semigroups.

Let E be a Banach space and H a separable Hilbert space such that $H \subset E$ is a continuous linear embedding. We say that a cylindrical measure ν on H admits a countably additive extension to E, if the cylindrical measure ν^E on E defined by the formula $\nu^E(C) = \nu(C \cap H)$, where C is a cylindrical subset of E, is countably additive. Note that for the Fourier transforms the following formula holds:

$$\widetilde{\nu^E}(l) = \widetilde{\nu}(l), \qquad \text{for all } l \in E^* \subset H^*. \tag{2.11}$$

Note also that (2.7) and (2.8) make sense if μ_t, $t > 0$, are merely cylindrical probability measures. We call a cylindrical measure on H *strong* or *strongly cylindrical* if its Fourier transform is continuous on H^*.

Theorem 2.5. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a separable Hilbert space H and let $(\mu_t)_{t\geq 0}$ be a family of strongly cylindrical probability measures on H satisfying (2.7). Let E be a Hilbert space so the properties of Corollary 1.4 are satisfied. Then*

(i) μ, μ_t admit countably additive extensions to E (denoted by μ^E, μ_t^E).

(ii) $(p_t)_{t\geq 0}$, defined as in (2.1) with $(T_t^E)_{t\geq 0}$ and $(\mu_t^E)_{t\geq 0}$, is a generalized Mehler semigroup on E.

(iii) If, in addition, $(t,l) \mapsto \tilde{\mu}_t(l)$ is continuous on $[0,\infty) \times H^*$, then $(\mu_t^E)_{t\geq 0}$ is weakly continuous and Lemma 2.1 applies.

Proof. Since $H \subset E$ is Hilbert-Schmidt, we obtain (i). From (2.11) and the fact that $(T_s^E)^* = T_s^*$ on $E^* \subset H^*$, we see that if (2.8) holds for $(\tilde{\mu}_t)_{t\geq 0}$ and $(T_t^*)_{t\geq 0}$ on H^*, then (2.8) also holds for $(\tilde{\mu}_t^E)_{t\geq 0}$ and $((T_t^E)^*)_{t\geq 0}$ on E^*. Therefore, by Proposition 2.2, (ii) also holds. Finally, assume that $(t,l) \mapsto \tilde{\mu}_t(l)$ is continuous on $[0,\infty) \times H^*$, and let $t_0 \geq 0$ and $t_n \to t_0$ as $n \to \infty$. Suppose that we can prove that $(\mu_{t_n}^E)_{n\in\mathbb{N}}$ is relatively compact in the weak topology, then by the assumption, it is easy to see that $\mu_{t_n}^E \to \mu_{t_0}^E$ weakly as $n \to \infty$. To show relative compactness, by [VTC 87, p. 365] it suffices to check that $(\tilde{\mu}_{t_n}^E)_{n\in\mathbb{N}}$ is equicontinuous at zero with respect to the Sazonov topology on E^* (cf. [VTC 87, p. 363]). Let $\epsilon > 0$. Since $(t,l) \mapsto \tilde{\mu}_t(l)$ is continuous on $[0,\infty) \times H^*$, there exists $\delta > 0$ such that for all $l \in \delta U_{H^*}$,

$$|\tilde{\mu}_{t_n}(l) - 1| \leq \epsilon \quad \text{ for all } n \in \mathbb{N}. \tag{2.12}$$

Let $Q := (\delta U_{H^*}) \cap E^*$. Then for all $l \in Q$,

$$|\tilde{\mu}^E_{t_n}(l) - 1| \le \epsilon \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

It remains to show that Q is a neighborhood of zero in the Sazonov topology on E^*. Note that $l \in Q$ if and only if $l \in E^*$ and $\|l\|_{H^*} \le \delta$. Since U_H is a Hilbert-Schmidt ellipsoid in E, by Remark 1.1 (i), (ii) there exists an injective, self-adjoint, non-negative Hilbert-Schmidt operator T on E such that $U_H = T(U_E)$. Hence for $l \in E^*$,

$$\|l\|_{H^*} = \sup_{h \in U_H} {}_{H^*}\langle l, h\rangle_H = \sup_{h \in U_E} {}_{E^*}\langle l, Th\rangle_E = \sup_{h \in U_E} {}_{E^*}\langle T^* l, h\rangle_E = \|T^* l\|_{E^*}. \tag{2.14}$$

Since T^* is a Hilbert-Schmidt operator on E^*, Q is a neighborhood of zero in the Sazonov topology on E^*. □

The following lemma provides both structural information and a construction method for $(\mu_t)_{t \ge 0}$ satisfying (2.7).

Lemma 2.6. *Let H be a separable real Hilbert space and $(T_t)_{t \ge 0}$ a strongly continuous semigroup on H.*

(i) Let $(\mu_t)_{t\ge 0}$ be a family of cylindrical probability measures on H. Assume that for all $l \in H^*$, $t \mapsto \tilde{\mu}_t(l)$ is differentiable at $t = 0$ and set $\lambda(l) := -\frac{d}{dt}\tilde{\mu}_t(l)|_{t=0}$. Assume also that $t \mapsto \tilde{\mu}_t(l)$ is locally absolutely continuous on $[0, \infty)$ and $s \mapsto \lambda(T_s^* l)$ is locally Lebesgue integrable. Then the following are equivalent:

$$\tilde{\mu}_{t+s}(l) = \tilde{\mu}_s(l)\tilde{\mu}_t(T_s^* l), \qquad \text{for all } l \in H^*,\ t, s \ge 0. \tag{2.15}$$

$$\tilde{\mu}_t(l) = \exp\left(-\int_0^t \lambda(T_s^* l)\, ds\right), \qquad \text{for all } l \in H^*,\ t \ge 0. \tag{2.16}$$

In this case, λ is a negative definite function (cf. [BeF 75]).

(ii) Suppose that $\lambda: H^* \to \mathbb{C}$ is a continuous, negative-definite function with $\lambda(0) = 0$. Then there exist $(\mu_t)_{t\ge 0}$ as in (i), satisfying (2.16) and hence (2.15).

Proof. (i) Suppose that (2.15) holds. Then by Remark 2.3, $\tilde{\mu}_0 \equiv 1$ and for all $l \in H^*$, $s, t \ge 0$ we have

$$\frac{\tilde{\mu}_{s+t}(l) - \tilde{\mu}_s(l)}{t} = \frac{\tilde{\mu}_t(T_s^* l) - 1}{t}\,\tilde{\mu}_s(l), \tag{2.17}$$

hence $\frac{d}{ds}\tilde{\mu}_s(l) = -\lambda(T_s^* l)\tilde{\mu}_s(l)$ and (2.16) follows. Conversely, if (2.16) holds, then for all $l \in H^*$, $s, t \ge 0$ we have

$$\begin{aligned}\tilde{\mu}_{t+s}(l) &= \exp\left(-\int_0^s \lambda(T_u^* l)\, du\right)\exp\left(-\int_s^{s+t} \lambda(T_u^* l)\, du\right) \\ &= \tilde{\mu}_s(l)\,\tilde{\mu}_t(T_s^* l).\end{aligned} \tag{2.18}$$

If (2.16) holds, then λ is negative definite since it is a limit of $(\tilde{\mu}_0 - \tilde{\mu}_t)/t$ as $t \to 0$, as $\tilde{\mu}_0 - \tilde{\mu}_t = 1 - \tilde{\mu}_t = \tilde{\mu}_t(0) - \tilde{\mu}_t$, as seen above, and [BeF 75; Chapter II, Corollary 7.7] applies.

(ii) By [BeF 75; Chapter II, Theorem 7.8] and the assumption, the function on the right hand side of (2.16) is positive definite. Its restrictions to finite dimensional subspaces of H^* are continuous, hence e.g. by [VTC 87; Chapter VI, Proposition 3.2 (c)], the family $(\mu_t)_{t\geq 0}$ as in (i), satisfying (2.16) exists. $\square$

Example 2.7. Let $H := L^2(T, \mathcal{B}, \mu)$ where $(T, \mathcal{B}, \mu)$ is a finite measure space. Let $a: \mathbb{R} \to \mathbb{C}$ be a negative-definite function with $a(0) = 0$. Define

$$\lambda(l) := \int_T a((R_H l)(t))\, \mu(dt), \quad l \in H^*, \tag{2.19}$$

where $R_H: H^* \to H$ denotes the Riesz identification. Then, since by [BeF 75; Chapter II, Corollary 7.16], there exists $c \in (0, \infty)$ such that $|a(s)| \leq c(1 + |s|^2)$ for all $s \in \mathbb{R}$, λ is well-defined, continuous on H^*, and negative-definite. Hence Lemma 2.6 (ii) applies. If we replace H by some Sobolev space of arbitrary order $r \in \mathbb{N}$ in $L^2(T; dx)$ with T an open subset of $\mathbb{R}^d$, and assume $a: \mathbb{R}^n \to \mathbb{C}$ to be continuous and negative-definite, where n is the number of multi-indices of length r, then Lemma 2.6 (ii) also applies for

$$\lambda(l) := \int_T a\big((\partial^\alpha (R_H l))_{|\alpha| \leq r}\big)\, dx, \quad l \in H^*. \tag{2.20}$$

Remark 2.8. (i) In the situation of Lemma 2.6 (i) we can calculate the generator of the associated generalized Mehler semigroup $(p_t)_{t\geq 0}$. It is given by the pseudo-differential operator which can be written as the sum of the pseudo-differential operator with symbol λ and the linear drift given by the generator of $(T_t^*)_{t\geq 0}$ (cf. [BLR 95], [FuhR 95]).

