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GENERALIZED MOTION OF NONCOMPACT HYPERSURFACES WITH VELOCITY HAVING ARBITRARY GROWTH ON THE CURVATURE TENSOR

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Abstract. In this note we study the generalized motion of noncompact hypersurfaces with normal velocity depending on the normal direction and the curvature tensor. This work extends the by-now-classical works of Evans and Spruck (for mean curvature) and Chen, Giga and Goto (for general motions with sublinear curvature dependence), because it allows general dependence on the curvature tensor. It also allows a general treatment of the generalized evolution including noncompact hypersurfaces. A number of results regarding no interior, convexity, etc. are also presented.

Introduction. During the past few years there has been a substantial progress in understanding the evolution of surfaces, moving with normal velocity depending on the curvature tensor and the normal direction, past the first time singularities occur. The so-called *level set approach*, which is based on characterizing the surfaces as a level set (for definiteness the zero level set) of the solution of certain fully nonlinear degenerate parabolic PDE's, was developed successfully by Evans and Spruck [ES] for motions by mean curvature and by Chen, Giga and Goto [CGG] for more general evolutions, in which, however, the normal velocity depends, at most linearly, on the curvature tensor. The basic tool of [ES] and [CGG] is the theory of viscosity solutions. We refer to the *User's Guide* by Crandall, Ishii and Lions [CIL] for a general discussion of the theory of viscosity solutions and its scope, to [ES] and [CGG] for the origin of the level set approach and to Soner [Son] and Barles, Soner and Souganidis [BSS] for alternative formulations, extensions, discussions, etc. Some of the most striking justifications of the generalized motion of hypersurfaces were provided by its use towards obtaining rigorous results regarding the asymptotic behavior of reaction-diffusion equations (see, for example, Evans, Soner and Souganidis [ESS] and Barles, Soner and Souganidis [BSS]) and, more recently, the hydrodynamic limits of particle systems in Katsoulakis and Souganidis [KS1], [KS2] (see also Souganidis [Sou]).

The purpose of this note is to extend the results of [CGG] to cases where the

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normal velocity is a *general continuous function* of the *normal vector* and the *curvature tensor*. Such evolutions arise very naturally in geometry, since they include, for example, the Gaussian curvature, as well as in applications like image processing (see, for example, Lions [L] and Alvarez, Guichard, Lions and Morel [AGLM]), etc. The main difficulty in studying such evolutions is that they give rise to PDE's with singularities of order higher than the one's considered by [ES], [CGG], etc. To overcome this difficulty, we extend the class of admissible test functions in the definition of viscosity solutions and then prove a comparison principle as well as an existence result in this class. A new feature of the level set approach here is that our uniqueness result concerning the zero level sets of solutions of nonlinear PDE's is sharp enough to treat the generalized evolutions of *noncompact* hypersurfaces. As a result, our arguments are slightly more natural than those in Ilmanen [I] concerning generalized evolutions of noncompact hypersurfaces.

The paper is organized as follows: In Section 1 we formulate the problem, give the definitions and recall basic facts from the theory of viscosity solutions adapted to our setting. We also recall the definition of the level set approach to the generalized motion of hypersurfaces. Finally, we present a number of examples of motions of hypersurfaces which can be put in our framework. In Section 2 we state and prove our main results, namely, a comparison principle for viscosity solutions as well as a general existence result. Finally, in Section 3 we state a number of results regarding the regularity properties of the generalized evolution.

At about the time when this work was completed, Goto [G] proved similar results but in the case of compact interfaces. Goto's approach, which is different from ours, is based on introduction of a notion of finite speed of propagation for the evolution.

The authors would like to thank the referee for pointing out an error in the original version of proof of Theorem 1.7 and for making suggestions for us to improve English expressions in this paper.

1. Formulation of the problem, definitions and basic facts. We consider the nonlinear equation

$$u_t + F(Du, D^2u) = 0 \quad \text{in} \quad Q_T = \Omega \times (0, T), \tag{1.1}$$

where $T > 0$, Ω is an open subset of $\boldsymbol{R}^N$, u_t, Du and D^2u denote the time derivative, the spatial gradient and the spatial Hessian of the unknown function $u\colon \Omega \times [0, T] \to \boldsymbol{R}$ respectively, $F\colon \boldsymbol{R}^N \times \boldsymbol{S}^N \to \boldsymbol{R}$ is a given function and $\boldsymbol{S}^N$ denotes the space of $N \times N$ symmetric matrices.

Throughout the paper we will be assuming that

$$F \in C(J_0), \qquad \text{where} \quad J_0 = (\boldsymbol{R}^N \setminus \{0\}) \times \boldsymbol{S}^N, \tag{1.2}$$

$$\begin{cases} F \text{ is elliptic, i.e., for all } p \in \boldsymbol{R}^N \setminus \{0\} \text{ and } X, Y \in \boldsymbol{S}^N, \\ \text{if } X \le Y, \text{ then } F(p, X) \ge F(p, Y), \end{cases} \tag{1.3}$$

and, finally,

$$(1.4)\qquad \begin{cases} F \text{ is geometric, i.e., for any } \lambda>0,\ \mu\in \boldsymbol{R} \text{ and } (p, X)\in J_0\,, \\ F(\lambda p, \lambda X+\mu p\otimes p)=\lambda F(p, X)\,. \end{cases}$$

An immediate consequence of (1.2) is that there exists a function $c\in C((0, \infty))$ such that

$$(1.5)\qquad -c(|p|)\le F(p, I)\le F(p, -I)\le c(|p|) \qquad (p\in \boldsymbol{R}^N\setminus\{0\})\,.$$

As mentioned in the Introduction, the correct class of weak solutions of the equation (1.1) is the class of viscosity solutions. Here we need to adapt their definition for the possible singularities of F at $p=0$.

To this end, denote by $\mathscr{F}$ the set of functions $f\in C^2([0, \infty))$ such that $f(0)=f'(0)=f''(0)=0$ and $f''(r)>0$ for $r>0$ which satisfy

$$\lim_{p\to 0}\frac{f'(|p|)}{|p|}F(p, I)=\lim_{p\to 0}\frac{f'(|p|)}{|p|}F(p, -I)=0\,.$$

It is obvious that $\mathscr{F}$ is a cone in $C^2([0, \infty))$ with vertex at the origin, i.e., if $f, g\in\mathscr{F}$ and $\alpha>0$, then $f+g\in\mathscr{F}$ and $\alpha f\in\mathscr{F}$. In the sequel, we may write $\mathscr{F}(F)$ for $\mathscr{F}$ to indicate which F we are concerned with.

We note that $\mathscr{F}\neq\varnothing$ provided F satisfies (1.2) and (1.3). Indeed, without any loss of generality, we may assume that the function c, given by (1.5), is actually in $C^1((0, \infty))$ and satisfies

$$c>0 \text{ and } (1/c)'>0 \text{ in } (0, 1] \text{ and } \lim_{r\downarrow 0} c(r)=\infty \text{ and } \lim_{r\downarrow 0}(1/c)'(r)=0\,.$$

Next define f: $[0, 1]\to \boldsymbol{R}$ by

$$f(r)=\begin{cases} \displaystyle\int_0^r \frac{s^2}{c(s)}\,ds & \text{if } 0<r\le 1\,, \\ 0 & \text{if } r=0\,. \end{cases}$$

It follows that, if $|p|\le 1$, then

$$-|p|=-\frac{f'(|p|)}{|p|}c(|p|)\le\frac{f'(|p|)}{|p|}F(p, I)\le\frac{f'(|p|)}{|p|}F(p, -I)\le\frac{f'(|p|)}{|p|}c(|p|)=|p|\,,$$

hence

$$\lim_{p\to 0}\frac{f'(|p|)}{|p|}F(p, I)=\lim_{p\to 0}\frac{f'(|p|)}{|p|}F(p, -I)=0\,.$$

An extension of f to $[0, \infty)$ in an appropriate fashion yields an $f\in\mathscr{F}(F)$.

Now let $\mathscr{O}$ be an open subset of Q_T and we introduce the class of *admissible test* *functions*. For future reference we will call this class $\mathscr{A}(F)$ to denote its dependence on the specific F under consideration.

DEFINITION 1.1. A function $\varphi \in C^2(\mathcal{O})$ is *admissible* if for any $\hat{z} \equiv (\hat{x}, \hat{t})$ in $\mathcal{O}$ such that $D\varphi(\hat{z}) = 0$, there is a constant $\delta > 0$ and functions $f \in \mathscr{F}$ and $\omega \in C([0, \infty))$ satisfying $\lim_{r \downarrow 0} \omega(r)/r = 0$ such that, for all $(x, t) \in B(\hat{z}, \delta)$

$$|\varphi(x, t) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t - \hat{t})| \leq f(|x - \hat{x}|) + \omega(|t - \hat{t}|).$$

Next recall that the *upper semicontinuous envelope* u^* and the *lower semicontinuous envelope* u_* of a function $u: \mathcal{O} \to \boldsymbol{R} \cup \{\pm\infty\}$ are defined by

$$u^*(z) = \lim_{r \downarrow 0} \sup\{u(\zeta) \mid |\zeta - z| \leq r\} \quad \text{and} \quad u_*(z) = \lim_{r \downarrow 0} \inf\{u(\zeta) \mid |\zeta - z| \leq r\},$$

respectively.

