Generalized Nash Equilibrium Problems

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Joint work with Anna von Heusinger, Axel Dreves, and Masao Fukushima

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A vector $x^* = \left(x^{*,1}, x^{*,2}, \dots, x^{*,N}\right) \in X_1 \times X_2 \times \dots \times X_N$ is called a Nash equilibrium (or simply a solution) of the NEP if

$$\theta_\nu(x^*) \leq \theta_\nu\left(x^{*,1}, \dots, x^{*,\nu-1}, x^\nu, x^{*,\nu+1}, \dots, x^{*,N}\right) \quad \forall x^\nu \in X_\nu$$

holds for all $\nu = 1, \dots, N$.

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- x^* is a Nash equilibrium if and only if no player can improve his cost function by unilaterally changing his strategy.

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- If $X = X_1 \times \ldots \times X_N$, the GNEP reduces to a standard NEP.

- The so-called normalized Nash equilibria form a subset of the set of all solutions of a GNEP. This set coincides with the set of all solutions in case the GNEP is a standard NEP.

Generalized Nash Equilibrium Problems: Applications

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- Environmental models (as formulated in the Kyoto protocol)
- First GNEP models introduced by Debreu (1952), Arrow and Debreu (1954), Rosen (1965)
- Alternative names for a GNEP: pseudo-game, social equilibrium problem, equilibrium programming, coupled constraint equilibrium problem, abstract economy

Assumptions

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- ▷ The cost functions $\theta_\nu(\cdot, x^{-\nu})$ are convex as a mapping of x^ν alone

- ▷ The common strategy space has a representation of the form

$$X := \{x \mid g(x) \leq 0\}$$

  with a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ whose components g_i are convex (later also assumed to be continuously differentiable)

(Regularized) Nikaido-Isoda Function and Merit Function

The Nikaido-Isoda-function of a GNEP (or NEP) is defined by

$$\Psi(x, y) := \sum_{\nu=1}^{N} \Big[\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu}) \Big] .$$

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Let

$$V_\alpha(x) := \max_{y \in X} \Psi_\alpha(x,y) = \Psi_\alpha\big(x, y_\alpha(x)\big)$$

where

$$y_\alpha(x) := \mathrm{argmax}_{y \in X} \Psi_\alpha(x,y)$$

denotes the uniquely defined maximizer.

Constrained Optimization Reformulations of Normalized Nash Equilibria

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Let $0 < \alpha < \beta$ be given and define the corresponding functions V_α, V_β as before. Let

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Modified Merit Function

Let $\alpha > 0$ be a given parameter. Recall the definition of the regularized Nikaido-Isoda-function

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and the corresponding merit function

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$$\hat{V}_\alpha(x) := \max_{y \in \Omega(x)} \Psi_\alpha(x, y) = \Psi_\alpha\big(x, \hat{y}_\alpha(x)\big),$$

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$$\Omega(x) := X_1(x^{-1}) \times \ldots \times X_N(x^{-N}).$$

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Unconstrained Optimization Reformulation Not Well-Defined

But: If $x \notin X$, then $\Omega(x)$ might be empty. Hence $\hat{V}_\alpha(x), \hat{V}_\beta(x)$ and $\hat{V}_{\alpha\beta}(x)$ are not necessarily defined in this case!!!

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Solution: All previous results remain true if we redefine $\hat{V}_\alpha(x), \hat{V}_\beta(x)$ and $\hat{V}_{\alpha\beta}(x)$ in the following way for the unconstrained reformulation:

$$
\begin{aligned}
\hat{V}_\alpha(x) &:= \max_{y \in \Omega(P_X(x))} \Psi_\alpha(x, y), \\
\hat{V}_\beta(x) &:= \max_{y \in \Omega(P_X(x))} \Psi_\beta(x, y), \\
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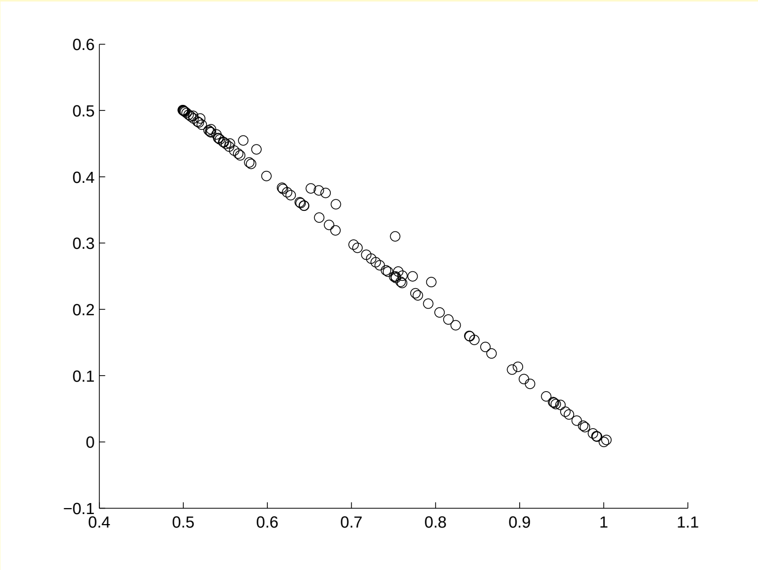
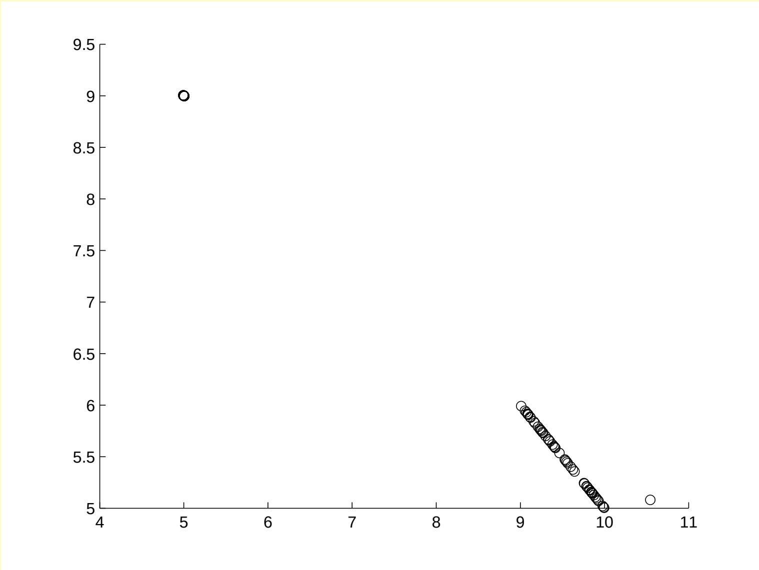
Numerical Example for Unconstrained Nonsmooth Reformulation





Example 1: $\{(\alpha, 1-\alpha) \mid \alpha \in [\frac{1}{2}, 1]\}$

Example 2: $\{(5,9)\} \cup \{(\alpha, 15-\alpha) \mid \alpha \in [9, 10]\}$

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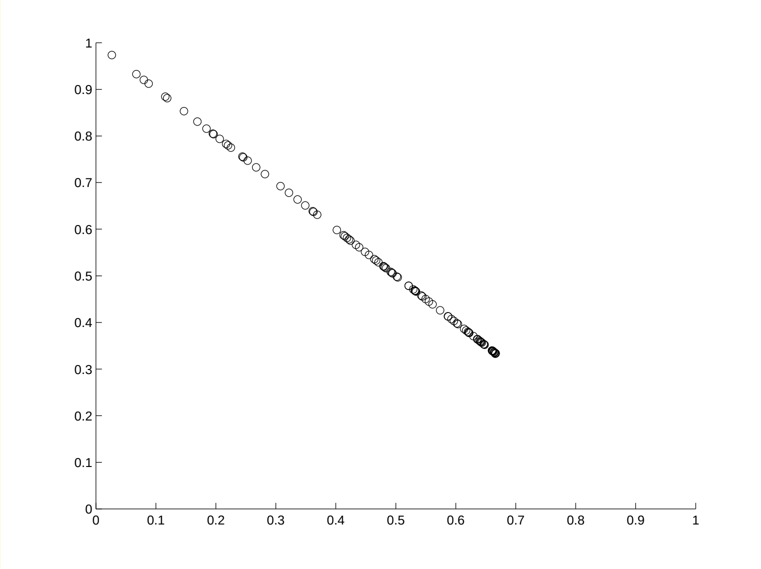