(ii) It is well-known that if μ is a Gaussian measure on a Hilbert space E_0 with Cameron–Martin space H, then every operator $T \in \mathcal{L}(H)$ admits a unique extension to a μ-measurable linear map $\widetilde{T}^{E_0}$ on E_0. In particular, any continuous semigroup (T_t) on H extends uniquely to a semigroup $(\widetilde{T}_t^{E_0})$ of measurable linear mappings on E_0. However, in contrast to our Theorem 1.6, one cannot always take such extensions to be continuous on E_0. What is even more important, the domain of the generator of (T_t^E) on our space E has full μ-measure. By the Hilbert–Schmidt character of the embedding $H \to D(A^E)$ this is true for the countably additive extension ν^E on E of any strong cylindrical measure ν on H. This observation leads to new existence results both for deterministic and stochastic evolution equations. Note that if E is chosen as mentioned in Remark 1.9 for given E_0, then $\widetilde{T}_t^{E_0} = T_t^E$ μ-a.e.

3. Corresponding Markov processes.

In this section we extend the results from [BR 95; Section 4] to generalized Mehler semigroups. Let E be a separable Banach space, $(T_t)_{t\geq 0}$ a strongly continuous semigroup on E, and $(\mu_t)_{t\geq 0}$ a family of probability measures on $\mathcal{B}(E)$. Let $(p_t)_{t\geq 0}$ be as specified in (2.1); where we assume that (2.7) holds, and that $t \mapsto \mu_t$ is continuous on $[0, \infty)$ in the weak topology. By the usual Kolmogorov scheme one can construct a normal Markov process $\mathbf{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ with transition semigroup $(p_t)_{t\geq 0}$. This process, however, is only of interest if one can prove certain regularity properties of its sample paths. Unfortunately, $\mathbf{M}$ is in general not a *right process* (i.e., is strong Markov and has right continuous sample paths) as will be seen in Section 7 below. We intend now to give conditions which imply that the sample paths of $\mathbf{M}$ are even continuous (P_z-a.s. for all $z \in E$). In particular, $\mathbf{M}$ is then *strong Markov* (since $(p_t)_{t\geq 0}$ is Feller), hence a *diffusion*.

Let μ be a fixed probability measure on E. By Kolmogorov's existence theorem there exists a unique probability measure $\tilde{P}_\mu$ on $(E^{[0,\infty)}, \mathcal{A})$ (where $\mathcal{A}$ is the σ-algebra generated by the canonical projections $\tilde{X}_t : E^{[0,\infty)} \to E$) such that for any $n \in I\!N$, $0 < t_1 < \cdots < t_n < \infty$, and $A_0, A_1, \ldots, A_n \in \mathcal{B}(E)$,

$$\begin{aligned}\tilde{P}_\mu[\tilde{X}_0 \in A_0, \tilde{X}_{t_1} \in A_1, \ldots, \tilde{X}_{t_n} \in A_n] \\ = \int_{A_0}\int_{A_1}\cdots\int_{A_n} p_{t_n - t_{n-1}}(z_{n-1}, dz_n)\cdots p_{t_1}(z_0, dz_1)\,\mu(dz_0).\end{aligned} \tag{3.1}$$

Let $\Omega = C([0,\infty), E)$, $X_t := \tilde{X}_t$ on Ω for $t \geq 0$ and let $\mathcal{F} := \sigma(X_t \mid t \in [0,\infty))$. Suppose that the following holds:

There exists a probability measure P in $(\Omega, \mathcal{F})$ having the same finite dimensional distributions as $\tilde{P}_\mu$, i.e., P satisfies (3.1). (3.2)

Remark 3.1. In Section 7 below, we shall prove a necessary and sufficient condition for (3.2) to hold, using the theory of Dirichlet forms (cf. Theorem 7.3). Condition (3.2) is not always true, a counterexample may be found in [BR 95; Example 6.6 (ii)].

Define $Y : \Omega \to \Omega$ and $\mathcal{T} : E \to \Omega$ (componentwise) by

$$Y_t := X_t - T_t X_0, \quad t \geq 0, \tag{3.3}$$

and for $z \in E$,

$$(\mathcal{T}z)_t := T_t z, \quad t \geq 0. \tag{3.4}$$

Clearly, Y is $\mathcal{F}/\mathcal{F}$-measurable and $\mathcal{T}$ is $\mathcal{B}(E)/\mathcal{F}$-measurable. Define for $z \in E$ the probability measure P_z on $(\Omega, \mathcal{F})$ by

$$P_z[F] := (P \circ Y^{-1})[F - \mathcal{T}z], \quad F \in \mathcal{F}. \tag{3.5}$$

Then we have the following result.

Theorem 3.2. *Assume that (3.2) holds and that $(P_z)_{z\in E}$ are as in (3.5). Then for all $z \in E$, and $0 < t_1 < \cdots < t_n < \infty$, $n \in I\!N$, and $A_1, \ldots, A_n \in \mathcal{B}(E)$,*

$$\begin{aligned}P_z[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n] \\ = \int_{A_1}\cdots\int_{A_n} p_{t_n - t_{n-1}}(z_{n-1}, dz_n)\cdots p_{t_1}(z, dz_1).\end{aligned} \tag{3.6}$$

In particular, $\mathbf{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ *is a (conservative) diffusion process (i.e., a conservative, normal, strong Markov process with state space* E *and continuous sample paths) having transition probabilities* $(p_t)_{t\geq 0}$*. Moreover, (3.2) holds with* μ *replaced by any other probability measure* ν *on* E*.*

Proof. Equation (3.6) is proved by calculating the Fourier transform of $P_z \circ (X_{t_1}, \ldots, X_{t_n})^{-1}$ (cf. [R 88b]). Indeed, let $z \in E$, $0 < t_1 < t_2 < \ldots < t_n < \infty$, $n \in \mathbb{N}$, $l_1, \ldots, l_n \in E^*$, and for $x_1, \ldots, x_n \in E$, define

$$f(x_1, \ldots, x_n) := \exp[i \sum_{j=1}^{n} {}_{E^*}\langle l_j, x_j \rangle_E]. \tag{3.7}$$

Obviously, it suffices to show that

$$\int f(X_{t_1}, \ldots, X_{t_n})\, dP_z = \int \cdots \int f(x_1, \ldots, x_n) p_{t_n - t_{n-1}}(x_{n-1}, dx_n) \cdots p_{t_2 - t_1}(x_1, dx_2) p_{t_1}(z, dx_1). \tag{3.8}$$

By definition,

$$\begin{aligned} \int f(X_{t_1}, \ldots, X_{t_n})\, dP_z &= \int f(X_{t_1} - T^E_{t_1} X_0 + T^E_{t_1} z, \ldots, X_{t_n} - T^E_{t_n} X_0 + T^E_{t_n} z)\, dP \\ &= \exp[i \sum_{j=1}^{n} {}_{E^*}\langle l_j, T^E_{t_j} z \rangle_E] \cdot D_\mu, \end{aligned} \tag{3.9}$$

where for a probability measure ν on $\mathcal{B}(E)$ we set

$$D_\nu := \int \nu(dx_0) \exp[i \sum_{j=1}^{n} {}_{E^*}\langle l_j, T^E_{t_j} x_0 \rangle_E] \cdot D(x_0), \tag{3.10}$$

where by induction for $k \in \{1, \ldots, n\}$, if $t_0 := 0 =: t_{-1}$,

$$\begin{aligned} D(x_0) &:= \int \cdots \int p_{t_1}(x_0, dx_1) \ldots p_{t_n - t_{n-1}}(x_{n-1}, dx_n) \exp[i \sum_{j=1}^{n} {}_{E^*}\langle l_j, x_j \rangle_E] \\ &= \int \cdots \int p_{t_1}(x_0, dx_1) \ldots p_{t_{(n-k)} - t_{(n-(k+1))}}(x_{(n-(k+1))}, dx_{(n-k)}) \\ &\qquad \exp[i \sum_{j=0}^{k-1} {}_{E^*}\langle l_{n-j}, T^E_{t_{n-j} - t_{n-k}} x_{n-k} \rangle_E] \cdot c_k \end{aligned} \tag{3.11}$$

with $c_k \in \mathbb{C}$ only depending on $t_n, \ldots, t_{n-k}$, $l_n, \ldots, l_{n-(k+1)}$. Thus for $k = n$,

$$D(x_0) = \exp[i \sum_{j=1}^{n} {}_{E^*}\langle l_j, T^E_{t_j} x_0 \rangle_E] \cdot c_n \tag{3.12}$$

and hence $D_\nu = c_n$, in particular, $D_\mu = D_{\varepsilon_z}$. But

$$\exp[i \sum_{j=1}^{n} {}_{E^*}\langle l_j, T_{t_j}^E z \rangle_E] \cdot D_{\varepsilon_z} = D(z) \tag{3.13}$$

which is exactly the right hand side of (3.6) $\square$

Remark 3.3. (i) A similar study of the corresponding Martin boundary as in [R 88b, 92] can be carried out in the more general situation of Theorem 3.2 above.