DEFINITION 1.2. (i) A function $u: \mathcal{O} \to \boldsymbol{R} \cup \{-\infty\}$ is a *viscosity subsolution* of (1.1) in $\mathcal{O}$ if $u^* < \infty$ in $\mathcal{O}$ and for all $\varphi \in \mathscr{A}(F)$ and all local finite maximum points z of $u^* - \varphi$,

$$\begin{cases} \varphi_t(z) + F(D\varphi(z), D^2\varphi(z)) \leq 0 & \text{if } \ D\varphi(z) \neq 0, \\ \varphi_t(z) \leq 0 & \text{otherwise}. \end{cases}$$

(ii) A function $u: \mathcal{O} \to \boldsymbol{R} \cup \{\infty\}$ is a *viscosity supersolution* of (1.1) in $\mathcal{O}$ if $u_* > -\infty$ in $\mathcal{O}$ and for all $\varphi \in \mathscr{A}(F)$ and all local finite minimum points z of $u_* - \varphi$,

$$\begin{cases} \varphi_t(z) + F(D\varphi(z), D^2\varphi(z)) \geq 0 & \text{if } \ D\varphi(z) \neq 0, \\ \varphi_t(z) \geq 0 & \text{otherwise}. \end{cases}$$

(iii) A *viscosity solution* of (1.1) in $\mathcal{O}$ is defined to be a function which is both a viscosity subsolution and supersolution of (1.1) in $\mathcal{O}$.

The word *admissible* may sound confusing. The introduction of the class of admissible test functions does not lessen but rather strengthen the usual requirements for functions to be viscosity solutions.

It is immediate that if $u \in C^2(\mathcal{O})$ satisfies

$$\begin{cases} u_t(z) + F(Du(z), D^2u(z)) \leq 0 & \text{if } \ Du(z) \neq 0, \\ u_t(z) \leq 0 & \text{if } \ Du(z) = 0, \end{cases}$$

or

$$\begin{cases} u_t(z) + F(Du(z), D^2u(z)) \geq 0 & \text{if } \ Du(z) \neq 0, \\ u_t(z) \geq 0 & \text{if } \ Du(z) = 0, \end{cases}$$

then u is, respectively, a viscosity subsolution or a viscosity supersolution of (1.1) in $\mathcal{O}$.

In the sequel we are only concerned with viscosity subsolution, supersolutions and solutions of (1.1). For brevity we will simply call them sub-, super- and solutions of (1.1), respectively.

Next we discuss a number of properties of viscosity solutions. When necessary we also briefly sketch their proofs.

PROPOSITION 1.3. *Let F and F_n ($n \in \boldsymbol{N}$) satisfy* (1.2)–(1.4). *Assume that $F_n \to F$ locally uniformly in J_0 and that $\mathscr{F}(F) \subset \mathscr{F}(F_n)$ for all $n \in \boldsymbol{N}$ and that for any $f \in \mathscr{F}(F)$,*

$$\liminf_{\substack{p \to 0 \\ n \to \infty}} \frac{f'(|p|)}{|p|} F_n(p, I) \geq 0 \quad \left(resp., \limsup_{\substack{p \to 0 \\ n \to \infty}} \frac{f'(|p|)}{|p|} F_n(p, -I) \leq 0 \right).$$

Let u_n ($n \in \boldsymbol{N}$) be subsolutions (*resp., supersolutions*) *of*

$$\frac{\partial u_n}{\partial t} + F_n(Du_n, D^2 u_n) = 0 \qquad in \quad \mathcal{O}\,,$$

and define $\bar{u}, \underline{u}\colon \mathcal{O} \to \boldsymbol{R} \cup \{\pm\infty\}$ by

$$\bar{u}(z) = \lim_{r \downarrow 0} \sup\left\{ u_n(\zeta) \,|\, |\zeta - z| \leq r, n > \frac{1}{r} \right\},$$

$$\underline{u}(z) = \lim_{r \downarrow 0} \inf\left\{ u_n(\zeta) \,|\, |\zeta - z| \leq r, n > \frac{1}{r} \right\}.$$

Assume that $\bar{u}(z) < \infty$ (*resp., $\underline{u}(z) > -\infty$*) *for all $z \in \mathcal{O}$. Then $\bar{u}$* (*resp., $\underline{u}$*) *is a subsolution* (*resp., a supersolution*) *of* (1.1) *in $\mathcal{O}$.*

PROOF. We only prove the subsolution case; the case of supersolution follows exactly in the same way.

Let $\varphi \in \mathscr{A}(F)$ and assume that $\bar{u} - \varphi$ has a strict local finite maximum at some $\hat{z} = (\hat{x}, \hat{t}) \in \mathcal{O}$. If $D\varphi(\hat{z}) \neq 0$, we conclude as in the standard case in the theory of viscosity solutions.

It only remains to show that $\varphi_t(\hat{z}) \leq 0$ when $D\varphi(\hat{z}) = 0$. Since φ is admissible, there are $\delta > 0$, $f \in \mathscr{F}$ and $\omega \in C(\boldsymbol{R})$ with $\omega(r) = o(r)$ as $r \to 0$ such that

$$|\varphi(x, t) - \varphi(\hat{x}, \hat{t}) - \varphi_t(\hat{x}, \hat{t})(t - \hat{t})| \leq f(|x - \hat{x}|) + \omega(t - \hat{t})$$

for all $(x, t) \in B(\hat{z}, \delta)$. Without loss of generality we assume that $\omega \in C^1(\boldsymbol{R})$ and $\omega(0) = \omega'(0) = 0$ and also that $\omega(r) > 0$ for $r \neq 0$.

Next choose a sequence $\{\omega_n\} \subset C^2(\boldsymbol{R})$ such that $\omega_n(r) \to \omega(r)$ and $\omega_n'(r) \to \omega'(r)$ locally uniformly in $\boldsymbol{R}$ as $n \to \infty$ and set

$$\psi(x, t) = \varphi_t(\hat{z})(t - \hat{t}) + 2f(|x - \hat{x}|) + 2\omega(t - \hat{t})\,,$$

$$\psi_n(x, t) = \varphi_t(\hat{z})(t - \hat{t}) + 2f(|x - \hat{x}|) + 2\omega_n(t - \hat{t})\,.$$

It is immediate that $\psi_n \in \mathscr{A}(F)$ and, moreover, $\bar{u} - \psi$ has a local strict maximum at $\hat{z}$.

Since $\psi_n \to \psi$ locally uniformly in $\mathcal{O}$, we may assume that $u_n^* - \psi_n$ attains a local finite maximum at some point (x_n, t_n), where $(x_n, t_n) \to \hat{z}$ as $n \to \infty$. Since u_n is a subsolution, we have

$$\varphi_t(\hat{z}) + 2\omega_n'(t_n - \hat{t}) + F\left(2f'(|x_n - \hat{x}|)\frac{x_n - \hat{x}}{|x_n - \hat{x}|}, \frac{2f'(|x_n - \hat{x}|)}{|x_n - \hat{x}|} I\right) \leq 0$$

if $x_n \neq \hat{x}$, and $\varphi_t(\hat{z}) + 2\omega_n'(t_n - \hat{t}) \leq 0$ if $x_n = \hat{x}$. Letting $n \to \infty$, we get $\varphi_t(\hat{z}) \leq 0$. □

The next two propositions, which are classical in the theory of viscosity solutions, (see, for example, [CIL]), follow by adapting their proofs as above; we therefore state them without proof.

PROPOSITION 1.4. *Assume that* (1.2)–(1.4) *hold. Let* $\mathcal{S}$ *be a collection of subsolutions of* (1.1) *in* $\mathcal{O}$. *Set*

$$u(z) = \sup\{v(z) \mid v \in \mathcal{S}\} \qquad \text{for} \quad z \in \mathcal{O}\,.$$

If u is locally bounded above in $\mathcal{O}$, i.e., $u^* < \infty$ in $\mathcal{O}$, then u is a subsolution of (1.1) *in* $\mathcal{O}$. *A similar assertion holds for supersolutions of* (1.1) *in* $\mathcal{O}$.

PROPOSITION 1.5. *Assume that* (1.2)–(1.4) *hold. Let g and h be a subsolution and a supersolution of* (1.1) *in $\mathcal{O}$, respectively, and assume that g and h are locally bounded in $\mathcal{O}$ and satisfy $g \leq h$ in $\mathcal{O}$. Finally define $u\colon \mathcal{O} \to \boldsymbol{R}$ by*

$$u(z) = \sup\{v(z) \mid v \text{ is a subsolution of (1.1) in } \mathcal{O},\ g \leq v \leq h \text{ in } \mathcal{O}\} \qquad \text{for} \quad z \in \mathcal{O}.$$

Then u is a solution of (1.1) *in* $\mathcal{O}$.

A straightforward adaptation of the proof of an analogous results of [CGG] yields:

PROPOSITION 1.6. *Assume* (1.2)–(1.4) *and let θ be a nondecreasing continuous function on $\boldsymbol{R}$ and u a subsolution* (*resp., supersolution*) *of* (1.1) *in $\mathcal{O}$. Then $\theta \circ u$ is a subsolution* (*resp., supersolution*) *of* (1.1) *in* $\mathcal{O}$.

REMARK. In the above and below we agree to understand that

$$\theta \circ u(z) = \lim_{r \to \pm\infty} \theta(r) \qquad \text{if} \quad u(z) = \pm\infty\,, \text{ respectively}\,.$$

Before stating our main results, we need to introduce the following notation:

$$\partial_p Q_T = (\Omega \times \{0\}) \cup (\partial\Omega \times [0, T)) \quad \text{and} \quad \mathcal{R}_T = \bar{\Omega} \times [0, T)\,.$$

THEOREM 1.7. *Assume that* (1.2)–(1.4) *hold. Let $u \in \mathrm{USC}(\mathcal{R}_T)$ and $v \in \mathrm{LSC}(\mathcal{R}_T)$ be a subsolution and a supersolution of* (1.1), *respectively. Assume that*

$$(1.6) \qquad \lim_{r \downarrow 0} \sup\{u(z) - v(\zeta) \mid (z, \zeta) \in (\partial_p Q_T \times \mathcal{R}_T) \cup (\mathcal{R}_T \times \partial_p Q_T), |z - \zeta| \leq r\} \leq 0\,.$$

Then $u \leq v$ *in* $\mathscr{R}_T$ *and moreover,*

$$\lim_{r \downarrow 0} \sup\{u(z) - v(\zeta) \mid z, \zeta \in \mathscr{R}_T, |z - \zeta| \leq r\} \leq 0 . \tag{1.7}$$

Now we consider the initial value problem

$$\begin{cases} u_t + F(Du, D^2u) = 0 & \text{in} \quad Q_T , \\ u = g & \text{on} \quad \boldsymbol{R}^N \times \{0\} , \end{cases} \tag{1.8}$$

where g is a given function on $\boldsymbol{R}^N$ and Q_T denotes the set $\boldsymbol{R}^N \times (0, T)$. As before we write

$$\mathscr{R}_T = \boldsymbol{R}^N \times [0, T) .$$

Finally, we denote by BUC(D) and UC(D) the sets of bounded uniformly continuous functions on D and uniformly continuous functions on D, respectively.

