Differential and Integral Equations, Volume 7, Number 2, March 1994, pp. 323–343.

GENERALIZED MOTION OF HYPERSURFACES WHOSE GROWTH SPEED DEPENDS SUPERLINEARLY ON THE CURVATURE TENSOR

SHUN'ICHI GOTO*
Department of Applied Science, Faculty of Engineering
Kyushu University, Fukuoka 812, Japan

(Submitted by: Y. Giga)

Abstract. We prove a comparison principle for viscosity solution with finite speed for its level set, which solves degenerate parabolic equations with discontinuity. We also prove the (global) existence of solution in the class of viscosity solution with finite speed for the initial value problem. Our comparison and existence results yield a unique global-in-time generalized solution to interface evolution equations whose speed grows superlinearly in curvature tensors.

1. Introduction. Let $\Gamma(t)$ be an interface bounding the whole space R^N $(N \geq 2)$ into two phases at time $t \geq 0$. To write down the equation of $\Gamma(t)$ we temporarily assume that $\Gamma(t)$ is the smooth boundary of the open set $D(t)$. The evolution of $\Gamma(t)$ that we consider here depends locally on its normal vector field and curvature tensors.

Let $\vec{n} = \vec{n}(t,x)$ denote the unit exterior normal vector field to $\Gamma(t) = \partial D(t)$ at $x \in \Gamma(t)$. It is convenient to extend $\vec{n}$ to a vector field, still denoted by $\vec{n}$, on a tubular neighborhood of $\Gamma(t)$ such that $\vec{n}$ is constant in the normal direction of $\Gamma(t)$. Let $V = V(t,x)$ denote the growth speed of $\Gamma(t)$ at $x \in \Gamma(t)$ in the exterior normal direction. In this paper, as a continuation of [5] and [14], we study the evolution equation of form

$$V = f(\vec{n}, \nabla\vec{n}) \quad \text{on } \Gamma(t), \quad t > 0. \tag{1.1a}$$

Here f is a given continuous function and ∇ stands for spatial derivatives. We are interested in constructing global-in-time solutions (family) $\{\Gamma(t)\}_{t\geq 0}$ to the evolution equation (1.1a) under the initial condition

$$\Gamma(t)|_{t=0} = \Gamma_0, \tag{1.1b}$$

where Γ_0 is an arbitrary given (compact) initial interface.

Received May 1993, in revised form August 1993.
*Partially supported by Grant-in-Aid for scientific Research (No. 05740104), Ministry of Education, Science and Culture.
AMS Subject Classifications: 35B05, 35K22, 35K65.

Since the solution of (1.1a,b) may develop singularities in a finite time, even for the mean curvature flow equation

$$V = -\operatorname{div} \vec{n} \tag{1.2}$$

with smooth initial data (see [20], and also [1], [2], [29]), we are forced to introduce a notion of generalized solution in order to track down the evolution of $\Gamma(t)$ at all time.

The generalized solution was introduced by Chen, Giga and the author (see [5], and also [14]), and independently by Evans and Spruck (see [9] where only (1.2) is discussed). The basic idea in both approaches is to describe the interface $\Gamma(t)$ in the form of a level set of some function u, called a *definition function* of $\Gamma(t)$, and then to discuss the evolution of u. We call this the *level set approach.*

If equation (1.1a) is degenerate parabolic and if f grows linearly in $\nabla\vec{n}$, then we can claim the unique global existence of the generalized solution for (1.1a,b) (see [5]). A typical example is the mean curvature flow equation (1.2). Our goal is to extend these results to the case of the function f which is superlinear in $\nabla\vec{n}$, for example

$$V = -(\operatorname{div} \vec{n})^3, \tag{1.3}$$

so we only assume here that (1.1a) is degenerate parabolic.

Other important examples were found in two-phase thermomechanics by Gurtin (see [21] and references therein):

$$\beta(\vec{n}, V)V = -\Big(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\frac{\partial H}{\partial p_i}(\vec{n}) + c\Big), \tag{1.4}$$

where β is a positive function, H is positively homogeneous of degree one and c is a constant. If $\beta \equiv 1$, $H = |p|$ and $c = 0$, then (1.4) is the mean curvature flow equation (1.2). When β is independent of V, equation (1.4) is linear in $\nabla\vec{n}$ (see [5] and [26]). We are interested in the case of β depending on V. For example, if $\beta = V^{-2/3}$, $H = |p|$ and $c = 0$, then equation (1.4) is equal to (1.3).

To explain the difficulty in solving (1.3), we recall the level set approach (see e.g. [14]). Let us assume the existence of the (smooth) interface $\Gamma(t)$ for all $t \in [0, \infty)$. Let u be a real valued (continuous) function over $[0, \infty) \times R^N$ such that

$$\Gamma(t) = \{x \in R^N : u(t,x) = 0\} \quad \text{and} \quad D(t) = \{x \in R^N : u(t,x) > 0\}. \tag{1.5}$$

If the function u is C^2 and $\nabla u \neq 0$ near $\Gamma(t)$, then the relation

$$\vec{n} = -\nabla u/|\nabla u| \quad \text{and} \quad \nabla\vec{n} = -Q_q(\nabla^2 u)/|\nabla u|; \quad q = \nabla u/|\nabla u| \tag{1.6}$$

holds on $\Gamma(t)$. Here and hereafter we use the notation

$$Q_q(X) = (I - q \otimes q)X(I - q \otimes q),$$

where X is an $N \times N$ matrix, I is the $N \times N$ identity matrix and $\otimes$ stands for the tensor product of two vectors. By using (1.6) and $V = \partial_t u/|\nabla u|$ on $\Gamma(t)$, equation (1.1a) is formally equivalent to

$$\partial_t u + F(\nabla u, \nabla^2 u) = 0 \quad \text{on } \Gamma(t),$$

called the *level set equation*, with

$$F(p,X) = -|p|f(-\overline{p}, -Q_{\overline{p}}(X)/|p|); \quad \overline{p} = p/|p|.$$

In our strategy, we first consider the initial value problem

$$\partial_t u + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0,\infty) \times R^N, \tag{1.7a}$$

$$u(0,x) = a(x) \quad \text{for } x \in R^N, \tag{1.7b}$$

for some continuous function a satisfying

$$\Gamma_0 = \{x \in R^N : a(x) = 0\} \quad \text{and} \quad D_0 = \{x \in R^N : a(x) > 0\}.$$

Here D_0 is a (initial) domain whose boundary is $\Gamma_0 = \partial D_0$ (or $\Gamma_0 \supset \partial D_0$ in general). Since equation (1.7a) is degenerate parabolic with singularity at $\nabla u = 0$, we apply the theory of viscosity solution to the initial value problem (1.7a,b). Viscosity solution is a kind of weak solution of fully nonlinear degenerate parabolic equations for which the comparison principle holds (see [7] and references therein).

When $F(p,X)$ is continuously extended to $(p,X) = (0,O)$, the comparison principle holds without any restrictive conditions (see [5], and also [17]). In the superlinear case, for example (1.3), F is not extended to $(p,X) = (0,O)$ continuously, because a direct calculation shows

$$F_*(0,O) = -\infty \quad \text{and} \quad F^*(0,O) = \infty,$$

where F_* and F^* denote, respectively, the lower semicontinuous and the upper semicontinuous relaxations (envelopes) of F. In this case, the comparison principle does not necessarily hold for all solutions which solve (1.7a) in the sense of viscosity solution, since the class of viscosity solutions is too big.

To overcome this difficulty we introduce a notion of *solution with finite speed* for its level set; in other words, each level set of the solution does not disappear suddenly. We establish a comparison principle by reductive absurdity. Although we use Crandall-Ishii's lemma (see e.g. [7]) instead of using sup and inf convolutions, the flavor of our proof is closer to that of [5] than that of [17]. We use a family of test functions parameterized by a vector of R^N. To handle $F(p,X)$ at the point $(p,X) = (0,O)$, we invoke a notion of finite speed together with a geometric lemma (due to Y.Giga) on balls touching a closed set.

We then establish the existence of solution in the class of viscosity solution with finite speed for the initial value problem (1.7a,b). Since it is not clear whether Perron's method applies to construct a solution with finite speed, we rather approximate the discontinuous function F by a continuous one F_k to get an approximate solution u^k. It turns out that our approximate solution u^k has a uniform bound of speed (independent of $k = 1, 2, \cdots$). Since a uniform limit of approximate solution u^k yields a viscosity solution u of (1.7a,b), this implies that u has a finite speed.

Our comparison and existence results extend the previous work in [5] to more general geometric equations of form (1.7a). Although we do not state explicitly, our theory applies to the case when F continuously depends on the time variable as in [5].

It is now rather standard that the comparison principle and the existence result yield a unique global-in-time generalized solution to our original problem (1.1a,b), via the relation (1.5). Since the equation (1.7a) is geometric, as shown in [5], the interface $\Gamma(t)$ is determined independently of the choice of the definition function a of Γ_0.

In Section 2 we introduce a notion of solution with finite speed. We state our main comparison and existence results of solutions with finite speed. In Section 3 we prepare to prove the comparison principle. Its proof is completed in Section 4 except a geometric lemma on balls touching a closed set which is proved in Section 6. In Section 5 we prove the existence of solution for the initial value problem (1.7a,b).