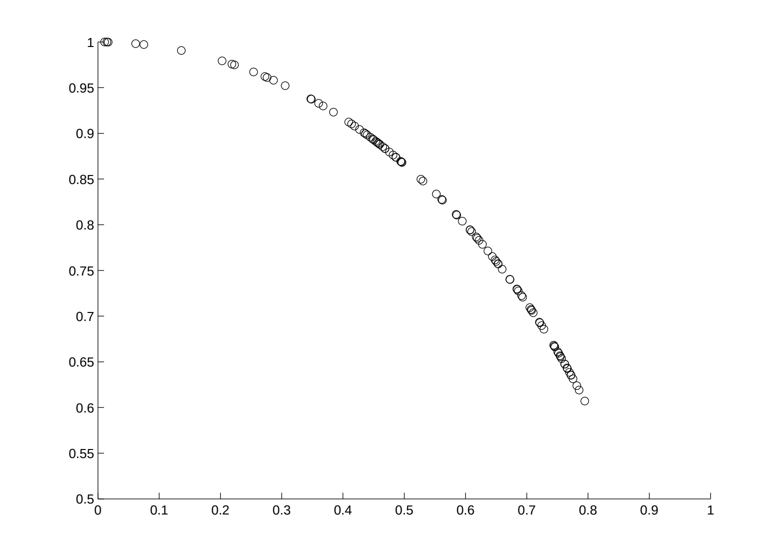

Example 3: $\{(\alpha, 1-\alpha) \mid \alpha \in [0, \frac{2}{3}]\}$

Example 4: $\{(\alpha, \sqrt{1-\alpha^2}) \mid \alpha \in [0, \frac{4}{5}]\}$

Fixed Point Characterization of Normalized Nash Equilibria

Recall that

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Note that the corresponding fixed point iteration (Picard iteration)

$$x^{k+1} := y_\alpha(x^k), \qquad k = 0, 1, 2, \ldots$$

is, usually, not convergent even under very favourable assumptions.

Relaxation Method

Modification of the Picard fixed point iteration leads to the Relaxation method by Uryasev and Rubinstein (1994):

Choose $\alpha = 0$ and

$$x^{k+1} := t_k y_\alpha(x^k) + (1 - t_k)x^k, \qquad k = 0, 1, 2, \ldots$$

Convergence shown under a number of (difficult to verify) assumptions provided that

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Comment: The natural choice $t_k := 1/(k+1)$ gives very slow convergence in practice. Other modifications exists which are either very expensive to compute or still have some heuristics in it.

Relaxation Method Viewed as Descent Method

Take $\alpha > 0$. The relaxation method

$$x^{k+1} := t_k y_\alpha(x^k) + (1 - t_k)x^k, \qquad k = 0, 1, 2, \ldots$$

can be rewritten as

$$x^{k+1} := x^k + t_k d^k, \qquad k = 0, 1, 2, \ldots$$

with the direction vector

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Under suitable (definiteness) assumptions, d^k has the descent property

$$\nabla V_\alpha(x^k)^T d^k < 0.$$

Hence t_k can be chosen by an inexact (Armijo-type) line search rule.

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(S.3) Compute $t_k = \max\{\beta^l \mid l = 0, 1, 2, \ldots\}$ such that

$$V_\alpha(x^k + t_k d^k) \leq V_\alpha(x^k) - \sigma t_k^2 \|d^k\|.$$

Relaxation Method with Inexact Line Search

(S.0) Choose $x^0 \in X, \beta, \sigma \in (0,1)$, and set $k := 0$.

(S.1) Check a suitable termination criterion $\big($like $V_\alpha(x^k) \leq \varepsilon$ for some $\varepsilon > 0\big)$.

(S.2) Compute $y_\alpha(x^k)$ and set $d^k := y_\alpha(x^k) - x^k$.

(S.3) Compute $t_k = \max\{\beta^l \mid l = 0, 1, 2, \ldots\}$ such that

$$V_\alpha(x^k + t_k d^k) \leq V_\alpha(x^k) - \sigma t_k^2 \|d^k\|.$$

(S.4) Set $x^{k+1} := x^k + t_k d^k$, $k \longleftarrow k+1$, and go to (S.1).

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- ▷ Every accumulation point of a sequence generated by the algorithm is a normalized Nash equilibrium of the GNEP
- ▷ Local rate of convergence unknown, but numerical examples indicate a (relatively) fast linear rate

Numerical Results: River Basin Pollution Game

This test problem is the river basin pollution game taken from Krawczyk and Uryasev (Environmental Modeling and Assessment 5, 2000, pp. 63–73). The cost functions are quadratic with linear constraints. The assumptions for convergence are satisfied.

k	x_1^k	x_2^k	x_3^k	$V_\alpha(x^k)$	stepsize
0	0.000000	0.000000	0.000000	90.878301693511	0.000
1	19.325863	17.174698	3.811533	0.118402581670	1.000
2	20.704303	16.105378	3.049526	0.003663469196	1.000
3	21.036699	16.036757	2.808432	0.000213429907	1.000
4	21.118197	16.029540	2.746408	0.000012918766	1.000
5	21.138222	16.028243	2.731024	0.000000789309	1.000
6	21.143173	16.027948	2.727213	0.000000047954	1.000
7	21.144471	16.027877	2.726212	0.000000001927	1.000
8	21.144714	16.027858	2.726025	0.000000000000	1.000

Numerical Results: Internet Switching Model

This test problem is an internet switching model introduced by Kesselman et al. and also analysed by Facchinei et al. We modify this example slightly and add the additional constraint $x^\nu \geq 0.01, \nu = 1, \ldots, N$ and use $N = 10$ players.

