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GENERALIZED ODE APPROACH TO IMPULSIVE RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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It is known that retarded functional differential equations (RFDEs) can be regarded as generalized ordinary differential equations (we write GODEs). See [2, 6, 7]. In this paper, we prove the equivalence between RFDEs with pre-assigned moments of impulse effects and a certain class of GODEs introduced in [8] using some ideas of [2, 6, 7]. We state results on the existence, uniqueness and continuous dependence of solutions for this class of GODEs and we use them to obtain fine results concerning the corresponding impulsive RFDEs.

1. Introduction

The beginning of the theory of impulsive ordinary differential equations (impulsive ODEs) goes back to 1960 in a paper by V.D. Mil'man and A.D. Myshkis [5]. The difficulties and peculiarities encountered in this theory such as "beating," "dying," "merging," noncontinuation of solutions, etc., were slowly overcome. In recent years, the qualitative analysis of impulsive ODEs has been studied extensively and even a significant progress has been made in the theory of impulsive retarded functional differential equations (RFDEs).

In order to generalize certain results on continuous dependence of solutions of ODEs with respect to parameters, J. Kurzweil introduced, in 1957,

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what he called generalized ordinary differential equations (GODE) for Euclidean and Banach space-valued functions (see [4]). The theory of GODEs is extensively described in [8].

The correspondence between GODEs and ODEs is simple. Indeed, it is known that the ODE

$$\dot{x} = f\,(x,t) \quad \left(\dot{x} = \frac{dx}{dt}\right) \tag{1.1}$$

is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau),\tau)d\tau, \tag{1.2}$$

when the integral exists in some sense. It is also known that, when the integral in (1.2) is in the sense of Riemann, Lebesgue (McShane) or Henstock-Kurzweil, then it can be approximated by a sum of the form

$$\sum_{i=1}^{m} f(x(\tau_i),\tau_i)[s_i - s_{i-1}]$$

or, alternatively, if we define $F(x,s) = \int_{s_0}^{s} f(x,\sigma)d\sigma$, then the integral in (1.2) can be approximated by

$$\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} f(x(\tau_i),\sigma)d\sigma = \sum_{i=1}^{m} \left[F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})\right], \tag{1.3}$$

where $t_0 = s_0 \le s_1 \le \ldots \le s_m = t$ is a fine partition of the interval $[t_0,t]$ and, for each i, σ_i is "*close*" to $[s_{i-1},s_i]$. In the second case, the right-hand side of (1.3) approximates the Kurzweil integral which, in turn, originates a differential equation in a wider sense when replaced in (1.2). Such a differential equation is known as a generalized ordinary differential equation (see Definitions 2.1 and 2.7 in the sequel). In this manner, a one-to-one relation between ODEs and GODEs can be established.

The correspondence between GODEs and RFDEs *without impulses* was first investigated in 1966 by C. Imaz and Z. Vorel and by F. Oliva and Z. Vorel under rather technical assumptions (see [6] and [7]). Later, M. Federson and P. Z. Táboas proved in [2] the same correspondence in a setting more close to [8]. Yet in [2] the theory of GODEs was useful in the investigation of topological dynamics of RFDEs. More specifically it was proved that, under very weak conditions where the limiting equations are no longer differential equations in the usual senses (retarded or even ordinary), it is possible to construct a local flow by means of GODEs.

To specify one advantage of the GODEs theory over the classical theory, we mention an article by Z. Artstein (see [1]). Even in the simple ordinary case, if we consider certain Carathéodory- and Lipschitz-type conditions, one is not always able to obtain results which require, for instance, the completeness of the space of functions (or equations, since we can identify $\dot{x} = f(x,t)$ with f) satisfying (*) and (**) below. Indeed, the space of ODEs fulfilling such conditions does not contain all its limiting equations. Indeed, consider (1.1) again and assume $f : \Omega \times \mathbb{R} \to \mathbb{R}^n$, with $\Omega \subset \mathbb{R}^n$ open, is measurable in t and continuous in x. Consider also the topology characterized by the convergence

$$f_k \to f_0, \qquad \text{if} \quad \int_0^t f_k(x,s)ds \to \int_0^t f_0(x,s)ds, \quad (x,t) \in \Omega \times \mathbb{R}$$

and the following properties:

(*) for each compact $A \subset \Omega$, there is a locally Lebesgue integrable function $M_A(t)$ such that $x \in A$ implies

$$|f(x,s)\,ds| \leq M_A(s),$$

where $\int_t^{t+h} M_A(s)\,ds$ is uniformly continuous in t;

(**) for each compact $A \subset \Omega$, there is a locally Lebesgue integrable function $L_A(t)$ such that $x_1, x_2 \in A$ implies

$$|f(x_1,s) - f(x_2,s)| \leq L_A(s)\,|x_1 - x_2|.$$

Let F_0 be a continuous nowhere differentiable function. Then there exists a sequence in C^1, say F_j, $j = 1, 2, \ldots$, which converges uniformly to F_0. Moreover, if f_j is the derivative of F_j, then $\int_0^t f_j(x,s)ds$ converges for all (x,t). However, the limit

$$F_0(x,t) = \lim \int_0^t f_j(x,s)ds$$

does not have an integral representation as $F_0(x,t) = \int_0^t f(x,s)ds$. Since we can associate an equation of type (1.1) with each f_j, it turns out that the solution of the equation is, up to a constant, the primitive F_j of f_j. Therefore the limiting solution F_0 is not a solution of an ODE. But this kind of problem is overcome when the GODEs theory is applied. Other examples of applications of GODEs theory can be found in [1], [2], [4] and [8], among others.

In the present paper, we consider the initial-value problem for an RFDE

$$\dot{y}(t) = f(y_t, t), \qquad y_{t_0} = \phi, \tag{1.4}$$

where $f(\phi, t) : PC([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, with $\sigma > 0$ and $r \geq 0$, and $t \mapsto f(y_t, t)$ is Lebesgue integrable and satisfies conditions similar to those used in [2]:

(*′) there is a Lebesgue integrable function $M : [t_0, t_0 + \sigma] \to \mathbb{R}$ such that for all $x \in PC_1$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$,

$$\left| \int_{u_1}^{u_2} f(x_s, s)\, ds \right| \leq \int_{u_1}^{u_2} M(s)\, ds;$$

(**′) there is a Lebesgue integrable function $L : [t_0, t_0 + \sigma] \to \mathbb{R}$ such that for all $x, y \in PC_1$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$,

$$\left| \int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\, ds \right| \leq \int_{u_1}^{u_2} L(s)\, \|x_s - y_s\|\, ds,$$

where PC_1 is a certain open subset of $PC([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$. In addition, we consider impulses

$$\Delta y(t_k) = I_k(y(t_k)), \; k = 1, \ldots, m,$$

where $t_k,\ k = 1, \ldots, m$ with $t_0 < t_1 < \ldots < t_k < \ldots < t_m \leq t_0 + \sigma$ are pre-assigned moments of impulse, $y \mapsto I_k(y)$ maps $\mathbb{R}^n$ into itself and

$$\Delta y(t_k) := y(t_k+) - y(t_k-) = y(t_k+) - y(t_k), \quad k = 1, 2, \ldots, m,$$

and we are mainly concerned with

- embedding RFDEs fulfilling (*′) and (**′) with impulse operators fulfilling similar conditions in a space of GODEs.

We prove that under pre-assigned moments of impulse effects, an RFDE can still be related to a "*non-impulsive*" GODE with values in a Banach space and there is a one-to-one relation between the solutions of an impulsive RFDE and the solutions of the corresponding GODE. We are also concerned with

- stating fundamental results for impulsive RFDEs by means of the GODEs theory.

Here we prove existence, uniqueness and continuous dependence of solutions on the initial data. We also discuss the maximal interval of existence.

Regarding continuous dependence, for instance, our result (namely Theorem 4.1) encompasses previous ones (see e.g. [2], Theorem 5.3 and [9], Theorem 4.2; the latter for the case when the impulse operator does not involve delays). In general, one can not expect that an impulsive delay differential system depends on the initial data. In [9], the authors present an elucidative discussion on continuous dependence of solutions of an impulsive delay differential equation whose impulse operators also involve delays. Our result does not take into account delays in the impulsive operators. On the other hand, we do not require that f be continuous. In fact, our assumptions rely on the indefinite integral of f instead.

One of the advantages of treating RFDEs with (or without) impulses by means of the theory of GODEs is that the theory of GODE is developed to a great extent. The assumptions usually concern the indefinite integral (in some sense) of the functions involved in the equations instead of the functions themselves. This leads to very fine and general results. Also, because impulsive RFDEs can be regarded as GODEs, it is possible to obtain good results with short proofs just by transferring the results from one space to the other through the relation between the solutions.

We organized this paper as follows. In Section 2 we present the basic knowledge concerning GODEs on the basis of [8]. In Section 3 the relation between impulsive RFDEs and GODEs is studied under relatively weak conditions on the entries of the impulsive RFDEs. A continuous dependence result for GODEs from [8] is used to get a result of this type for RFDEs in Section 4. Since the results from Sections 3 and 4 are presented for solutions on compact intervals and have a local character, in Section 5 the elements of studying global (maximal) solutions of RFDEs are shortly described.

2. Generalized ordinary differential equations

A *tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $(\tau_i, [s_{i-1}, s_i])$, where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a,b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, k$.

A *gauge* on a set $E \subset [a,b]$ is any function $\delta : E \to (0, +\infty)$.

Given a gauge δ on $[a,b]$, a tagged division $d = (\tau_i, [s_{i-1}, s_i])$ is *δ-fine* if, for every i, $[s_{i-1}, s_i] \subset \{t \in [a,b]\,;\, |t - \tau_i| < \delta(\tau_i)\}$.

Let X be a Banach space. In the sequel we will use integration specified by the following definition.

Definition 2.1. A function $U(\tau, t) : [a,b] \times [a,b] \to X$ is *Kurzweil integrable over the interval* $[a,b]$ if there is a unique element $I \in X$ ($I = \int_a^b DU(\tau, t)$) such that, given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a,b]$, we have

$$\|S(U,d) - I\| < \varepsilon,$$

where $S(U,d) = \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$.