THEOREM 1.8. *Assume that* $g \in \mathrm{BUC}(\boldsymbol{R}^N)$ *and that* (1.2)–(1.4) *hold. Then there is a unique solution* $u \in \mathrm{BUC}(\mathscr{R}_T)$ *of* (1.8).

Next we recall briefly the level set approach to motions of hypersurfaces. For the details we refer to [ES], [CGG], [BSS], etc.

To this end, we denote by $\mathscr{E}$ the collection of triples (Γ, D^+, D^-) consisting of a closed subset Γ and two open subsets $D^\pm$ of $\boldsymbol{R}^N$ such that

$$\Gamma \cup D^+ \cup D^- = \boldsymbol{R}^N \quad \text{and} \quad \Gamma, D^+, D^- \quad \text{are mutually disjoint} . \tag{1.9}$$

We note that a triple (Γ, D^+, D^-) of a closed subset Γ and two open subsets $D^\pm$ of $\boldsymbol{R}^N$ satisfies (1.9) if and only if there is a function $g \in \mathrm{BUC}(\boldsymbol{R}^N)$ such that

$$\begin{cases} \Gamma = \{x \in \boldsymbol{R}^N \mid g(x) = 0\} , \\ D^+ = \{x \in \boldsymbol{R}^N \mid g(x) > 0\} , \\ D^- = \{x \in \boldsymbol{R}^N \mid g(x) < 0\} . \end{cases} \tag{1.10}$$

Fix a $(\Gamma_0, D_0^+, D_0^-) \in \mathscr{E}$ and choose a function $g \in \mathrm{BUC}(\boldsymbol{R}^N)$ satisfying (1.10) with (Γ_0, D_0^+, D_0^-) in place of (Γ, D^+, D^-). Theorem 1.8 yields the existence of a unique solution $u \in \mathrm{BUC}(\mathscr{R}_T)$ of (1.8). For each $t \in (0, T)$ define Γ_t, D_t^+ and D_t^- by

$$\Gamma_t = \{x \in \boldsymbol{R}^N \mid u(x, t) = 0\} , \tag{1.11}$$

and

$$D_t^\pm = \{x \in \boldsymbol{R}^N \mid u(x, t) \gtrless 0\} . \tag{1.12}$$

Then, by the arbitrariness of $T > 0$ and the uniqueness of the solution u of (1.8), the definition of (Γ_t, D_t^+, D_t^-) can be extended for all $t \geq 0$.

An important issue here is whether the triples (Γ_t, D_t^+, D_t^-) depend on the choice of g or not. The following theorem answers this question:

THEOREM 1.9. *Assume that* (1.2)–(1.4) *hold and let* $g_1, g_2 \in \mathrm{BUC}(\boldsymbol{R}^N)$ *satisfy*

$$\{g_1>0\}=\{g_2>0\}\,, \quad \{g_1<0\}=\{g_2<0\} \quad (\textit{and hence } \{g_1=0\}=\{g_2=0\})\,.$$

Let $u_1, u_2 \in \mathrm{BUC}(\mathscr{R}_T)$ *be the solutions of* (1.8), *respectively, with* g_1 *and* g_2 *in place of* g. *Then*

$$\{u_1>0\}=\{u_2>0\}\,, \quad \{u_1<0\}=\{u_2<0\} \quad \textit{and} \quad \{u_1=0\}=\{u_2=0\}\,.$$

In the above and henceforth we use the notational convention: for a function: $D \to \boldsymbol{R}$ and $\gamma \in \boldsymbol{R}$, we write

$$\{f=\gamma\}\,, \quad \{f<\gamma\}\,, \text{ etc.}$$

for

$$\{x \in D \,|\, f(x)=\gamma\}\,, \quad \{x \in D \,|\, f(x)<\gamma\}\,, \text{ etc.}$$

From Theorem 1.9 we see that for each $t \geq 0$ the procedure described above defines a mapping

$$E_t\colon \mathscr{E} \ni (\Gamma_0, D_0^+, D_0^-) \mapsto (\Gamma_t, D_t^+, D_t^-) \in \mathscr{E}. \tag{1.13}$$

Moreover, it is easily seen that the collection $\{E_t \,|\, t \geq 0\}$ has the semigroup property:

$$E_0=\mathrm{id}_{\mathscr{E}}\,, \quad E_t \circ E_s = E_{t+s} \qquad \text{for} \quad t, s \geq 0\,. \tag{1.14}$$

We continue with an alternative way to define a generalized evolution of fronts or hypersurfaces. This approach, which is of course equivalent to the level set approach under certain conditions, is based on using the *signed distance function* to the front and was introduced by Soner [Son] for mean curvature evolution and further developed by Barles, Soner and Souganidis [BSS] for more general motions.

We say that $(\Gamma_t, D_t^+, D_t^-) \in \mathscr{E}$ is a generalized evolution of $(\Gamma_0, D_0^+, D_0^-) \in \mathscr{E}$ if and only if the signed distance function

$$d(x,t)=\begin{cases} \mathrm{dist}(x,\Gamma_t) & \text{if} \quad x \in D_t^+ \cup \Gamma_t\,, \\ -\mathrm{dist}(x,\Gamma_t) & \text{if} \quad x \in D_t^-\,, \end{cases}$$

is such that

$$\begin{cases} d \vee 0 \text{ is a supersolution of (1.1)}\,, \\ d \wedge 0 \text{ is a subsolution of (1.1)}\,, \end{cases} \tag{1.15}$$

where $d \vee 0 = \max(d, 0)$ and $d \wedge 0 = \min(d, 0)$.

We conclude this section by listing a number of examples which are included in the theory developed here.

In the whole generality we consider the motion of surfaces Γ_t with normal velocity

$$V=f(\kappa_1, \kappa_2, \ldots, \kappa_{N-1}, n) \qquad \text{on} \quad \Gamma_t\,, \tag{1.16}$$

where n is the normal vector and $\kappa_1 \le \cdots \le \kappa_{N-1}$ are the principal curvatures of Γ_t. It follows from Giga and Goto ([GG]) that the geometric pde in the level set approach describing (1.16) is given by (1.1) with

$$F(p, X) = -|p| f\left(k_1(p, X), \ldots, k_{N-1}(p, X), -\frac{p}{|p|}\right), \tag{1.17}$$

where $k_1(p, X) \le k_2(p, X) \le \cdots \le k_{N-1}(p, X)$ are the eigenvalues of the linear mapping on the orthogonal complement of the vector p (assuming $p \neq 0$) in $\boldsymbol{R}^N$ induced by the matrix

$$\frac{1}{|p|}\left(I - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) X .$$

(Note that $I - p \otimes p / |p|^2$ is the orthogonal projection of $\boldsymbol{R}^N$ onto the orthogonal complement of p.)

It is immediate that F's defined by (1.17) satisfy (1.2) and (1.4), the first as long as f in (1.16) is a continuous function. A sufficient condition for F to be degenerate elliptic is the following:

$$\begin{cases} \text{For } i = 1, \ldots, N-1, p \in S^{N-1} \text{ and each } (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{N-1}) \in \boldsymbol{R}^{N-2} \\ \text{the function } \lambda_i \mapsto f(\lambda_1, \ldots, \lambda_{N-1}, p) \text{ is nondecreasing in } \boldsymbol{R} . \end{cases} \tag{1.18}$$

Let us just mention some examples where $f(\lambda_1, \ldots, \lambda_{N-1}, p) = f(\lambda_1, \ldots, \lambda_{N-1})$ is independent of p. The case where

$$f(\lambda_1, \ldots, \lambda_{N-1}) = \lambda_1 + \cdots + \lambda_{N-1}$$

corresponds to motions of surfaces by their mean curvature; then

$$F(p, X) = -\operatorname{tr}\left(I - \frac{p \otimes p}{|p|^2}\right) X .$$

Condition (1.18) is satisfied and, moreover,

$$|F(p, X)| \le \|X\| \qquad \text{for all} \quad (p, X) \in J_0 ,$$

from which follows that $F^*(0, 0) = F_*(0, 0) = 0$. This is the case covered by the theory of [ES] and [CGG].

We obtain a simple generalization of a motion of surfaces by mean curvature by setting

$$f(\lambda_1, \ldots, \lambda_{N-1}) = g(\lambda_1 + \cdots + \lambda_{N-1}) \qquad \text{with} \quad g \in C(\boldsymbol{R}) .$$

Then

$$F(p, X) = -|p| g\left(\frac{1}{|p|} \operatorname{tr}\left(I - \frac{p \otimes p}{|p|^2}\right) X\right) .$$

If g is nondecreasing in $\boldsymbol{R}$, then F satisfies (1.2), (1.3) and (1.4). If, for instance, $g(r)=r^\alpha$ and $\alpha>1$ is an odd natural number, then

$$F^*(0,0)=\infty \quad \text{and} \quad F_*(0,0)=-\infty\,; \tag{1.19}$$

this is a situation where the theory of [ES] and [CGG] is not applicable.