(ii) Replacing Ω resp. $(X_t)_{t\geq 0}$, in condition (3.2) by the set Ω' consisting of all cadlag sample paths on $[0,\infty)$ resp. $X'_t := \tilde{X}_t$ on Ω', Theorem 3.2 (with the same proof) obviously remains true in the sense that we obtain a conservative normal strong Markov process $\mathbf{M}'$ with merely cadlag sample paths. In Section 7 below we shall see, however, that in fact also in this case the sample paths are continuous P_z-a.s. for all $z \in E$.

4. The general Gaussian case.

Let H be a separable real Hilbert space and $(T_t)_{t\geq 0}$ a strongly continuous semigroup on H. Let $(\mu_t)_{t\geq 0}$ be a family of centred Gaussian strong cylindrical measures on H, and let $(B_t)_{t\geq 0}$ the corresponding covariance operators, i.e., $B_t \in \mathcal{L}(H^*)$ for all $t > 0$, such that

$$\tilde{\mu}_t(l) = \exp\bigl(-(1/2)\|B_t l\|_{H^*}^2\bigr), \quad l \in H^*,\ t \geq 0. \tag{4.1}$$

Proposition 4.1. *The family $(\mu_t)_{t\geq 0}$ satisfies (2.7) on H^* if and only if for all $l \in H^*$,*

$$\|B_{t+s} l\|_{H^*}^2 - \|B_s l\|_{H^*}^2 = \|B_t T_s^* l\|_{H^*}^2, \quad s,t \geq 0. \tag{4.2}$$

In this case, $(t,l) \mapsto \|B_t l\|_{H^*}^2$ is continuous on $[0,\infty) \times H^*$, and Theorem 2.5 (i)–(iii) applies.

Proof. The equivalence is obvious by Proposition 2.2, and the proof of part (ii) is a exercise in real analysis, so is included in the appendix. $\square$

Remark 4.2. (i) Let $l \in H^*$ such that (4.2) holds. Then $t \mapsto \|B_t l\|_{H^*}^2$ is a distribution function of some measure m^l on $\mathcal{B}([0,\infty))$. If this holds for all $l \in H^*$, then (4.2) just says that the image measure of m^l under shift by s, is exactly the measure $m^{T_s^* l}$. One should note the resemblance with the defining property of additive functionals of Markov processes.

(ii) Note that Theorem 4.1 applies also if each μ_t is even a Gaussian probability measure on H. In this case $(\mu_t)_{t\geq 0}$ gives rise to a generalized Mehler semigroup (2.1) on H.

We now want to apply Lemma 2.6 (i) to gain some structural information about $(B_t)_{t\geq 0}$.

Proposition 4.3. *Assume that (4.2) holds and that $t \mapsto \|B_t l\|_{H^*}^2$ is differentiable at $t = 0$ for all $l \in H^*$. Then there exists $C^* \in \mathcal{L}(H^*)$ such that for all $l \in H^*$, $\frac{d}{dt}\|B_t l\|_{H^*}^2|_{t=0} = \|C^* l\|_{H^*}^2$ and*

$$\|B_t l\|_{H^*}^2 = \int_0^t \|C^* T_s^* l\|_{H^*}^2 \, ds, \quad t \geq 0. \tag{4.3}$$

Proof. Since the mapping that takes $l \in H^*$ to $\frac{d}{dt}\|B_t l\|_{H^*}^2|_{t=0} = \lim_{t\to 0} \downarrow (1/t)\|B_t l\|_{H^*}^2$, is non-negative, satisfies the parallelogram law, and is defined on all of H^*, there exists $C^* \in \mathcal{L}(H^*)$ such that for all $l \in H^*$,

$$\frac{d}{dt}\|B_t l\|_{H^*}^2|_{t=0} = \|C^* l\|_{H^*}^2 \tag{4.4}$$

It is not hard (cf. Section 8) to show that (4.2) combined with differentiability at $t = 0$ implies, in fact, that $t \mapsto \|B_t l\|_{H^*}^2$ is absolutely continuous and so (4.3) follows by Lemma 2.6 (i). □

Invariant measures.

Let E be a Hilbert space such that the properties in Corollary 1.4 are satisfied. Let $(\mu_t)_{t\geq 0}$ be a family of centred Gaussian strong cylindrical measures on H, and $(B_t)_{t\geq 0}$ be the corresponding covariance operators as above, and assume that (4.2) is satisfied. Let $(p_t)_{t\geq 0}$ be the corresponding generalized Mehler semigroup defined as in Theorem 2.5, which exists by Proposition 4.1. For the rest of this section, we make the following crucial assumption

$$\sup_{t\geq 0} \|B_t l\|_{H^*}^2 < \infty \quad \text{for all } l \in H^*. \tag{4.5}$$

Condition (4.5) is related to the existence of an invariant measure for $(p_t)_{t\geq 0}$. The proof of the corresponding theorem below is an adaptation of [DPZ 92; Theorem 11.7] to our situation. We nevertheless include the proof for completeness.

Theorem 4.4. *(i) Condition (4.5) holds if and only if there exists a probability measure ν on $\mathcal{B}(E)$, which is a strong cylindrical measure on H, and which is* invariant *for $(p_t)_{t\geq 0}$, i.e.,*

$$\int p_t f \, d\nu = \int f \, d\nu \quad \textit{for all } t \geq 0, \, f \in \mathcal{B}_b(E). \tag{4.6}$$

(ii) If (4.5) holds, there exists a self-adjoint operator $B \in \mathcal{L}(H^*)$ such that $\sup_{t\geq 0} \|B_t l\|_{H^*}^2 = \|Bl\|_{H^*}^2$ for all $l \in H^*$. Let μ be the centred Gaussian measure on $\mathcal{B}(E)$ with Fourier transform

$$\tilde{\mu}(l) = \exp\big(-(1/2)\|Bl\|_{H^*}^2\big), \quad l \in E^*. \tag{4.7}$$

Then a probability ν on $\mathcal{B}(E)$ is invariant for $(p_t)_{t\geq 0}$ if and only if there exists a probability measure σ on $\mathcal{B}(E)$ such that $\sigma \circ (T_t^E)^{-1} = \sigma$ for all $t \geq 0$, and $\nu = \mu * \sigma$.

Proof. We first note that for $t \geq 0$ and a probability measure ν on $\mathcal{B}(E)$, by an elementary calculation,

$$\begin{aligned}\nu p_t = \nu &\iff \big(\nu \circ (T_t^E)^{-1}\big) * \mu_t = \nu \\ &\iff \tilde{\nu}\big((T_t^E)^* l\big)\tilde{\mu}_t(l) = \tilde{\nu}(l) \text{ for all } l \in E^*.\end{aligned} \tag{4.8}$$

(i): Now suppose that (4.5) holds. Since $t \mapsto \|B_t l\|_{H^*}^2$, $l \in H^*$ is increasing by (4.2), it follows that there exists a self-adjoint $B \in \mathcal{L}(H^*)$ such that

$$\|Bl\|_{H^*}^2 = \lim_{t\to\infty} \|B_t l\|_{H^*}^2 \quad \text{for all } l \in H^*. \tag{4.9}$$

Letting $t \to \infty$ in (4.2) we obtain

$$\|Bl\|_{H^*}^2 - \|BT_s^* l\|_{H^*}^2 = \|B_s l\|_{H^*}^2 \quad \text{for all } l \in H^*, s \geq 0. \tag{4.10}$$

Since $H \subset E$ is Hilbert-Schmidt, there exists a probability μ on $\mathcal{B}(E)$ satisfying (4.7). Clearly, μ is a strong cylindrical measure on H, and by (4.8), (4.10) it is an invariant measure for $(p_t)_{t\geq 0}$.

Conversely, if ν is a probability measure on $\mathcal{B}(E)$, which is a strong cylindrical measure on H, satisfying (4.6), then the equation involving the Fourier transforms in (4.8) extends to all of H^*. Suppose $l \in H^*$ such that $\sup_{t\geq 0} \|B_t l\|_{H^*}^2 = \infty$. Fix $s \in \mathbb{R} \setminus \{0\}$. Then $\lim_{t\to\infty} \tilde{\mu}_t(sl) = 0$. Hence by (4.8), since $t \mapsto \tilde{\nu}(s(T_t^E)^* l)$ is bounded, $\tilde{\nu}(sl) = 0$. Letting $s \to 0$ it follows that $1 = \tilde{\nu}(0) = 0$. This contradiction proves (4.5).

(ii): The first part of the assertion was already proved above. To show the second, let σ be as in the assertion. Then if $\nu := \mu * \sigma$, obviously, since $\tilde{\sigma}((T_t^E)^* l) = \tilde{\sigma}(l)$ for all $l \in E^*$ and $t \geq 0$,

$$\tilde{\nu}((T_t^E)^* l)\tilde{\mu}_t(l) = \tilde{\nu}(l) \quad \text{for all } l \in E^*. \tag{4.11}$$

Hence ν is an invariant probability measure for $(p_t)_{t\geq 0}$ by (4.8). Conversely, if ν is an invariant probability measure for $(p_t)_{t\geq 0}$, then by (4.8) for all $l \in E^*$,

$$\lim_{t\to\infty} \tilde{\nu}((T_t^E)^* l) = \tilde{\nu}(l) \exp\big((1/2)\|Bl\|_{H^*}^2\big). \tag{4.12}$$

Since the right hand side of (4.12) is a function on E^* which is continuous in the Sazonov topology (cf. [VTC 87; p. 363]), by the Bochner-Sazonov theorem (cf. [VTC 87; Chapter VI, Theorem 1.1]) the left hand side of (4.12) is the Fourier transform of a measure σ on $\mathcal{B}(E)$. Clearly, $\sigma \circ (T_t^E)^{-1} = \sigma$ for all $t \geq 0$ and the proof is complete. □

Remark 4.5. (i) The existence of an invariant measure (which is strongly cylindrical on H) imposes extra regularity on $t \mapsto \|B_t l\|_{H^*}^2$ via (4.10). For $l \in D(A^*)$, the map $t \mapsto \|B_t l\|_{H^*}^2$ is continuously differentiable with

$$\frac{d}{dt}\|B_t l\|_{H^*}^2 = -2(BA^*T_t^* l, BT_t^* l)_{H^*}, \quad t \geq 0, \tag{4.13}$$

and so

$$\|B_t l\|_{H^*}^2 = 2\int_0^t (-BA^*T_s^* l, BT_s^* l)_{H^*}\, ds. \tag{4.14}$$

If the quadratic form $(-BA^*k, Bl)_{H^*} + (Bk, -BA^*l)_{H^*}$ extends to all of H^*, then the square root of the corresponding generator would be exactly the C^* in Proposition 4.3 and we would be in that case.