This type of integration is due to Jaroslav Kurzweil and it was described extensively in Chapter I of [8] for the case $X = \mathbb{R}^n$ (see Definition 1.2n in [8]).

Checking the results concerning this integration in [8], it can be easily seen that the results presented there can be transferred without any changes to the case of X-valued functions $U(\tau, t) : [a, b] \times [a, b] \to X$. Let us mention a few of them. The integral has the usual properties of linearity, additivity with respect to adjacent intervals, etc.

An important result which will be used later concerns the integrability on subintervals and is stated next (see Theorem 1.10 in [8]).

Lemma 2.2. *Let $U(\tau, t) : [a, b] \times [a, b] \to X$ be integrable over $[a, b]$. Then $\int_c^d DU(\tau, t)$ exists, for each interval $[c, d] \subset [a, b]$.*

The following result is known as the Saks-Henstock lemma (see Lemma 1.13 in [8]).

Proposition 2.3 (Saks-Henstock lemma)**.** *Let $U(\tau, t) : [a, b] \times [a, b] \to X$. If, for every $\varepsilon > 0$, δ is a gauge of $[a, b]$ such that for every δ-fine tagged division $d = (\tau_i, s_i)$ of $[a, b]$,*

$$\Big\| \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - \int_{[a,b]} DU(\tau, t) \Big\| < \varepsilon$$

then, for $a \le c_1 \le \eta_1 \le d_1 \le c_2 \le \eta_2 \le d_2 \le \dots \le c_l \le \eta_l \le d_l \le b$, with $\eta_j \in [c_j, d_j] \subset [\eta_j - \delta(\eta_j), \eta_j + \delta(\eta_j)]$, $j = 1, 2, \dots, l$,

$$\Big\| \sum_j \Big[U(\eta_j, d_j) - U(\eta_j, c_j) - \int_{[c_j, d_j]} DU(\tau, t) \Big] \Big\| < \varepsilon.$$

The following result is an important Hake-type theorem (see Theorem 1.14 in [8]).

Lemma 2.4. *Let a function $U : [a, b] \times [a, b] \to X$ be given such that U is integrable over $[a, c]$ for every $c \in [a, b)$ and let the limit*

$$\lim_{c \to b-} \Big[\int_a^c DU(\tau, t) - U(b, c) + U(b, b) \Big] = I \in X$$

exist. Then the function U is integrable over $[a, b]$ and

$$\int_a^b DU(\tau, t) = I.$$

Similarly, if the function U is integrable over $[c,b]$ for every $c \in (a,b]$ and the limit

$$\lim_{c\to a+}\Big[\int_c^b DU(\tau,t)+U(a,c)-U(a,a)\Big]=I\in X$$

exists, then the function U is integrable over $[a,b]$ and

$$\int_a^b DU(\tau,t)=I.$$

This leads to the following (see Theorem 1.16 in [8]).

Lemma 2.5. *Let $U:[a,b]\times[a,b]\to X$ be integrable over $[a,b]$ and $c\in[a,b]$. Then*

$$\lim_{s\to c}\Big[\int_a^s DU(\tau,t)-U(c,s)+U(c,c)\Big]=\int_a^c DU(\tau,t).$$

Remark 2.6. Lemma 2.5 shows that the function given by

$$s\in[a,b]\mapsto\int_a^s DU(\tau,t)\in X,$$

i.e., the *indefinite integral of U*, may not be continuous in general. The indefinite integral is continuous at a point $c\in[a,b]$ if and only if the function $U(c,\cdot):[a,b]\to X$ is continuous at the point c.

Note that if $U:[a,b]\times[a,b]\to X$ is integrable over $[a,b]$, then by Lemma 2.2 the indefinite integral of the function U is well defined on the whole interval $[a,b]$.

Having the concept of Kurzweil integrability of a function $U:[a,b]\times[a,b]\to X$, we are able to define the notion of a generalized ordinary differential equation.

Let an open set $\Omega\subset X\times\mathbb{R}$ be given. Assume that $G:\Omega\to X$ is a given X-valued function $G(x,t)$ defined for $(x,t)\in\Omega$.

Definition 2.7. A function $x:[\alpha,\beta]\to X$ is called *a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau}=DG(x,t) \tag{2.1}$$

on the interval $[\alpha,\beta]\subset\mathbb{R}$ if $(x(t),t)\in\Omega$ for all $t\in[\alpha,\beta]$ and if the equality

$$x(v)-x(\gamma)=\int_\gamma^v DG(x(\tau),t) \tag{2.2}$$

holds for every γ, $v\in[\alpha,\beta]$.

The integral on the right-hand side of (2.2) has to be understood as the Kurzweil integral introduced by Definition 2.1.

Given an initial condition $(z_0, t_0) \in \Omega$ the following definition of a solution of the initial-value problem for the equation (2.1) will be used.

Definition 2.8. A function $x : [\alpha, \beta] \to X$ is a *solution of the generalized ordinary differential equation* (2.1) *with the initial condition* $x(t_0) = z_0$ *on the interval* $[\alpha, \beta] \subset \mathbb{R}$ if $t_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - z_0 = \int_{t_0}^{v} DG(x(\tau), t) \tag{2.3}$$

holds for every $v \in [\alpha, \beta]$.

Remark 2.9. Let $U(\tau, t) = G(x(\tau), t)$. In the definition of $\int_{[a,b]} DG\left(x\left(\tau\right), t\right)$ there are only differences such as

$$U\left(\tau_i, s_i\right) - U\left(\tau_i, s_{i-1}\right) = G\left(x\left(\tau_i\right), s_i\right) - G\left(x\left(\tau_i\right), s_{i-1}\right).$$

Thus, adding to $G\left(x(\tau), t\right)$ a function varying only in x, the solutions of (2.1) do not change. In particular, subtracting $G\left(x(\tau), t_0\right)$ from $G\left(x(\tau), t\right)$, we obtain a *normalized representation* G_1 of G fulfilling $G_1\left(z, t_0\right) = 0$ for every z.

Definitions 2.7 or 2.8 do not provide too much information about the properties of the function $x : [\alpha, \beta] \to X$ which is a solution of (2.1). The only fact we know implicitly is that the integral $\int_{\gamma}^{v} DG(x(\tau), t)$ exists for every γ, $v \in [\alpha, \beta]$. Nevertheless, using Lemma 2.5 the following can be derived (see Proposition 3.6 in [8]).

Lemma 2.10. *If* $x : [\alpha, \beta] \to X$ *is a solution of the generalized ordinary differential equation* (2.1) *on* $[\alpha, \beta]$, *then*

$$\lim_{s \to \sigma} [x(s) - G(x(\sigma), s) + G(x(\sigma), \sigma)] = x(\sigma)$$

for every $\sigma \in [\alpha, \beta]$.

This lemma shows that if $x : [\alpha, \beta] \to X$ is a solution of (2.1), then for every fixed $\sigma \in [\alpha, \beta]$ the value of $x(s)$ can be approximated by $x(\sigma) + G(x(\sigma), s) - G(x(\sigma), \sigma)$ provided $s \in [\alpha, \beta]$ is sufficiently close to σ.

Now we introduce a class of functions $G : \Omega \to X$ for which it is possible to get more specific information about the solutions of (2.1).

Let $(a, b) \subset \mathbb{R}$ be an interval with $-\infty < a < b < \infty$ and let us set

$$\Omega = O \times [a, b],$$

where $O \subset X$ is an open set (e.g. $O = B_c = \{x \in X; \|x\| < c\}$ for some $c > 0$). We will use the set $\Omega \subset X \times \mathbb{R}$ in our subsequent study of generalized differential equations (2.1). Assume that $h : [a, b] \to \mathbb{R}$ is a nondecreasing function defined on $[a, b]$.

Definition 2.11. A function $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$ if

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{2.4}$$

for all (x, s_2), $(x, s_1) \in \Omega$ and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)| \tag{2.5}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

For functions $G \in \mathcal{F}(\Omega, h)$ we are coming to more specific information about solutions of the generalized differential equation (2.1). We have the following (see Lemma 3.10 in [8]).

Lemma 2.12. *Assume that $G : \Omega \to X$ satisfies the condition* (2.4)*. If $[\alpha, \beta] \subset [a, b]$ and $x : [\alpha, \beta] \to X$ is a solution of* (2.1)*, then the inequality*

$$\|x(s_1) - x(s_2)\| \leq |h(s_2) - h(s_1)|$$

holds for every $s_1, s_2 \in [\alpha, \beta]$.

Let $\operatorname{var}_\alpha^\beta x$ be the variation of a function $x : [\alpha, \beta] \to X$ and let $BV([\alpha, \beta])$ be the space of functions $x : [\alpha, \beta] \to X$ of bounded variation. Lemma 2.12 gives easily the following property of solutions of (2.1).

Corollary 2.13. *Assume that $G : \Omega \to X \times \mathbb{R}$ satisfies the condition* (2.4)*. If $[\alpha, \beta] \subset (a, b)$ and $x : [\alpha, \beta] \to X$ is a solution of* (2.1)*, then x is of bounded variation on $[\alpha, \beta]$ and*

$$\operatorname{var}_\alpha^\beta x \leq h(\beta) - h(\alpha) < +\infty.$$

Every point in $[\alpha, \beta]$ at which the function h is continuous is a continuity point of the solution $x : [\alpha, \beta] \to X$.

Moreover, we have the following (see Lemma 3.12 in [8]).

Lemma 2.14. *If $x : [\alpha, \beta] \to X$ is a solution of* (2.1) *and $G : \Omega \to X \times \mathbb{R}$ satisfies the condition* (2.4)*, then*

$$x(\sigma+) - x(\sigma) = \lim_{s \to \sigma+} x(s) - x(\sigma) = G(x(\sigma), \sigma+) - G(x(\sigma), \sigma)$$

for $\sigma \in [\alpha, \beta)$ and

$$x(\sigma) - x(\sigma-) = x(\sigma) - \lim_{s \to \sigma-} x(s) = G(x(\sigma), \sigma) - G(x(\sigma), \sigma-)$$

for $\sigma \in (\alpha, \beta]$, *where*

$$G(x, \sigma+) = \lim_{s \to \sigma+} G(x, s) \text{ for } \sigma \in [\alpha, \beta)$$

and

$$G(x, \sigma-) = \lim_{s \to \sigma-} G(x, s) \text{ for } \sigma \in (\alpha, \beta].$$

Up to this moment, we do not have any information about the existence of a solution of (2.1). The following result gives us an answer.

