JOURNAL OF SPACECRAFT AND ROCKETS
Vol. 42, No. 1, January–February 2005

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed six manuscript pages and three figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Generalized Model for Solar Sails

L. Rios-Reyes* and D. J. Scheeres†
University of Michigan, Ann Arbor, Michigan 48105-2140

Introduction

WE define a new methodology for the analytic description of the force and moment generated by a solar sail of arbitrary shape and surface optical properties. We find that the total force and moment generated by the sail can be completely defined by the computation of a number of constants of the sail, which are only functions of the sail geometry and independent of the incident light. Given these constants, some standard sail properties, and the incident light pressure and direction, the total force and moment acting on the sail can be computed using simple formulas. Specifically, we can characterize the force generated by a solar sail of arbitrary sail geometry under general illumination conditions with 18 numbers. To characterize the moment requires an additional 36 numbers. The advantages of this description are that these constants can be computed off-line and are defined for arbitrarily shaped sails, meaning that only one formalism must be coded to deal with all types of sails.

There are a number of limits to this approach. First, this model tacitly assumes that the sail will not change shape as its attitude varies, thus their utility might be somewhat limited or constrained to relatively small angular motions. This limitation will be addressed in the future. Second, it assumes that there is no self-shadowing, that is, that every element of the sail surface will "see" an incident ray. This is a reasonable assumption given the likely mode in which a solar sail will be operated.

Solar Radiation Pressure

McInnes[1] develops the solar pressure caused by the sun's finite disk on an ideal sail normal to the sun, which includes the force exerted on the sail caused by impinging and reflected photons. The radiation pressure at a distance r from the sun as a result of a finite solar disk is[1]

$$P(r) = P^*(r)F(r) \tag{1}$$

where $P^*(r)$ is the radiation pressure of a point source given by[1]

$$P^*(r) = (I_0\pi/c)(R_s/r)^2 \tag{2}$$

$$F(r) = \tfrac{2}{3}(r/R_s)^2\left\{1 - \left[1 - (R_s/r)^2\right]^{\frac{3}{2}}\right\} \tag{3}$$

where c is the speed of light, r is the distance from the sun, R_s is the sun's radius, and I_0 is the frequency integrated specific intensity. $F(r)$ is a correction function to account for the sun's finite disk. With this formalism we assume the solar radiation travels in parallel rays when it reaches the sail.

Sail Optical Parameters and Differential Forces

The total force acting on the solar sail is caused by a combination of forces that result from photons impinging on and reflecting from the sail surface, as shown in Fig. 1. The sail is assumed to be opaque so that the transmissivity is zero; thus, the sum of the reflectivity ρ and the absorptivity a must be unity. Also, ρ and a might be dependent on the angle between the incident light source direction and the surface normal α and the wavelength; we do not account for these dependencies. The dependency on wavelength is not a major problem; as the incident radiation contains the whole electromagnetic spectrum, it is sufficient to use an integrated averaged value over the spectrum for both ρ and a. The dependence on α is more complex. Its effects on the sail are dependent on the law that governs this change, which might be different for different sail materials. At this stage we assume that the values of ρ and a do not depend on α.

The first differential force is caused by the impact of photons on the sail $\mathrm{d}\boldsymbol{F}_a$. The differential force caused by reflection $\mathrm{d}\boldsymbol{F}_r$ is composed of two components: $\mathrm{d}\boldsymbol{F}_{\mathrm{rs}}$, the fraction caused by specular reflection acting along the normal and transverse directions, and $\mathrm{d}\boldsymbol{F}_{\mathrm{rd}}$, the fraction caused by diffuse or uniform reflection acting along the normal direction. According to Meyer-Arendt,[2] a Lambertian surface has the same radiance in all directions. The force caused by emission $\mathrm{d}\boldsymbol{F}_e$ is as a result of absorbed photons that are radiated as heat and acts along the normal direction. When the sail absorbs photons, its temperature increases to an equilibrium temperature where the absorbed energy is equal to the radiated energy. Performing an energy balance, it can be shown that the equilibrium temperature of the sail is given by[1]

$$T^4 = \frac{(1-\rho)cP\cos(\alpha)}{\sigma(\epsilon_f + \epsilon_b)} \tag{4}$$

where ϵ is the surface emissivity, the subscripts f and b denote the front and back surfaces, respectively, and σ is the Stefan–Boltzmann constant.