We obtain another example by replacing mean curvature by Gaussian curvature

$$f(\lambda_1,\ldots,\lambda_{N-1})=\lambda_1\cdots\lambda_{N-1}\,.$$

Then

$$F(p,X)=-|p|\det\left(\frac{1}{|p|}\left(I-\frac{p\otimes p}{|p|^2}\right)X+\frac{p\otimes p}{|p|^2}\right).$$

This F does not satisfy (1.3), however. A related F which satisfies (1.3) is introduced by

$$f(\lambda_1,\ldots,\lambda_{N-1})=\lambda_1^+\ldots\lambda_{N-1}^+\,.$$

More generally, let $N>1$ and P_m with $m\in\{1,\ldots,N-1\}$ denote the m-th elementary symmetric polynomial of the variables $\lambda_1,\ldots,\lambda_{N-1}$. It is well-known (see [M], [T], for instance) that for $m>1$ there is a closed convex cone K_m in $\boldsymbol{R}^{N-1}$ with vertex at the origin such that $f=P_m$ satisfies (1.18) as long as $(\lambda_1,\ldots,\lambda_{N-1})\in K_m$, such that $K_m\supset[0,\infty)^{N-1}$ and such that $P_m(\lambda_1,\ldots,\lambda_{N-1})>0$ in the interior K_m° and $P_m(\lambda_1,\ldots,\lambda_{N-1})=0$ on ∂K_m. In particular, if $m=N-1$, then $K_m=[0,\infty)^{N-1}$. Define $\hat{P}_m\in C(\boldsymbol{R}^{N-1})$ by

$$\hat{P}_m(\lambda_1,\ldots,\lambda_{N-1})=\begin{cases}P_m(\lambda,\ldots,\lambda_{N-1}) & \text{if}\quad(\lambda_1,\ldots,\lambda_{N-1})\in K_m\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Again, if $m=N-1$, then $\hat{P}_m(\lambda_1,\ldots,\lambda_{N-1})=\lambda_1^+\cdots\lambda_{N-1}^+$. Corresponding to $f=\hat{P}_m$, we have

$$F(p,X)=-|p|\,\hat{P}_m(k_1(p,X),\ldots,k_{N-1}(p,X))\,,$$

where the k_i are as in (1.17), which satisfies (1.2)–(1.4).

A little more complicated examples of f's are given by the ratio of P_m and P_l with $0<l<m\le N$. That is,

$$f(\lambda_1,\ldots,\lambda_{N-1})=\begin{cases}\dfrac{P_m(\lambda,\ldots,\lambda_{N-1})}{P_l(\lambda_1,\ldots,\lambda_{N-1})} & \text{if}\quad(\lambda_1,\ldots,\lambda_{N-1})\in K_m\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

It is known (see [M]) that f satisfies (1.18). Thus the function

$$F(p,X)=-|p|\,f(k_1(p,X),\ldots,k_{N-1}(p,X))$$

on J_0 satisfies (1.2)–(1.4). Also, if $g\in C(\boldsymbol{R})$ is nondecreasing, then

$$F(p, X) = -|p| g \circ f(k_1(p, X), \ldots, k_{N-1}(p, X))$$

on J_0 satisfies (1.2)–(1.4). If $N \geq 3$, $l = N-2$ and $m = N-1$ and if $\kappa_1, \ldots, \kappa_{N-1}$ denote the principal curvatures of a surface Γ, then

$$f(\kappa_1, \ldots, \kappa_{N-1}) = \left(\sum_{i=1}^{N-1} \frac{1}{\kappa_i} \right)^{-1}$$

represents the harmonic curvature of Γ.

2. Proofs of main theorems and generalizations.

PROOF OF THEOREM 1.7. In view of Proposition 1.6, we may assume that u and v are bounded on $\mathscr{R}_T$. It is convenient to extend the domain of definition of u, v to $\bar{Q}_T$ by setting

$$u(x, T) = \lim_{r \downarrow 0} \sup\{u(y, s) \,|\, (y, s) \in \mathscr{R}_T, |y - x| + |s - T| \leq r\},$$

and

$$v(x, T) = \lim_{r \downarrow 0} \inf\{v(y, s) \,|\, (y, s) \in \mathscr{R}_T, |y - x| + |s - T| \leq r\}.$$

The function u is still upper semicontinuous and bounded in $\bar{Q}_T$. Similarly, v is lower semicontinuous and bounded in $\bar{Q}_T$. Moreover, u and v are, respectively, a subsolution and a supersolution of (1.1) in $\Omega \times (0, T]$ in the following sense: If $\varphi \in C^2(\mathcal{O})$ is admissible for some open neighborhood $\mathcal{O}$ of $\Omega \times (0, T]$ and $u - \varphi$ (resp., $v - \varphi$) has a local maximum (resp., minimum) at some $z \in \Omega \times (0, T]$, then

$$\begin{cases} \varphi_t(z) + F(D\varphi(z), D^2\varphi(z)) \leq 0 & \text{if } D\varphi(z) \neq 0, \\ \varphi_t(z) \leq 0 & \text{otherwise}, \end{cases}$$

(resp.,

$$\begin{cases} \varphi_t(z) + F(D\varphi(z), D^2\varphi(z)) \geq 0 & \text{if } D\varphi(z) \neq 0, \\ \varphi_t(z) \geq 0 & \text{otherwise}.) \end{cases}$$

Below we only check the claim for u; the argument for v is similar. To do so, we may assume that $u - \varphi$ has a strict maximum at $z = (y, T)$ with $y \in \Omega$. Then, for any $n \in N$ large enough, the function $(x, t) \mapsto u(x, t) - \varphi(x, t) - 1/[n(T-t)]$ attains a local maximum at a point $z_n = (y_n, t_n) \in Q_T$, where $z_n \to z$ as $n \to \infty$. Since u is a subsolution of (1.1) in Q_T, for all $n \in N$ large enough, we have

$$\varphi_t(z_n) + F(D\varphi(z_n), D^2\varphi(z_n)) \leq \varphi_t(z_n) + \frac{1}{[n(T-t_n)]^2} + F(D\varphi(z_n), D^2\varphi(z_n)) \leq 0$$

if $D\varphi(z) \neq 0$ and $\varphi_t(z_n) \leq \varphi_t(z_n) + 1/[n(T-t_n)]^2 \leq 0$ otherwise. Sending $n \to \infty$, we conclude that u is a subsolution of (1.1) in $\Omega\times(0, T]$.

Now, we set

$$\theta_0=\lim_{r\downarrow 0}\sup\{u(z)-v(\zeta)\,|\,(z,\zeta)\in\bar{Q}_T^2,\,|z-\zeta|\le r\},$$

and will show that $\theta_0\le 0$. To this end, we assume that $\theta_0>0$, and will get a contradiction.

Fix any $\varepsilon>0$ so that

$$\theta_1\equiv\lim_{r\downarrow 0}\sup\{u(x,t)-v(y,s)-\varepsilon t-\varepsilon s\,|\,(x,t,y,s)\in\bar{Q}_T^2,\,|x-y|\vee|t-s|\le r\}>0\,.$$

Let $f\in\mathscr{F}(F)$ and let $\alpha>0$ be a constant to be fixed later on. We define a function Φ on $\bar{Q}_T^2$ by

$$\Phi(x,t;y,s)=u(x,t)-v(y,s)-\alpha f(|x-y|)-\alpha(t-s)^2-\varepsilon t-\varepsilon s,$$

and set $\theta=\sup_{\bar{Q}_T^2}\Phi$. Note that

$$\theta\ge\hat{\theta}\equiv\lim_{r\downarrow 0}\sup\{\Phi(x,t;y,s)\,|\,(x,t),(y,s)\in\bar{Q}_T,\,|x-y|\le r\}\ge\theta_1\,.$$

Let $M>0$ be a constant satisfying

$$u(z)-v(\zeta)\le M\qquad\text{for all}\quad z,\zeta\in\bar{Q}_T,$$

and observe that if $\Phi(x,t;y,s)\ge 0$, then

$$M\ge\alpha f(|x-y|)+\alpha(t-s)^2+\varepsilon t+\varepsilon s,$$

i.e.,

$$\text{(2.1)}\qquad |x-y|\le f^{-1}\left(\frac{M}{\alpha}\right),\qquad |t-s|\le\left(\frac{M}{\alpha}\right)^{1/2}.$$

Also, in view of (1.6), there is $\gamma>0$ such that

$$\sup\{\Phi(z;\zeta)\,|\,(z,\zeta)\in(\partial_p\bar{Q}_T\times\bar{Q}_T)\cup(\bar{Q}_T\times\partial_p\bar{Q}_T),\,|z-\zeta|\le\gamma\}\le\frac{\theta_1}{2}\,.$$

Here and henceforth $\partial_p\bar{Q}_T$ denotes the set $(\Omega\times\{0\})\cup(\partial\Omega\times[0,T])$. Fix $\alpha>0$ so that $f^{-1}(M/\alpha)\vee(M/\alpha)^{1/2}\le\gamma$. It follows from (2.1) that if $z,\zeta\in\bar{Q}_T$ and $\Phi(z;\zeta)>\theta_1/2$, then

$$\text{(2.2)}\qquad z,\zeta\in\Omega\times(0,T]\,.$$

Assume that $\hat{\theta}=\theta$. In view of the definition of $\hat{\theta}$, there is a sequence $\{(x_n,t_n,y_n,s_n)\}\subset\bar{Q}_T^2$ such that

$$\Phi(x_n,t_n;y_n,s_n)>\frac{\theta_1}{2}\vee\left(\theta_1-\frac{1}{n}\right)\quad\text{and}\quad|x_n-y_n|\le\frac{1}{n}\,.$$