(ii) Suppose that (4.5) holds. Since $t \mapsto \|B_t l\|_{H^*}^2$ is increasing, the derivative in (4.13) must be positive. In particular, at $t = 0$, we get

$$-(BA^*l, Bl)_{H^*} \geq 0, \qquad \text{for all } l \in D(A^*). \tag{4.15}$$

Conversely, if we are given $B \in \mathcal{L}(H^*)$ satisfying (4.15), then we may define B_t by the left-hand side of (4.10) and obtain a family $(B_t)_{t\geq 0}$ which satisfies (4.2) and (4.5). Thus any such B leads to a Gaussian generalized Mehler semigroup with invariant measure μ given by (4.7).

5. Strong solutions for the associated stochastic differential equations on enlargements.

As is well-known, the Gaussian Mehler semigroups studied in the preceding section arise as transition probabilities of processes solving linear stochastic differential equations of type (0.1). Using Theorem 1.8 we can solve (0.1) in a very strong sense on an enlarged state space E. This can be done even if the diffusion operators are unbounded.

Let H be a separable real Hilbert space and $(T_t)_{t\geq 0}$, $(S_t)_{t\geq 0}$ strongly continuous commuting semigroups on H with generators $(A, D(A))$ resp. $(G, D(G))$.

Theorem 5.1. *Let $C \in \mathcal{L}(H)$ and $(CW_t)_{t\geq 0}$ be the cylindrical centred Gaussian process on H having covariance $(t \wedge s)(C^*h_1, C^*h_2)_{H^*}$, $h_1, h_2 \in H^*$, $s, t \geq 0$, (i.e., $(CW_t)_{t\geq 0}$ is the cylindrical Wiener process on H if $C =$ identity). Let E be a Hilbert space such that the properties of Theorem 1.8 are satisfied. Then*

(i) $(CW_t)_{t\geq 0}$ is a continuous Gaussian process on D_{A^E,G^E}.

(ii) For each $x \in E$, there exists a continuous Gaussian process $(X_t^x)_{t\geq 0}$ in E which solves equation (0.1) (with $G_E C$ replacing C) in the following sense

$$X_t^x = x + G^E CW_t + A^E\left(\int_0^t X_s^x\, ds\right). \tag{5.1}$$

It is given by

$$X_t^x = T_t^E x + G^E CW_t + \int_0^t A^E T_{t-s}^E G^E CW_s\, ds, \quad t \geq 0. \tag{5.2}$$

In particular, if $x \in D(A^E)$, then $(X_t^x)_{t\geq 0}$ takes values in $D(A^E)$ and A^E can be interchanged with the integral in (5.1).

Proof. Assertion (i) follows immediately from standard results about Gaussian processes on Hilbert spaces, since the embedding $H \subset D_{A^E,G^E}$ is Hilbert-Schmidt.

Therefore it remains to prove assertion (ii). Let $t \mapsto F_t$ be any continuous map from $[0,\infty)$ to D_{A^E,G^E} with $F_0 = 0$. Define for $x \in E$,

$$X_t^x := T_t^E x + F_t + \int_0^t A^E T_{t-s}^E F_s \, ds. \tag{5.3}$$

Since $A^E T_{t-s}^E F_s = T_{t-s}^E A^E F_s$, the integrand is a continuous map with values in E. Hence $t \mapsto X_t^x$ is continuous from $[0,\infty)$ to E, and integrating (5.3) from 0 to t we obtain by a simple computation

$$\int_0^t X_s^x \, ds = \int_0^t T_s^E x \, ds + \int_0^t T_{t-s}^E F_s \, ds. \tag{5.4}$$

Hence $\int_0^t X_s^x \, ds \in D(A^E)$, and we can apply A^E to (5.4). Since $(A^E, D(A^E))$ is closed on E and $s \mapsto T_{t-s}^E F_s$ is continuous as a $D(A^E)$-valued map, A^E can be interchanged with the integral on the right hand side of (5.4), and so we obtain

$$X_t^x = x + F_t + A^E\Big(\int_0^t X_s^x \, ds\Big). \tag{5.5}$$

For each fixed $\omega \in \Omega$, taking $F_t := G^E C W_t(\omega)$ for $t \geq 0$ we obtain (5.1). $\square$

Remark 5.2. (i) Note that if

$$X_t = x + G^E C W_t + \int_0^t A^E X_s \, ds, \quad t \geq 0, \tag{5.6}$$

(where we assume implicitly that $X_s \in D(A^E)$ for all $s \geq 0$ and also $x \in D(A^E)$), i.e.: $(X_t)_{t\geq 0}$ solves (0.1), then it is well-known and easy to see that $(X_t)_{t\geq 0}$ satisfies (5.2). Therefore, at least if $x \in D(A^E)$, the solution in Theorem 5.1 is unique.

(ii) In the situation of Theorem 5.1, the Gaussian strong cylindrical measure ν_0 on H whose Fourier transform is given by

$$\widetilde{\nu}_0(l) := \exp(-1/2\|C^* l\|_{H^*}^2), \quad l \in H^*, \tag{5.7}$$

extends to a Gaussian probability measure on D_{A^E,G^E}. Let ν denote the image of this measure on E under the map G^E. Then by Remark 1.5

$$\widetilde{\nu}(l) = \exp\big(-1/2\|C^* G^* l\|_{H^*}^2\big), \quad \text{for all } l \in D((G^E)^*). \tag{5.8}$$

Hence by Sazonov's theorem (cf. [VTC 87; p. 363]) there exists a Hilbert-Schmidt operator Λ^* on E^* such that

$$\|C^* G^* l\|_{H^*}^2 = \|\Lambda^* l\|_{E^*}^2 \quad \text{for all } l \in D((G^E)^*). \tag{5.9}$$

(iii) A theorem corresponding to Theorem 5.1 can be found in [FD 95], but their operators T_t^E, S_t^E were *only μ-measurable* extensions of $T_t, S_t, t \geq 0$ (see also [FD 91, 94]) which are not "true" semigroups.

Proposition 5.3. *Consider the situation of Theorem 5.1 and let Λ^* be as in Remark 5.2 (ii). Define Hilbert-Schmidt operators B_t, $t \geq 0$, on E^* by*

$$\|B_t l\|_{E^*}^2 = \int_0^t \|\Lambda^*(T_s^E)^* l\|_{E^*}^2 \, ds, \quad l \in E^* \tag{5.10}$$

and let $(\mu_t)_{t\geq 0}$ be centred Gaussian probability measures on E with Fourier transform

$$\tilde{\mu}_t(l) = \exp\bigl(-1/2\|B_t l\|_{E^*}^2\bigr), \quad l \in E^*. \tag{5.11}$$

Let $(p_t)_{t\geq 0}$ be the generalized Mehler semigroup defined by

$$p_t f(x) := (\mu_t * f)(T_t^E x), \quad x \in E, f \in \mathcal{B}_b(E). \tag{5.12}$$

Then for all $f \in \mathcal{B}_b(E)$

$$p_t f(x) = E[f(X_t^x)] \quad x \in E, t \geq 0. \tag{5.13}$$

Proof. First note that $(\mu_t)_{t\geq 0}$ satisfies (2.7) by Proposition 4.1, hence $(p_t)_{t\geq 0}$ is indeed a generalized Mehler semigroup. Fix $x \in E$, $t \geq 0$, and $f \in \mathcal{B}_b(E)$. We may assume that $f = e^{il}$ for some $l \in D_{(A^E)^*,(G^E)^*}$. It is well-known that by Itô's product formula, equation (5.1) can be rewritten as

$$X_t^x = T_t^E x + \int_0^t T_{t-s}^E \, d(G^E C W_s). \tag{5.14}$$

Hence

$$E[\exp\{il(X_t^x)\}] = \exp\{il(T_t^E x)\} E[\exp\{il(\int_0^t T_{t-s}^E \, d(G^E C W_s))\}]. \tag{5.15}$$

We now replace the stochastic integral by a Riemannian sum and obtain that

$$\begin{aligned} &E[\exp\{il(\sum_{k=1}^n T_{t-s_k}^E (G^E C W_{s_{k+1}} - G^E C W_{s_k}))\}] \\ &= \prod_{k=1}^n E[\exp\{{}_{E^*}\langle (T_{t-s_k}^E)^* l, G^E C W_{s_{k+1}} - G^E C W_{s_k} \rangle_E\}] \\ &= \prod_{k=1}^n \exp\{-(1/2)(s_{k+1} - s_k)\|G^E C^* T_{t-s_k}^* l\|_{H^*}^2\} \\ &= \exp\{-(1/2) \sum_{k=1}^n \|\Lambda^* T_{t-s_k}^* l\|_{H^*}^2 (s_{k+1} - s_k)\} \end{aligned} \tag{5.16}$$

(cf. Remark 5.2 (ii)). Consequently, taking the limit over partitions of $[0,t]$, and changing variables we get

$$\begin{aligned} E[\exp\{il(X_t^x)\}] &= \exp\{il(T_t^E x) - (1/2)\int_0^t \|\Lambda^* T_s^* l\|_{H^*}^2 \, ds\} \\ &= \exp\{il(T_t^E x) - (1/2)\|B_t l\|_{H^*}^2\} \\ &= p_t f(x). \end{aligned} \tag{5.17}$$

Thus (5.7) is shown. □

Remark 5.4. In Theorem 5.1 we solved (0.1) for a diffusion operator which can be decomposed into a bounded operator on H^* and a possibly unbounded one, but which is a generator of a semigroup that commutes with the one generated by the drift. In Section 7 below, using the theory of Dirichlet forms, we shall construct a solution for (0.1) where the diffusion operator might not have such a representation.