Defining the unit normal vector $\hat{\boldsymbol{n}}$ perpendicular to a surface of an area $\mathrm{d}A$ and the transverse vector $\hat{\boldsymbol{t}}$, perpendicular to $\hat{\boldsymbol{n}}$ and in the

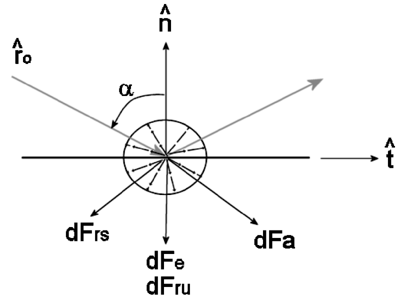

Fig. 1 Force directions.

Presented as AAS Paper 2004-286 at the AAS/AIAA 14th Space Flight Mechanics Meeting, Maui, HI, 8–12 February 2004; received 13 March 2004; revision received 8 September 2004; accepted for publication 9 September 2004.

*Graduate Student, Department of Aerospace Engineering, 1320 Beal Avenue; leonelr@umich.edu. Student Member AIAA.
†Associate Professor, Department of Aerospace Engineering, 1320 Beal Avenue; scheeres@umich.edu. Associate Fellow AIAA.

plane of the incident light and the surface normal, the forces acting on a differential area of the sail along these directions are given by Eqs. (5–8). They represent the contribution from radiation impacting the sail $d\boldsymbol{F}_a$, reflected specularly $d\boldsymbol{F}_{rs}$, and diffusively $d\boldsymbol{F}_{ru}$ from it, and emitted by radiation from the sail $d\boldsymbol{F}_e$, respectively[1]:

$$d\boldsymbol{F}_a = P(r)\cos\alpha[-\cos\alpha\hat{\boldsymbol{n}} + \sin\alpha\hat{\boldsymbol{t}}]\,dA \tag{5}$$

$$d\boldsymbol{F}_{rs} = P(r)\cos\alpha\rho s[-\cos\alpha\hat{\boldsymbol{n}} - \sin\alpha\hat{\boldsymbol{t}}]\,dA \tag{6}$$

$$d\boldsymbol{F}_{ru} = -P(r)\cos\alpha B_f\rho(1-s)\hat{\boldsymbol{n}}\,dA \tag{7}$$

$$d\boldsymbol{F}_e = -P(r)\cos\alpha(1-\rho)\frac{\epsilon_f B_f - \epsilon_b B_b}{\epsilon_f + \epsilon_b}\hat{\boldsymbol{n}}\,dA \tag{8}$$

where B_f and B_b are the sail front and back surface Lambertian coefficients.

Regrouping these elements, the differential force normal to the sail element can be expressed as

$$d\boldsymbol{F}_n = -P(r)[a_1\cos^2\alpha + a_2\cos\alpha]\,dA\hat{\boldsymbol{n}} \tag{9}$$

where $a_1 = 1 + \rho s$, $a_2 = B_f(1-s)\rho + (1-\rho)[(\epsilon_f B_f - \epsilon_b B_b)/(\epsilon_f + \epsilon_b)]$. The differential transverse force is given by

$$d\boldsymbol{F}_t = P(r)a_3\cos\alpha\sin\alpha\,dA\hat{\boldsymbol{t}} \tag{10}$$

where $a_3 = 1 - \rho s$. The sun's position unit vector is specified as $\hat{\boldsymbol{r}}_0$ and points from the sun to the sail. Thus, the angle α is defined by $\cos\alpha = -\hat{\boldsymbol{r}}_0 \cdot \hat{\boldsymbol{n}}$.

The total force as a result of these normal and transverse components is found by integrating these expressions over the sail surface:

$$\boldsymbol{F} = \int_A (d\boldsymbol{F}_n + d\boldsymbol{F}_t) \tag{11}$$

where we note that the normal and transverse directions will change as one moves over a general sail shape.

To derive the total force, we note that Eqs. (9) and (10) require knowledge of $\cos\alpha$, $\sin\alpha$, and $\hat{\boldsymbol{t}}$, which can be obtained from

$$\cos\alpha = -\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}}_0 \tag{12}$$

$$\sin\alpha = \|\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times -\hat{\boldsymbol{r}}_0)\| \tag{13}$$

$$\hat{\boldsymbol{t}} = -\frac{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{r}}_0)}{\|\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{r}}_0)\|} \tag{14}$$

The force equations stated in this form lead to difficulties when trying to carry out the surface integrations analytically. If the sail surface, and therefore its normal vector, is not simple, analytic solutions cannot be found in general. Also, the integrals are strongly dependent on the sun's position, apparently making it very difficult to generalize the integrals for any sail attitude.