After this work was completed, Ishii and Souganidis introduced a restrictive condition for solution of the same problem (1.7a,b) (see [25]), which is different from ours. Their results can be applied to the case of noncompact interfaces; on the other hand, our comparison principle holds for more general nongeometric F than theirs. They assume that F is positively homogeneous of degree one.

After the level set approach was introduced, the generalized motion of hypersurfaces was studied by many authors using this method. Evans and Spruck proved several interesting results (see [10], [11], [12]) for generalized solutions of the mean curvature flow equation, and Ilmanen defined the level set flow on a manifold (see [22], [23]). We refer to [1], [2] and [29] for breaking out of singularities. Recently, Sternberg and Ziemer studied the Dirichlet problem for the mean curvature flow equation (see [30]). Giga and Sato proved the comparison principle under the Neumann boundary condition for the generalized equations of form (1.7a) (see [18], [19] and [27]). We also refer to [3], [15], [16], [26] and [28] for more developments.

Recently, three types of generalized solutions, Brakke's solution (see [4]), a singular limit solution of the Allen-Cahn equation and a solution by the level set approach are well-compared (see [8], [24]).

There are several examples of motion by speed depending superlinearly on principal curvatures. For example, Tso discussed the motion of convex surfaces by the Gauss-Kronecker curvature (see [31], also see [13] and [32] for the other examples of curvature flows). However, these works restrict themselves to classical evolutions.

2. Statement of results. In this paper we are concerned with the evolution in time for an interface $\Gamma(t)$ in R^N $(N \geq 2)$ satisfying

$$V = f(\vec{n}, \nabla\vec{n}) \quad \text{on } \Gamma(t), \quad t > 0, \tag{2.1a}$$

$$\Gamma(t)|_{t=0} = \Gamma_0. \tag{2.1b}$$

According to the level set approach, $\Gamma(t)$ is described by the (zero) level set of the definition function $u : [0,\infty) \to R$. We then study the initial value problem for the level set equation

$$\partial_t u + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0,\infty) \times R^N, \tag{2.2a}$$

$$u(0,x) = a(x) \quad \text{for } x \in R^N. \tag{2.2b}$$

Here F is defined by

$$F(p,X) = -|p| f(-\overline{p}, -Q_{\overline{p}}(X)/|p|); \quad \overline{p} = p/|p| \tag{F0}$$

and the initial data a satisfies

$$\Gamma_0 = \{x \in R^N : a(x) = 0\} \quad \text{and} \quad D_0 = \{x \in R^N : a(x) > 0\}, \tag{2.3}$$

where D_0 is a domain whose boundary is $\Gamma_0 = \partial D_0$ or contained to Γ_0 in general, i.e., $\Gamma_0 \supset \partial D_0$. This paper studies the case when Γ_0 is compact, then we may assume that $a \in K_\alpha(R^N)$ for some $\alpha < 0$, i.e.,

$$a(x) - \alpha \text{ is a continuous function with compact support in } R^N.$$

We introduce a weak notion of solution in order to get a global-in-time solution $\{\Gamma(t)\}_{t\geq 0}$ of the problem (2.1a,b). Since $F(p, X)$ is not continuously extended to $(p, X) = (0, O)$ in general, we introduce a restrictive condition for viscosity solution of (2.2a,b).

Definition (Finite speed condition). Let u be a function on $Q = (0, T) \times \Omega$, where $T > 0$ and Ω is a domain in R^N. Suppose that for each $R > 0$ there is $\nu = \nu(R) \geq 0$ such that

$$\sup_{(t,x)\in\triangle_\nu} u(t, x) \leq c \quad \text{with } \triangle_\nu = \{(t, x) : t \geq t_0, |x - x_0| \leq R - \nu(R)(t - t_0)\}$$

whenever $c \in R$ and $(t_0, x_0) \in Q$ satisfy $\sup_{|x-x_0|\leq R} u(t_0, x) \leq c$. We then say u *has an upper speed bound* $\nu(R)$ (for its level set). If u and $-u$ have upper speed bounds, we say u *has finite speed.*

Definition (Generalized solution). Let Γ_0 be a compact set in R^N. Let $a \in K_\alpha(R^N)$ for some $\alpha < 0$ satisfying (2.3). If u is a viscosity solution with finite speed (for its level set) of the initial value problem (2.2a,b) such that $u \in K_\alpha([0, T] \times R^N)$ for all $T > 0$, then $\{\Gamma(t)\}_{t\geq 0}$ defined by (1.5) is called a *generalized solution* of the original problem (2.1a,b).

We remark that this notion of the generalized solution requires $\Gamma(t)$ to be a closed set, not necessarily a hypersurface.

Our final goal in this paper is to prove

Theorem 2.1. *Suppose that* (2.1a) *is degenerate parabolic and f is continuous. If Γ_0 is compact, then there exists a global-in-time unique generalized solution $\{\Gamma(t)\}_{t\geq 0}$ of the initial value problem* (2.1a,b).

The key tool is the comparison principle for viscosity solution of (2.2a). We recall properties of the function F, which naturally follow from the hypotheses on the function f of Theorem 2.1, via the relation (F0).

$$F = F(p, X) : (R^N \setminus \{0\}) \times S^N \to R \quad \text{is continuous,} \tag{F1}$$

where S^N is the space of $N \times N$ real symmetric matrices. F is degenerate elliptic, i.e.,

$$F(p, X) \leq F(p, Y) \quad \text{if } X \geq Y. \tag{F2}$$

F is geometric, i.e.,

$$F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p, X) \quad \text{for all } \lambda > 0 \text{ and } \sigma \in R. \tag{F3}$$

We state the comparison principle under (F1), (F2) and

$$\begin{cases} \lim\limits_{\lambda\downarrow 0} \sup\{F(\lambda p, \lambda I) : m \leq |p| \leq M\} = 0 & \text{for all } m, M > 0, \\ \lim\limits_{\lambda\downarrow 0} \inf\{F(\lambda p, -\lambda I) : m \leq |p| \leq M\} = 0 & \text{for all } m, M > 0. \end{cases} \tag{F3$'$}$$

The property (F3$'$) is fulfilled if F is geometric.

Theorem 2.2 (Comparison principle). *Assume* (F1), (F2) *and* (F3$'$). *Suppose that* $T > 0$ *and* Ω *is a bounded domain in* R^N. *Let* u *and* v *be real valued functions over* Q *and a viscosity subsolution and a supersolution, respectively, of* (2.2a) *in* Q. *Suppose that* u^* *and* $-v_*$ *have upper speed bounds for all points in* Q. *If* $u^* \le v_*$ *on* $\partial_p Q$, *then* $u^* \le v_*$ *on* $\overline{Q}$.

Here $\partial_p Q$ is the parabolic boundary of $Q = (0,T) \times \Omega$ denoted by $\partial_p Q = [0,T] \times \partial\Omega \cup \{0\} \times \Omega$. Our comparison principle can be applied to nongeometric equations, for example

$$u_t = u_{xx}/|u_x|^\alpha \quad (0 \le \alpha < 1).$$

Note that this yields a porous medium type equation, $v_t = ((1-\alpha)^{-1} v^{1-\alpha})_{xx}$, with fast diffusion if we set $v = u_x$.

In Section 3 we recall the definition of viscosity solution and prepare to prove Theorem 2.2 whose proof is completed in Section 4.

Proposition. *The conditions* (F1)–(F3) *yield*

$$F(p,-I) \le c_-(|p|) \quad \textit{and} \quad F(p,I) \ge -c_+(|p|) \tag{F4}$$

for some positive functions $c_\pm \in C^1(0,\infty)$.

Proof. By (F1) and (F2) we see that $M(\sigma) = \sup_{|q|=1} F(q, -I/\sigma)$ is a lower semicontinuous and nonincreasing function over $(0,\infty)$. We find a positive function $M' \in C^1(0,\infty)$ with $M \le M'$. By setting $c_-(\sigma) = \sigma M'(\sigma)$ and (F3) we see

$$F(p,-I) = |p| F(p/|p|, -I/|p|) \le |p| \sup_{|q|=1} F(q, -I/|p|) \le c_-(|p|).$$

Similarly, $F(p,I) \ge -c_+(|p|)$ holds for some $c_+ \in C^1(0,\infty)$. $\square$

We remark that the functions $c_\pm$ control the growth speed of the level set of spherically symmetric solutions of (2.2a).

In Section 5 we show the existence result.

Theorem 2.3 (Existence). *Suppose that* F *satisfies* (F1)–(F3). *Let* $a \in K_\alpha(R^N)$ *for some* $\alpha \in R$. *Then there exists a global-in-time viscosity solution* u *of* (2.2a,b) *satisfying* $u \in K_\alpha([0,T] \times R^N)$ *for all* $T > 0$. *Moreover,* u *has finite speed depending only on* $c_\pm$.

The support of $u(t,\cdot) - \alpha$ is compact in a (finite) time interval $[0,T]$, so contained in a sufficiently large ball. Then we can apply Theorem 2.2 to the solution u of Theorem 2.3, which implies that the viscosity solution of (2.2a,b) with finite speed is unique.

The generalized solution $\{\Gamma(t)\}_{t\ge0}$ of (2.1a,b), defined by (1.5), is independent of the choice of the definition function a of Γ_0. Indeed, if u has an upper speed bound, so does $\theta(u)$ for nondecreasing, continuous $\theta : R \to R$. Since we have the comparison Theorem 2.2, this together with [5, Theorem 5.6] yields the uniqueness of $\{\Gamma(t)\}_{t\ge0}$ as in the proof of [5, Theorem 7.1]. The proof of Theorem 2.1 is now complete.