k	x_1^k	x_2^k	$V_\alpha(x^k)$	stepsize
0	0.100000	0.100000	0.026332722333	0.000
1	0.087172	0.087172	0.002241194298	0.250
2	0.090379	0.090379	0.000039775125	0.250
3	0.089905	0.089905	0.000002517609	0.250
4	0.090024	0.090024	0.000000156756	0.250
5	0.089994	0.089994	0.000000010751	0.250
6	0.090002	0.090002	0.000000000671	0.250
7	0.090000	0.090000	0.000000000000	0.250

Numerical Results: Oligopoly Model

This is the Cournot oligopoly problem with shared constraints and nonlinear cost functions as described in Outrata, Kocvara, and Zowe (1998). We use the parameter $P = 100$ (total production activity).

k	x_1^k	x_2^k	x_3^k	$V_\alpha(x^k)$	stepsize
0	10.000000	10.000000	10.000000	1836.050150600377	0.000
1	17.833057	19.050570	20.189450	4.898567426891	1.000
2	15.207025	18.069382	20.605731	0.389727842587	1.000
3	14.408253	17.849904	20.795588	0.033154445717	1.000
4	14.161948	17.805303	20.868540	0.002976203103	1.000
5	14.085260	17.797975	20.894315	0.000278156683	1.000
6	14.061205	17.797524	20.903000	0.000026779751	1.000
7	14.053616	17.797912	20.905860	0.000002633959	1.000
8	14.051210	17.798178	20.906771	0.000000263170	1.000
9	14.050445	17.798303	20.907059	0.000000026572	1.000
10	14.050201	17.798354	20.907149	0.000000000000	1.000

Newton's Method Based on Fixed Point Formulation

Recall that

$$
\begin{aligned}
x^* \text{ is a normalized Nash equilibrium} \quad &\Longleftrightarrow \quad x^* \text{ is a fixed point of } y_\alpha, \text{ i.e. } x^* = y_\alpha(x^*) \\
&\Longleftrightarrow \quad x^* \text{ is a solution of } F_\alpha(x) = 0,
\end{aligned}
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where $F_\alpha(x) := x - y_\alpha(x)$.

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where $F_\alpha(x) := x - y_\alpha(x)$. Apply a (suitable!) nonsmooth Newton method to the nonlinear system of equations $F_\alpha(x) = 0$:

$$x^{k+1} := x^k - H_k^{-1} F_\alpha(x^k) \quad \forall k = 0, 1, 2, \ldots \quad \text{with} \quad H_k \approx F'_\alpha(x^k).$$

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$$x^{k+1} := x^k - H_k^{-1} F_\alpha(x^k) \quad \forall k = 0, 1, 2, \dots \quad \text{with} \quad H_k \approx F'_\alpha(x^k).$$

Then

- ▷ The method is locally quadratically convergent under very weak assumptions.
- ▷ The method finds the exact solution locally in just one iteration for quadratic games.

Numerical Results: River Basin Pollution Game

This test problem is the river basin pollution game taken from Krawczyk and Uryasev (Environmental Modeling and Assessment 5, 2000, pp. 63–73). The cost functions are quadratic with linear constraints. The assumptions for convergence are satisfied.

k	x_1^k	x_2^k	x_3^k	$\|y_\alpha(x^k) - x^k\|$	InnerIt
0	10.000000	10.000000	10.000000	12.0479757781438828	0
1	21.144791	16.027846	2.725969	0.0000000000000000	6

Numerical Results: Internet Switching Model

This test problem is an internet switching model introduced by Kesselman et al. and also analysed by Facchinei et al. We modify this example slightly and add the additional constraint $x^\nu \geq 0.01, \nu = 1, \ldots, N$ and use $N = 10$ players.

k	x_1^k	x_2^k	$\|y_\alpha(x^k) - x^k\|$	InnerIt
0	0.100000	0.100000	0.1622713514699797	0
1	0.090238	0.090238	0.0037589337871505	4
2	0.090000	0.090000	0.0000000000000000	3

Numerical Results: Oligopoly Model

This is the Cournot oligopoly problem with shared constraints and nonlinear cost functions as described in Outrata, Kocvara, and Zowe (1998). We use the parameter $P = 100$ (total production activity).

k	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	$\|y_\alpha(x^k) - x^k\|$	InnerIt
0	10.000000	10.000000	10.000000	10.000000	10.000000	22.5856681233344716	0
1	14.742243	17.889842	20.649363	22.776440	23.942112	0.5830566903965523	7
2	14.050339	17.798223	20.907147	23.111451	24.132840	0.0002091129151843	5
3	14.050091	17.798381	20.907187	23.111428	24.132914	0.0000000000000000	2

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- ▷ We presented two smooth optimization reformulations of normalized NE
- ▷ We presented two nonsmooth optimization reformulations of NE
- ▷ We gave a fixed-point formulation of normalized NE and re-interpreted the relaxation method as a descent method.
- ▷ We gave a nonsmooth Newton-type method for the computation of normalized NE.

Many thanks for your attention!