Derivation of the Generalized Sail Force Equation

We have found that the integration of Eq. (11) can be reduced to an integration over the sail, independent of the incident light direction and magnitude (under an ideal solar sail assumption where the structure is fixed).

Let the normal vector at any point on the sail be defined as $\hat{\boldsymbol{n}} = [\hat{n}_1\ \hat{n}_2\ \hat{n}_3]^T$ and define the cross product as the dot product of a dyad and a vector[3]:

$$\hat{\boldsymbol{n}} \times -\hat{\boldsymbol{r}}_0 = -\tilde{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}}_0 \tag{15}$$

where in the standard basis

$$\tilde{\boldsymbol{n}} = \begin{bmatrix} 0 & -\hat{n}_3 & \hat{n}_2 \\ \hat{n}_3 & 0 & -\hat{n}_1 \\ -\hat{n}_2 & \hat{n}_1 & 0 \end{bmatrix} \tag{16}$$

We note the following cross-product identities written in dyadic form:

$$\tilde{\boldsymbol{a}} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \tilde{\boldsymbol{b}} = -\tilde{\boldsymbol{b}} \cdot \boldsymbol{a} = -\boldsymbol{b} \cdot \tilde{\boldsymbol{a}} \tag{17}$$

Note that Eqs. (13) and (14) can be multiplied to obtain

$$\sin\alpha\hat{\boldsymbol{t}} = -\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{r}}_0) = -\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}}_0 \tag{18}$$

Following similar simplifications, Eqs. (9) and (10) become

$$d\boldsymbol{F}_n = -P(r)[a_1(\hat{\boldsymbol{r}}_0 \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}}_0) - a_2(\hat{\boldsymbol{r}}_0 \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}]\,dA \tag{19}$$

$$d\boldsymbol{F}_t = P(r)a_3(\hat{\boldsymbol{r}}_0 \cdot \hat{\boldsymbol{n}})\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}}_0\,dA \tag{20}$$

Some of the terms in the preceding expressions can be simplified by the introduction of a dyadic (and triadic) notation.[3] It is possible to define the dyadic product of the normal vector as

$$\bar{\bar{\boldsymbol{n}}} = \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} \tag{21}$$

and the triadic as

$$\bar{\bar{\bar{\boldsymbol{n}}}} = \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}\hat{\boldsymbol{n}} \tag{22}$$

These are rank 2 and 3 tensors and can also be specified as $\bar{\bar{n}}_{ij} = \hat{n}_i\hat{n}_j$ and $\bar{\bar{\bar{n}}}_{ijk} = \hat{n}_i\hat{n}_j\hat{n}_k$, where the indices range from 1 to 3. Also, making use of the identity

$$\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{n}} = -\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}\bar{\bar{\boldsymbol{U}}} + \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} = -\bar{\bar{\boldsymbol{U}}} + \bar{\bar{\boldsymbol{n}}} \tag{23}$$

where $\bar{\bar{\boldsymbol{U}}}$ is the identity dyadic ($\bar{\bar{\boldsymbol{U}}}_{ij} = 0$ if $i \neq j$, $\bar{\bar{\boldsymbol{U}}}_{ij} = 1$ if $i = j$), the differential forces can be stated as

$$d\boldsymbol{F}_n = -P(r)a_1\hat{\boldsymbol{r}}_0 \cdot \bar{\bar{\bar{\boldsymbol{n}}}}\,dA \cdot \hat{\boldsymbol{r}}_0 + P(r)a_2\hat{\boldsymbol{r}}_0 \cdot \bar{\bar{\boldsymbol{n}}}\,dA \tag{24}$$

$$d\boldsymbol{F}_t = -P(r)a_3\hat{\boldsymbol{r}}_0 \cdot \hat{\boldsymbol{n}}\bar{\bar{\boldsymbol{U}}}\,dA \cdot \hat{\boldsymbol{r}}_0 + P(r)a_3\hat{\boldsymbol{r}}_0 \cdot \bar{\bar{\bar{\boldsymbol{n}}}}\,dA \cdot \hat{\boldsymbol{r}}_0 \tag{25}$$