3. Preliminaries. For the reader's convenience, we state here the definition of viscosity solution for our problem (see [7] for the details). We consider the degenerate parabolic equation with singularity at $\nabla u = 0$, i.e.,

$$\partial_t u + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } Q = (0,T) \times \Omega, \tag{3.1}$$

where T is a positive number and Ω is a (bounded) domain in R^N.

Let $L \subset R^d$ be a set and h_k be a real valued function over L ($k = 1, 2, \cdots$). For the sequence $\{h_k\}_{k=1,2,\cdots}$, we define

$$(\operatorname*{lim_*}_{k\to\infty} h_k)(z) = \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \inf_{l\geq k} \inf\{h_l(\zeta) : |z-\zeta| < \varepsilon, \zeta \in L\} \quad \text{for } z \in \overline{L}.$$

In particular, when $h_k = h$ for all k, the limit $h_* = \operatorname*{lim_*}_{k\to\infty} h_k$ is called a lower semicontinuous relaxation (envelope) of h to $\overline{L}$. Clearly, we have

$$h_*(z) = \lim_{\varepsilon\to 0} \inf\{h(\zeta) : |z-\zeta| < \varepsilon, \zeta \in L\} \quad \text{for } z \in \overline{L}.$$

Similarly, we define

$$(\operatorname*{lim^*}_{k\to\infty} h_k)(z) = \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \sup_{l\geq k} \sup\{h_l(\zeta) : |z-\zeta| < \varepsilon, \zeta \in L\} \quad \text{for } z \in \overline{L},$$

and an upper semicontinuous relaxation (envelope) $h^* = \operatorname*{lim^*}_{k\to\infty} h_k$, when $h = h_k$ for all k. Clearly, we have

$$h^*(z) = \lim_{\varepsilon\to 0} \sup\{h(\zeta) : |z-\zeta| < \varepsilon, \zeta \in L\} \quad \text{for } z \in \overline{L}.$$

Definition (Viscosity solution). A function $u : Q \to R$ is called a viscosity sub- (resp. super-) solution of (3.1) on Q if $u^* < \infty$ (resp. $u_* > -\infty$) on $\overline{Q}$ and

$$\tau + F(p, X) \leq 0 \quad \text{for all } (t,x) \in Q \text{ and } (\tau, p, X) \in \mathcal{P}_Q^{2,+} u^*(t,x) \text{ with } p \neq 0$$
$$(\text{resp. } \tau + F(p, X) \geq 0 \quad \text{for all } (t,x) \in Q \text{ and } (\tau, p, X) \in \mathcal{P}_Q^{2,-} u_*(t,x) \text{ with } p \neq 0).$$

Moreover, if u is both a viscosity sub- and a supersolution of (3.1) on Q, u is called a viscosity solution of (3.1) on Q.

Here $\mathcal{P}_Q^{2,+}$ and $\mathcal{P}_Q^{2,-}$ denote the spaces of the parabolic super and sub 2-jets, respectively, i.e., for a function v defined near $(t,x) \in Q$, $\mathcal{P}_Q^{2,+}v(t,x)$ (resp. $\mathcal{P}_Q^{2,-}v(t,x)$) is the set of all $(\tau, p, X) \in R \times R^N \times S^N$ satisfying, as $(s,y) \to (t,x)$ in Q,

$$v(s,y) \leq v(t,x) + \tau(s-t) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t| + |y-x|^2)$$

(resp.

$$v(s,y) \geq v(t,x) + \tau(s-t) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t| + |y-x|^2)).$$

In what follows, we assume that the hypothesis of Theorem 2.2 holds. Replacing u (resp. v) by u^* (resp. v_*) we may assume that u (resp. v) is upper (resp. lower) semicontinuous on $\overline{Q}$. We remark here, since u and v are originally defined over Q, that its value at $t = T$ is denoted by relaxation, i.e.,

$$\begin{cases} u(T,x) = \lim_{\varepsilon \downarrow 0} \sup\{u(t,y) : T - \varepsilon \le t < T, |x-y| \le \varepsilon, y \in \Omega\}, \\ v(T,x) = \lim_{\varepsilon \downarrow 0} \inf\{v(t,y) : T - \varepsilon \le t < T, |x-y| \le \varepsilon, y \in \Omega\}. \end{cases} \tag{3.2}$$

We study about the point which attains the (locally) maximum of an upper semicontinuous function. Setting $w(t,x,y) = u(t,x) - v(t,y)$ for $(t,x,y) \in U = (0,T)\times\Omega\times\Omega$ and

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(t,x,y); |x-y| < \theta, (t,x,y) \in \overline{U}\},$$

since w is upper semicontinuous and $\overline{U}$ is compact, we see $\alpha < \infty$. We also set for $\varepsilon > 0$ and $\gamma > 0$,

$$\Phi(t,x,y) = w(t,x,y) - \phi(t,x,y); \quad \phi(t,x,y) = \frac{|x-y|^2}{2\varepsilon} + \frac{\gamma}{T-t}.$$

The function ϕ controls the maximum point of Φ over $\overline{U}$.

Proposition 3.1. *Suppose that $\alpha > 0$. Then,*

(i) *there exists a constant $\gamma_0 > 0$ such that $\sup_{\overline{U}} \Phi > \alpha/2$ holds for all $\varepsilon > 0$ and $0 < \gamma < \gamma_0$.*

(ii) *There exists a constant $\varepsilon_0 > 0$ such that Φ attains a (positive) maximum over $\overline{U}$ at an interior point of U for all $0 < \varepsilon < \varepsilon_0$ and $0 < \gamma < \gamma_0$.*

Proof. Since w is upper semicontinuous, by the definition of α and (3.2) there exists a point $(t_0, x_0, x_0) \in \overline{U}$ such that $t_0 < T$ and $w(t_0,x_0,x_0) > 3\alpha/4$. If γ_0 satisfies $0 < \gamma_0 < (T-t_0)\alpha/4$, then we see

$$\Phi(t_0,x_0,x_0) = w(t_0,x_0,x_0) - \frac{|x_0-x_0|^2}{2\varepsilon} - \frac{\gamma}{T-t_0} > \frac{3\alpha}{4} - 0 - \frac{\alpha}{4} = \frac{\alpha}{2}$$

for all $\varepsilon > 0$ and $0 < \gamma < \gamma_0$, which proves (i).

Let $(\hat{t},\hat{x},\hat{y}) \in \overline{U}$ be a maximum point of Φ, i.e., $\sup_{\overline{U}} \Phi = \Phi(\hat{t},\hat{x},\hat{y})$. By the definition of Φ we easily see $\hat{t} < T$. Let M be an upper bound of w over $\overline{U}$. Since $\Phi(\hat{t},\hat{x},\hat{y})$ is positive (by (i)), it follows that

$$M \ge w(\hat{t},\hat{x},\hat{y}) > \frac{|\hat{x}-\hat{y}|^2}{2\varepsilon} + \frac{\gamma}{T-\hat{t}} > \frac{|\hat{x}-\hat{y}|^2}{2\varepsilon}.$$

This leads to $|\hat{x}-\hat{y}| < \sqrt{2M\varepsilon}$ uniformly in $0 < \gamma < \gamma_0$. By the hypothesis, $w \le 0$ on $\partial_p Q$, of Theorem 2.2 and boundedness of Ω we get a *modulus function* m (i.e., $m : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and $m(0) = 0$) such that

$$w(t,x,y) \le m(|x-y|) \quad \text{on } \partial_p U.$$

If there are sequences $\{\varepsilon_j\}$ with $\varepsilon_j \downarrow 0$ and $\{\gamma_j\} \subset (0, \gamma_0)$ such that Φ (for ε_j and γ_j) attains the maximum over $\overline{U}$ at $(\hat{t}_j, \hat{x}_j, \hat{y}_j) \in \partial_p U$, then it follows that

$$\frac{\alpha}{2} < \Phi(\hat{t}_j, \hat{x}_j, \hat{y}_j) < w(\hat{t}_j, \hat{x}_j, \hat{y}_j) \le m(|\hat{x}_j - \hat{y}_j|) \le m(\sqrt{2M\varepsilon_j}) \to 0 \text{ (as } j \to \infty).$$

This contradicts the assumption, $\alpha > 0$, of this proposition, then (ii) is proved. □

Section 4 also needs more general cases in order to prove Theorem 2.2. We set for $\varepsilon > 0$, $\gamma > 0$, $\delta \ge 0$, $\eta \in R^N$ and $\hat{t} \in (0, T)$,

$$\Phi_\eta(t, x, y) = w(t, x, y) - \phi_\eta(t, x, y); \quad \phi_\eta(t, x, y) = \frac{|x - y - \eta|^2}{2\varepsilon} + \frac{\gamma}{T - t} + \delta(t - \hat{t})^2.$$

Proposition 3.2. *Suppose that $\alpha > 0$. Then,*

(i) *there are constants $\gamma_0 > 0$, $\delta_0 > 0$ and $\kappa_\varepsilon > 0$ (depending on ε) such that $\sup_{\overline{U}} \Phi_\eta > \alpha/2$ holds for all $\varepsilon > 0$, $0 < \gamma < \gamma_0$, $0 \le \delta < \delta_0$ and $\eta \in R^N$ with $|\eta| < \kappa_\varepsilon$.*