Here we may assume that $\{t_n\}$ and $\{s_n\}$ converge to some points $\hat{t}, \hat{s} \in [0, T]$, respectively. We write $\beta = \alpha + 1$ for notational simplicity. Noting that $\lim_{r \to \infty} f(r) = \infty$, we can choose a maximum point (ξ_n, τ_n) of the function

$$(x, t) \mapsto u(x, t) - \beta f(|x - y_n|) - \alpha(t - s_n)^2 - (t - \hat{t})^2 - \varepsilon t \qquad \text{on} \quad \bar{Q}_T,$$

and a maximum point (η_n, σ_n) of the function

$$(y, s) \mapsto -v(y, s) - \beta f(|x_n - y|) - \alpha(t_n - s)^2 - (s - \hat{s})^2 - \varepsilon s \qquad \text{on} \quad \bar{Q}_T.$$

If follows that

$$\begin{aligned} &u(x_n, t_n) - v(y_n, s_n) - \beta f(|x_n - y_n|) - \alpha(t_n - s_n)^2 - (t_n - \hat{t})^2 - \varepsilon t_n \\ &\quad \le u(\xi_n, \tau_n) - v(y_n, s_n) - \beta f(|\xi_n - y_n|) - \alpha(\tau_n - s_n)^2 - (\tau_n - \hat{t})^2 - \varepsilon \tau_n . \end{aligned}$$

Accordingly we have

$$f(|\xi_n - y_n|) + (\tau_n - \hat{t})^2 \le \theta - \Phi(x_n, t_n; y_n, s_n) + f(|x_n - y_n|) + (t_n - \hat{t})^2,$$

and

$$\Phi(x_n, t_n; y_n, s_n) \le \Phi(\xi_n, \tau_n; y_n, s_n) + f(|x_n - y_n|) + (t_n - \hat{t})^2 .$$

The former of the above inequalities yields that $\xi_n - y_n \to 0$ and $\tau_n \to \hat{t}$ as $n \to \infty$. In view of the latter we may assume that $\Phi(\xi_n, \tau_n; y_n, s_n) > \theta_1/2$ for all n. In the same way, we deduce that $x_n - \eta_n \to 0$ and $\sigma_n \to \hat{s}$ as $n \to \infty$ and we may assume that $\Phi(x_n, t_n; \eta_n, \sigma_n) > \theta_1/2$ for all n.

Note that (2.2) yields $(\xi_n, \tau_n) \in \Omega \times (0, T]$ for all $n \in N$ large enough. If $(\xi_n, \tau_n) \in \Omega \times (0, T]$ and if we set

$$\varphi(x, t) = \beta f(|x - y_n|) + \alpha(t - s_n)^2 + (t - \hat{t})^2 + \varepsilon t \quad \text{and} \quad p_n = \xi_n - y_n,$$

then, since u is a subsolution of (1.1) in $\Omega \times (0, T]$,

$$\begin{aligned} 0 &\ge \varphi_t(\xi_n, \tau_n) + F\left(\beta f'(|p_n|)\frac{p_n}{|p_n|}, \beta f'(|p_n|)\frac{1}{|p_n|}I\right) \\ &= 2\alpha(\tau_n - s_n) + 2(\tau_n - \hat{t}) + \varepsilon + \frac{\beta f'(|p_n|)}{|p_n|}F(p_n, I), \end{aligned}$$

if $p_n \ne 0$, and

$$0 \ge 2\alpha(\tau_n - s_n) + 2(\tau_n - \hat{t}) + \varepsilon,$$

if $p_n = 0$. Sending $n \to \infty$, we obtain

$$0 \ge 2\alpha(\hat{t} - \hat{s}) + \varepsilon \ge 2\alpha(\hat{t} - \hat{s}) + \varepsilon .$$

Since v is a supersolution, we similarly obtain

$$0 \le 2\alpha(\hat{t} - \hat{s}) - \varepsilon .$$

Subtracting this from the above, we get $0 \geq 2\varepsilon$, which is a contradiction.

If $\hat{\theta} < \theta$, then choose $\rho > 0$ so that

$$\theta_2 \equiv \sup\{\Phi(x, t; y, s) \,|\, (x, t), (y, s) \in \bar{Q}_T, |x - y| \leq \rho\} < \theta ,$$

and define Ψ on $\bar{Q}_T^2$ by

$$\Psi(x, t; y, s) = \Phi(x, t; y, s) - \delta |x|^2 - \delta |y|^2 \qquad (\delta > 0) .$$

It is clear that Ψ attains a maximum at some point $(\hat{x}, \hat{t}, \hat{y}, \hat{s}) \in \bar{Q}_T^2$. Observe also that $\hat{\theta} \leq \theta_2$. We henceforth assume that δ is sufficiently small so that $\sup_{(\bar{Q}_T)^2} \Psi > \theta_2$, which yields

$$\Phi(\hat{x}, \hat{t}; \hat{y}, \hat{s}) \geq \sup_{\bar{Q}_T^2} \Psi > \theta_2 ,$$

and in turn $(\hat{x}, \hat{t}, \hat{y}, \hat{s}) \in (\Omega \times (0, T])^2$, by (2.2). The above together with (2.1) guarantees that

$$\rho < |\hat{x} - \hat{y}| \leq f^{-1}(M/\alpha) .$$

Moreover, since $0 \leq \Psi(\hat{x}, \hat{t}; \hat{y}, \hat{s}) \leq M - \delta |\hat{x}|^2 - \delta |\hat{y}|^2$, we have $\delta(|\hat{x}| + |\hat{y}|) \to 0$ as $\delta \downarrow 0$.

We set

$$u_\delta(x, t) = u(x, t) - \delta |x|^2 \quad \text{and} \quad v_\delta(y, s) = v(y, s) + \delta |y|^2$$

and recall (cf. [CIL]) that there is $X \in \boldsymbol{S}^N$ such that

$$\left(\alpha f'(|\hat{p}|) \frac{\hat{p}}{|\hat{p}|}, 2\alpha(\hat{t} - \hat{s}) + \varepsilon, X\right) \in \bar{\mathscr{P}}^{2,+} u_\delta(\hat{x}, \hat{t}) ,$$

and

$$\left(\alpha f'(|\hat{p}|) \frac{\hat{p}}{|\hat{p}|}, 2\alpha(\hat{t} - \hat{s}) - \varepsilon, X\right) \in \bar{\mathscr{P}}^{2,-} v_\delta(\hat{y}, \hat{s}) ,$$

where $\hat{p} = \hat{x} - \hat{y}$. Here we rely on [CIL] for the definitions of $\bar{\mathscr{P}}^{2,\pm}$. If we set

$$w(x, t, y, s) = \alpha f(|x - y|) + \alpha(t - s)^2 + \varepsilon t + \varepsilon s ,$$

then

$$D^2 w(x, t, y, s) \leq \alpha \left[\left(f''(|x - y|) + \frac{f'(|x - y|)}{|x - y|}\right) \vee 2\right] \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} ,$$

where I denotes the identity matrix of order $N + 1$. Therefore we may assume that

$$\|X\| \leq C\alpha \left(f''(|\hat{p}|) + \frac{f'(|\hat{p}|)}{|\hat{p}|} + 1\right) ,$$

where C is an absolute constant. Since $\rho < |\hat{p}| \leq f^{-1}(M/\alpha)$, we have $\|X\| \leq C_1$, where

C_1 is a constant depending only on α, M, ρ and f. It follows that

$$\left(\alpha f'(|\hat{p}|)\frac{\hat{p}}{|\hat{p}|}+2\delta\hat{x},\ 2\alpha(\hat{t}-\hat{s})+\varepsilon,\ X+2\delta I\right)\in\bar{\mathscr{P}}^{2,+}u(\hat{x},\hat{t}),$$

and

$$\left(\alpha f'(|\hat{p}|)\frac{\hat{p}}{|\hat{p}|}-2\delta\hat{y},\ 2\alpha(\hat{t}-\hat{s})-\varepsilon,\ X-2\delta I\right)\in\bar{\mathscr{P}}^{2,-}v(\hat{y},\hat{s}).$$

Using the fact that $|\hat{p}|>\rho$, we may also assume, choosing δ small enough, that

$$\alpha f'(|\hat{p}|)\frac{\hat{p}}{|\hat{p}|}+2\delta\hat{x}\neq 0, \quad \text{and} \quad \alpha f'(|\hat{p}|)\frac{\hat{p}}{|\hat{p}|}-2\delta\hat{y}\neq 0.$$

Now the definition of viscosity solution yields

$$2\alpha(\hat{t}-\hat{s})+\varepsilon+F\left(\alpha f'(|\hat{p}|)\frac{\hat{p}}{|\hat{p}|}+2\delta\hat{x},\ X+2\delta I\right)\leq 0,$$

and

$$2\alpha(\hat{t}-\hat{s})-\varepsilon+F\left(\alpha f'(|\hat{p}|)\frac{\hat{p}}{|\hat{p}|}-2\delta\hat{y},\ X-2\delta I\right)\geq 0.$$

Sending $\delta\downarrow 0$, we get

$$2\alpha(\bar{t}-\bar{s})+\varepsilon+F\left(\alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|},\ Y\right)\leq 0, \tag{2.3}$$

and

$$2\alpha(\bar{t}-\bar{s})-\varepsilon+F\left(\alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|},\ Y\right)\geq 0, \tag{2.4}$$

for some $\bar{p}\in\boldsymbol{R}^N\setminus\{0\}$, $\bar{t}, \bar{s}\in[0, T)$ and $Y\in S^N$. Subtracting one of the above inequalities from the other, we again obtain a contradiction. □

For $u\colon Q_T\to\boldsymbol{R}\cup\{-\infty,\infty\}$ define $K^+u(Q_T)$ and $K^-u(Q_T)$ by

$$K^{\pm}u(Q_T)=\{(p, X)\in J_0\,|\,(p, b, X)\in\mathscr{P}^{2,\pm}u(x, t) \quad \text{for some} \quad (x, t, b)\in Q_T\times\boldsymbol{R}\}.$$

Using this notation, if we note that

$$\left(\alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|},\ Y\right)\in\overline{K^+u(Q_T)}\cap\overline{K^-v(Q_T)},$$

where the left hand side is from (2.3) or, equivalently, from (2.4), then we conclude:

THEOREM 2.1. *Let F and G satisfy* (1.2)–(1.4). *Let $u\in\mathrm{USC}(\mathscr{R}_T)$ and $v\in\mathrm{LSC}(\mathscr{R}_T)$*

be a subsolution of (1.1) *and a supersolution of* (1.1) *with G in place of F, respectively. Assume that*

$$F(p, X) \geq G(p, X) \qquad \textit{for all} \quad (p, X) \in K^+u(Q_T) \cap K^-v(Q_T)$$

and that (1.6) *holds. Then* $u \leq v$ *in* Q_T *and, moreover,* (1.7) *holds.*

PROOF OF THEOREM 1.8. In view of Proposition 1.5, in order to show the existence of solutions it suffices to construct appropriate sub- and super-solutions of (1.8).