The products of these tensors and the sun's position unit vector can be stated in terms of the summation convention as

$$\hat{\boldsymbol{r}}_0 \cdot \bar{\bar{\bar{\boldsymbol{n}}}} \cdot \hat{\boldsymbol{r}}_0 = \hat{n}_i\hat{n}_j\hat{n}_k\hat{r}_{0_i}\hat{r}_{0_k}, \qquad \hat{\boldsymbol{r}}_0 \cdot \bar{\bar{\boldsymbol{n}}} = \hat{n}_i\hat{n}_j\hat{r}_{0_i}, \qquad \hat{\boldsymbol{r}}_0 \cdot \bar{\boldsymbol{n}} = \hat{n}_i\hat{r}_{0_i}$$

where $\hat{n}_i$, $\hat{n}_j$, and $\hat{n}_k$ are the ith, the jth, and the kth scalar components of the normal vector, and equal indices imply summation, that is,

$$a_ib_i = \sum_{i=1}^{3} a_ib_i$$

Adding Eqs. (24) and (25), and integrating over the sail surface, the total force is

$$\boldsymbol{F} = P(r)\left[\int_A a_2\bar{\bar{\boldsymbol{n}}}\,dA \cdot \hat{\boldsymbol{r}}_0 + \hat{\boldsymbol{r}}_0 \cdot \left(-2\int_A \rho s\bar{\bar{\bar{\boldsymbol{n}}}}\,dA - \bar{\bar{\boldsymbol{U}}}\int_A a_3\hat{\boldsymbol{n}}\,dA\right) \cdot \hat{\boldsymbol{r}}_0\right] \tag{26}$$

The integrands of all of these expressions are independent of the solar incidence direction $\hat{\boldsymbol{r}}_0$ and can be computed off-line for a given sail shape, reused over a range of solar incidence directions, and ideally can accommodate nonuniformities in the sail optical properties.

We can introduce a more systematic notation for these integrals. Define the surface normal distribution integrals as

$$\boldsymbol{J}^m = \int_A \hat{\boldsymbol{n}}^m\,dA \tag{27}$$

$$= \int_A \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}\ldots\hat{\boldsymbol{n}}\,dA \tag{28}$$

where $\boldsymbol{J}^m$ is a rank-m tensor, computed by integrating the outer product of the normal vectors over the surface area of the sail.

Assuming constant sail optical properties, the force can now be rewritten as

$$\boldsymbol{F} = P(r)\left[a_2\boldsymbol{J}^2 \cdot \hat{\boldsymbol{r}}_0 - 2\rho s\hat{\boldsymbol{r}}_0 \cdot \boldsymbol{J}^3 \cdot \hat{\boldsymbol{r}}_0 - a_3(\boldsymbol{J}^1 \cdot \hat{\boldsymbol{r}}_0)\hat{\boldsymbol{r}}_0\right] \quad (29)$$

With this formalism, it is also possible to accommodate sails with varying optical properties; these integrals would be redefined by keeping the sail constants a_i inside the integrals. Thus, we have arrived at a completely analytic formula for the force acting on an arbitrary solar sail.

To properly specify $\boldsymbol{J}^m$ as a tensor, we can use the notation

$$J^m_{i_1 i_2 \ldots i_m} = \int_A \hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_m}\,\mathrm{d}A \quad (30)$$

$$i_j = 1, 2, 3 \quad (31)$$

where the entries $\hat{n}_i$ are just the elements of the normal vector evaluated at the surface element dA. We note that these tensors are completely symmetric in their indices, that is, $J^m_{i_1 i_2 \ldots i_m} = J^m_{i_2 i_1 \ldots i_m}$, and so on for any two indices. Thus, for a rank-3 tensor, which could have up to 27 entries, we only need to compute nine independent values. In general, a tensor $\boldsymbol{J}^m$ as just defined will only have $3m$ unique terms among its 3^m entries. Thus, the three integrals in Eq. (29) are specified by $3+6+9=18$ numbers for the general case. If we are dealing with a simplified model, such as a flat-plate solar sail model where $\hat{\boldsymbol{n}} = [0, 0, 1]^T$, the number of independent numbers would reduce to 3, one for each $\boldsymbol{J}^m$.

Because these tensors are defined as integrations, it is always possible to add additional sail elements by adding the $\boldsymbol{J}^m$ term for that additional piece, so long as they are computed relative to the same coordinate frame.