(ii) *There is a constant $\varepsilon_0 > 0$ such that Φ_η attains a (positive) maximum over $\overline{U}$ at an interior point of U for all $0 < \varepsilon < \varepsilon_0$, $0 < \gamma < \gamma_0$, $0 \le \delta < \delta_0$ and $\eta \in R^N$ with $|\eta| < \kappa_\varepsilon$.*

Proof. For a point $(t_0, x_0, x_0) \in \overline{U}$ satisfying $t_0 < T$ and $w(t_0, x_0, x_0) > 3\alpha/4$, we choose γ_0 and κ_ε such that $\gamma_0/(T - t_0) < \alpha/16$ and $\kappa_\varepsilon^2/2\varepsilon < \alpha/16$. If $\delta_0 < T^{-2}\alpha/8$, then we see

$$\begin{aligned}\Phi_\eta(t_0, x_0, x_0) &= w(t_0, x_0, x_0) - \frac{|x_0 - x_0 - \eta|^2}{2\varepsilon} - \frac{\gamma}{T - t_0} - \delta(t_0 - \hat{t})^2 \\ &> \frac{3\alpha}{4} - \frac{\alpha}{16} - \frac{\alpha}{16} - \frac{\alpha}{8} = \frac{\alpha}{2}\end{aligned}$$

for all $\varepsilon > 0$, $0 < \gamma < \gamma_0$, $0 \le \delta < \delta_0$ and $\eta \in R^N$ with $|\eta| < \kappa_\varepsilon$, which proves (i). The proof of (ii) is the same as Proposition 3.1, so it is omitted here. □

In the notion of viscosity solution, it is basic to study the maximum point of Φ (and also Φ_η). If $\sup_{\overline{U}} \Phi = \Phi(\hat{t}, \hat{x}, \hat{y})$ and $(\hat{t}, \hat{x}, \hat{y})$ is an interior point of U, then

$$(\partial_t \phi(\hat{t}, \hat{x}, \hat{y}), \nabla_{x,y}\phi(\hat{t}, \hat{x}, \hat{y}), \nabla^2_{x,y}\phi(\hat{t}, \hat{x}, \hat{y})) \in \mathcal{P}^{2,+}_U w(\hat{t}, \hat{x}, \hat{y}).$$

The following is a variant of Crandall-Ishii's lemma (cf. Lemma 2.10 in [17], and also see [7]).

Lemma 3.3. *Let u_i $(i = 1, \cdots, k)$ be an upper semicontinuous function on $(0, T) \times \Omega_i$, where Ω_i is an open set in R^{N_i}. Let w be a function on $U = (0, T) \times \Omega_1 \times \cdots \times \Omega_k$ given by*

$$w(t, x) = u_1(t, x_1) + \cdots + u_k(t, x_k); \quad x = (x_1, \cdots, x_k).$$

Let $F_i : R^{N_i} \times S^{N_i} \to R \cup \{\pm\infty\}$ $(i = 1, \cdots, k)$ be a lower semicontinuous function. For $(s, z) \in U$, suppose that $(\tau, p, A) \in \mathcal{P}^{2,+}_U w(s, z)$, where $p = (p_1, \cdots, p_k)$ and $z = (z_1, \cdots, z_k)$. If u_i is a viscosity subsolution of (3.1) for F_i in a neighborhood *of* $(s, z_i) \in (0,T) \times \Omega_i$ *for* $i = 1, \cdots, k$. *Then for each* $\lambda > 0$ *there exist* $X_i \in S^{N_i}$ *such that*

$$\tau + \sum_{i=1}^{k} F_i(p_i, X_i) \leq 0$$

and

$$-(1/\lambda + |A|)I \leq \begin{pmatrix} X_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & X_k \end{pmatrix} \leq A + \lambda A^2,$$

where I *is the* $N \times N$ *identity matrix* $(N = N_1 + \cdots + N_k)$ *and* $|A|$ *denotes the operator norm of* A.

4. Proof of Theorem 2.2. The basic strategy is similar to [5]. Suppose $\alpha > 0$, which means the conclusion, $u \leq v$ on $\overline{Q}$, of Theorem 2.2 is false. Here $Q = (0,T) \times \Omega$ for a bounded domain Ω in R^N, $U = (0,T) \times \Omega \times \Omega$ and

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(t,x,y) = u(t,x) - v(t,y) : |x-y| < \theta, (t,x,y) \in \overline{U}\}.$$

To get a contradiction, we find a nice (parabolic) super 2-jet of the function w at some point in U. For $\varepsilon > 0$ and $\gamma > 0$ we set

$$\Phi(t,x,y) = w(t,x,y) - \phi(t,x,y); \quad \phi(t,x,y) = \frac{|x-y|^2}{2\varepsilon} + \frac{\gamma}{T-t}.$$

By Proposition 3.1 we see that Φ attains a (positive) maximum over $\overline{U}$ at an interior point $(\hat{t}, \hat{x}, \hat{y}) \in U$ for all $0 < \varepsilon < \varepsilon_0$ and $0 < \gamma < \gamma_0$. Then it holds that

$$(\hat{\phi}_t, \hat{\phi}_{x,y}, A) \in \mathcal{P}_U^{2,+} w(\hat{t}, \hat{x}, \hat{y}) \quad \text{for all } A \in S^N \text{ with } \nabla^2_{x,y}\phi(\hat{t}, \hat{x}, \hat{y}) \leq A,$$

where $\hat{\phi}_t = \partial_t \phi(\hat{t}, \hat{x}, \hat{y})$, $\hat{\phi}_x = \nabla_x \phi(\hat{t}, \hat{x}, \hat{y})$, $\hat{\phi}_y = \nabla_y \phi(\hat{t}, \hat{x}, \hat{y})$ and $\hat{\phi}_{x,y} = (\hat{\phi}_x, \hat{\phi}_y)$. It is obvious that

$$\hat{\phi}_t = \gamma/(T-\hat{t})^{-2} \quad \text{and} \quad \hat{\phi}_x = -\hat{\phi}_y = \varepsilon^{-1}(\hat{x} - \hat{y}).$$

Case 1: We first discuss the case $\hat{x} \neq \hat{y}$. Lemma 3.3 states that there exist X and $Y \in S^N$ such that

$$\gamma/(T-\hat{t})^{-2} + F(\varepsilon^{-1}(\hat{x}-\hat{y}), X) - F(\varepsilon^{-1}(\hat{x}-\hat{y}), -Y) \leq 0, \tag{4.1}$$

and

$$-(\lambda^{-1} + |A|)I \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq A + \lambda A^2, \tag{4.2}$$

since u and v are, respectively, viscosity sub- and supersolutions of

$$\partial_t u + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } Q. \tag{4.3}$$

By choosing

$$A = \nabla^2_{x,y}\phi(\hat{t}, \hat{x}, \hat{y}) = \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and using (4.2) we see that $X \le -Y$. Applying (F2) to (4.1), we obtain

$$0 \ge \gamma/(T-\hat{t})^{-2} + F(\varepsilon^{-1}(\hat{x}-\hat{y}), X) - F(\varepsilon^{-1}(\hat{x}-\hat{y}), -Y) \ge \gamma T^{-2},$$

which leads to a contradiction.

Case 2: When $\hat{x} = \hat{y}$, we cannot use (4.1) to get a contradiction, because $F(p, X)$ has a singularity at $p = 0$. For $\varepsilon > 0$, $\gamma > 0$, $\delta > 0$ and $\eta \in R^N$ we set

$$\Phi_\eta(t,x,y) = w(t,x,y) - \phi_\eta(t,x,y); \quad \phi_\eta(t,x,y) = \frac{|x-y-\eta|^2}{2\varepsilon} + \frac{\gamma}{T-t} + \delta(t-\hat{t})^2.$$

By Proposition 3.2 we see that Φ_η attains a (positive) maximum over $\overline{U}$ at an interior point $(t_\eta, x_\eta, y_\eta) \in U$ for all $0 < \varepsilon < \varepsilon_0$, $0 < \gamma < \gamma_0$, $0 \le \delta < \delta_0$ and $|\eta| < \kappa_\varepsilon$.

Case 2a: When there is a sequence $\{\eta_j\} \subset R^N$ such that $\eta_j \to 0$ and $x_{\eta_j} - y_{\eta_j} \ne \eta_j$ for all $0 < |\eta_j| < \kappa_\varepsilon$, we get a contradiction similarly to Case 1.

Case 2b: In the opposite case to Case 2a, there is a positive constant $\kappa < \kappa_\varepsilon$ such that $x_\eta - y_\eta = \eta$ for all $|\eta| < \kappa$. Since Φ has a positive maximum at $(\hat{t}, \hat{x}, \hat{x})$, we get $w(\hat{t}, \hat{x}, \hat{x}) > 0$. Suppose that

$$w(\hat{t}, x, x) = w(\hat{t}, \hat{x}, \hat{x}) \quad \text{for all } x \in \Omega. \tag{4.4}$$

Then for each sequence $\{x_l\} \subset \Omega$ and $\overline{x} \in \partial\Omega$ satisfying $x_l \to \overline{x}$, we see that

$$0 < \limsup_{l\to\infty} w(\hat{t}, x_l, x_l) \le w(\hat{t}, \overline{x}, \overline{x}),$$

since w is upper semicontinuous. This contradicts the hypothesis, $w \le 0$ on $\partial_p Q$, of Theorem 2.2.