Theorem 2.15. *Let* $G : \Omega \to X$ *belong to the class* $\mathcal{F}(\Omega, h)$, *where the function* h *is continuous from the left* ($h(t-) = h(t)$ *for* $t \in (a, b]$). *Then for every* $(\widetilde{x}, t_0) \in \Omega$ *such that for* $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)$, *we have* $(\widetilde{x}_+, t_0) \in \Omega$ *and there exists a* $\Delta > 0$ *such that on the interval* $[t_0, t_0 + \Delta]$ *there exists a unique solution* $x : [t_0, t_0 + \Delta] \to X$ *of the generalized ordinary differential equation* (2.1) *for which* $x(t_0) = \widetilde{x}$.

A sketch of the proof. At first, let t_0 be a point of continuity of the function h; i.e. $h(t_0+) = h(t_0)$. Assume that $\Delta > 0$ is such that $[t_0, t_0 + \Delta] \subset (a, b)$, $h(t_0 + \Delta) - h(t_0) < \frac{1}{2}$ and that $\|x - \widetilde{x}\| \leq h(t_0 + \Delta) - h(t_0)$ implies $x \in O$.

Let Q be the set of functions $z : [t_0, t_0+\Delta] \to X$ such that $z \in BV([t_0, t_0+\Delta])$ and $\|z(t) - \widetilde{x}\| \leq h(t) - h(t_0)$ for $t \in [t_0, t_0 + \Delta]$.

It is easy to show that the set $Q \subset BV([t_0, t_0 + \Delta])$ is closed.

For $s \in [t_0, t_0 + \Delta]$ and $z \in Q$, define

$$Tz(s) = \widetilde{x} + \int_{t_0}^{s} DG(z(\tau), t).$$

The integral on the right-hand side exists (see Corollary 3.16 in [8]) and for $s \in [t_0, t_0 + \Delta]$, (2.4) implies

$$\|Tz(s) - \widetilde{x}\| = \Big\| \int_{t_0}^{s} DG(z(\tau), t) \Big\| \leq h(s) - h(t_0)$$

and it follows that T maps Q into itself.

Take $t_0 \leq s_1 < s_2 \leq t_0 + \Delta$ and $z_1, z_2 \in Q$. Then, using (2.5), we obtain

$$\begin{aligned}
&\|Tz_2(s_2) - Tz_1(s_2) - [Tz_2(s_1) - Tz_1(s_1)]\| \\
&= \Big\| \int_{s_1}^{s_2} D[G(z_2(\tau), t) - G(z_1(\tau), t)] \Big\| \leq \Big\| \int_{s_1}^{s_2} D\|z_2(\tau) - z_1(\tau)\| h(t) \Big\| \\
&\leq \sup_{\tau \in [s_1, s_2]} \|z_2(\tau) - z_1(\tau)\| \cdot (h(s_2) - h(s_1))
\end{aligned}$$

$$\leq \sup_{\tau\in[t_0,t_0+\Delta]} \|z_2(\tau) - z_1(\tau)\| \cdot (h(s_2) - h(s_1))$$
$$\leq \|z_2 - z_1\|_{BV([t_0,t_0+\Delta])} \cdot (h(s_2) - h(s_1)).$$

Note that $\|z\|_{BV([t_0,t_0+\Delta])} = \|z(t_0)\| + \operatorname{var}_{t_0}^{t_0+\Delta} z$ defines a norm in $BV([t_0, t_0+\Delta])$. Hence,

$$\|Tz_2 - Tz_1\|_{BV([t_0,t_0+\Delta])} \leq \|z_2 - z_1\|_{BV([t_0,t_0+\Delta])} \cdot (h(t_0 + \Delta) - h(t_0))$$
$$< \frac{1}{2}\|z_2 - z_1\|_{BV([t_0,t_0+\Delta])}$$

and T is a contraction. By the Banach fixed-point theorem the result follows.

Now we consider when t_0 is not a point of continuity of h. Take $\widetilde{h}(t) = h(t)$ for $t \leq t_0$ and $\widetilde{h}(t) = h(t) - h(t_0+)$ for $t > t_0$. Then the function $\widetilde{h}$ is continuous at t_0, continuous from the left and nondecreasing. Defining $\widetilde{G}(x,t) = G(x,t)$ for $t \leq t_0$ and $\widetilde{G}(x,t) = G(x,t) - [G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)]$ for $t > t_0$ it is easy to check that $\widetilde{G} \in \mathcal{F}(\Omega, \widetilde{h})$ and, as above, a solution z of $\dfrac{dz}{d\tau} = D\widetilde{G}(z,t)$ with $z(t_0) = \widetilde{x}_+$ exists. Defining $x(t_0) = \widetilde{x}$ and $x(t) = z(t)$ for $t > t_0$ we have a solution of (2.1) for which $x(t_0) = \widetilde{x}$. □

Remark 2.16. The assumption of the continuity from the left of the function h in Theorem 2.15 shows that the solutions of (2.1) are also continuous from the left (cf. Lemma 2.12). Given a solution x of (2.1), the limit $x(\sigma-)$ exists for every σ in the domain of x. This follows again by Lemma 2.12 and, by Lemma 2.14, we have the relation

$$x(\sigma) = x(\sigma-) + G(x(\sigma), \sigma) - G(x(\sigma), \sigma-)$$

which describes the discontinuity of the given solution.

We close the short survey of results on generalized differential equations with the following simple convergence result (see Theorem 8.2 in [8]).

Theorem 2.17. *Assume that $G_p : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$ for $p = 0, 1, \ldots$ and that*

$$\lim_{p\to\infty} G_p(x,t) = G_0(x,t)$$

for $(x,t) \in \Omega$. Let $x_p : [\alpha, \beta] \to X$, $p = 1, 2, \ldots$ be a solution of the generalized differential equation

$$\frac{dx}{d\tau} = DG_p(x,t)$$

on $[\alpha,\beta]\subset(a,b)$ *such that*

$$\lim_{p\to\infty} x_p(s) = x(s), \quad s\in[\alpha,\beta],$$

and $(x(s),s)\in\Omega$ *for* $s\in[\alpha,\beta]$. *Then* $x:[\alpha,\beta]\to X$ *is of bounded variation on* $[\alpha,\beta]$ *and it is a solution of the generalized differential equation*

$$\frac{dx}{d\tau} = DG_0(x,t) \quad \text{on } [\alpha,\beta].$$

3. Impulsive retarded differential equations in the frame of generalized ordinary differential equations

Consider the following initial-value problem for a retarded functional differential equation with impulses:

$$\begin{cases} \dot{y}(t) = f(y_t,t), \; t\neq t_k \\ \Delta y(t_k) = I_k(y(t_k)), \; k=1,\ldots,m \\ y_{t_0} = \phi, \end{cases} \tag{3.1}$$

where $t_k,\ k=1,\ldots,m$ with $t_0<t_1<\ldots<t_k<\ldots<t_m\le t_0+\sigma$, $\sigma>0$, are pre-assigned moments of impulse, $y\mapsto I_k(y)$ maps $\mathbb{R}^n$ into itself and

$$\Delta y(t_k) := y(t_k+) - y(t_k-) = y(t_k+) - y(t_k), \quad k=1,2,\ldots,m;$$

that is, we suppose y is left continuous at $t=t_k$ and that the lateral limit $y(t_k+)$ exists, $k=1,2,\ldots,m$.

We write $PC([a,b],X)$ to denote the space of piecewise continuous functions from an interval $[a,b]\subset\mathbb{R}$ to a Banach space X.

We consider $PC([a,b],X)$ equipped with the usual supremum norm, $\|\cdot\|$, and we assume $\phi\in PC([-r,0],\mathbb{R}^n)$, $r\ge 0$, and that $f(\phi,t)$ maps some open subset of $PC\big([-r,0],\mathbb{R}^n\big)\times[t_0,t_0+\sigma]$ to $\mathbb{R}^n$.

Given a function $y:[t_0-r,t_0+\sigma]\to\mathbb{R}^n$, we consider $y_t:[-r,0]\to\mathbb{R}^n$ given by

$$y_t(\theta) = y(t+\theta), \quad \theta\in[-r,0],\ t\in[t_0,t_0+\sigma].$$

Now we introduce some notation. Let PC_{t_0} be the set of all functions $y:[t_0,t_0+\sigma]\to\mathbb{R}^n$ such that for $k=1,2,\ldots,m$, y is continuous at $t\neq t_k$, y is left continuous at $t=t_k$ and the right limit $y(t_k+)$ exists. Given $\phi\in PC([-r,0],\mathbb{R}^n)$, we also define

$$PC_{\phi,t_0} = \Big\{ y:[t_0-r,t_0+\sigma]\to\mathbb{R}^n\,;\ y_{t_0}=\phi,\ y|_{[t_0,t_0+\sigma]}\in PC_{t_0} \Big\}$$

These two spaces are complete under the norm induced by

$$PC([t_0-r,t_0+\sigma],\mathbb{R}^n).$$

Let us recall the concept of a solution to the problem (3.1).

Definition 3.1. A function $y \in PC_{\phi,t_0}$ such that $(y_t, t) \in PC([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma]$ for $t_0 \le t < t_0 + \sigma$ and moreover

(i) $\dot{y}(t) = f(y_t, t)$, for almost every t, $t \ne t_k$,

(ii) $y(t_k+) = y(t_k) + I_k(y(t_k))$, $k = 1, 2, \dots, m$,

is called a *solution of* (3.1) *in* $[t_0, t_0 + \sigma]$ (or sometimes also in $[t_0 - r, t_0 + \sigma]$) *with the initial condition* (ϕ, t_0).