Because we are dealing with tensors, it is also possible to transform a given $\boldsymbol{J}^m$ defined in one coordinate frame into a different coordinate frame. Suppose we have a $\boldsymbol{J}^m$ defined for a panel of our sail, computed in the panel-fixed frame. Also assume we have a transformation matrix T that takes a vector from the panel-fixed frame into the sail-fixed frame. Thus, to transform a normal vector $\hat{\boldsymbol{n}}$ from the panel-fixed frame to the sail-fixed frame we just perform a matrix multiply, $\hat{\boldsymbol{n}}' = T\hat{\boldsymbol{n}}$, where the $'$ signifies that the vector is specified in the new frame. Using tensor notation, this same transformation would be expressed as $\hat{n}'_j = T^i_j\hat{n}_i$, where the i index signifies the column number for the T matrix and j signifies the row number, and the summation convention is assumed again i.e.,

$$T^i_j\hat{n}_i = \sum_{i=1}^{3} T^i_j\hat{n}_i$$

Then the following operations would transform the $\boldsymbol{J}^m$ tensor computed relative to the panel frame into the sail-fixed frame, where they could be directly added to obtain the sail's complete $\boldsymbol{J}^m$ tensors. As these transformation matrices are known in general, this would be a simple operation to define and extremely simple to carry out in an algorithm. Given a transformation T, then the general coordinate transformation result for a tensor $\boldsymbol{J}^m$ is

$$J^{m'}_{j_1 j_2 \ldots j_m} = T^{i_1}_{j_1}T^{i_2}_{j_2}\ldots T^{i_m}_{j_m}J^m_{i_1 i_2 \ldots i_m} \quad (32)$$

Derivation of the Generalized Sail Moment Equation

The total moment acting on the sail can be found by integrating the expression

$$\mathrm{d}\boldsymbol{M} = \boldsymbol{r} \times \mathrm{d}\boldsymbol{F} = \tilde{\boldsymbol{r}} \cdot \mathrm{d}\boldsymbol{F} \quad (33)$$

over the entire sail. This is equivalent to

$$\mathrm{d}\boldsymbol{M} = P(r)\tilde{\boldsymbol{r}} \cdot [a_2\bar{\bar{\boldsymbol{n}}}\,\mathrm{d}A \cdot \hat{\boldsymbol{r}}_0 + \hat{\boldsymbol{r}}_0 \cdot (-2\rho s\bar{\bar{\bar{\boldsymbol{n}}}}\,\mathrm{d}A - a_3\hat{\boldsymbol{n}}\bar{\bar{\boldsymbol{U}}}\,\mathrm{d}A) \cdot \hat{\boldsymbol{r}}_0] \quad (34)$$

where $\boldsymbol{r}$ is the position of the differential element dA with respect to a given reference frame on the sail. Integrating yields the total moment about the origin of the sail reference frame:

$$\boldsymbol{M} = P(r)\left[\int_A a_2\tilde{\boldsymbol{r}} \cdot \bar{\bar{\boldsymbol{n}}}\,\mathrm{d}A \cdot \hat{\boldsymbol{r}}_0 - 2\hat{\boldsymbol{r}}_0 \cdot \left(\int_A \rho s\tilde{\boldsymbol{r}} \cdot \bar{\bar{\bar{\boldsymbol{n}}}}\,\mathrm{d}A \cdot \hat{\boldsymbol{r}}_0\right) - \hat{\boldsymbol{r}}_0 \cdot \int_A a_3\hat{\boldsymbol{n}}\tilde{\boldsymbol{r}} \cdot \bar{\bar{\boldsymbol{U}}} \cdot \hat{\boldsymbol{r}}_0\,\mathrm{d}A\right] \quad (35)$$

Defining the moment surface normal distribution integrals as

$$\boldsymbol{K}^m = \int_A \tilde{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}^m\,\mathrm{d}A \quad (36)$$

$$\boldsymbol{L} = \int_A \hat{\boldsymbol{n}}\boldsymbol{r}\,\mathrm{d}A \quad (37)$$

and assuming constant optical properties, the moment can be rewritten as

$$\boldsymbol{M} = P(r)\left[a_2\boldsymbol{K}^2 \cdot \hat{\boldsymbol{r}}_0 - 2\rho s\hat{\boldsymbol{r}}_0 \cdot \left(\boldsymbol{K}^3 \cdot \hat{\boldsymbol{r}}_0\right) - a_3\hat{\boldsymbol{r}}_0 \cdot \boldsymbol{L} \cdot \tilde{\hat{\boldsymbol{r}}}_0\right] \quad (38)$$