It is enough to prove (4.4) in Case 2b. We set

$$f(\eta) = \sup\{w(t_\eta, x, y) - \frac{\gamma}{T-t_\eta} - \delta(t_\eta - \hat{t})^2; x - y = \eta\}.$$

Since (t_η, x_η, y_η) is a maximum point of Φ_η and $x_\eta - y_\eta = \eta$, it follows that

$$w(t,x,y) - \frac{|x-y-\eta|^2}{2\varepsilon} - \frac{\gamma}{T-t} - \delta(t-\hat{t})^2 \le w(t_\eta, x_\eta, y_\eta) - \frac{\gamma}{T-t_\eta} - \delta(t_\eta - \hat{t})^2$$

for all $(t,x,y) \in U$ and $|\eta| < \kappa$. This yields $|f(\xi) - f(\eta)| \le |\xi - \eta|^2/2\varepsilon$ by taking $t = t_\xi$ and $x - y = \xi$ for $|\xi| < \kappa$. Hence, $f(\eta)$ is a constant for $|\eta| < \kappa$, which implies

$$\begin{aligned} &\sup_{|x-y|<\kappa} \{w(t_\eta, x, y) - \frac{\gamma}{T-t_\eta} - \delta(t_\eta - \hat{t})^2\} \\ &= \sup_{x\in\Omega}\{w(\hat{t}, x, x) - \frac{\gamma}{T-\hat{t}}\} = w(\hat{t}, \hat{x}, \hat{x}) - \frac{\gamma}{T-\hat{t}}, \end{aligned}$$

since $t_\eta \to \hat{t}$ as $\eta \to 0$. We set $\Sigma_\kappa = \{(x,y) \in \Omega \times \Omega; |x-y| < \kappa\}$. Since (t_η, x_η, y_η) attains a maximum of Φ_η on $(0,T) \times \Sigma_\kappa$ for $|\eta| < \kappa$ and $x_\eta - y_\eta = \eta$, we have

$$\sup_{(0,T)\times\Sigma_\kappa} \{w(t,x,y) - \frac{\gamma}{T-t} - \delta(t-\hat{t})^2\} = w(\hat{t},\hat{x},\hat{x}) - \frac{\gamma}{T-\hat{t}}. \tag{4.5}$$

We now set

$$A = \{(x,y) \in \Sigma_\kappa; w(\hat{t},x,y) = w(\hat{t},\hat{x},\hat{x})\}.$$

Since $(\hat{x},\hat{x}) \in A$ and w is upper semicontinuous, it follows from (4.5) that A is a nonempty and closed subset of Σ_κ. To prove (4.4), it suffices to show that $A = \Sigma_\kappa$. Assuming $A \neq \Sigma_\kappa$, we will deduce a contradiction.

To this end we prepare a geometric lemma on balls touching a closed set. Here a closed (hyper)ball B is called *touching* a closed set A if $\operatorname{int} B \cap A = \emptyset$ and $\partial B \cap A \neq \emptyset$, where $\operatorname{int} B$ is the interior of B. When the dimension of the base space is even, i.e., $A, B \subset R^{2N}$, and $B = B_r(\overline{x},\overline{y})$ is touching A, the ball B is called *obliquely touching* A provided that $x' \neq \overline{x}$ and $y' \neq \overline{y}$ for all $(x',y') \in \partial B \cap A$.

Lemma 4.1. *Let Σ be a connected open set in R^{2N} and let A be a closed subset of Σ such that $A \neq \emptyset$ and $A \neq \Sigma$. Then there exists a closed (hyper)ball $B = B_r(\overline{x},\overline{y}) \subset \Sigma$ satisfying one of the following properties:*

(I) *B is obliquely touching A.*

If there are no balls obliquely touching A,

(IIa) *B is touching A at a point $(\overline{x},y')$ and $(x,y') \in A$ for all x with $|x-\overline{x}| < r$.*

(IIb) *B is touching A at a point $(x',\overline{y})$ and $(x',y) \in A$ for all y with $|y-\overline{y}| < r$.*

We postpone the proof of Lemma 4.1 in Section 6. According to Lemma 4.1 with $\Sigma = \Sigma_\kappa$, we have the three cases in Case 2b.

Case 2b(I): There is a closed ball $B = B_r(\overline{x},\overline{y})$ obliquely touching A. We may assume that B is touching at a single point (x',y') with $x' \neq \overline{x}$ and $y' \neq \overline{y}$ by taking B smaller. For $\gamma > 0$, $\delta > 0$ and $\lambda \geq 0$ we set

$$\Psi_\lambda(t,x,y) = w(t,x,y) - \frac{\gamma}{T-t} - \delta(t-\hat{t})^2 - \lambda(|x-\overline{x}|^2 + |y-\overline{y}|^2).$$

By $(\overline{x},\overline{y}) \notin A$ and (4.5) we see $w(\hat{t},\overline{x},\overline{y}) < w(\hat{t},\hat{x},\hat{x})$. By setting $w(\hat{t},\hat{x},\hat{x}) - w(\hat{t},\overline{x},\overline{y}) = \theta > 0$ and choosing $\overline{\lambda}$ with $0 < \overline{\lambda} < (|\hat{x}-\overline{x}|^2 + |\hat{x}-\overline{y}|^2)^{-1}\theta/2$, we see that

$$\Psi_\lambda(\hat{t},\hat{x},\hat{x}) - \Psi_\lambda(\hat{t},\overline{x},\overline{y}) > \theta/2 \quad \text{for all } 0 \leq \lambda < \overline{\lambda}.$$

This yields that $(\hat{t},\overline{x},\overline{y})$ is not a maximum point of Ψ_λ. Let $(t_\lambda, x_\lambda, y_\lambda)$ be a maximum point of Ψ_λ.

By (4.5) we see that $(\hat{t},\hat{x},\hat{x})$ is a maximum point of Ψ_0 and Ψ_0 attains a maximum only at $t = \hat{t}$. This implies that $(t_\lambda, x_\lambda, y_\lambda) \to (\hat{t}, x'', y'')$ as $\lambda \to 0$ for some point $(x'',y'') \in \Sigma_\kappa$. By the definition of Ψ_λ we see $(x_\lambda, y_\lambda) \in B$ and then $(x'',y'') = (x',y') \in \partial B \cap A$. Since B is obliquely touching A, there exist $\theta' > 0$ and λ' with $0 < \lambda' \leq \overline{\lambda}$ such that

$$|x_\lambda - \overline{x}| \quad \text{and} \quad |y_\lambda - \overline{y}| \geq \theta' \quad \text{for all } 0 \leq \lambda < \lambda'.$$

Then we get a contradiction similarly to Case 1.

Case 2b(IIa): There is a closed ball $B = B_r(\overline{x}, \overline{y})$ touching A at a point $(\overline{x}, y')$ and $(x, y') \in A$ for all $|x - \overline{x}| < r$. First, we see that

$$\begin{aligned}\sup_{(0,T)\times\Sigma_\kappa} \Psi_0 &= \Psi_0(\hat{t}, \overline{x}, y') \qquad (= w(\hat{t}, \overline{x}, y') - \frac{\gamma}{T - \hat{t}}) \\ &= \Psi_0(\hat{t}, x, y') \quad \text{for all } |x - \overline{x}| < r\end{aligned}$$

by (4.5) and the definition of the set A. This implies that there exists a constant C such that $u(\hat{t}, x) = C$ for all $|x - \overline{x}| < r$. We now invoke our finite speed assumption.

Lemma 4.2. *Suppose that u has an upper speed bound ν. If $u(t_0, x) \geq C$ for $|x - x_0| < \delta_0$, then there exists for all δ, a positive number $r = r(\delta)$ such that*

$$\sup_{|x - x_0| \leq \delta} u(t, x) \geq C \quad \textit{for } \ t_0 - r \leq t \leq t_0.$$

Proof. Suppose that there is $\delta > 0$ such that for all numbers $k \geq k_\delta$,

$$\sup_{|x - x_0| < \delta} u(t_k, x) = m_k < C \quad \text{with } t_k = t_0 - k^{-1},$$

where k_δ satisfies $\delta > \nu(\delta) k_\delta^{-1}$. By setting $R(t) = \delta - \nu(\delta)(t - t_k)$ we see that $R(t_0) = \delta - \nu(\delta) k^{-1} > 0$. The finite speed condition yields

$$\sup_{|x - x_0| < R(t_0)} u(t_0, x) \leq m_k < C,$$

which contradicts the assumption. $\square$

We now return to discuss Case 2b(IIa) of the proof of Theorem 2.2. By Lemma 4.2 there exists $s = s(r) > 0$ such that

$$\sup_{|x - \overline{x}| < r} u(t, x) \geq C \quad \text{for all } \ \hat{t} - s \leq t \leq \hat{t}.$$

Since $\Psi_0(t, x, y) \leq \Psi_0(\hat{t}, \overline{x}, y')$, we see that

$$\begin{aligned} v(t, y) - v(\hat{t}, y') &\geq u(t, x) - u(\hat{t}, \overline{x}) - \delta(t - \hat{t})^2 - \frac{\gamma}{T - t} + \frac{\gamma}{T - \hat{t}} \\ &\geq \sup_{|x - \overline{x}| < r} u(t, x) - C - \delta(t - \hat{t})^2 - \frac{\gamma}{T - t} + \frac{\gamma}{T - \hat{t}} \\ &\geq -\delta(t - \hat{t})^2 - \frac{\gamma}{T - t} + \frac{\gamma}{T - \hat{t}} \end{aligned}$$

for all $\hat{t} - s \leq t \leq \hat{t}$. Hence, there is $r' > 0$ such that

$$\inf_{(\hat{t} - s, \hat{t}) \times V} \{v(t, y) + \frac{\gamma}{T - t} + \delta(t - \hat{t})^2\} = v(\hat{t}, y') + \frac{\gamma}{T - \hat{t}}, \tag{4.6}$$

where $V = \{y : |y - y'| < r'\}$.