The impulsive system (3.1) is known to be equivalent to the "integral" equation

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\, ds + \sum_{t_0 < t_k \le t} I_k(y(t_k)),\ t \in [t_0, t_0 + \sigma], \quad y_{t_0} = \phi,$$

when the integral exists in the Lebesgue sense (cf. [3]).

For $T \in (t_0, \infty)$ define the left continuous Heaviside function concentrated at T as follows:

$$H_T(t) = \begin{cases} 0 & \text{for } t_0 \le t \le T \\ 1 & \text{for } T < t. \end{cases}$$

Then

$$\sum_{t_0 < t_k \le t} I_k(y(t_k)) = \sum_{k=1}^{m} I_k(y(t_k)) H_{t_k}(t)$$

and the system (3.1) is equivalent to

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\, ds + \sum_{k=1}^{m} I_k(y(t_k)) H_{t_k}(t),\ t \in [t_0, t_0 + \sigma], \quad y_{t_0} = \phi. \tag{3.2}$$

Let $PC_1 \subset PC_{\phi,t_0}$ be an open set (in the topology of $PC([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$) with the following property: if $y = y(t)$, $t \in [t_0 - r, t_0 + \sigma]$, is an element of PC_1 and $\bar{t} \in [t_0 - r, t_0 + \sigma]$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \le t \le \bar{t} \\ y(\bar{t}_+), & \bar{t} < t \le t_0 + \sigma \end{cases}$$

also belongs to PC_1.

Denote by $|\cdot|$ a norm in $\mathbb{R}^n$. We consider $f(\phi, t) : PC([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, $t \mapsto f(y_t, t)$ to be Lebesgue integrable and the following conditions are fulfilled:

(A) there is a Lebesgue integrable function $M : [t_0, t_0+\sigma] \to \mathbb{R}$ such that for all $x \in PC_1$ and all $u_1, u_2 \in [t_0, t_0+\sigma]$,

$$\Big| \int_{u_1}^{u_2} f(x_s, s)\, ds \Big| \le \int_{u_1}^{u_2} M(s)\, ds;$$

(B) there is a Lebesgue integrable function $L : [t_0, t_0+\sigma] \to \mathbb{R}$ such that for all $x, y \in PC_1$ and all $u_1, u_2 \in [t_0, t_0+\sigma]$,

$$\Big| \int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\, ds \Big| \le \int_{u_1}^{u_2} L(s)\, \|x_s - y_s\|\, ds.$$

Concerning the impulse functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \dots, m$, we assume the following conditions:

(A') there is a constant $K_1 > 0$ such that for all $k = 1, \dots, m$ and all $x \in \mathbb{R}^n$

$$|I_k(x)| \le K_1;$$

(B') there is a constant $K_2 > 0$ such that for all $k = 1, \dots, m$ and all $x, y \in \mathbb{R}^n$

$$|I_k(x) - I_k(y)| \le K_2 |x - y|.$$

Suppose $f(\phi, t) : PC([-r, 0], \mathbb{R}^n) \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ and for each $y \in PC_1$ the mapping $t \mapsto f(y_t, t)$ is integrable in the Lebesgue sense. For $y \in PC_1$, let

$$F(y,t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0 \text{ or } t_0 - r \le t \le t_0 \\ \int_{t_0}^{\vartheta} f(y_s, s)\, ds, & t_0 \le \vartheta \le t \le t_0 + \sigma; \\ \int_{t_0}^{t} f(y_s, s)\, ds, & t_0 \le t \le \vartheta \le t_0 + \sigma. \end{cases} \tag{3.3}$$

Then given $(y, t) \in PC_1 \times [t_0 - r, t_0 + \sigma]$, equation (3.3) defines an element $F(y,t)$ of $C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and $F(y,t)(\tau) \in \mathbb{R}^n$ is the value of $F(y,t)$ at a point $\tau \in [t_0 - r, t_0 + \sigma]$; that is,

$$F : PC_1 \times [t_0 - r, t_0 + \sigma] \to C([t_0 - r, t_0 + \sigma], \mathbb{R}^n),$$

where $C([a,b], \mathbb{R}^n)$ denotes the Banach space of continuous functions from $[a,b]$ to $\mathbb{R}^n$ with the supremum norm. (A proof of this fact is a straightforward adaptation of [2], Proposition 2.1.)

The idea of defining the function F by (3.3) comes from the pioneering work of Z. Vorel, C. Imaz and F. Oliva ([6], [7]) where RFDEs have been related to GODEs for the first time.

Assume that conditions (A) and (B) are satisfied for the map $f(\phi, t) : PC([-r, 0], \mathbb{R}^n) \times [t_0, t_0+\sigma] \to \mathbb{R}^n$.

Given $x \in PC_1$ and $t_0 \le s_1 < s_2 < t_0 + \sigma$ we have for $F : PC_1 \times [t_0 - r, t_0 + \sigma] \to C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ given by (3.3) the following:

$$F(x, s_2)(\vartheta) - F(x, s_1)(\vartheta) = \begin{cases} 0, & \vartheta \in [t_0 - r, s_1], \\ \int_{s_1}^{\vartheta} f(x_s, s)\, ds, & \vartheta \in [s_1, s_2], \\ \int_{s_1}^{s_2} f(x_s, s)\, ds, & \vartheta \in [s_2, t_0 + \sigma]. \end{cases} \tag{3.4}$$

Hence, for an arbitrary $x \in PC_1$ and for $t_0 \le s_1 < s_2 < t_0 + \sigma$, we have by (A)

$$\|F(x, s_2) - F(x, s_1)\| = \sup_{\vartheta \in [t_0 - r, t_0 + \sigma]} |F(x, s_2)(\vartheta) - F(x, s_1)(\vartheta)| = \tag{3.5}$$

$$= \sup_{\vartheta \in [s_1, s_2]} |F(x, s_2)(\vartheta) - F(x, s_1)(\vartheta)| = \sup_{\vartheta \in [s_1, s_2]} \Big| \int_{s_1}^{\vartheta} f(x_s, s)\, ds \Big|$$

$$\le \sup_{\vartheta \in [s_1, s_2]} \int_{s_1}^{\vartheta} M(s) ds = \int_{s_1}^{s_2} M(s) ds.$$

Similarly, using (3.4) and (B), we get that if $x, y \in PC_1$ and $t_0 \le s_1 < s_2 < t_0 + \sigma$ then

$$\|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)\| \tag{3.6}$$

$$= \sup_{\vartheta \in [s_1, s_2]} \Big| \int_{s_1}^{\vartheta} [f(x_s, s) - f(y_s, s)\, ds] \Big| \le \int_{s_1}^{s_2} L(s) \|x_s - y_s\| ds$$

$$\le \sup_{\vartheta \in [s_1 - r, s_2]} |x(\vartheta) - y(\vartheta)| \int_{s_1}^{s_2} L(s) ds \le \|x - y\| \int_{s_1}^{s_2} L(s) ds.$$

Define $h_1 : [t_0, t_0 + \sigma] \to \mathbb{R}$ by

$$h_1(t) = \int_{t_0}^{t} [M(s) + L(s)] ds, \ t \in [t_0, t_0 + \sigma].$$

The function h_1 is (absolutely) continuous and nondecreasing since $M, L : [t_0, t_0 + \sigma] \to \mathbb{R}$ are nonnegative almost everywhere.

According to (3.5) and (3.6) we have

$$\|F(x, s_2) - F(x, s_1)\| \le |h_1(s_2) - h_1(s_1)| \tag{3.7}$$

for all (x, s_2), $(x, s_1) \in PC_1 \times [t_0, t_0 + \sigma]$ and

$$\|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)\| \le \|x - y\| |h_1(s_2) - h_1(s_1)| \tag{3.8}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in PC_1 \times [t_0, t_0 + \sigma]$.

Let us consider the impulsive terms of the problem (3.1).

Given an arbitrary $y \in PC_1$, let

$$J(y,t)(\vartheta) = \sum_{k=1}^{m} H_{t_k}(t) H_{t_k}(\vartheta) I_k(y(t_k)) \tag{3.9}$$

for $\vartheta \in [t_0 - r, t_0 + \sigma]$ and $t \in [t_0, t_0 + \sigma]$. ($H_{t_k}$ is the left continuous Heaviside function concentrated at t_k.)

If $t_0 \le s_1 < s_2 \le t_0 + \sigma$, we have

$$J(y,s_2)(\vartheta) - J(y,s_1)(\vartheta) = \sum_{k=1}^{m} [H_{t_k}(s_2) - H_{t_k}(s_1)] H_{t_k}(\vartheta) I_k(y(t_k)) \tag{3.10}$$

for $\vartheta \in [t_0 - r, t_0 + \sigma]$. So for $t_0 \le s_1 < s_2 \le t_0 + \sigma$ and $x \in PC_1$ we have

$$\begin{aligned} \|J(x,s_2) - J(x,s_1)\| &= \sup_{\vartheta \in [t_0 - r, t_0 + \sigma]} \sum_{k=1}^{m} [H_{t_k}(s_2) - H_{t_k}(s_1)] H_{t_k}(\vartheta) |I_k(x(t_k))| \\ &\le \sum_{k=1}^{m} [H_{t_k}(s_2) - H_{t_k}(s_1)] K_1 \end{aligned} \tag{3.11}$$

and, similarly, if $x, y \in PC_1$ we get

$$\|J(x,s_2) - J(x,s_1) - J(y,s_2) + J(x,s_1)\| \le \sum_{k=1}^{m} [H_{t_k}(s_2) - H_{t_k}(s_1)] K_2 \|x - y\|_{PC}. \tag{3.12}$$

Define $h_2 : [t_0, t_0 + \sigma] \to \mathbb{R}$ by

$$h_2(t) = \max(K_1, K_2) \cdot \sum_{k=1}^{m} H_{t_k}(t).$$

Then h_2 is left continuous and nondecreasing while by (3.11) and (3.12) we get

$$\|J(x,s_2) - J(x,s_1)\| \le h_2(s_2) - h_2(s_1) \tag{3.13}$$

and

$$\|J(x,s_2) - J(x,s_1) - J(y,s_2) + J(x,s_1)\| \le \|x - y\| (h_2(s_2) - h_2(s_1)) \tag{3.14}$$

provided $x, y \in PC_1$ and $t_0 \le s_1 < s_2 \le t_0 + \sigma$.