6. Associated generators and Dirichlet forms.

Let H be a separable real Hilbert space and $(T_t)_{t\geq 0}$ a strongly continuous semigroup on H. Let $(\mu_t)_{t\geq 0}$ be a family of centred Gaussian strong cylindrical measures on H, and let $(B_t)_{t\geq 0}$ the corresponding covariance operators. Let E be a Hilbert space satisfying the conditions in Corollary 1.4. We assume that (4.2) holds and let $(p_t)_{t\geq 0}$ be the generalized Mehler semigroup as in Theorem 2.5 (iii). We also assume that (4.5) holds, and let μ be the Gaussian invariant measure whose Fourier transform is given by (4.7).

It follows from the μ-invariance that $(p_t)_{t\geq 0}$ gives rise to a contraction semigroup $(P_t)_{t\geq 0}$ on $L^2(E;\mu)$. We can now calculate the generator $(L, D(L))$ on $L^2(E;\mu)$ corresponding to this semigroup $(P_t)_{t\geq 0}$. Although the semigroup $(P_t)_{t\geq 0}$ acts on real L^2-space, for computational convenience, we will apply it to a space $\mathcal{C}$ of complex-valued functions. To this end, define the following subspace of complex $L^2(E;\mu)$,

$$\mathcal{C} := \big\{u = \sum_{j=1}^{n} \alpha_j \exp[il_j] \;\; \mu\text{-a.e.}; \; n \in \mathbb{N}, \, l_j \in D((A^E)^*), \alpha_j \in \mathbb{C}\big\}. \tag{6.1}$$

Let us define $\mathcal{C}'$ to be the subspace of real $L^2(E;\mu)$ which is the linear span of $\{\sin(l), \cos(l) \mid l \in D((A^E)^*)\}$, and note that $\mathcal{C}'$ is a subspace of $\mathcal{C}$. We also recall (cf. Remark 1.5) that $A^* = (A^E)^*$ on $D((A^E)^*) \subset D(A^*)$.

Proposition 6.1. *$\mathcal{C}'$ is dense in $L^2(E;\mu)$ and the generator $(L, D(L))$ of $(P_t)_{t\geq 0}$ is the closure of $(L, \mathcal{C}')$, where for $u = \sum_{j=1}^{n} \alpha_j \exp[il_j] \in \mathcal{C}'$,*

$$Lu = \sum_{j=1}^{n} \alpha_j \exp[il_j]\big(i((A^E)^* l_j) + (BA^* l_j, Bl_j)_{H^*}\big). \tag{6.2}$$

Proof. From (2.1) and (4.1), for any l in E^*, if $u = \exp[il]$, then

$$P_t u = \exp[i(T_t^E)^* l] \exp[-(1/2)\|B_t l\|_{H^*}^2]. \tag{6.3}$$

Hence, if $l \in D((A^E)^*)$, then $t \mapsto P_t u$ is differentiable in L^2 with (cf. (4.13))

$$\begin{aligned} \frac{d}{dt} P_t(\exp[il]) &= \exp[i(T_t^E)^* l] \, (i((A^E)^* l)) \exp[-(1/2)\|B_t l\|_{H^*}^2] \\ &\quad + \exp[i(T_t^E)^* l] \exp[-(1/2)\|B_t l\|_{H^*}^2](BA^* T_t^* l, BT_t^* l)_{H^*}. \end{aligned} \tag{6.4}$$

Thus, $\exp[il] \in D(L)$ and,

$$L(\exp[il]) = \exp[il]\big(i((A^E)^* l) + (BA^* l, Bl)_{H^*}\big). \tag{6.5}$$

By linearity, (6.2) holds for all of $\mathcal{C}$, in particular $\mathcal{C}'$ is contained in $D(L)$, and equation (6.2) is valid there. Using a monotone class argument, it is not hard to show and well-known that $\mathcal{C}'$ is dense in $L^2(E;\mu)$. In addition, $(T_t^E)^*$ maps $D((A^E)^*)$ back into itself, and so taking real and imaginary parts in (6.3), implies that P_t maps $\mathcal{C}'$ back into itself. Consequently, we can apply the core theorem [ReS 75; X. 49], and conclude that L is equal to the closure of $(L, \mathcal{C}')$, i.e., L is uniquely determined by its values on $\mathcal{C}'$. □

We now calculate the corresponding quadratic form $\mathcal{E}$ on $\mathcal{C}' \times \mathcal{C}'$. Let $l, k \in D((A^E)^*)$ and $u = \exp[il]$ and $v = \exp[ik]$. Then, since μ is Gaussian, using integration by parts we obtain

$$\begin{aligned} &\int (-Lu)\, v \, d\mu \\ &\quad = -i \int ((A^E)^* l) \exp[i(l+k)] \, d\mu - (BA^* l, Bl)_{H^*} \int \exp[i(l+k)] \, d\mu \\ &\quad = -\Big(i^2 (BA^* l, B(l+k))_{H^*} + (BA^* l, Bl)_{H^*} \Big) \exp\{-\|B(l+k)\|_{H^*}^2/2\} \\ &\quad = (BA^* l, Bk)_{H^*} \exp\{-\|B(l+k)\|_{H^*}^2/2\}. \end{aligned} \tag{6.6}$$

On the other hand, $u' = ilu$ and $v' = ikv$, where u', and v' are the Frechet derivatives of u and v on E. This gives $(BA^* u', Bv')_{H^*} = -(BA^* l, Bk)_{H^*} uv$, and then by integrating this equation over E we obtain

$$\mathcal{E}(u, v) := \int (-Lu)\, v \, d\mu = \int (-BA^* u', Bv')_{H^*} \, d\mu. \tag{6.7}$$

By linearity, this holds for all of $\mathcal{C}$, in particular for all u, v in the real space $\mathcal{C}'$. □

Condition 6.2. There is a constant K so that for $k, l \in D(A^*)$ we have

$$|(-BA^* k, Bl)_{H^*}|^2 \le K^2 \, (BA^* k, Bk)_{H^*} (BA^* l, Bl)_{H^*}. \tag{6.8}$$

For the definition of a non-symmetric Dirichlet form we refer to [MR 92; Chapter I, Definition 4.5].

Theorem 6.3. *Under condition 6.2, the form $(\mathcal{E}, \mathcal{C}')$ is closable, and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form satisfying the local property, that is,*

$$\textit{if } u, v \in D(\mathcal{E}) \textit{ and } uv = 0 \ \mu - a.e., \textit{ then } \mathcal{E}(u, v) = 0. \tag{6.9}$$

Proof. Note first that, since $(L, D(L))$ generates a contraction semigroup or because of (4.15), $(\mathcal{E}, \mathcal{C}')$ is positive definite. Condition 6.2 guarantees that $(\mathcal{E}, \mathcal{C}')$ satisfies the sector condition, and so by [MR 92; Chapter I, Proposition 3.3] it is closable. The generator $(\tilde{L}, D(\tilde{L}))$ of the closure $(\mathcal{E}, D(\mathcal{E}))$ i.e., the Friedrichs extension $(L, \mathcal{C}')$, generates the L^2-semigroup corresponding to $(\mathcal{E}, D(\mathcal{E}))$. On the other hand, we already know that $(L, D(L))$, the closure of $(L, \mathcal{C}')$, generates a strongly continuous L^2-semigroup and since $(\tilde{L}, D(\tilde{L}))$ extends $(L, D(L))$, these two operators must coincide. It now follows that the semigroup corresponding to $(\mathcal{E}, D(\mathcal{E}))$ is, in fact, the semigroup $(P_t)_{t \geq 0}$ that we began with.