It is not hard to see that there is a function $f \in \mathscr{F}$ such that

$$\sup_{r>0} f'(r) < \infty .$$

Fix such a function $f \in \mathscr{F}$. Then

$$\sup_{p \in \mathbf{R}^N} \frac{f'(|p|)}{|p|} |F(p, I)| \vee |F(p, -I)|$$
$$= \sup_{p \in \mathbf{R}^N} f'(|p|) \left| F\left(\frac{p}{|p|}, \frac{1}{|p|} I\right)\right| \vee \left| F\left(\frac{p}{|p|}, -\frac{1}{|p|} I\right)\right| < \infty .$$

For each $0 < \varepsilon < 1$ we choose $A(\varepsilon) > 0$ and $B(\varepsilon) > 0$ so that for all $p \in \mathbf{R}^N$,

$$|g(x) - g(y)| \leq \varepsilon + A(\varepsilon) f(|x - y|)$$

and

$$A(\varepsilon) \frac{f'(|p|)}{|p|} |F(p, I)| \vee |F(p, -I)| \leq B(\varepsilon) .$$

We define the functions $V^\pm$ on $\mathscr{R}_T$ by

$$V^\pm(x, t; \varepsilon, y) = g(y) \pm \varepsilon \pm A(\varepsilon) f(|x - y|) \pm B(\varepsilon) t , \qquad (y \in \mathbf{R}^N)$$

which turn out to be super- and sub-solutions of (1.1), respectively. Moreover, for all $x, y \in \mathbf{R}^N$, $0 \leq t < T$, and $0 < \varepsilon < 1$, we have $V^-(x, t; \varepsilon, y) \leq g(x) \leq V^+(x, t; \varepsilon, y)$, and $\sup_{0<\varepsilon<1, y \in \mathbf{R}^N} V^-(x, 0; \varepsilon, y) = g(x) = \inf_{0<\varepsilon<1, y \in \mathbf{R}^N} V^+(x, 0; \varepsilon, y)$. □

A natural generalization of Theorems 1.7 and 1.8 is the following:

THEOREM 2.2. *Assume that* $g \in \mathrm{UC}(G_R)$ *for each* $R > 0$, *where* $G_R = \{|g| < R\}$, *and that* (1.2)–(1.4) *hold. Then there is a unique solution u of* (1.8) *such that* $u \in \mathrm{UC}(U_R)$ *for all* $R > 0$, *where* $U_R = \{|u| < R\}$.

Theorem 2.2 follows from the following lemma:

LEMMA 2.3. *Let* $u \in \mathrm{BUC}(\mathscr{R}_T)$ *be a solution of* (1.1) *in* $Q_T = \mathbf{R}^N \times (0, T)$ *and* $a \in \mathbf{R}$. *Assume that* $u(x, 0) < a$ (*resp.*, $u(x, 0) > a$) *for all* $x \in \mathbf{R}^N$. *Then* $u(z) < a$ (*resp.*, $u(z) > a$) *for all* $z \in \mathscr{R}_T$.

PROOF. Fix $f\in\mathscr{F}$ so that $f'(r)\leq 1$ for all $r\geq 0$ and set

$$B=\sup_{p\neq 0}\frac{f'(|p|)}{|p|}|F(p,I)|\vee|F(p,-I)|.$$

For $\varepsilon>0$ and $y\in\boldsymbol{R}^N$ we define $w_\varepsilon\in(C^2\cap \mathrm{UC})(\mathscr{R}_T)$ by

$$w_\varepsilon(x,t)=a+\varepsilon(-BT+Bt+f(|x-y|)).$$

It is easily seen that w_ε is a supersolution of (1.1). Choose $R>0$ so that $f(R)>BT$ and observe that the assumption on $u(\cdot,0)$ yields

$$u(x,0)<a\leq w_\varepsilon(x,0) \qquad \text{for} \quad x\in\boldsymbol{R}^N\setminus B(0,R).$$

Finally let $\varepsilon>0$ be so small that

$$u(x,0)\leq w_\varepsilon(x,0) \qquad \text{for} \quad x\in B(0,R).$$

Applying Theorem 1.7, we conclude that $u\leq w_\varepsilon$ in $\mathscr{R}_T$, and thus

$$u(y,t)\leq w_\varepsilon(y,t)=a+\varepsilon B(t-T)<a$$

for all $t\in[0,T)$. Since $y\in\boldsymbol{R}^N$ is arbitrary, we see that $u(z)<a$ for all $z\in\mathscr{R}_T$.

An argument parallel to the above shows that if $u(x,0)>a$ for all $x\in\boldsymbol{R}^N$, then $u(z)>a$ for all $z\in\mathscr{R}_T$.

We need the following lemma for the proof of Theorem 1.9:

LEMMA 2.4. *Assume that* (1.2)–(1.4) *hold. Let* $g, g_n\in \mathrm{BUC}(\boldsymbol{R}^N)$ *be such that*

$$g_n(x)\uparrow g(x) \qquad \textit{for all} \quad x\in\boldsymbol{R}^N \quad \textit{as} \quad n\to\infty.$$

Finally let u_n *and* u *be the solutions of* (1.8), *with initial data* g_n *and* g, *respectively. Then*

$$u_n(z)\uparrow u(z) \qquad \textit{for all} \quad z\in\mathscr{R}_T \quad \textit{as} \quad n\to\infty.$$

PROOF. The choice of the g_n's and Theorem 1.7 yield

$$u_n\leq u_{n+1}\leq u \qquad \text{in} \quad \mathscr{R}_T.$$

Define $v:\mathscr{R}_T\to\boldsymbol{R}$ by

$$v(z)=\lim_{n\to\infty}u_n(z)=\sup_{n\in\boldsymbol{N}}u_n(z) \qquad (z\in\mathscr{R}_T).$$

Since $u_n\in C(\mathscr{R}_T)$ for all $n\in\boldsymbol{N}$, it is clear that $v\in \mathrm{LSC}(\mathscr{R}_T)$. Using the monotonicity of the sequence $\{v_k\}$, we obtain

$$\begin{aligned} u_n(z) &= \lim_{r\downarrow 0} \inf\{u_n(\zeta) \mid \zeta \in \mathscr{R}_T, |\zeta - z| \le r\} \\ &\le \lim_{r\downarrow 0} \inf\left\{u_k(\zeta) \mid \zeta \in \mathscr{R}_T, |\zeta - z| \le r, k > \frac{1}{r}\right\} \\ &\le \lim_{r\downarrow 0} \inf\{v(\zeta) \mid \zeta \in \mathscr{R}_T, |\zeta - z| \le r\} = v(z) \end{aligned}$$

for any $z \in \mathscr{R}_T$ and $n \in \boldsymbol{N}$. This shows that

$$v(z) = \lim_{r\downarrow 0} \inf\left\{u_n(\zeta) \mid \zeta \in \mathscr{R}_T, |\zeta - z| \le r, n > \frac{1}{r}\right\}.$$

Therefore, from Proposition 1.3 we see that v is a supersolution of (1.1).

Fix any $\varepsilon > 0$ and $y \in \boldsymbol{R}^N$. Fix $f \in \mathscr{F}$ so that $\sup_{[0,\infty)} f' < \infty$. The proof of Lemma 2.3 yields the existence of constants $A(\varepsilon) > 0$ and $B(\varepsilon) > 0$, which are independent of y, such that the function $w: \mathscr{R}_T \to \boldsymbol{R}$ defined by

$$w(x, t) = g(y) - 2\varepsilon - A(\varepsilon) f(|x - y|) - B(\varepsilon) t$$

is a subsolution of (1.8) and such that $g(x) - \varepsilon \ge w(x, 0)$ for all $x \in \boldsymbol{R}^N$.