Now, consider $F(y,t)$ from (3.3) and $J(y,t)$ from (3.9) and let

$$G(y,t) = F(y,t) + J(y,t) \tag{3.15}$$

for $y \in PC_1$ and $t \in [t_0 - r, t_0 + \sigma]$. Then $G(y,t)$ belongs to $PC([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$; that is,

$$G : PC_1 \times [t_0 - r, t_0 + \sigma] \to PC([t_0 - r, t_0 + \sigma], \mathbb{R}^n).$$

By (3.7) and (3.13), we have

$$\|G(x,s_2) - G(x,s_1)\| \le \|F(x,s_2) - F(x,s_1)\| + \|J(x,s_2) - J(x,s_1)\| \quad (3.16)$$

$$\le h_1(s_2) - h_1(s_1) + h_2(s_2) - h_2(s_1) = h(s_2) - h(s_1),$$

where $h(t) = h_1(t) + h_2(t)$ is nondecreasing and continuous from the left.

Similarly, (3.8) and (3.14) yield

$$\|G(x,s_2) - G(x,s_1) - G(y,s_2) + G(y,s_1)\| \le \|x - y\|(h(s_2) - h(s_1)). \quad (3.17)$$

The inequalities (3.16) and (3.17) show the following.

Proposition 3.2. *If the conditions* (A), (B), (A'), (B') *are satisfied then the function G given by* (3.15) *belongs to the class $\mathcal{F}(\Omega, h)$, where $\Omega = PC_1 \times [a,b]$ for any $[a,b] \subset [t_0, t_0 + \sigma]$.*

Consider the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG\,(x,t)\,, \quad (3.18)$$

where G is given by (3.15). We will work now with a specific initial-value problem for the equation (3.18).

Let $\phi \in PC\,([-r,0], \mathbb{R}^n)$ be given.

A function $x\,(t)$ defined on the interval $[t_0 - r, t_0 + \sigma]$ and taking values in PC_1 is a *solution of the generalized ordinary differential equation* (3.18) in the interval $[t_0, t_0 + \sigma]$, with *initial condition* $x\,(t_0) \in PC_1$ given for $\phi \in PC\,([-r,0], \mathbb{R}^n)$ by

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0) & \text{for } \vartheta \in [t_0 - r, t_0], \\ x(t_0)(t_0) & \text{for } \vartheta \in [t_0, t_0 + \sigma] \end{cases}$$

if for every $v \in [t_0, t_0 + \sigma]$, we have

$$\begin{aligned} x\,(v) &= x\,(t_0) + \int_{t_0}^{v} DG\,(x\,(\tau)\,, t) \\ &= x\,(t_0) + \int_{t_0}^{v} DF\,(x\,(\tau)\,, t) + \int_{t_0}^{v} DJ(x(\tau), t). \end{aligned}$$

Lemma 3.3. *Let $x(t)$ be a solution of* (3.18) *in the interval* $[t_0, t_0+\sigma]$, *with G given by* (3.15) *and with initial condition $x(t_0) \in PC_1$ given by* $x(t_0)(\vartheta) = \phi(\vartheta)$ *for* $\vartheta \in [t_0 - r, t_0]$, $x(t_0)(\vartheta) = x(t_0)(t_0)$ *for* $\vartheta \in [t_0, t_0+\sigma]$. *Then if* $v \in [t_0, t_0+\sigma]$, *we have*

$$x(v)(\vartheta) = x(v)(v), \quad \vartheta \geq v,\ \vartheta \in [t_0 - r, t_0+\sigma] \tag{3.19}$$

and

$$x(v)(\vartheta) = x(\vartheta)(\vartheta), \quad v \geq \vartheta,\ \vartheta \in [t_0 - r, t_0+\sigma]. \tag{3.20}$$

Proof. Assume that $\vartheta \geq v$. Since x is a solution of (3.18), we have

$$x(v)(v) = x(t_0)(v) + \int_{t_0}^{v} DG(x(\tau), t)(v)$$

and similarly

$$x(v)(\vartheta) = x(t_0)(\vartheta) + \int_{t_0}^{v} DG(x(\tau), t)(\vartheta).$$

Since $x(t_0)(\vartheta) = x(t_0)(v)$ by the properties of the initial condition, we have

$$x(v)(\vartheta) - x(v)(v) = \int_{t_0}^{v} DG(x(\tau), t)(\vartheta) - \int_{t_0}^{v} DG(x(\tau), t)(v).$$

Since the integral $\int_{t_0}^{v} DG(x(\tau), t)$ exists, for every $\varepsilon > 0$ there is a gauge δ on $[t_0, t_0+\sigma]$ such that if $(\tau_i, [s_{i-1}, s_i])$ is a δ-fine division of $[t_0, v]$, then

$$\Big\| \sum_i [G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})] - \int_{t_0}^{v} DG(x(\tau), t) \Big\| < \varepsilon.$$

Therefore, we have

$$\begin{aligned} |x(v)(\vartheta) - x(v)(v)| < 2\varepsilon + \Big| &\sum_i [G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})](\vartheta) \\ &- \sum_i [G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})](v) \Big|. \end{aligned}$$

By the definition of G in (3.15), the form of F given in (3.3) and of J in (3.9), it is a matter of routine to check that for every i we have

$$G(x(\tau_i), s_i)(\vartheta) - G(x(\tau_i), s_{i-1})(\vartheta) = G(x(\tau_i), s_i)(v) - G(x(\tau_i), s_{i-1})(v)$$

and this implies by the last inequality above that

$$|x(v)(\vartheta) - x(v)(v)| < 2\varepsilon.$$

Since this holds for an arbitrary $\varepsilon > 0$, the relation (3.19) is satisfied.

For the second relation assume that $\vartheta \leq v$.

By the definition of a solution of (3.18), we have similarly as in the first part of the proof

$$x(v)(\vartheta) = x(t_0)(\vartheta) + \int_{t_0}^{v} DG(x(\tau), t)(\vartheta)$$

and

$$x(\vartheta)(\vartheta) = x(t_0)(\vartheta) + \int_{t_0}^{\vartheta} DG(x(\tau), t)(\vartheta).$$

Hence,

$$x(v)(\vartheta) - x(\vartheta)(\vartheta) = \int_{\vartheta}^{v} DG(x(\tau), t)(\vartheta).$$

If now $(\tau_i, [s_{i-1}, s_i])$ is an arbitrary tagged division of $[\vartheta, v]$, it is again straightforward to check by (3.3) and (3.9) that for every i we have

$$G(x(\tau_i), s_i)(\vartheta) - G(x(\tau_i), s_{i-1})(\vartheta) = 0.$$

But this means that $\int_{\vartheta}^{v} DG(x(\tau), t)(\vartheta) = 0$ and that $x(v)(\vartheta) = x(\vartheta)(\vartheta)$ holds. Hence, (3.20) is proved. □

For a similar lemma, see [7], Lemma 2.1.

Let us now study the relation between the impulsive retarded differential equation (3.1) and the generalized ordinary differential equation (3.18), provided the conditions (A), (B), (A') and (B') are fulfilled.

Theorem 3.4. *Consider equation* (3.1)*, where*

$$f : PC\left([-r, 0], \mathbb{R}^n\right) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n, \quad t \mapsto f\left(y_t, t\right)$$

is Lebesgue integrable over $[t_0, t_0 + \sigma]$ *and* (A), (B), (A'), (B') *are fulfilled. Let* $y\left(t\right)$ *be a solution of the problem* (3.1) *in the interval* $[t_0, t_0 + \sigma]$*. Given* $t \in [t_0 - r, t_0 + \sigma]$*, let*

$$x\left(t\right)\left(\vartheta\right) = \begin{cases} y\left(\vartheta\right), & \vartheta \in [t_0 - r, t] \\ y\left(t\right), & \vartheta \in [t, t_0 + \sigma]. \end{cases} \tag{3.21}$$

Then $x(t) \in PC\left([t_0 - r, t_0 + \sigma], \mathbb{R}^n\right)$ *is a solution of* (3.18) *in* $[t_0 - r, t_0 + \sigma]$*.*

Proof. We will show that for every $v \in [t_0, t_0+\sigma]$, the integral $\int_{t_0}^{v} DG(x(\tau), t)$ exists and

$$x\left(v\right) - x\left(t_0\right) = \int_{t_0}^{v} DG\left(x\left(\tau\right), t\right).$$

Let an arbitrary $\varepsilon > 0$ be given. Since y is a solution of (3.1), the relations (3.2) concerning the equivalent "integral" form are satisfied and it is easy to see that the function $y : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is the sum of an absolutely continuous function and a simple left continuous step function.

Therefore, for every $\tau \in [t_0, t_0 + \sigma]$ there is a $\delta(\tau) > 0$ such that

$$|y(\rho) - y(\tau)| < \varepsilon \quad \text{for every} \quad \rho \in [\tau - \delta(\tau), \tau] \tag{3.22}$$

and

$$|y(\rho) - y(\tau+)| < \varepsilon \quad \text{for every} \quad \rho \in (\tau, \tau + \delta(\tau)]. \tag{3.23}$$

We write $y(\tau+) = \lim_{\rho \to \tau+} y(\rho)$. In this way, a gauge δ on $[t_0, t_0 + \sigma]$ is given. Further, let the gauge δ be such that if $\tau \in [t_0, t_0 + \sigma]$, then

$$\Big| \int_u^v L(s) ds \Big| < \frac{\varepsilon}{(m+1)(K_1 + 1)}, \quad \text{for every} \quad [u, v] \subset (\tau - \delta(\tau), \tau + \delta(\tau)), \tag{3.24}$$

where m is the number of impulse points and K_1 is the constant from (A'). Such a choice is possible because the function $L : [t_0, t_0 + \sigma] \to \mathbb{R}$ from (B) is assumed to be Lebesgue integrable.