Formula (6.7) tells us how the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ behaves on the space $\mathcal{C}'$, which is the linear span of $\{\sin(l), \cos(l) \mid l \in D((A^E)^*)\}$. The space $\mathcal{C}'$ is useful for identification purposes, but for computations, the following space (6.11) is more useful. Using fairly standard techniques in functional calculus for Dirichlet forms [S 92], you can show that $(\mathcal{E}, D(\mathcal{E}))$ can also be realized as the closure of

$$\mathcal{E}(u, v) = \int (-BA^* u', Bv')_{H^*} \, d\mu, \quad u, v \in \mathcal{F}C_b^\infty(D((A^E)^*)), \tag{6.10}$$

where

$$\mathcal{F}C_b^\infty(D((A^E)^*)) = \{f(l_1, \ldots, l_k) \mid \ k \in I\!N, \ l_1, \ldots, l_k \in D((A^E)^*), \ f \in C_b^\infty(I\!R^k)\}. \tag{6.11}$$

To show the Dirichlet property we shall use [MR 92; Chapter I, Propositions 4.7 and 4.10]. So, let $\varphi_\epsilon : I\!R \to [-\epsilon, 1 + \epsilon]$ be a smooth function such that $\varphi_\epsilon(t) = t$ for all $t \in [0, 1]$, and $0 \leq \varphi_\epsilon(t_2) - \varphi_\epsilon(t_1) \leq t_2 - t_1$ for $t_1 \leq t_2$. Then by the chain rule, for all $u \in \mathcal{F}C_b^\infty(D((A^E)^*))$ we have $\varphi_\epsilon(u) \in \mathcal{F}C_b^\infty(D((A^E)^*))$ and

$$\mathcal{E}(u \pm \varphi_\epsilon(u), u \mp \varphi_\epsilon(u)) = \int \big(1 \pm \varphi_\epsilon'(u)\big)\big(1 \mp \varphi_\epsilon'(u)\big)(-BA^* u', Bu') \, d\mu \geq 0, \tag{6.12}$$

by (4.15), since $0 \leq \varphi_\epsilon' \leq 1$. Hence [MR 92; Chapter I, Proposition 4.10] implies that $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form on $L^2(E; \mu)$. The local property follows immediately from [S 95; Proposition 2.3]. $\square$

Remark 6.4. (i) For $l, k \in D((A^E)^*)$ and $u := \exp[il]$, $v := \exp[ik]$

$$\int v p_t u \, d\mu = \exp[(-1/2)\|Bl\|_{H^*}^2] \exp[(-1/2)\|Bk\|_{H^*}^2] \exp[-(Bk, BT_t^* l)_{H^*}]. \tag{6.13}$$

Hence $(p_t)_{t\geq 0}$ is symmetric on $L^2(E;\mu)$, (or equivalently $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form), if and only if $(k,l) \mapsto (Bk, BT_t^* l)_{H^*}$ is symmetric for all $t \geq 0$. This situation has been studied in [BR 95]. Note that, in [BR 95], $B_t := \sqrt{1 - T_{2t}^*}\, B$ for all $t \geq 0$, where $(T_t^*)_{t\geq 0}$ was assumed to be a strongly continuous symmetric contraction semigroup on H^*, which commutes with B.

(ii) There is a result due to M. Fuhrman [Fuh 93] related to Theorem 6.3 above proved in a different framework under more restrictive assumptions.

(iii) Let $(\hat{P}_t)_{t\geq 0}$ be the dual semigroup to $(P_t)_{t\geq 0}$ on $L^2(E;\mu)$. Then the fact that μ is an invariant measure for $(p_t)_{t\geq 0}$ (cf. Theorem 4.4) implies that $\hat{P}_t 1 = 1$, $t \geq 0$. Since each $\hat{P}_t$ is positivity preserving (because each P_t is so), we obtain that $(\hat{P}_t)_{t\geq 0}$ is sub-Markovian (i.e., $0 \leq u \leq 1$ implies $0 \leq \hat{P}_t u \leq 1$ for all $t \geq 0$) $0 \leq u \leq 1$ implies $0 \leq \hat{P}_t u \leq 1$ for all $t \geq 0$) as is $(P_t)_{t\geq 0}$ by definition. Hence (instead of the last argument in the preceding proof) the Dirichlet property of $(\mathcal{E}, D(\mathcal{E}))$ also follows from [MR 92, Chap. I, Theorem 4.4 and Proposition 4.3].

7. Quasi-regularity, continuity of sample paths, and weak solutions of the associated stochastic differential equations via Dirichlet forms.

Let E, μ be as in Section 6, in particular, Condition 6.2 is assumed to hold. We recall the following notions from [MR 92].

Definition 7.1. Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(E;\mu)$.

(i) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$*-nest* if $\cup_{k\geq 1}\{u \in D(\mathcal{E}) \mid u = 0 \ \mu\text{-a.e. on } E \setminus F_k \text{ for some } k \in \mathbb{N}\}$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})$. Here $\mathcal{E}_1 := \mathcal{E} + (\ ,\ )_{L^2(E;\mu)}$ and $\tilde{\mathcal{E}}_1$ is its symmetric part.

(ii) A subset $N \subset E$ is called $\mathcal{E}$*-exceptional* if $N \subseteq \cap_{k\geq 1} F_k^c$ for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$. A property of points in E holds $\mathcal{E}$*-quasi-everywhere* (abbreviated $\mathcal{E}$-q.e.), if the property holds outside some $\mathcal{E}$-exceptional set.

(iii) An $\mathcal{E}$-q.e. defined function $f : E \to \mathbb{R}$ is called $\mathcal{E}$*-quasi-continuous* if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ so that $f|_{F_k}$ is continuous for each $k \in \mathbb{N}$.

Definition 7.2. A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;\mu)$ is called *quasi-regular* if:

(QR1) There exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets.

(QR2) There exists an $\tilde{\mathcal{E}}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous μ-versions.

(QR3) There exist $u_n \in D(\mathcal{E}), n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n, n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n \,|\, n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Now we are prepared to prove the first main result of this section. Let $(p_t)_{t\geq 0}$ and

$(\mathcal{E}, D(\mathcal{E}))$ be as in Section 6. We also adopt the notation of Section 3.

Theorem 7.3. *The following assertions are equivalent:*

(i) Condition (3.2) holds.

(ii) $(p_t)_{t\geq 0}$ *is the transition function of a (conservative) diffusion process.*

(iii) Condition (3.2) holds with Ω *resp.* $X_t, t \geq 0$*, replaced by the set* Ω' *of all cadlag paths from* $[0,\infty)$ *to* E *resp.* $X'_t :=$ *evaluation at* t *on* Ω' , $t \geq 0$.

(iv) $(\mathcal{E}, D(\mathcal{E}))$ *is quasi-regular.*

Proof.
(i) $\Rightarrow$ (ii): This is the last part of Theorem 3.2.
(ii) $\Rightarrow$ (iii): Trivial.
(iii) $\Rightarrow$ (iv): Since $(p_t)_{t\geq 0}$ is Feller, it follows by [Dy 71; Satz 5.10] that the normal Markov process $\mathbf{M}'$ in Remark 3.3 (ii) is strong Markov, hence a *right process.* Clearly, $\mathbf{M}'$ is *associated with* $(\mathcal{E}, D(\mathcal{E}))$ (cf. [MR 92; Chapter IV, Section 2]). Therefore, using [MR 92; Chapter IV, Theorems 1.15 and 5.4 (including Remark 5.5)] and also [MR 92; Chapter III, Proposition 2.11 (i)] we conclude that (QR1) holds. Hence $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.
(iv) $\Rightarrow$ (i): Since $(\mathcal{E}, D(\mathcal{E}))$ has the local property by Theorem 6.3, [MR 92; Chapter V, Theorem 1.11] (see also [AMR 93]) implies that there exists a diffusion process $\mathbf{M}=(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ having $(p_t)_{t\geq 0}$ as transition semigroup. (Note that the lifetime ζ is identically equal to $+\infty$, since $p_t 1 = 1$, $t \geq 0$.) Hence $P := \int P_x\, \mu(dx)$ satisfies (3.2). □

Remark 7.4. (i) Note that if $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular, by [MR 92; Chapter V, Theorem 1.11] there always exists a diffusion process such that for its transition semigroup $(\tilde{p}_t)_{t\geq 0}$ we have that $\tilde{p}_t f$ is a $\mathcal{E}$-quasi-continuous μ-version of $P_t f$ for all $f \in L^2(E;\mu)$, $t \geq 0$. Theorem 6.3, however, implies that we can even find "better versions", namely $(p_t)_{t\geq 0}$ given by (2.1) and these $(p_t)_{t\geq 0}$ are even Feller.

(ii) We emphasize that $(\mathcal{E}, D(\mathcal{E}))$ is not always quasi-regular and refer to [BR 95; Example 6.6 (ii)] for a counterexample. A sufficient condition for quasi-regularity is given in the following proposition (cf. [BR 95; Proposition 6.5] for a special case of this.)