We set $K = \{y \in V; v(\hat{t}, y) = v(\hat{t}, y')\}$. Since $y' \in K$ and v is lower semicontinuous, it follows from (4.6) that K is a nonempty and closed subset of V. In the beginning of Case 2b(IIa) we proved that $u(\hat{t}, x)$ was a constant near $\overline{x}$. If $v(\hat{t}, y)$ is also a constant near y', we get a contradiction to $(\overline{x}, y') \in \partial A$. Then we see $K \neq V$, which implies that there is a closed ball $B_\rho(\overline{y}) \subset V$ touching K at a single point (see Lemma 6.4). For $\gamma > 0$, $\delta > 0$ and $\lambda > 0$ we set that

$$\psi_\lambda(t, y) = v(t, y) + \frac{\gamma}{T - t} + \delta(t - \hat{t})^2 + \lambda|y - \overline{y}|^2.$$

Let ψ_λ attain a minimum at (t_λ, y_λ) over $(\hat{t} - s, \hat{t}) \times V$. We observe that $t_\lambda \to \hat{t}$ and $y_\lambda \to y'' \in B_\rho(\overline{y}) \cap K$ as $\lambda \to 0$, in particular, there is a positive constant θ satisfying $|y_\lambda - \overline{y}| \geq \theta$ for small λ. Since (t_λ, y_λ) is an interior point and v is a viscosity supersolution of (4.3), we see that

$$-\gamma/(T - t_\lambda)^2 - 2\delta(t_\lambda - \hat{t}) + F(-2\lambda(y_\lambda - \overline{y}), -2\lambda I) \geq 0.$$

This implies a contradiction, $-\gamma T^{-2} \geq 0$ (as $\lambda \to 0$), by using the condition (F3′).

In Case 2b(IIb), which is the final case of the proof, we also get a contradiction similar to Case 2b(IIa). Then, the proof of Theorem 2.2 is complete except for the proof of Lemma 4.1.

5. Construction of solutions. In this section we construct a viscosity solution with finite speed for the initial value problem

$$\partial_t u + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0, \infty) \times R^N, \tag{5.1a}$$
$$u(0, x) = a(x) \quad \text{for } x \in R^N. \tag{5.1b}$$

Here we assume that $F = F(p, X)$ satisfies (F1)–(F3), which implies for some positive functions $c_\pm \in C^1(0, \infty)$,

$$F(p, -I) \leq c_-(|p|) \quad \text{and} \quad F(p, I) \geq -c_+(|p|), \tag{F4}$$

and $a \in K_\alpha(R^N)$ for some $\alpha \in R$, i.e.,

$$a(x) - \alpha \text{ is a continuous function with compact support in } R^N.$$

Our goal in this section is to prove

Theorem 5.1. *Suppose that F satisfies* (F1)–(F3). *Let $a \in K_\alpha(R^N)$. Then there exists a global-in-time viscosity solution u of* (5.1a,b) *satisfying $u \in K_\alpha([0, T] \times R^N)$ for all $T > 0$. Moreover, u has finite speed depending only on $c_\pm$.*

We begin with the approximated initial value problem

$$\partial_t u^k + F_k(\nabla u^k, \nabla^2 u^k) = 0 \quad \text{in } (0, \infty) \times R^N, \tag{5.2a}$$
$$u^k(0, x) = a(x) \quad \text{for } x \in R^N, \tag{5.2b}$$

where F_k is denoted by

$$F_k(p, X) = (F(p, X) \wedge k|p|) \vee (-k|p|) \quad \text{for } k = 1, 2, \cdots, \tag{5.3}$$

and $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. The function $F_k(p, X)$ is continuously extended to $(p, X) = (0, X)$ for all $X \in S^N$, i.e.,

$$F_k : R^N \times S^N \to R \quad \text{is continuous,} \tag{F1$'$}$$

and also satisfies (F2)–(F3) and

$$F_k(p, -I) \leq c_-^k(|p|) \quad \text{and} \quad F_k(p, I) \geq -c_+^k(|p|) \tag{F4$'$}$$

for each k with $c_\pm^k(\sigma) = c_\pm(\sigma) \wedge (k\sigma)$. When $F(p, X)$ is continuously extended to $(p, X) = (0, O)$, the comparison and existence results hold without the finite speed condition (see [5], and also [17]).

Proposition 5.2. *Let u and v be a viscosity subsolution and a supersolution, respectively, of* (5.2a) *in $Q = (0, T) \times \Omega$, where $T > 0$ and Ω is a bounded domain in R^N. If $u^* \leq v_*$ on $\partial_p Q$, then $u^* \leq v_*$ on $\overline{Q}$.*

Proposition 5.3. *There exists a global-in-time viscosity solution u^k of* (5.2a,b) *satisfying $u^k \in K_\alpha([0, T] \times R^N)$ for all $T > 0$.*

We first remark that the support of $u^k(t, \cdot) - \alpha$ grows independently of k, whose speed bounds a value depending only on $c_\pm(1) = c_\pm(1) \wedge k$ for sufficiently large k (see Lemma 6.5 in [5]). In other words, for each $T > 0$ there exists $R^* > 0$ such that

$$\operatorname{supp}(u^k(t, \cdot) - \alpha) \subset B_{R^*}(0) \quad \text{for all } t \in (0, T).$$

We must show that u^k (and also $-u^k$) has an upper speed bound independent of k, i.e., for each $R > 0$ we will find $\nu = \nu(R) \geq 0$ independent of k such that

$$\sup_{(t,x) \in \triangle_\nu} u^k(t, x) \leq C \quad \text{with } \triangle_\nu = \{(t, x) : t \geq t_0, |x - x_0| \leq R - \nu(R)(t - t_0)\} \tag{5.4}$$

whenever $C \in R$ and $(t_0, x_0) \in (0, \infty) \times R^N$ satisfies $\sup_{|x - x_0| \leq R} u^k(t_0, x) \leq C$. It suffices to discuss the case when $(t_0, x_0) \in (0, T) \times B_{R^*}(0)$ for each $T > 0$.

Without loss of generality we may assume that $(t_0, x_0) = (0, 0)$. Let b^k be a continuous and radial function satisfying

$$u^k(0, x) \leq b^k(x) \quad \text{and} \quad b^k(x) = C \quad \text{if } |x| \leq R. \tag{5.5}$$

A direct calculation shows that

$$v^k(t, x) = t + \int_0^{|x|} \frac{\sigma}{c_+^k(\sigma)} d\sigma \tag{5.6}$$

is a viscosity supersolution of (5.2a). For each ξ, let h_ξ be a real valued continuous and nondecreasing function over R satisfying $h_\xi(0) = b^k(\xi)$. By using the properties of viscosity solution we see that

$$g^k(t, x) = \inf\{h(v^k(t, x - \xi)); h = h_\xi, \xi \in R^N\} \quad \text{with } g^k(0, x) = b^k(x)$$

is also a viscosity supersolution of (5.2a) (see Proposition 6.4 in [5]). Proposition 5.2 and (5.5) yield $u^k(t, x) \leq g^k(t, x)$ for all $t \geq 0$.

Hence, it suffices to find an upper speed bound for each v^k, which is uniform in k.

Proposition 5.4. *The function v^k has an upper speed bound $\mu(R) = 4\gamma(R)/R$, where*

$$\gamma(R) = \max\{c_+(\sigma); R/2 \le \sigma \le R^* + R\}.$$

Proof. For each $R > 0$, let $C \in R$ and $(t_0, x_0) \in (0, T) \times B_{R^*}(0)$ satisfying

$$\sup_{|x-x_0|\le R} v^k(t_0, x) \le C.$$

By setting $R_0 = |x_0|$ $(R_0 \le R^*)$ and (5.6) we see that

$$v^k(t_0, x) \le t_0 + \int_0^{R_0+R} \frac{\sigma}{c_+^k(\sigma)} d\sigma \quad \text{if } |x - x_0| \le R. \tag{5.7}$$

Since the equality sign of (5.7) is attained at some point whose length is equal to $R_0 + R$, it follows that

$$t_0 + \int_0^{R_0+R} \frac{\sigma}{c_+^k(\sigma)} d\sigma \le C.$$

Let $\mu \ge 0$ be a number. For all $0 < t' \le R/\mu$, we obtain

$$v^k(t_0 + t', x) \le t_0 + t' + \int_0^{R_0+R-\mu t'} \frac{\sigma}{c_+^k(\sigma)} d\sigma \quad \text{if } |x - x_0| \le R - \mu t'.$$

This equality sign is also attained at some point. Note that $\mu = \mu(R)$ is an upper speed bound for v^k, if and only if

$$t_0 + t' + \int_0^{R_0+R-\mu t'} \frac{\sigma}{c_+^k(\sigma)} d\sigma \le t_0 + \int_0^{R_0+R} \frac{\sigma}{c_+^k(\sigma)} d\sigma \tag{5.8}$$

holds for $0 < t' \le R/\mu$.