The tensors $\boldsymbol{K}^m$ and $\boldsymbol{L}$ (which are rank-m and rank-2 tensors, respectively) do not have the same complete symmetry as the $\boldsymbol{J}^m$, and more numbers are needed to specify them. For example, $\boldsymbol{L}$ and $\boldsymbol{K}^2$ require at least nine numbers, and $\boldsymbol{K}^3$ requires 18. (It is symmetric in its second and third index.) Despite this, they are well defined and can be used to characterize the moments acting on the body as the sail attitude changes, which can always be recast as a change in the incident light direction of the sun $\hat{\boldsymbol{r}}_0$. Transformations from different coordinate frames might be necessary in some cases. For these situations the use of Eq. (32) will still be appropriate, so long as the transformation is a pure rotation and does not involve translation. If the panel is to be translated as well, an additional term $\boldsymbol{r}_t \times \boldsymbol{F}$ must be subtracted, where $\boldsymbol{r}_t$ is the translation vector and $\boldsymbol{F}$ is the total force acting on that panel.

Center of Pressure

One particular application of the current result is to find the center of pressure of the sail $\boldsymbol{r}_p$. In general, this vector is defined by the condition

$$\boldsymbol{M} = \boldsymbol{r}_p \times \boldsymbol{F} \quad (39)$$

$$= \boldsymbol{r}_p \cdot \tilde{\boldsymbol{F}} \quad (40)$$

where $\boldsymbol{F}$ is the total computed force and $\boldsymbol{M}$ is the total computed moment for a given origin. Let us dot both sides on the right with $-\tilde{\boldsymbol{F}}$ to obtain

$$-\boldsymbol{r}_p \cdot \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}} = -\boldsymbol{M} \cdot \tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}} \cdot \boldsymbol{M} \quad (41)$$

Dividing by the force magnitude squared F^2, this equation becomes

$$\boldsymbol{r}_p \cdot [\bar{\bar{\boldsymbol{U}}} - \hat{\boldsymbol{F}}\hat{\boldsymbol{F}}] = (1/F^2)\tilde{\boldsymbol{F}} \cdot \boldsymbol{M} \quad (42)$$

The terms in the brackets are dyads (rank-2 tensors) that project the center-of-pressure vector into a vector perpendicular to the force line. Taking the pseudoinverse of this operator yields the vector from the coordinate origin to the center of pressure, which has an arbitrary component along the line of action for the sail force:

$$\boldsymbol{r}_p = (1/F^2)\boldsymbol{F} \times \boldsymbol{M} + \sigma\hat{\boldsymbol{F}} \quad (43)$$

where σ is an arbitrary distance.

Finally, if we are also given the center of mass of the sail $\boldsymbol{r}_{\mathrm{CM}}$ and the center of pressure $\boldsymbol{r}_p$, we can compute the total moment acting on the sail about its center of mass:

$$\boldsymbol{M}_{\mathrm{CM}} = (\boldsymbol{r}_p - \boldsymbol{r}_{\mathrm{CM}}) \times \boldsymbol{F} \quad (44)$$

$$= \boldsymbol{M} - \boldsymbol{r}_{\mathrm{CM}} \times \boldsymbol{F} \quad (45)$$

which is of interest for attitude dynamics computations.

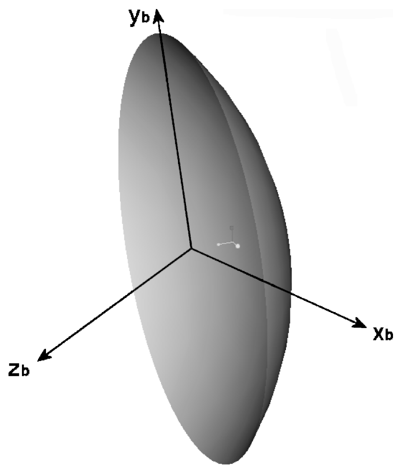

Fig. 2 Circular sail geometry.