Proposition 7.5. *Suppose that there exists* $c \in (0,\infty)$ *such that* $(-BA^*l, Bl)_{H^*} \leq c\|l\|^2_{E^*}$ *for all* $l \in D((A^E)^*)$*. Then* $(\mathcal{E}, D(\mathcal{E}))$ *is quasi-regular. In particular, this is always true for a Hilbert space* E *satisfying the properties in Theorem 1.6.*

Proof. We want to apply [RS 95; Theorem 3.4]. To this end, let $l_j \in D((A^E)^*)$, $j \in \mathbb{N}$, such that $\|l_j\|_{E^*} \leq 1$ for all $j \in \mathbb{N}$, and $\|z\|_E = \sup_j l_j(z)$ for all $z \in E$. Let $\varphi \in C_b^1(\mathbb{R})$ such that $\varphi(0) = 0$, φ is strictly increasing, and φ' is both decreasing and bounded by 1. Then $\rho_1(z,x) := \varphi(\|z-x\|_E)$ is a bounded metric on E that is uniformly equivalent with the usual metric $\rho(z,x) = \|z-x\|_E$. Let $\{x_i \mid i \in \mathbb{N}\}$ be a countable dense subset of E, and define for $i, j \in \mathbb{N}$

$$f_{ij}(z) := \varphi(l_j(z - x_i)). \tag{7.1}$$

Then $f_{ij} \in \mathcal{F}C_b^\infty(D((A^E)^*))$ for every $i, j \in I\!N$ and

$$f'_{ij}(z) = \varphi'(l_j(z - x_i))l_j, \tag{7.2}$$

and so for all $z \in E$,

$$\begin{aligned} \sup_{i,j}(-BA^* f'_{ij}, Bf'_{ij})_{H^*} =& \sup_{i,j}(\varphi'(l_j(z - x_i)))^2 \ (-BA^* l_j, l_j)_{H^*} \\ \leq& c\|l_j\|^2_{E^*} \leq c. \end{aligned} \tag{7.3}$$

Furthermore, for every fixed $i \in I\!N$

$$\begin{aligned} \sup_j f_{ij}(z) &= \varphi(\sup_j l_j(z - x_i)) \\ &= \varphi(\|z - x_i\|_E) \\ &= \rho_1(z, x_i) \end{aligned} \tag{7.4}$$

for every $z \in E$. Hence the conditions of [RS 95; Theorem 3.4] with $\Gamma^h(u, v) = (-BA^*u', Bv')_{H^*}$, $u, v \in \mathcal{F}C_b^\infty(D((A^E)^*))$ and $h \equiv 1$, are satisfied and hence $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. To prove the last part of the assertion, let $l_n \in D((A^E)^*)$, $n \in I\!N$, such that $l_n \to 0$ in E^* as $n \to \infty$. Clearly, it suffices to prove that $A^* l_n \to 0$ weakly in H^* as $n \to \infty$. So, let $l \in H^*$. Then, if $R_H: H^* \to H$ denotes the Riesz isomorphism,

$$\begin{aligned} (A^* l_n, l)_{H^*} &= {}_{H^*}\langle A^* l_n, R_H l\rangle_H \\ &= {}_{E^*}\langle (A^E)^* l_n, R_H l\rangle_E \\ &= {}_{E^*}\langle l_n, A^E R_H l\rangle_E \\ &\to 0 \text{ as } n \to \infty \end{aligned} \tag{7.5}$$

where we used that, by construction in the situation of Theorem 1.6, we have $H \subset D(A^E)$.

Assumption 7.6. We assume from now on that one of the equivalent conditions in Theorem 7.3 holds.

Let $\mathbf{M}= (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ be the (conservative) diffusion process introduced in Theorem 3.2. Since $(p_t)_{t\geq 0}$ is the corresponding transition semigroup, by [MR 92; Chapter IV, Theorem 5.1], $\mathbf{M}$ is properly associated (see ibidem) to the quasi-regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. We intend to show that under P_x (for $\mathcal{E}$-q.e. $x \in E$), $(X_t)_{t\geq 0}$ solves an equation of type (0.1) where $(CW_t)_{t\geq 0}$ will be a Gaussian process whose covariance is determined by a *not-necessarily bounded* linear operator C^* on H^*. This operator exists by virtue of the sector condition (6.8) and is defined as follows:

Let

$$q(k, l) := 2(-BA^*k, Bl)_{H^*}, \quad k, l \in D(A^*). \tag{7.6}$$

Then, since B is bounded on H^*, it is known (e.g. by [MR 92; Chapter I, Propositions 3.3 and 3.5]) that $(q, D(A^*))$ is closable and the closure $(q, D(q))$ is a closed coercive form. Therefore, its symmetric part $(\tilde{q}, D(q))$ is also closed. Let $(C_1, D(C_1))$ be the corresponding self-adjoint operator, i.e., the unique negative definite self-adjoint operator on H^* such that $D(C_1) \subset D(q)$ and $\tilde{q}(k,l) = (-C_1 k, l)_{H^*}$ for all $k \in D(C_1), l \in D(q)$. Define $C^* := \sqrt{-C_1}$. Note that C_1 is negative definite by (4.15), that $D(A^*) \subset D(C^*)$, and that

$$(C^* l, C^* l)_{H^*} = 2(-BA^* l, Bl)_{H^*} \text{ for all } l \in D(A^*). \tag{7.7}$$

For $l \in E^*$ set

$$u_l(x) := {}_{E^*}\langle l, x\rangle_E, \quad x \in E. \tag{7.8}$$

Note that for $l \in E^*$, $\int u_l^2 \, d\mu = \|Bl\|_{H^*}^2$, and hence $u_l \in L^2(E;\mu)$. Moreover, we have

Lemma 7.7. *Let $l \in D((A^E)^*)$. Then $u_l \in D(L)$ and $Lu_l = u_{A^*l}$.*

Proof. We will approximate u_l with the trigonometric function $u_n := n\sin(u_l/n)$. Elementary calculations show that $(x - n\sin(x/n))^2 \le x^6/(36n^4)$ for all $x \in \mathbb{R}$. Thus, we obtain

$$\int_E (u_l(x) - u_n(x))^2 \, \mu(dx) \le (1/(36n^4)) \int_E u_l(x)^6 \, \mu(dx) \to 0, \tag{7.9}$$

as $n \to \infty$, since μ is Gaussian and $\int u_l^6 \, d\mu < \infty$. Furthermore, taking imaginary parts in (6.5) shows that $u_n \in D(L)$ and that

$$Lu_n = \cos(u_l/n)(u_{A^*l}) + (1/n)\sin(u_l/n)(-BA^*l, Bl)_{H^*}. \tag{7.10}$$

Taking the limit as $n \to \infty$ shows that Lu_n converges to u_{A^*l} in $L^2(E;\mu)$, and since L is a closed operator, the result follows. $\square$

Proposition 7.8. *Let $l \in D((A^E)^*)$. Then for $\mathcal{E}$-q.e. $x \in E$ we have P_x-a.s. that*

$$u_l(X_t) - u_l(X_0) = M_t^{[u_l]} + \int_0^t u_{A^*l}(X_s)\, ds, \quad t \ge 0, \tag{7.11}$$

where $(M_t^{[u_l]})_{t\ge0}$ is a martingale additive functional of finite energy (cf. [Fu 80; §5.1, §5.2]) resp. [MR 92; Chapter VI, Section 2]). Furthermore, $\langle M^{[u_l]}\rangle_t = t\|C^*l\|_{H^*}^2$, $t \ge 0$; in particular, $(M_t^{[u_l]})_{t\ge0}$ is a (time-scaled) one-dimensional $(\mathcal{F}_t)_{t\ge0}$-Brownian motion starting at zero, where $(\mathcal{F}_t)_{t\ge0}$ is the minimum completed admissible filtration corresponding to $\mathbf{M}$ (cf. [Fu 80; §4.1]).

Proof. This follows by [MR 92; Chapter VI, Theorem 2.5] and [Fu 80; p. 138]. Since $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular, by [MR 92; Chapter VI, Theorem 2.5], the last part is proved in exactly the same way as Proposition 4.5 in [AR 91]. $\square$

For the rest of this section we assume that E is as in Theorem 1.6. In particular, we know that $\mu(D(A^E)) = 1$ and that Assumption 7.6 holds. Set

$$A^E x := 0 \quad \text{if } x \in E \setminus D(A^E). \tag{7.12}$$

Theorem 7.9. *Let E be as in Theorem 1.6. Then there exists a map $CW:\Omega \to C([0,\infty))$ such that for $\mathcal{E}$-q.e. $x \in E$, under P_x the process $CW = (CW_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion on E starting at zero with covariance $\|C^* \cdot\|_{H^*}^2$ such that for $\mathcal{E}$-q.e. $x \in E$,*

$$X_t = z + CW_t + \int_0^t A^E(X_s)\, ds, \quad t \geq 0,\ P_x - a.e.. \tag{7.13}$$

Proof. Since μ is a mean zero Gaussian measure on $D(A^E)$, we know by Fernique's theorem (cf. [Fe 70], [Fe 75; Théorème 1.3.2]) that $\int_E \|A^E x\|_E^2\, \mu(dx) < \infty$. Thus, the result is proved in exactly the same way as Theorem 6.10 in [AR 91]. □

Remark 7.10. (i) Equation (7.13) says that $\mathbf{M}$ solves (0.1) weakly (in the sense of probability theory). The reader is warned that (7.13) does not imply that $(X_t)_{t\geq 0}$ takes values in $D(A^E)$, but merely that for $\mathcal{E}$-q.e. $x \in E$, and for P_x-a.e. $\omega \in \Omega$, $X_s(\omega) \in D(A^E)$ for ds-a.e. $s \in [0,\infty)$ (cf. the proof of Theorem 6.10 in [AR 91]). Therefore, (7.13) in general *cannot* be rewritten in the form (5.2) (cf. Remark 5.2 (i)). In particular, in contrast to the process given by (5.2), the process $(X_t)_{t\geq 0}$ in (7.13) is maybe not adapted to the filtration generated by $(CW_t)_{t\geq 0}$.

(ii) By Proposition 6.1 the solution constructed in Theorem 7.10 is unique (in the weak sense). This follows by [AR 95; Theorem 3.5]. We also refer to [AR 95] with respect to the definition of "uniqueness" in this case.