We set $\gamma_1 = \max\{c_+^k(\sigma); R/2 \le \sigma \le R^* + R\}$ and $\mu_1 = 2\gamma_1/R$. The number $\gamma_1 > 0$ is independent of k for sufficiently large k. For $0 < t' \le R/2\mu_1$, we see that

$$\begin{aligned}\int_{R_0+R-\mu_1 t'}^{R_0+R} \frac{\sigma}{c_+^k(\sigma)} d\sigma &\ge \frac{1}{2\gamma_1}[(R_0 + R)^2 - (R_0 + R - \mu_1 t')^2] \\ &= \frac{1}{2\gamma_1}[2(R_0 + R) - \mu_1 t']\mu_1 t' \ge \frac{1}{2\gamma_1} \cdot \frac{3R}{2} \cdot \mu_1 t' \ge t'.\end{aligned}$$

Hence, (5.8) holds for all $0 < t' \le R/2\mu_1$, so (5.4) holds for v^k and the trapezoid set

$$\triangle'_{\mu_1} = \{(t, x); t_0 \le t \le t_0 + R/2\mu_1, |x - x_0| \le R - \mu_1(R)(t - t_0)\}.$$

By choosing $\mu = 2\mu_1$, we see $\triangle_\mu \subset \triangle'_{\mu_1}$. This completes the proof. □

By Proposition 5.4 we get an upper speed bound for u^k independent of k. Similarly, we also find an upper speed bound for $-u^k$ independent of k.

Now, the relation (5.3) implies that

$$\lim_{k\to\infty}{}_* F_k(p, X) \ge F(p, X) \quad \text{and} \quad \lim_{k\to\infty}{}^* F_k(p, X) \le F(p, X). \tag{5.9}$$

Under the assumption (5.9) the following stability result is known (cf. Proposition 2.4 in [5]).

Proposition 5.5. *Suppose* (5.9). *Let* u^k *be a viscosity sub- (resp. super-) solution of* (5.2a), *and let*

$$\overline{u} = \limsup_{k\to\infty}{}^{*} u^k \quad (\textit{resp. } \underline{u} = \liminf_{k\to\infty}{}_{*} u^k).$$

If $\overline{u} < \infty$ (*resp.* $\underline{u} > -\infty$), *then* $\overline{u}$ (*resp.* $\underline{u}$) *is a viscosity sub- (resp. super-) solution of* (5.1a).

Proposition 5.2 implies that the viscosity solution u^k of (5.2a,b) satisfies the uniform estimate

$$|u^k(t,x)| \le \sup_{x\in R^N} |a(x)| \quad \text{for all } k.$$

The functions $\overline{u}$ and $\underline{u}$, defined in Proposition 5.5, satisfy $\overline{u} < \infty$ and $\underline{u} > -\infty$, respectively. Then $\overline{u}$ and $\underline{u}$ are, respectively, a viscosity sub- and supersolution of (5.1a).

We next show that $\overline{u}$ and $\underline{u}$ satisfy the initial condition (5.1b) and the hypothesis of Theorem 2.2. By (5.2b) we see that $(\lim_{k\to\infty}^* u^k)(0,x) \ge a(x)$. Since u^k has an upper speed bound ν, it follows that for all $x \in R^N$ and $\varepsilon > 0$,

$$\sup_{(s,y)\in\triangle_\varepsilon} u^k(s,y) \le \sup_{|x-y|\le\varepsilon} a(y) \quad \text{with } \triangle_\varepsilon = \{(s,y) : s \ge 0, |x-y| \le \varepsilon - \nu(\varepsilon)s\}.$$

Obviously, it follows that

$$\begin{aligned}(\lim_{k\to\infty}{}^{*} u^k)(0,x) &= \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \sup_{l\ge k}\sup\{u^l(s,y) : |s| \le \varepsilon, |x-y| \le \varepsilon\} \\ &\le \lim_{\varepsilon\downarrow 0} \sup_{|x-y|\le\varepsilon} a(y) = a(x),\end{aligned}$$

since a is continuous. Hence, we see that $\overline{u}(0,x) = a(x)$ and, similarly, $\underline{u}(0,x) = a(x)$. Proposition 5.3 implies that $\overline{u}$ and $\underline{u}$ belong to $K_\alpha([0,T]\times R^N)$ for all $T > 0$. Then the hypothesis of Theorem 2.2 on the parabolic boundary is fulfilled. Since u^k and $-u^k$ have upper speed bounds independent of k, it follows that $\overline{u}$ and $-\underline{u}$ have the same bounds.

By Theorem 2.2 we see $\overline{u} \le \underline{u}$. Since, obviously, $\overline{u} \ge \underline{u}$, we see that $\underline{u} = \overline{u}$; so the function $u = \underline{u} = \overline{u}$ is a (unique) viscosity solution of (5.1a,b), which has finite speed. This completes the proof of Theorem 5.1.

6. Lemma on balls touching a closed set. This section is devoted to proving Lemma 4.1, which I learned from Y. Giga.

Let $k \ge 2$ be an integer, $r_j > 0$ $(j = 1, 2)$ and $(x_0, y_0) \in R^N \times R^N$. We set

$$E^k_{r_1,r_2}(x_0,y_0) = \{(x,y) \in R^N \times R^N : (|x-x_0|/r_1)^k + (|y-y_0|/r_2)^k \le 1\}.$$

When $r_1 = r_2 = r$, $B^k_r(x_0,y_0) = E^k_{r,r}(x_0,y_0)$ is called a closed L^k-ball. Especially, a closed L^2-ball $B^2_r(x_0,y_0)$ is a usual (hyper)ball denoted by $B_r(x_0,y_0)$. We also use the following notations:

$$\begin{aligned}&\mathcal{B}_k = \{B^k_r(x_0,y_0) : (x_0,y_0) \in R^N \times R^N, r > 0\} \quad (k = 2,3,\cdots),\\ &\mathcal{E} = \{E_{r_1,r_2}(x_0,y_0) = E^2_{r_1,r_2}(x_0,y_0) : (x_0,y_0) \in R^N \times R^N, r_j > 0\ (j = 1,2)\},\\ &\mathcal{B} = \bigcup_{k\ge 2} \mathcal{B}_k \cup \mathcal{E}.\end{aligned}$$

Definition. Let $A \subset R^{2N}$ and $B \in \mathcal{B}$. (i) B is called *touching* A if $\operatorname{int} B \cap A = \emptyset$ and $\partial B \cap A \neq \emptyset$, where $\operatorname{int} B$ is the interior of B. (ii) When B is touching A, B is called *obliquely touching* A provided that $\overline{x} \neq x_0$ and $\overline{y} \neq y_0$ for all $(\overline{x}, \overline{y}) \in \partial B \cap A$.

When $B \in \mathcal{B}$ is touching a set $A \subset R^{2N}$, there is a ball $B' \subset B$ such that $\partial B' \cap A$ contains just a single point. Indeed, for a touching point $(\overline{x}, \overline{y}) \in \partial B \cap A$ the new radius is sufficiently small and the center moves to some point in $(\overline{x}, \overline{y})$-direction.

By similar argument we have the following two propositions. The proof of the propositions is easy, so it is omitted here.

Proposition 6.1. *If $B \in \mathcal{B}$ is touching a set $A \subset R^{2N}$, then there is $B' \in \mathcal{B}_2$ such that $B' \subset B$ and B' is touching A.*

Proposition 6.2. *Let $A \subset R^{2N}$ and $B = B^k_r(x_0, y_0)$ (resp. $E_{r_1,r_2}(x_0, y_0)$).* (i) *Suppose that* $\operatorname{int} B \cap A \neq \emptyset$ *and*

$$(\operatorname{int} B \cap A) \cap (\{(x_0, y) : y \in R^N\} \cup \{(x, y_0) : x \in R^N\}) = \emptyset.$$

Then there exists a constant λ with $0 < \lambda < 1$ such that $B^k_{\lambda r}(x_0, y_0)$ (respectively $E_{\lambda r_1, \lambda r_2}(x_0, y_0)$) is obliquely touching A. (ii) *When* $\operatorname{int} B \cap A = \emptyset$, *there is $B' \in \mathcal{B}$ such that $B' \subset B$ and B' is obliquely touching A at a single point provided that there exists $(\overline{x}, \overline{y}) \in \partial B \cap A$ with $\overline{x} \neq x_0$ and $\overline{y} \neq y_0$.*

Our goal in this section is to prove

Theorem 6.3. *Let U be a connected open set in R^{2N} and let A be a closed subset of U such that $A \neq \emptyset$ and $A \neq U$. Then there is a ball $B = B_r(x_0, y_0) \subset U$ satisfying one of the following properties:*

(I) *B is obliquely touching A.*

If there are no balls obliquely touching A,

(IIa) *B is touching A at a point $(x_0, \overline{y})$ and $(x, \overline{y}) \in A$ for all x with $|x - x_0| < r$.*

(IIb) *B is touching A at a point $(\overline{x}, y_0)$ and $(\overline{x}, y) \in A$ for all y with $|y - y_0| < r$.*

We also use "touching" when the dimension of the base space is not even, and set $B_\rho(z_0) = \{z \in R^d ; |z - z_0| \leq \rho\}$ for $\rho > 0$ and $z_0 \in R^d$.