Moreover, assume that the gauge δ satisfies

$$0 < \delta(\tau) < \min \Big\{ \frac{t_k - t_{k-1}}{2}; \ k = 1, \ldots, m \Big\} \tag{3.25}$$

and

$$0 < \delta(\tau) < \min \{ d(\tau, t_k), d(\tau, t_{k-1}); \ \tau \in (t_{k-1}, t_k), \ k = 1, \ldots, m \}, \tag{3.26}$$

where $d(\tau, t_k)$ denotes the distance of τ to t_k and similarly for $d(\tau, t_{k-1})$.

The condition (3.25) assures that if a point-interval pair $(T, [s_1, s_2])$ is δ-fine, then the interval $[s_1, s_2]$ contains at most one of the points t_k, $k = 1, \ldots, m$, while (3.26) implies $T = t_k$ whenever $t_k \in [s_1, s_2]$.

Assume now that $(\tau_i, [s_{i-1}, s_i])$ is a δ-fine division of the interval $[t_0, v]$.

Using the definition (3.21) of x and (3.2) it can be easily shown that

$$[x(s_i) - x(s_{i-1})](\vartheta) = \tag{3.27}$$

$$= \begin{cases} 0, \quad \vartheta \in [t_0 - r, s_{i-1}] \\ \int_{s_{i-1}}^{\vartheta} f(y_s, s)\, ds + \sum_{k=1}^m I_k(y(t_k))[H_{t_k}(\vartheta) - H_{t_k}(s_{i-1})], \ \vartheta \in [s_{i-1}, s_i] \\ \int_{s_{i-1}}^{s_i} f(y_s, s) ds + \sum_{k=1}^m I_k(y(t_k))[H_{t_k}(s_i) - H_{t_k}(s_{i-1})], \ \vartheta \in [s_i, t_0 + \sigma]. \end{cases}$$

Using the definition of G from (3.15), (3.3) and (3.9) we obtain

$$[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})](\vartheta) = [F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})](\vartheta)$$

$$+ \sum_{k=1}^m [H_{t_k}(s_i) - H_{t_k}(s_{i-1})] H_{t_k}(\vartheta) I_k(x(\tau_i)(t_k)) \tag{3.28}$$

$$= \left\{ \begin{array}{l} 0,\ \vartheta \in [t_0 - r, s_{i-1}] \\ \int_{s_{i-1}}^{\vartheta} f\left(x\left(\tau_i\right)_s, s\right) ds,\ \vartheta \in [s_{i-1}, s_i] \\ \int_{s_{i-1}}^{s_i} f\left(x\left(\tau_i\right)_s, s\right) ds,\ \vartheta \in [s_i, t_0 + \sigma] \end{array} \right\}$$

$$+ \sum_{k=1}^{m} [H_{t_k}(s_i) - H_{t_k}(s_{i-1})] H_{t_k}(\vartheta) I_k(x(\tau_i)(t_k)).$$

Using the properties (3.25) and (3.26) of the gauge δ and the corresponding properties of the division $(\tau_i, [s_{i-1}, s_i])$, there are two possibilities for a given point-interval pair $(\tau_i, [s_{i-1}, s_i])$:

(i) there is exactly one $t_l \in [s_{i-1}, s_i)$,

(ii) $[s_{i-1}, s_i)$ does not contain any point of impulse; i.e., $[s_{i-1}, s_i) \cap \{t_1, \ldots, t_m\} = \emptyset$.

In case (i), we have

$$\sum_{k=1}^{m} I_k(y(t_k))[H_{t_k}(\vartheta) - H_{t_k}(s_{i-1})] = I_l(y(t_l)) H_{t_l}(\vartheta)$$

and, since $\tau_i = t_l$, we get by the definition of x

$$\begin{aligned} \sum_{k=1}^{m} [H_{t_k}(s_i) - H_{t_k}(s_{i-1})] H_{t_k}(\vartheta) I_k(x(\tau_i)(t_k)) &= I_l(x(\tau_i)(t_l)) H_{t_l}(\vartheta) \\ &= I_l(y(t_l)) H_{t_l}(\vartheta). \end{aligned}$$

By (3.27) and (3.28), we have

$$\begin{aligned} &[x\left(s_i\right) - x\left(s_{i-1}\right)](\vartheta) - [G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})](\vartheta) \\ &= \left\{ \begin{array}{l} 0,\ \vartheta \in [t_0 - r, s_{i-1}] \\ \int_{s_{i-1}}^{\vartheta} f\left(y_s, s\right) ds - \int_{s_{i-1}}^{\vartheta} f\left(x\left(\tau_i\right)_s, s\right) ds,\ \vartheta \in [s_{i-1}, s_i] \\ \int_{s_{i-1}}^{s_i} f\left(y_s, s\right) ds - \int_{s_{i-1}}^{s_i} f\left(x\left(\tau_i\right)_s, s\right) ds,\ \vartheta \in [s_i, t_0 + \sigma] \end{array} \right. \\ &= \left\{ \begin{array}{l} 0,\ \vartheta \in [t_0 - r, s_{i-1}] \\ \int_{s_{i-1}}^{\vartheta} [f\left(y_s, s\right) - f\left(x\left(\tau_i\right)_s, s\right)] ds,\ \vartheta \in [s_{i-1}, s_i] \\ \int_{s_{i-1}}^{s_i} [f\left(y_s, s\right) - f\left(x\left(\tau_i\right)_s, s\right)] ds,\ \vartheta \in [s_i, t_0 + \sigma]. \end{array} \right. \end{aligned} \tag{3.29}$$

In case (ii), we have

$$\sum_{k=1}^{m} I_k(y(t_k))[H_{t_k}(\vartheta) - H_{t_k}(s_{i-1})] = 0$$

and also

$$\sum_{k=1}^{m}[H_{t_k}(s_i)-H_{t_k}(s_{i-1})]H_{t_k}(\vartheta)I_k(x(\tau_i)(t_k))=0.$$

By (3.27) and (3.28) we again obtain the relation (3.29).

Using (3.29), consider now

$$\|x\,(s_i)-x\,(s_{i-1})-[G(x(\tau_i),s_i)-G(x(\tau_i),s_{i-1})]\|$$

$$=\sup_{\vartheta\in[t_0-r,t_0+\sigma]}|\,[x\,(s_i)-x\,(s_{i-1})]\,(\vartheta)-[G(x(\tau_i),s_i)-G(x(\tau_i),s_{i-1})](\vartheta)|$$

$$=\sup_{\vartheta\in[t_0-r,t_0+\sigma]}\begin{cases}|\int_{s_{i-1}}^{\vartheta}[f\,(y_s,s)-f\,(x\,(\tau_i)_s\,,s)]ds|, & \vartheta\in[s_{i-1},s_i]\\ |\int_{s_{i-1}}^{s_i}[f\,(y_s,s)-f\,(x\,(\tau_i)_s\,,s)]ds|, & \vartheta\in[s_i,t_0+\sigma]\end{cases}$$

$$=\sup_{\vartheta\in[s_{i-1},s_i]}|\int_{s_{i-1}}^{\vartheta}[f\,(y_s,s)-f\,(x\,(\tau_i)_s\,,s)]ds|.$$

By the definition of x from (3.21), we have for the case (i)

$$\int_{s_{i-1}}^{\vartheta}[f\,(y_s,s)-f\,(x\,(\tau_i)_s\,,s)]ds=\int_{t_l}^{\vartheta}[f\,(y_s,s)-f\,(x\,(t_l)_s\,,s)]ds$$

for $\vartheta\in[t_l,s_i]$, and

$$\int_{s_{i-1}}^{\vartheta}[f\,(y_s,s)-f\,(x\,(\tau_i)_s\,,s)]ds=0$$

for $\vartheta\in[s_{i-1},t_l]$. By condition (B) we have

$$\left|\int_{t_l}^{\vartheta}[f\,(y_s,s)-f\,(x\,(t_l)_s\,,s)]ds\right|\le\int_{t_l}^{\vartheta}L(s)\|y_s-x\,(t_l)_s\,\|ds.$$

Using (3.22) and (B') we have

$$\|y_s-x\,(t_l)_s\,\|=\sup_{\rho\in[-r,0]}|y(s+\rho)-x(t_l)(s+\rho)|$$

$$=\sup_{\rho\in[t_l,s]}|y(\rho)-y(t_l)|=\sup_{\rho\in[t_l,s]}|y(\rho)-y(t_l+)+y(t_l+)-y(t_l)|$$

$$=\sup_{\rho\in[t_l,s]}|y(\rho)-y(t_l+)+I_ly(t_l)|\le\varepsilon+K_1.$$

Therefore by the property (3.24) of the gauge δ we get

$$\|x\,(s_i)-x\,(s_{i-1})-[G(x(\tau_i),s_i)-G(x(\tau_i),s_{i-1})]\|\le(\varepsilon+K_1)\int_{t_l}^{s_i}L(s)ds$$

$$\leq \varepsilon \int_{t_l}^{s_i} L(s)ds + K_1 \frac{\varepsilon}{(m+1)(K_1+1)} < \varepsilon \int_{t_l}^{s_i} L(s)ds + \frac{\varepsilon}{(m+1)}.$$

Similarly, for the case (ii) we have

$$\int_{s_{i-1}}^{\vartheta} [f(y_s, s) - f(x(\tau_i)_s, s)]ds = \int_{\tau_i}^{\vartheta} [f(y_s, s) - f(x(\tau_i)_s, s)]ds$$

for $\vartheta \in [\tau_i, s_i]$ and

$$\int_{s_{i-1}}^{\vartheta} [f(y_s, s) - f(x(\tau_i)_s, s)]ds = 0$$

for $\vartheta \in [s_{i-1}, \tau_i]$.