(iii) It follows from Theorem 7.9 that $(CW_t)_{t\geq 0}$ has nuclear covariance on E^*. Since by the continuity of its sample paths $(X_t)_{t\geq 0}$ is predictable with respect to $(\mathcal{F}_t)_{t\geq 0}$, it is then easy to see that it is in fact a "mild solution" to (0.1) (resp. (7.13)) in the sense of [DPZ 92]. In particular, $(X_t)_{t\geq 0}$ can be expressed in terms of a stochastic integral with respect to $(CW_t)_{t\geq 0}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$.

(iv) To solve (0.1) (with given C and A) using the above scheme one has to find $B \in \mathcal{L}(H^*)$ such that (7.7) (which essentially is an analogue of condition (d) in Theorem 4.1 in [ZSn 70]) and (6.8) hold. Then one obtains $(B_t)_{t\geq 0}$ from (4.14) (with $(T_t)_{t\geq 0}$ being the semigroup generated by A on H) and (4.2) automatically holds. Hence Proposition 4.1 and Theorem 4.4 (ii) provide us with a generalized Mehler semigroup $(p_t)_{t\geq 0}$ with invariant measure μ to which all results in Sections 6 and 7, (in particular, Theorem 7.9) apply to give a solution of (0.1) on the space E constructed in Theorem 1.6.

8. Appendix.

Lemma 8.1. *Let H be a separable Hilbert space and $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on H. Suppose that $(B_t)_{t\geq 0}$ is a family in $\mathcal{L}(H)$ that satisfies*

$$\|B_{t+s}l\|_H^2 - \|B_s l\|_H^2 = \|B_t T_s l\|_H^2, \quad \textit{for all } s,t \geq 0,\ l \in H. \tag{8.1}$$

Then the map $(t,l) \to \|B_t l\|_H^2$ *is continuous on* $[0,\infty) \times H$.

Proof. We first show that $g(t) := \|B_t l\|_H^2$ is continuous on $[0,\infty)$ for each fixed $l \in H$. By (8.1) we see that g is increasing, and so for each $l \in H$, and $T > 0$

$$\sup_{0\le t\le T} \|B_t l\|_H = \|B_T l\|_H < \infty, \tag{8.2}$$

which, by the uniform boundedness principle, means that

$$\sup_{0\le t\le T} \|B_t\|_{\mathcal{L}(H)} =: c_1 < \infty. \tag{8.3}$$

Similarly, by the strong continuity of $(T_t)_{t\ge 0}$, we get $\sup_{0\le t\le T} \|T_t\|_{\mathcal{L}(H)} =: c_2 < \infty$. Define a family $(f_t)_{0<t<T}$ of real-valued functions on $[0,T]$ by

$$f_t(s) := \|B_{t+s} l\|_H^2 - \|B_s l\|_H^2. \quad 0 \le s \le T. \tag{8.4}$$

Then using (8.1) we see that for $0 < t < T$ and $0 \le s, u \le T$ we have

$$\begin{aligned}
|f_t(s) - f_t(u)| &= \Big| \|B_t T_s l\|_H^2 - \|B_t T_u l\|_H^2 \Big| \\
&= \big(\|B_t T_s l\|_H + \|B_t T_u l\|_H\big) \big| \|B_t T_s l\|_H - \|B_t T_u l\|_H \big| \\
&\le (2c_1c_2)c_1 \|l\|_H \, \|T_s l - T_u l\|_H \\
&\le (2c_1c_2)c_1 \|l\|_H \, \|T_{s\wedge u}(T_{|u-s|} - I) l\|_H \\
&\le (2c_1^2c_2^2) \|l\|_H \, \|(T_{|u-s|} - I) l\|_H.
\end{aligned} \tag{8.5}$$

Because of the strong continuity of $(T_t)_{t\ge 0}$, (8.5) shows that $(f_t)_{0<t<T}$ is equicontinuous on $[0,T]$. Now for each $s \in [0,T]$, as $t \downarrow 0$ we have $f_t(s) \to g(s+) - g(s) =: f(s)$. By Ascoli's theorem the convergence is uniform and so f is continuous. But g is increasing and so f is equal to zero, except possibly at countably many points. Thus $f \equiv 0$ and g is right-continuous on $[0,T]$. Since T is arbitrary, g is right-continuous on $[0,\infty)$. Similarly, consider the family $(h_t)_{0<t<1/T}$ in $C([1/T,T])$ given by

$$h_t(s) := f_t(s-t) = \|B_s l\|_H^2 - \|B_{s-t} l\|_H^2, \quad 1/T \le s \le T. \tag{8.6}$$

As before, $(h_t)_{0<t<1/T}$ is equicontinuous and converges pointwise to $h(s) := g(s) - g(s-)$ as $t \downarrow 0$. As before, $h \equiv 0$ and so g is left-continuous on $[1/T,T]$. Since T is arbitrary, we conclude that g is left-continuous on $(0,\infty)$.

Now consider the family of continuous functions $(g_t)_{0\le t\le T}$ defined on H by $g_t(l) = \|B_t l\|_H^2$. We have

$$|g_t(l) - g_t(k)| \le c_1^2 \big(\|l\|_H + \|k\|_H\big) \|l - k\|_H, \tag{8.7}$$

and so the family is equicontinuous at each point in H. Let $t_n \to t \in [0,T]$ and $l_n \to l \in H$ and let K be the compact set $K := \{l_1, l_2, \dots\} \cup \{l\} \subset H$. Since, by the first part,

$g_{t_n}(l) \to g_t(l)$ for all $l \in H$, the Ascoli theorem says that the convergence is uniform on compact sets. In particular, for $\varepsilon > 0$ there exists N so that if $n \geq N$, then

$$\sup_{k\in K} |g_{t_n}(k) - g_t(k)| \leq \varepsilon. \tag{8.8}$$

For $n \geq N$, then

$$\begin{aligned} |g_{t_n}(l_n) - g_t(l)| &\leq |g_{t_n}(l_n) - g_t(l_n)| + |g_t(l_n) - g_t(l)| \\ &\leq \varepsilon + |g_t(l_n) - g_t(l)|. \end{aligned} \tag{8.9}$$

Thus $\limsup_{n\to\infty} |g_{t_n}(l_n) - g_t(l)| \leq \varepsilon$ by the continuity of g_t. Since ε is arbitrary, $g_{t_n}(l_n) \to g_t(l)$ which establishes the joint continuity. □

Lemma 8.2. *Let H be a separable Hilbert space and $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on H. Suppose that $(B_t)_{t\geq 0}$ is a family in $\mathcal{L}(H)$ that satisfies (8.1). If the map $t \to \|B_t l\|_H^2$ is differentiable at zero for all $l \in H$, then $t \to \|B_t l\|_H^2$ is continuously differentiable for all $l \in H$.*

Proof. Fix $l \in H$ and define $g(t) := \|B_t l\|_H^2$. Dividing (8.1) by $t > 0$ and letting $t \downarrow 0$ shows that g is also right differentiable at s with

$$D^+ g(s) = \lim_{t\downarrow 0} (1/t)\|B_t(T_s l)\|_H^2 =: \|C^*(T_s l)\|_H^2. \tag{8.10}$$

By assumption $((1/t)\|B_t l\|_H^2)_{0\leq t\leq 1}$ is bounded for each $l \in H$, so the uniform boundedness principle says that $\sup_{0\leq t\leq 1} \|B_t/\sqrt{t}\|_{\mathcal{L}(H)} =: c < \infty$. Thus the family of functions $f_t(l) := (1/t)\|B_t l\|_H^2$ for $0 \leq t \leq 1$ is equicontinuous on H, and so the convergence $f_t(l) \to \|C^* l\|_H^2$ is uniform on compacts. To check left differentiability, consider $0 < t < s$, and look at the ratio

$$\frac{\|B_s l\|_H^2 - \|B_{s-t} l\|_H^2}{t} = \frac{1}{t}\|B_t(T_{s-t} l)\|_H^2. \tag{8.11}$$

For fixed $s > 0$, $l \in H$ the set $K := (T_{s-t} l)_{0\leq t\leq s}$ is compact. So for any $\varepsilon > 0$, there exists $0 < t_0 \leq 1$ so that $t \leq t_0$ implies

$$\sup_{k\in K} \left|(1/t)\|B_t k\|_H^2 - \|C^* k\|_H^2\right| \leq \varepsilon. \tag{8.12}$$

For such t we have,

$$\begin{aligned} &\left|(1/t)\|B_t(T_{s-t} l)\|_H^2 - \|C^*(T_s l)\|_H^2\right| \\ &\qquad \leq \left|\|C^*(T_{s-t} l)\|_H^2 - \|C^*(T_s l)\|_H^2\right| + \varepsilon \\ &\qquad \leq 2c^2 \|l\|_H^2 \, \|T_{s-t} l - T_s l\|_H + \varepsilon. \end{aligned} \tag{8.13}$$

This implies that

$$\limsup_{t\downarrow 0} \left|(1/t)\|B_t(T_{s-t} l)\|_H^2 - \|C^*(T_s l)\|_H^2\right| \leq \varepsilon, \tag{8.14}$$

and since ε is arbitrary, we conclude that the left derivative $D^- g(s)$, is also equal to $\|C^*(T_s l)\|_H^2$. This shows that, in fact, g is continuously differentiable. □

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