Lemma 6.4. *Under the hypothesis of Theorem 6.3 (in R^d), there is a ball B such that $B \subset U$ and B is touching A.*

Proof. Since $A \neq \emptyset$ and $A \neq U$, we find a point $\overline{z} \in \partial A \cap U$. For a sufficiently small $\rho > 0$ satisfying $B_\rho(\overline{z}) \subset U$, we choose $z_0 \in B_{\rho/2}(\overline{z}) \setminus A$. Since $\operatorname{dist}(z_0, \partial B_\rho(\overline{z})) \geq \rho/2$, we see $B_{\rho/2}(z_0) \subset B_\rho(\overline{z})$. By $\overline{z} \in \partial A \cap B_{\rho/2}(z_0)$ we see that $B_{\rho/2}(z_0) \cap A$ is a nonempty closed set. Set $r = \operatorname{dist}(z_0, \partial(B_{\rho/2}(z_0) \cap A)) > 0$, then $B_r(z_0) \subset U$ is touching A. $\square$

When $d = 2N$ in Lemma 6.4, by replacing $\rho/2$ by $\rho/(2\sqrt{2N})$ in the proof of Lemma 6.4, we also see that

$$\{(x, y) \in R^N \times R^N ; |x - x_0| < r, |y - y_0| < r\} \subset U \quad ((x_0, y_0) = z_0).$$

We can now prove Theorem 6.3. By Lemma 6.4 we get a ball $B = B_r(x_0, y_0) \subset U$ touching A. We may assume that B is touching A at a single point $(\overline{x}, \overline{y}) \in \partial B \cap A$.

Suppose that there are no balls satisfying (I). Then $\overline{x} = x_0$, $\overline{y} \neq y_0$ or $\overline{x} \neq x_0$, $\overline{y} = y_0$.

Suppose that $\overline{x} = x_0$ and $\overline{y} \neq y_0$. We will show the existence of a ball satisfying (IIa). We first prove that

$$(x,y) \notin A \quad \text{if } |x - x_0| < r \text{ and } |y - y_0| < r. \tag{6.1}$$

Indeed, (6.1) clearly holds if $(x,y) \in B$. When $(x,y) \notin B$, we see that $(x,y) \in \operatorname{int} B_r^k(x_0,y_0)$ for some $k > 2$, and so

$$(\operatorname{int} B_r^k(x_0,y_0) \cap A) \cap (\{(x_0,y); y \in R^N\} \cup \{(x,y_0); x \in R^N\}) = \emptyset.$$

Suppose that (6.1) is false. Then $\operatorname{int} B_r^k(x_0,y_0) \cap A \neq \emptyset$. By Proposition 6.2 (i) there is a constant λ with $0 < \lambda < 1$ such that $B_{\lambda r}^k(x_0,y_0)$ is obliquely touching A, which implies, by Proposition 6.1, the existence of a ball satisfying (I).

We next prove that

$$(x,\overline{y}) \in A \quad \text{if } |x - x_0| < r. \tag{6.2}$$

To do this, we use a generalization of (6.1), whose proof is similar to that of (6.1), so it is omitted here.

Lemma 6.5. *Under the hypothesis of Theorem* 6.3, *let $E_{r_1,r_2}^2(x_0,y_0) \subset U$ be touching A at a single point $(\overline{x},\overline{y})$. Suppose that $\overline{x} = x_0$ and $\overline{y} \neq y_0$ (or $\overline{x} \neq x_0$ and $\overline{y} = y_0$). If $|x - x_0| < r_1$ and $|y - y_0| < r_2$, then $(x,y) \notin A$.*

We set $Q = \{x \in R^N; |x - x_0| < r\}$ and $V = \{x \in Q; (x,\overline{y}) \in A\}$. Since $(x_0,\overline{y}) \in A$ and A is a closed set, it follows that V is a nonempty and closed subset of Q. Hence, it is enough to show that V is also an open subset of Q. For each $x_1 \in V$, let $\rho > 0$ satisfying $B_{2\rho}(x_1) \subset Q$. Let $x_2 \in B_\rho(x_1)$ and suppose that $x_2 \notin V$. Since $E_{\rho,r/2}^2(x_2,\hat{y}) \cap A = \emptyset$ ($\hat{y} = (y_0 + \overline{y})/2$) by Lemma 6.5, there is a constant $\varepsilon > 0$ such that $E_{\rho+\varepsilon,r/2+\varepsilon}^2(x_2,\hat{y})$ is touching A. We may assume that this touching is not oblique. By Lemma 6.5 we see

$$(x,y) \notin A \quad \text{if} \quad |x - x_2| < \rho + \varepsilon \quad \text{and} \quad |y - \hat{y}| < r/2 + \varepsilon.$$

This contradicts $x_1 \in V$, so $x_2 \in V$ or $B_\rho(x_1) \subset V$, which implies that V is an open set of Q.

By (6.1) and (6.2) there exists a ball satisfying (IIa). When $\overline{x} \neq x_0$ and $\overline{y} = y_0$ are supposed, the same argument also implies the existence of a ball satisfying (IIb). The proof of Theorem 6.3 is complete.

Acknowledgment. The author is grateful to Professor Yoshikazu Giga for calling my attention to the problem treated in this paper and for valuable comments and constant encouragement. In particular, Theorem 6.3 and its original proof are entirely due to him. Thanks are also due to Professor Hitoshi Ishii for useful comments and some simplification of the proof of Theorem 6.3.

REFERENCES

[1] S. Altschuler, S. Angenent and Y. Giga, *Generalized motion by mean curvature for surface of rotation,* Advanced Studies in Pure Mathematics, to appear.

[2] S. Altschuler, S. Angenent and Y. Giga, *Mean curvature flow through singularities for surfaces of rotation,* J. Geom. Anal., to appear.

[3] G. Barles, H. M. Soner and P. E. Souganidis, *Fronts propagations and phase field theory,* SIAM J. Cont. Optim., 31 (1993), 439–469.

[4] K. A. Brakke, "The Motion of a Surface by Its Mean Curvature," Princeton Univ. Press, Princeton, 1978.

[5] Y.-G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,* J. Diff. Geom., 33 (1991), 749–786.

[6] Y.-G. Chen, Y. Giga and S. Goto, *Remarks on viscosity solutions for evolution equations,* Proc. Japan Acad., 67 (1991), 323–328.

[7] M.G. Crandall, H. Ishii and P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations,* Bull. Amer. Math. Soc., 27 (1992), 1–67.

[8] L.C. Evans, H.M. Soner and P.E. Souganidis, *Phase transitions and generalized motion by mean curvature,* Comm. Pure Appl. Math., 45 (1992), 1097–1123.

[9] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature I,* J. Diff. Geom., 33 (1991), 635–681.

[10] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature II,* Trans. Amer. Math. Soc., 330 (1992), 321–332.

[11] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature III,* J. Geom. Anal., 2 (1992), 121–150.

[12] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature IV,* J. Geom. Anal., to appear.

[13] C. Gerhardt, *Flow of nonconvex hypersurfaces into spheres,* J. Diff. Geom., 32 (1990), 299–314.

[14] Y. Giga and S. Goto, *Motion of hypersurfaces and geometric equations,* J. Math. Soc. Japan, 44 (1992), 99–111.

[15] Y. Giga and S. Goto, *Geometric evolution of phase-boundaries,* in "On the Evolution of Phase Boundaries," edited by M.E. Gurtin and G.B. McFadden, the IMA volumes in mathematics and its applications 43, Springer-Verlag, New York (1992), 51–65.

[16] Y. Giga, S. Goto and H. Ishii, *Global existence of weak solutions for interface equations coupled with diffusion equations,* SIAM J. Math. Anal., 23 (1992), 821–835.

[17] Y. Giga, S. Goto, H. Ishii and M.-H. Sato, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains,* Indiana Univ. Math. J., 40 (1991), 443–470.

[18] Y. Giga and M. Sato, *Generalized interface evolution with the Neumann boundary condition,* Proc. Japan Acad., 67 (1991), 263–266.

[19] Y. Giga and M. Sato, *Neumann problem for singular degenerate parabolic equations,* Differential and Integral Equations, 6 (1993), 1217–1230.

[20] M.A. Grayson, *A short note on the evolution of a surface by its mean curvature,* Duke Math. J., 58 (1989), 555–558.

[21] M.E. Gurtin, "Thermomechanics of Evolving Phase Boundaries in the Plane," Clarendon Press, Oxford, 1993.

[22] T. Ilmanen, *Generalized flow of sets by mean curvature on a manifold,* Indiana Univ. Math. J., 41 (1992), 671–705.

[23] T. Ilmanen, *The level-set flow on a manifold,* preprint.

[24] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature,* J. Diff. Geom., 38 (1993), 417–461.

[25] H. Ishii and P.E. Souganidis, in preparation.

[26] M. Ohnuma and M.-H. Sato, *Singular degenerate parabolic equations with applications to geometric evolutions,* Differential and Integral Equations, to appear.

[27] M.-H. Sato, *Interface evolution with Neumann boundary condition,* Advances in Mathematical Sciences and Applications, to appear.

[28] H.M. Soner, *Motion of a set by the curvature of its boundary,* J. Diff. Eqns., 101 (1993), 313–372.

[29] H.M. Soner and P.E. Souganidis, *Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature*, preprint.

[30] P. Sternberg and W.P. Ziemer, *Generalized motion by curvature with a Dirichlet condition*, preprint.

[31] K. Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure Appl. Math., 38 (1985), 867–882.

[32] J.I. E. Urbas, *An expansion of convex hypersurfaces*, J. Diff. Geom., 33 (1991), 91–125.