Dini's theorem yields that $g_n \to g$ locally uniformly in $\boldsymbol{R}^N$ as $n \to \infty$. We may assume in view of Proposition 1.6 that the g_n's are uniformly bounded below on $\boldsymbol{R}^N$. Hence, there is an $l \in \boldsymbol{N}$ such that

$$g_n \ge w(\cdot, 0) \quad \text{in} \quad \boldsymbol{R}^N \quad \text{for all} \quad n \ge l.$$

But then Theorem 1.7 yields that $u_n \ge w$ for all $n \ge l$, from which it follows that $v \ge w$ in $\mathscr{R}_T$. Therefore,

$$u(x, t) - v(y, s) \le u(x, t) - w(y, s) \le u(x, t) - g(y) + 2\varepsilon + B(\varepsilon) s$$

for all $(x, t) \in \mathscr{R}_T$ and $0 \le s < T$. Since $u \in \mathrm{BUC}(\mathscr{R}_T)$ and $u = g$ on $\boldsymbol{R}^N \times \{0\}$, we can choose $\delta \in (0, \varepsilon/B(\varepsilon))$ so that if $(x, t) \in \mathscr{R}_T$, $|x - y| \le \delta$ and $t \le \delta$, then $u(x, t) - g(y) \le \varepsilon$. Now, if $(x, t) \in \mathscr{R}_T$, $|x - y| \le \delta$, $t \le \delta$ and $0 \le s \le \delta$, then $u(x, t) - v(y, s) \le 4\varepsilon$. Noting that $\varepsilon > 0$ and $y \in \boldsymbol{R}^N$ are arbitrary, we thus conclude that

$$\lim_{r\downarrow 0} \sup\{u(x, t) - v(y, s) \mid (x, t), (y, s) \in \mathscr{R}_T, |x - y| \le r, t \vee s \le r\} \le 0.$$

Using again Theorem 1.7, we see that $u \le v$ in $\mathscr{R}_T$. □

PROOF OF THEOREM 1.9. By symmetry it is enough to check that

$$D_1^+ \equiv \{z \in \mathscr{R}_T \mid u_1(z) > 0\} \subset D_2^+ \equiv \{z \in \mathscr{R}_T \mid u_2(z) > 0\}.$$

Recall that if u is a solution of (1.1), then, for any continuous nondecreasing function $\theta: \boldsymbol{R} \to \boldsymbol{R}$, the function $\theta \circ u$ is also a solution of (1.1). Noting that $u_i^+ = \theta \circ u_i$ for $i = 1, 2$, where

$$\theta(r)=\begin{cases} r & \text{for } r\geq 0\,, \\ 0 & \text{for } r<0\,, \end{cases}$$

we observe that u_i^+ ($i=1,2$) is the solution of (1.8) with g_i^+ in place of g.

For $n\in \boldsymbol{N}$, set

$$h_n=g_1^+\wedge(ng_2^+)$$

and let v_n be the solution of (1.8) with h_n in place of g. It is clear that $h_n(x)\uparrow g_1^+(x)$ for all $x\in \boldsymbol{R}^N$ as $n\to\infty$. From Lemma 2.4 we see that $v_n(z)\uparrow u_1^+(z)$ for all $z\in\mathscr{R}_T$ as $n\to\infty$. Since $h_n\leq ng_2^+$ and $nu_2^+=(n\theta)\circ u_2$ is a solution of (1.8) with g replaced by ng_2^+, Theorem 1.7 yields that $v_n\leq nu_2^+$ in $\mathscr{R}_T$.

Now let $z\in D_1^+$. Then $u_1^+(z)>0$ and hence $v_n(z)>0$ for some $n\in \boldsymbol{N}$. Therefore from the inequality that $v_n\leq nu_2^+$ in $\mathscr{R}_T$, we conclude that $u_2^+(z)>0$, i.e., $z\in D_2^+$. □

3. Some properties of the generalized evolutions. In this section we list a number of properties of the generalized evolutions obtained by the level set approach. Since their proofs are more or less direct adaptations of the corresponding results of [ES], [CGG], [BSS], etc., we omit most of them.

We begin by recalling that some examples of the velocity law (1.16) give PDE's (1.1) which do not satisfy (1.3) and are needed to be modified, so that the corresponding PDE's (1.1) satisfy (1.3). A natural question is whether we can impose some condition on Γ_0, so that the generalized evolution obtained by the level set approach does not depend on such modifications and hence it is in fact generated by the original velocity law. This is the first topic of the following discussions.

PROPOSITION 3.1. *Assume that* (1.2)–(1.4) *hold. Let* $g\in \mathrm{BUC}(\boldsymbol{R}^N)$ *satisfy*

$$F(Dg, D^2g)\leq\gamma \qquad \text{in} \quad \{|g|<\delta\} \tag{3.1}$$

in the viscosity sense for some $\gamma\in \boldsymbol{R}$, $\delta>0$. *Let* $u\in \mathrm{BUC}(\mathscr{R}_T)$ *be the solution of* (1.8). *Then, for each* $x\in\boldsymbol{R}^N$, *the function*: $t\mapsto u(x,t)+\gamma t$ *is nondecreasing on any subinterval of the set* $\{t\in(0,T)|\,|u(x,t)|<\delta\}$. *Moreover, if* $|u(x,t)|<\delta$, $(p,a,X)\in\mathscr{P}^{2,+}u(x,t)$ *and* $p\neq 0$, *then* $a+\gamma\geq 0$ *and* $F(p,X)\leq\gamma$.

To be precise, let us give the definition of viscosity subsolutions (just when they are continuous for simplicity) of

$$F(Dv, D^2v)=\gamma(x) \qquad \text{in} \quad \Omega\,, \tag{3.2}$$

where Ω is an open subset of $\boldsymbol{R}^N$ and $\gamma\in C(\Omega)$ is a given function. A function $\varphi\in C(\Omega)$ is *admissible* if the function $\tilde\varphi\in C^2(\Omega\times(0,T))$ given by $\tilde\varphi(x,t)=\varphi(x)$ is an admissible test function, i.e., $\tilde\varphi\in\mathscr{A}(F)$. Now, a function $v\in C(\Omega)$ is a *viscosity subsolution* (and will be simply called a subsolution) of (3.2), provided whenever $\varphi\in C^2(\Omega)$ is admissible and

$v-\varphi$ attains a local maximum at some $\hat{x}\in\Omega$, then

$$\begin{cases} F(D\varphi(\hat{x}), D^2\varphi(\hat{x}))\leq\gamma(\hat{x}) & \text{if } D\varphi(\hat{x})\neq 0\,, \\ 0\leq\gamma(\hat{x}) & \text{if } D\varphi(\hat{x})=0\,. \end{cases}$$

It is easily seen that if $g\in C(\Omega)$ satisfies (3.1) (in the viscosity sense), then the function $v\in C(Q_T)$ defined by $v(x,t)=g(x)+\gamma t$ solves (1.1).

Although the above proposition can be proved along the lines of the proof of Theorem 7.3 in [BSS], below we present a simpler one, which does not need local existence of smooth solutions. Before we give the proof of the proposition we state some immediate consequences.

COROLLARY 3.2. *Under the hypotheses of Proposition* 3.1 *let F, g and u be as in the proposition. Let G be a function on* J_0 *which satisfies the conditions* (1.2)–(1.4). *Assume that for any* $(p,X)\in J_0$,

$$G(p,X)=F(p,X) \qquad \textit{if} \quad F(p,X)\leq\gamma\,,$$

and

$$G(p,X)>\gamma \qquad \textit{if} \quad F(p,X)>\gamma\,.$$

Let $v\in\mathrm{BUC}(\mathscr{R}_T)$ *be the solution of* (1.8) *with G in place of F. Then*

$$\{|u|<\delta\}=\{|v|<\delta\}\,,\quad \{u\geq\delta\}=\{v\geq\delta\}\,,\quad \{u\leq-\delta\}=\{v\leq-\delta\}$$

and

$$u(x,t)=v(x,t) \qquad \textit{for all} \quad (x,t)\in\{|u|<\delta\}\,.$$

Let $F, G\in C(J_0)$ satisfy the conditions (1.3) and (1.4). As in (1.13), F and G define the generalized evolutions $E_t^F:\mathscr{E}\to\mathscr{E}$ and $E_t^G:\mathscr{E}\to\mathscr{E}$, with $t\geq 0$, respectively.

COROLLARY 3.3. *In addition to* (1.2)–(1.4), *let F and G satisfy*

$$\{F\leq\gamma\}=\{G\leq\gamma\} \quad (\textit{and hence, } \{F>\gamma\}=\{G>\gamma\})\,,$$

and

$$F(p,X)=G(p,X) \qquad \textit{for all} \quad (p,X)\in\{F\leq\gamma\}\,.$$

Let $(\Gamma, D^+, D^-)\in\mathscr{E}$. *Assume that there is a solution* $g\in\mathrm{BUC}(\boldsymbol{R}^N)$ *of* (3.1) *for some* $\delta>0$ *such that*

$$D^+=\{g>0\}\,,\quad D^-=\{g<0\} \quad \textit{and} \quad \Gamma=\{g=0\}\,. \tag{3.3}$$

Then

$$E_t^F(\Gamma,D^+,D^-)=E_t^G(\Gamma,D^+,D^-) \qquad \textit{for all} \quad t\geq 0\,.$$

The following is a typical sufficient condition to check if, for given $(\Gamma, D^+, D^-)\in\mathscr{E}$, there is a solution $g \in \mathrm{BUC}(\boldsymbol{R}^N)$ of (3.1) which satisfies (3.3): Γ is a compact C^2 hypersurface and the signed distance function d satisfies

$$F(Dd(x), D^2d(x)) \leq \gamma - \varepsilon \qquad \text{for all} \quad x \in \Gamma$$

pointwise for some $\varepsilon > 0$. Indeed, since $d \in C^2(\{|d| < \delta\})$ and $Dd(x) \neq 0$ for all $x \in \{|d| < \delta\}$ for some $\delta > 0$, by continuity d solves (3.1) in the classical sense for some $\delta > 0$.