Condition (B) also implies

$$\left| \int_{\tau_i}^{\vartheta} [f(y_s, s) - f(x(\tau_i)_s, s)]ds \right| \leq \int_{\tau_i}^{\vartheta} L(s) \|y_s - x(\tau_i)_s\| ds,$$

where

$$\|y_s - x(\tau_i)_s\| = \sup_{\rho \in [\tau_i, s]} |y(\rho) - y(\tau_i)| \leq \varepsilon$$

by the property (3.23) of the gauge δ. Hence,

$$\|x(s_i) - x(s_{i-1}) - [G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})]\| \leq \varepsilon \int_{\tau_i}^{s_i} L(s)ds.$$

Using the results obtained above and the fact that the case (i) occurs in at most m intervals, we get

$$\left\| x(v) - x(t_0) - \sum_i [G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})] \right\|$$

$$= \left\| \sum_i \{x(s_i) - x(s_{i-1}) - [G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})]\} \right\|$$

$$\leq \sum_i \|x(s_i) - x(s_{i-1}) - [G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})]\|$$

$$\leq \sum_{i; t_l \in [s_{i-1}, s_i)} \varepsilon \int_{t_l}^{s_i} L(s)ds + \frac{\varepsilon}{(m+1)} + \sum_i \varepsilon \int_{\tau_i}^{s_i} L(s)ds$$

$$< 2\varepsilon \int_{t_0}^{t_0+\sigma} L(s)ds + m \frac{\varepsilon}{(m+1)} < 2\varepsilon \int_{t_0}^{t_0+\sigma} L(s)ds + \varepsilon.$$

Hence, for every $v \in [t_0, t_0+\sigma]$ the integral $\int_{t_0}^{v} DG\left(x\left(\tau\right), t\right)$ exists and

$$x\left(v\right) - x\left(t_0\right) = \int_{t_0}^{v} DG\left(x\left(\tau\right), t\right).$$

This proves the result. □

Theorem 3.4 improves Theorem 2.1 presented in [2] with a different proof.

Theorem 3.5. *Let $x\left(t\right)$ be a solution of* (3.18)*, with G given by* (3.15)*, in the interval $[t_0 - r, t_0 + \sigma]$ satisfying the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), \ t_0 - r \le \vartheta \le t_0, \\ x(t_0)(t_0), \ t_0 \le \vartheta \le t_0 + \sigma. \end{cases}$$

For every $\vartheta \in [t_0 - r, t_0 + \sigma]$, let

$$y\left(\vartheta\right) = \begin{cases} x\left(t_0\right)\left(\vartheta\right), \ t_0 - r \le \vartheta \le t_0 \\ x\left(\vartheta\right)\left(\vartheta\right), \quad t_0 \le \vartheta \le t_0 + \sigma. \end{cases} \tag{3.30}$$

Then $y\left(\vartheta\right)$ is a solution of the problem (3.1) *in $[t_0 - r, t_0 + \sigma]$ and $y\left(\vartheta\right) = x\left(t_0 + \sigma\right)\left(\vartheta\right)$, $\vartheta \in [t_0 - r, t_0 + \sigma]$.*

Proof. According to (3.2), it suffices to prove that for every $\eta > 0$ and any $v \in [t_0, t_0 + \sigma]$, we have

$$\left| y\left(v\right) - y\left(t_0\right) - \int_{t_0}^{v} f\left(y_s, s\right) ds - \sum_{k=1}^{m} I_k(y(t_k)) H_{t_k}(v) \right| < \eta \tag{3.31}$$

and $y_{t_0} = \phi$. The last equality is clear by (3.30).

Assume that a gauge $\delta : [t_0, t_0 + \sigma] \to (0, +\infty)$ satisfies for $\tau \in [t_0, t_0 + \sigma]$ the following:

$$0 < \delta(\tau) < \min\left\{ \frac{t_k - t_{k-1}}{2};\ k = 1, \ldots, m \right\} \tag{3.32}$$

and

$$0 < \delta(\tau) < \min\left\{ d(\tau, t_k), d(\tau, t_{k-1}) \ \text{ for } \ \tau \in (t_{k-1}, t_k),\ k = 1, \ldots, m \right\}, \tag{3.33}$$

where $d(\tau, t_k)$ is the distance of τ to t_k and similarly for $d(\tau, t_{k-1})$.

As in the proof of Theorem 3.4, the requirement (3.32) assures that if a point-interval pair $(T, [s_1, s_2])$ is δ-fine, then the interval $[s_1, s_2]$ contains at most one of the points t_k, $k = 1, \ldots, m$, while (3.33) implies $T = t_k$ for $t_k \in [s_1, s_2]$.

If $(\tau_i, [s_{i-1}, s_i])$ is an arbitrary δ-fine division of $[t_0, v]$, then by (3.10), when $t_l \in [s_{i-1}, s_i)$, we have

$$J(x(\tau_i), s_i)(\vartheta) - J(x(\tau_i), s_{i-1})(\vartheta) = \sum_{k=1}^{m} [H_{t_k}(s_2) - H_{t_k}(s_1)] H_{t_k}(\vartheta) I_k(x(\tau_i)(t_k))$$
$$= H_{t_l}(\vartheta) I_k(x(t_l)(t_l)) = H_{t_l}(\vartheta) I_k(y(t_l))$$

for $\vartheta \in [t_0 - r, t_0 + \sigma]$, and if $[s_{i-1}, s_i)$ does not contain any of the points $t_1, \dots, t_m$, then

$$J(x(\tau_i), s_i)(\vartheta) - J(x(\tau_i), s_{i-1})(\vartheta)$$
$$= \sum_{k=1}^{m} [H_{t_k}(s_2) - H_{t_k}(s_1)] H_{t_k}(\vartheta) I_k(x(\tau_i)(t_k)) = 0.$$

This implies that the integral $\int_{t_0}^{v} DJ(x(\tau), t)$ exists and

$$\left(\int_{t_0}^{v} DJ(x(\tau), t) \right)(v) = \sum_{k=1}^{m} H_{t_k}(v) I_k(y(t_k)). \tag{3.34}$$

By (3.30), (3.19) and the fact that x is a solution of (3.18) we get

$$y(v) - y(t_0) = x(v)(v) - x(t_0)(t_0) = x(v)(v) - x(t_0)(v) \tag{3.35}$$
$$= \left(\int_{t_0}^{v} DG(x(\tau), t) \right)(v) = \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v) + \left(\int_{t_0}^{v} DJ(x(\tau), t) \right)(v),$$

for $v \in [t_0, t_0 + \sigma]$. Using this and (3.34), we have

$$y\,(v) - y\,(t_0) - \int_{t_0}^{v} f\,(y_s, s)\, ds - \sum_{k=1}^{m} I_k(y(t_k)) H_{t_k}(v) \tag{3.36}$$
$$= \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v) + \left(\int_{t_0}^{v} DJ(x(\tau), t) \right)(v) - \int_{t_0}^{v} f\,(y_s, s)\, ds$$
$$- \sum_{k=1}^{m} I_k(y(t_k)) H_{t_k}(v) = \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v) - \int_{t_0}^{v} f\,(y_s, s)\, ds.$$

The existence of the integrals $\int_{t_0}^{v} DG(x(\tau), t)$ and $\int_{t_0}^{v} DJ(x(\tau), t)$ implies the existence of $\int_{t_0}^{v} DF(x(\tau), t)$.

Let $\varepsilon > 0$ be given. Assume the gauge $\delta(\tau) > 0$ satisfies (3.32), (3.33) and also

$$|h(\rho) - h(\tau)| < \varepsilon \ \text{ for every } \ \rho \in [\tau - \delta(\tau), \tau], \tag{3.37}$$

and

$$|h(\rho) - h(\tau+)| < \varepsilon \quad \text{for every} \quad \rho \in (\tau, \tau + \delta(\tau)], \tag{3.38}$$

where $h(t) = h_1(t) + h_2(t)$ is the nondecreasing, left continuous function described as in (3.16) and (3.17).

Further, let the gauge δ be such that if $\tau \in [t_0, t_0 + \sigma]$, then

$$\Big| \int_u^v L(s)ds \Big| < \frac{\varepsilon}{(m+1)(K_1+1)} \quad \text{for every} \quad [u,v] \subset [\tau - \delta(\tau), \tau + \delta(\tau)], \tag{3.39}$$

where m is the number of impulse points and K_1 is the constant from (A'). Such a choice is possible because the function $L : [t_0, t_0 + \sigma] \to \mathbb{R}$ from (B) is assumed to be Lebesgue integrable. Moreover, for the gauge δ we have (by the existence of the integral $\int_{t_0}^v DF(x(\tau), t)$) the inequality

$$\Big\| \int_{t_0}^v DF(x(\tau), t) - \sum_i \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \right] \Big\| < \varepsilon \tag{3.40}$$

for every δ-fine division $(\tau_i, [s_{i-1}, s_i])$ of the interval $[t_0, v]$. Hence

$$\Big| \int_{t_0}^v DF(x(\tau), t)(v) - \sum_i \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \right](v) \Big| < \varepsilon \tag{3.41}$$

for every δ-fine division $(\tau_i, [s_{i-1}, s_i])$ of the interval $[t_0, v]$.