Example: Circular Sail with Billow

Consider a circular sail with a fixed nonplanar surface, defined by (Fig. 2)

$$z_b = -(\alpha_{\max}/2R_0)\left(x_b^2 + y_b^2\right) + \alpha_{\max} R_0/2 \tag{46}$$

where R_0 is the sail radius, $\alpha_{\max}$ the surface slope at the rim (and must be negative for this case), and x_b, y_b, z_b are the sail coordinates in the body-fixed frame as shown in Fig. 2. The slope varies linearly with distance from center. The normal vector is obtained by setting Eq. (46) equal to zero and taking the gradient to obtain

$$\hat{\boldsymbol{n}} = \left[1 \bigg/ \sqrt{1 + \left(\frac{\alpha_{\max}}{R_0}\right)^2 x_b^2 + \left(\frac{\alpha_{\max}}{R_0}\right)^2 y_b^2}\,\right] \begin{bmatrix} \dfrac{\alpha_{\max}}{R_0} x_b \\ \dfrac{\alpha_{\max}}{R_0} y_b \\ 1 \end{bmatrix} \tag{47}$$

With this information, the force coefficient integrals can be computed analytically and are given by $J_1^1 = J_2^1 = 0$, $J_3^1 = \pi R_0^2$; $J_{ij}^2 = 0$ for $i \neq j$ and $J_{11}^2 = J_{22}^2 = \pi R_0^2[2 + (-2+\alpha_{\max})\sqrt{(1+\alpha_{\max}^2)}]/(3\alpha_{\max}^2)$, $J_{33}^2 = 2\pi R_0^2[\sqrt{(1+\alpha_{\max}^2)} - 1]/\alpha_{\max}^2$; $J_{ijk}^3 = 0$ except for $J_{311}^3 = J_{131}^3 = J_{322}^3 = J_{232}^3 = J_{113}^3 = J_{223}^3 = \pi R_0^2[\alpha_{\max}^2 - \log(1+\alpha_{\max}^2)]/(2\alpha_{\max}^2)$ and $J_{333}^3 = \pi R_0^2 \log(1+\alpha_{\max}^2)/\alpha_{\max}^2$.

The moment coefficients with respect to the origin can also be computed analytically and are given by $L_{ij} = 0$ for $i \neq j$ and $L_{ii} = \pi R_0^3 \alpha_{\max}/4$; $K_{ij}^2 = 0$ except for $K_{12}^2 = -K_{21}^2 = \pi R_0^3[6 - 5\alpha_{\max}^2 - \sqrt{(1+\alpha_{\max}^2)}(6 - 8\alpha_{\max}^2 + \alpha_{\max}^4)]/(15\alpha_{\max}^3)$; $K_{ijk}^3 = 0$ except for $K_{231}^3 = K_{213}^3 = -K_{123}^3 = -K_{132}^3 = -\pi R_0^3[\alpha_{\max}^2(-2+\alpha_{\max}^2) - 2(-1+\alpha_{\max}^2)\log(1+\alpha_{\max}^2)]/(8\alpha_{\max})$.

Using the coefficients in Eqs. (29) and (38), it is possible to compute the exact force and moment acting on this particular sail shape as a function of solar illumination. Using Eq. (43), the center of pressure can be computed.

Conclusions

The forces and moments acting on a solar sail of arbitrary shape and optical properties can be determined analytically by computing a series of coefficients. If the optical properties are constant, then the surface integrals $\boldsymbol{J}^1, \boldsymbol{J}^2, \boldsymbol{J}^3, \boldsymbol{L}, \boldsymbol{K}^2$, and $\boldsymbol{K}^3$ define the sail specific coefficients for forces and moments. With this formalism, force and moment computations for completely different sail models can be handled by using a generic set of analytic equations. These distributions are nominally constant in the sail-fixed frame, and thus cannot directly represent the effect of changing sail structure geometry. Future research will address this limitation.

Acknowledgment

This research was supported by a grant from the Jet Propulsion Laboratory, California Institute of Technology, which is under contract with NASA.

References

[1]McInnes, C. R., *Solar Sailing: Technology, Dynamics and Mission Applications*, Springer-Verlag, Chichester, England, U.K., 1999, pp. 43–49.

[2]Meyer-Arendt, J. R., *Introduction to Classical and Modern Optics*, 3rd ed., Prentice–Hall, Upper Saddle River, NJ, 1989, pp. 371, 372.

[3]Greenwood, D. T., *Principles of Dynamics*, 2nd ed., Prentice–Hall, Upper Saddle River, NJ, 1988, pp. 315–318.

A. Ketsdever
Associate Editor