Now, let P_m be the m-th elementary symmetric polynomial of the variables $\lambda_1, \ldots, \lambda_{N-1}$ and $K_m \subset \boldsymbol{R}^{N-1}$ the closed convex cone with vertex at the origin as in the discussions of Section 1 concerning examples of functions which determine normal velocity. Let $m > 1$. Define $F_m \in C(J_0)$ by $F_m(p, X) = -|p| \hat{P}_m(k_1(p, X), \ldots, k_{N-1}(p, X))$, where $\hat{P}_m$ and the k_i's are the functions defined in Section 1. If $(\Gamma, D^+, D^-) \in \mathscr{E}$ is such that there is a fuction $g \in \mathrm{BUC}(\boldsymbol{R}^N)$ for which (3.3) holds and which satisfies

$$\hat{F}_m(Dg, D^2g) \leq -\gamma \qquad \text{in} \quad \{|g| < \delta\} \tag{3.4}$$

in the viscosity sense for some $\gamma > 0$ and some $\delta > 0$, then from Corollary 3.3 we see that (1.8) with $F = \hat{F}_m$ naturally determines a generalized evolution in $\mathscr{E}$ issued from (Γ, D^+, D^-). In other words, this generalized evolution issued from (Γ, D^+, D^-) does not depend on how to extend $P_m|K_m$ to J_0. (Note that if $P \in C(J_0)$ is an extension of $P_m|K_m$ to J_0 which satisfies (1.18), then $P \leq \hat{P}_m$.) Instead of $\hat{P}_m$, if we extend $P_m|K_m$ to J_0 in a way so that the resulting function $\tilde{P}_m$ satisfies $\tilde{P}_m(\lambda) < 0$ for all $\lambda \in \boldsymbol{R}^{N-1} \setminus K_m$, if we can find a solution $g \in \mathrm{BUC}(\boldsymbol{R}^N)$ of (3.4) with $\gamma = 0$ for which (3.3) holds and if we put $\hat{F}_m(p, X) = -|p| \tilde{P}_m(k_1(p, X), \ldots, k_{N-1}(p, X))$ for $(p, X) \in J_0$, then Corollary 3.3 tells us that (1.8) with $F = \tilde{F}_m$ defines naturally a generalized evolution in $\mathscr{E}$ issued from (Γ, D^+, D^-). For instance, if we define

$$\tilde{P}_m(\lambda) = \begin{cases} P_m(\lambda) & \text{if} \quad \lambda \in K_m, \\ -\mathrm{dist}(\lambda, K_m) & \text{if} \quad \lambda \in \boldsymbol{R}^{N-1} \setminus K_m, \end{cases}$$

similar remarks are valid also for the ratio P_m/P_l with $m > l$.

Let $\tilde{P}_{N-1}$ be as above and define $\tilde{F}_{N-1} \in C(J_0)$ by

$$\tilde{F}_{N-1}(\lambda) = -|p| \tilde{P}_{N-1}(k_1(p, X), \ldots, k_{N-1}(p, X))) \,.$$

Then the convexity of g guarantees the condition (3.1) with $F = \tilde{F}_{N-1}$ and $\gamma = 0$. Let $(\Gamma, D^+, D^-) \in \mathscr{E}$ be such that D^- is a nonempty convex set and $\Gamma = \partial D^-$. For each $y \in \Gamma$ choose $\xi \in \boldsymbol{R}^{N-1}$, with $|\xi| = 1$, such that $(x - y) \cdot \xi < 0$ for all $x \in D^-$, and define $g_y \colon \boldsymbol{R}^N \to \boldsymbol{R}$ by $g_y(x) = (x - y) \cdot \xi$. Then g_y, with $y \in \Gamma$, solves (3.1) with $F = \tilde{F}_{N-1}$ and $\gamma = 0$ in $\boldsymbol{R}^N$. Therefore, if we set $g(x) = \sup\{g_y(x) \mid y \in \Gamma\}$, then g solves (3.1) with $F = \tilde{F}_{N-1}$ and $\gamma = 0$ in $\boldsymbol{R}^N$ and satisfies (3.3). Thus, in this case the condition that D^- is a nonempty convex set and $\Gamma = \partial D^-$ gives a sufficient condition for the existence of $g \in \mathrm{BUC}(\boldsymbol{R}^N)$ which satisfies (3.1) and (3.3).

Proof of Proposition 3.1. Fix $\varepsilon \in (0, \delta)$, consider the function $\theta \colon \boldsymbol{R} \to \boldsymbol{R}$ given by

$$\theta(s)=\begin{cases} s & \text{if } |s|\le\varepsilon, \\ \varepsilon & \text{if } s>\varepsilon, \\ -\varepsilon & \text{if } s<-\varepsilon, \end{cases}$$

and observe that the function

$$v(x,t)=\theta(g(x)-\gamma t) \qquad \text{in} \quad \mathbf{R}^N\times(0,h)$$

solves (1.1) in $Q_T=\mathbf{R}^N\times(0,T)$ provided $h>0$ and $|\gamma|h\le\delta-\varepsilon$.

Let $\tilde{u}=\theta\circ u$; since $\tilde{u}$ solves (1.1) and $\tilde{u}\le v$ on $\mathbf{R}^N\times\{0\}$, Theorem 1.7 yields $v\le\tilde{u}$ on $\mathbf{R}^N\times[0,h\wedge T)$.

Finally, fix any $h\in(0,T)$ so that $|\gamma|h\le\delta-\varepsilon$. Noting that the function $\tilde{w}\colon\mathscr{R}_T\to\mathbf{R}$ given by

$$\tilde{w}(x,t)=\theta(u(x,t)-\gamma h)$$

solves (1.1), we conclude that if $\tilde{z}(x,t)=\tilde{u}(x,t+h)$, then $\tilde{w}\le\tilde{z}$ on $\mathbf{R}^N\times[0,T-h)$.

In view of the definition of θ, the last inequality yields

$$u(x,t)-\gamma h\le u(x,t+h)\,, \quad \text{i.e.,} \quad u(x,t)+\gamma t\le u(x,t+h)+\gamma(t+h)$$

if $|u(x,t)|<\delta$ and $|h|\ll 1$.

Now let $(x,t)\in\{|u|<\delta\}$ and $(p,a,X)\in\mathscr{P}^{2,+}u(x,t)$ with $p\neq 0$. The monotonicity in t we have just proved implies that $a+\gamma\ge 0$. Since u is a solution of (1.8), we have $a+F(p,X)\le 0$. We thus conclude that $F(p,X)\le\gamma$. □

Proof of Corollary 3.2. Fix $\varepsilon\in(0,\delta)$ and define $\theta\in C(\mathbf{R})$ as above. Define $\tilde{u},\tilde{v}\in$ BUC$(\mathscr{R}_T)$ by $\tilde{u}=\theta\circ u$ and $\tilde{v}=\theta\circ v$. By Proposition 3.1 we see that if we set $K=\{(p,X)\in J_0\,|\,F(p,X)\le\gamma\}$, then

$$K^+\tilde{u}(Q_T)\subset K \quad \text{and} \quad K^+\tilde{v}(Q_T)\subset K\,.$$

Applying Theorem 2.2, we see that $\tilde{u}=\tilde{v}$ on Q_T. Noting that this implies

$$\{u\ge\varepsilon\}=\{v\ge\varepsilon\} \quad \text{and} \quad \{u\le-\varepsilon\}=\{v\le-\varepsilon\}\,, \quad \text{with } \varepsilon\in(0,\delta)\,,$$

we conclude the proof. □

Proof of Corollary 3.3. Fix a solution $g\in$ BUC$(\mathbf{R}^N)$ of (3.1) so that (3.3) holds. Let $u,v\in$ BUC$(\mathscr{R}_T)$ be the solutions of (1.8) with $F=F$ and $F=G$, respectively. It is immediate from Corollary 3.2 that

$$\{u>0\}=\{v>0\}\,,\ \{u<0\}=\{v<0\} \quad \text{and} \quad \{u=0\}=\{v=0\}\,,$$

i.e., $E_t^F(\Gamma,D^+,D^-)=E_t^G(\Gamma,D^+,D^-)$ for all $t\ge 0$. □

A very important issue related to the level set approach is whether the level sets of solutions of (1.1) will develop the interior or not. The background of this issue is beyond the scope of this paper. We refer interested readers to [ES], [CGG], [BSS], etc., for relevant discussions.

Next we state a result which gives a general sufficient condition for no-interior. We omit its proof since it goes along the lines of the proofs of Proposition 3.1 above and Theorem 7.3 of [BSS].

To formulate the result we need to make the following additional assumption on F:

$$F(\mu Q^t p, \mu^2 Q^t X Q) = \mu^m F(p, X) \quad \text{for all } \mu > 0, (p, X) \in J_0 \text{ and } Q \in O(N), \tag{3.5}$$

where $m \in \boldsymbol{R}$ is a constant, Q^t denotes the adjoint of Q and $O(N)$ denotes the group of $N \times N$ orthogonal matrices.

PROPOSITION 3.4. *Assume that* (1.2)–(1.4) *and* (3.5) *hold. Let* (Γ_t, D_t^+, D_t^-), *with* $t \geq 0$, *be a generalized evolution determined by* (1.1). *Assume that* Γ_0 *is of class* C^2 *and compact and that there exist nonnegative constants* c_i $(i = 1, 2, 3)$, *a skew symmetric matrix* H *and* $x_0 \in \boldsymbol{R}^N$ *such that*

$$c_1(x - x_0) \cdot Dd(x) + c_2 H(x - x_0) \cdot Dd(x) - c_3 F(Dd(x), D^2 d(x)) \neq 0 \quad \text{on } \Gamma_0, \tag{3.6}$$

where d *is the signed distance to* Γ_0. *Then the set* $\bigcup_{t>0}(\Gamma_t \times \{t\})$ *has empty interior in* $\boldsymbol{R}^N \times (0, \infty)$.

We conclude by giving a sufficient condition for solutions $u(x, t)$ of (1.1) to be concave in x. A corresponding assertion concerning convexity of solutions will be easily deduced from the result. This result is proved exactly as in Giga, Goto, Ishii and Sato [GGIS] with appropriate modifications to take care of the singularities. We therefore omit the proof.

We need the following assumption:

$$F(p, X) \text{ is convex in } X \text{ for all } p \in \boldsymbol{R}^N \setminus \{0\}. \tag{3.7}$$

PROPOSITION 3.5. *Assume* $\Omega = \boldsymbol{R}^N$, (1.2)–(1.4) *and* (3.7), *and let* u *be a solution of* (1.8) *with* g *concave in* $\boldsymbol{R}^N$. *Moreover, assume that for each* $R > 0$, u *is uniformly continuous in* $\{(x, t) \in \mathscr{R}_T \mid |u(x, t)| \leq R\}$. *Then for each* $t \in (0, T)$ *the function* $u(\cdot, t)$ *is concave in* $\boldsymbol{R}^N$.

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