By (3.36) and (3.41), we have

$$\begin{aligned}
&\Big| y(v) - y(t_0) - \int_{t_0}^v f(y_s, s)\, ds - \sum_{k=1}^m I_k(y(t_k)) H_{t_k}(v) \Big| \qquad (3.42)\\
&= \Big| \left(\int_{t_0}^v DF(x(\tau), t) \right)(v) - \int_{t_0}^v f(y_s, s)\, ds \Big| \\
&< \varepsilon + \Big| \sum_i \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \right](v) - \int_{t_0}^v f(y_s, s)\, ds \Big| \\
&= \varepsilon + \Big| \sum_i \Big\{ \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \right](v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\, ds \Big\} \Big|.
\end{aligned}$$

The definition of F given in (3.3) yields

$$\left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \right](v) = \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s, s)\, ds.$$

By (3.20), we have $x(\tau_i)(\vartheta) = x(\vartheta)(\vartheta) = y(\vartheta)$ provided $\vartheta \leq \tau_i$. Therefore,

$$\int_{s_{i-1}}^{s_i} f\left(x(\tau_i)_s, s\right) ds - \int_{s_{i-1}}^{s_i} f\left(y_s, s\right) ds = \int_{s_{i-1}}^{s_i} [f\left(x(\tau_i)_s, s\right) - f\left(y_s, s\right)] ds$$

$$= \int_{\tau_i}^{s_i} [f\left(x(\tau_i)_s, s\right) - f\left(y_s, s\right)] ds.$$

For $\vartheta \in [\tau_i, s_i]$ we have again by (3.20) the equality $y(\vartheta) = x(\vartheta)(\vartheta) = x(s_i)(\vartheta)$ and therefore

$$\int_{\tau_i}^{s_i} [f\left(x(\tau_i)_s, s\right) - f\left(y_s, s\right)] ds = \int_{\tau_i}^{s_i} [f\left(x(\tau_i)_s, s\right) - f\left(x(s_i)_s, s\right)] ds.$$

Using the relations above and the assumption (B), we obtain

$$\left| \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\right](v) - \int_{s_{i-1}}^{s_i} f\left(y_s, s\right) ds \right| \tag{3.43}$$

$$= \left| \int_{\tau_i}^{s_i} [f\left(x(\tau_i)_s, s\right) - f\left(x(s_i)_s, s\right)] ds \right| \leq \int_{\tau_i}^{s_i} L(s) \|x(\tau_i)_s - x(s_i)_s\| ds.$$

Let us look now to $\|x(\tau_i)_s - x(s_i)_s\|$. By definition and by the fact that $x(s_i)(\tau_i) = x(\tau_i)(\tau_i)$ (see (3.20)), we have for every i the following:

$$\|x(\tau_i)_s - x(s_i)_s\| = \sup_{\vartheta \in [-r,0]} |x(s_i)(s+\vartheta) - x(\tau_i)(s+\vartheta)|$$

$$= \sup_{\rho \in [\tau_i, s_i]} |x(s_i)(\rho) - x(\tau_i)(\rho)| = \sup_{\rho \in (\tau_i, s_i]} |x(s_i)(\rho) - x(\tau_i)(\rho)|$$

$$= \sup_{\rho \in (\tau_i, s_i]} |x(s_i)(\rho) - x(\tau_i+)(\rho) + x(\tau_i+)(\rho) - x(\tau_i)(\rho)|$$

$$\leq \sup_{\rho \in (\tau_i, s_i]} \{|x(s_i)(\rho) - x(\tau_i+)(\rho)| + |x(\tau_i+)(\rho) - x(\tau_i)(\rho)|\}$$

$$\leq \|x(s_i) - x(\tau_i+)\| + \|G(x(\tau_i), \tau_i+) - G(x(\tau_i), \tau_i)\|$$

$$\leq h(s_i) - h(\tau_i+) + K_1 < \varepsilon + K_1,$$

where the last inequalities come from Lemma 2.12 and from the definition of G in (3.15).

Hence, by (3.43) we get for every i the inequality

$$\left| \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\right](v) - \int_{s_{i-1}}^{s_i} f\left(y_s, s\right) ds \right| \tag{3.44}$$

$$\leq \int_{\tau_i}^{s_i} L(s) \|x(\tau_i)_s - x(s_i)_s\| ds < (\varepsilon + K_1) \int_{\tau_i}^{s_i} L(s) ds.$$

Now by (3.42) and (3.44) we obtain

$$\left| y\,(v) - y\,(t_0) - \int_{t_0}^{v} f\,(y_s, s)\, ds - \sum_{k=1}^{m} I_k(y(t_k)) H_{t_k}(v) \right| \tag{3.45}$$

$$< \varepsilon + \sum_i \left| \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \right](v) - \int_{s_{i-1}}^{s_i} f\,(y_s, s)\, ds \right|$$

$$< \varepsilon + (\varepsilon + K_1) \sum_i \int_{\tau_i}^{s_i} L(s) ds$$

$$\leq \varepsilon + \varepsilon \int_{t_0}^{v} L(s) ds + K_1 \sum_{i; [s_{i-1}, s_i) \cap \{t_1, \ldots, t_m\} \neq \emptyset} \int_{\tau_i}^{s_i} L(s) ds$$

$$\leq \varepsilon (1 + \int_{t_0}^{v} L(s) ds) + m \cdot K_1 \frac{\varepsilon}{(m+1)(K_1+1)} \leq \varepsilon (1 + \int_{t_0}^{v} L(s) ds) + \varepsilon$$

by the property (3.39) of the gauge δ. Taking $\varepsilon > 0$ such that

$$\varepsilon (2 + \int_{t_0}^{v} L(s) ds) < \eta$$

we obtain (3.31) and the theorem is proved. □

4. Continuous dependence

Consider the following sequence of initial-value problems for impulsive RFDEs :

$$\begin{cases} \dot{y}\,(t) = f_p\,(y_t, t)\,,\ t \neq t_k \\ \Delta y\,(t_k) = I_k^p\,(y\,(t_k))\,,\ k = 1, \ldots, m \\ y_{t_0} = \phi_p, \end{cases} \tag{4.1}$$

where $p = 0, 1, 2, \ldots$.

As in the introduction to Section 3, for every $p = 0, 1, \ldots$ the system (4.1) is equivalent to

$$\begin{cases} y\,(t) = y(t_0) + \int_{t_0}^{t} f_p\,(y_s, s)\, ds + \sum_{k=1}^{m} I_k^p(y(t_k)) H_{t_k}(t),\ t \in [t_0, t_0 + \sigma], \\ y_{t_0} = \phi_p. \end{cases} \tag{4.2}$$

Let us assume that for $p = 0, 1, \ldots$ we have $\phi_p \in PC([-r, 0], \mathbb{R}^n)$ and the entries f_p, I_k^P satisfy conditions (A), (B), (A') and (B') from Section 3 with the same M, L, K_1, K_2 for all $p = 0, 1, \ldots$.

Defining for $p = 0, 1, \ldots$ and $y \in PC_1$ the functions

$$F_p(y,t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0 \text{ or } t_0 - r \le t \le t_0 \\ \int_{t_0}^{\vartheta} f_p(y_s, s)\, ds, & t_0 \le \vartheta \le t \le t_0 + \sigma; \\ \int_{t_0}^{t} f_p(y_s, s)\, ds, & t_0 \le t \le \vartheta \le t_0 + \sigma \end{cases} \tag{4.3}$$

and

$$J_p(y,t)(\vartheta) = \sum_{k=1}^{m} H_{t_k}(t) H_{t_k}(\vartheta) I_k^p(y(t_k)) \tag{4.4}$$

for $\vartheta \in [t_0 - r, t_0 + \sigma]$ and $t \in [t_0, t_0 + \sigma]$, we obtain by Proposition 3.2 that the functions

$$G_p(y,t) = F_p(y,t) + J_p(y,t) \tag{4.5}$$

belong to the same class $\mathcal{F}(\Omega, h)$ with

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)] ds + \max(K_1, K_2) \sum_{k=1}^{m} H_{t_k}(t), \; t \in [t_0, t_0 + \sigma],$$

where $\Omega = PC_1 \times [t_0, t_0 + \sigma]$.

According to the results given in Theorems 3.4 and 3.5 for every $p = 0, 1, \ldots$, there is a one-to-one correspondence between the solutions of the problem (4.1) and the solutions of the initial-value problem for the generalized differential equation

$$\frac{dx}{d\tau} = DG_p(x,t) \tag{4.6}$$

in the sense presented in Section 3 after equation (3.18).

Theorem 4.1. *Assume that for $p = 0, 1, \ldots$ we have $\phi_p \in PC([-r, 0], \mathbb{R}^n)$ and f_p, I_k^p satisfy conditions* (A), (B), (A') *and* (B') *from Section 3 with the same M, L, K_1, K_2 for all $p = 0, 1, \ldots$. Let the relations*

$$\lim_{p \to \infty} \sup_{\vartheta \in [t_0, t_0 + \sigma]} \left| \int_{t_0}^{\vartheta} [f_p(x_s, s) - f_0(x_s, s)] ds \right| = 0 \tag{4.7}$$

for every $x \in PC_1$ and

$$\lim_{p \to \infty} I_k^p(x) = I_k^0(x) \tag{4.8}$$

for every $x \in \mathbb{R}^n$, $k = 1, \ldots, m$ be satisfied. Assume that $y_p : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ for $p = 1, 2, \ldots$ is a solution of problem (4.1) *on $[t_0, t_0 + \sigma]$ such that*

$$\lim_{p \to \infty} y_p(s) = y(s) \quad \text{uniformly on } [t_0, t_0 + \sigma]. \tag{4.9}$$

Then $y : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is a solution of the problem

$$\begin{cases} \dot{y}(t) = f_0(y_t, t), \ t \neq t_k \\ \Delta y(t_k) = I_k^0(y(t_k)), \ k = 1, \dots, m \\ y_{t_0} = \phi_0. \end{cases} \tag{4.10}$$

Proof. Given $t \in [t_0 - r, t_0 + \sigma]$, let

$$x_p(t)(\tau) = \begin{cases} y_p(\vartheta), \ \vartheta \in [t_0 - r, t] \\ y_p(t), \ \vartheta \in [t, t_0 + \sigma] \end{cases} \tag{4.11}$$

for $p = 1, 2, \dots$ and

$$x(t)(\tau) = \begin{cases} y(\vartheta), \ \vartheta \in [t_0 - r, t] \\ y(t), \ \vartheta \in [t, t_0 + \sigma]. \end{cases} \tag{4.12}$$

Then $x_p(t) \in PC([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ is a solution of (4.6) in $[t_0, t_0 + \sigma]$ for $p = 1, 2, \dots$ by Theorem 3.4.

By (4.9), it is easy to check that for $s \in [t_0, t_0 + \sigma]$ we have

$$\lim_{p \to \infty} x_p(s) = x(s) \tag{4.13}$$

in $PC([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and $x(s) \in PC_1$ for $s \in [t_0, t_0 + \sigma]$.

By (4.7) and (4.8), it can be shown that

$$\lim_{p \to \infty} G_p(x, t) = G(x, t) \tag{4.14}$$

for $(x, t) \in PC_1 \times [t_0 - r, t_0 + \sigma]$.

Theorem 2.17 shows now that $x : [t_0, t_0 + \sigma] \to PC_1$ is a solution of

$$\frac{dx}{d\tau} = DG_0(x, t) \tag{4.15}$$

and Theorem 3.5 yields that the function $y : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is a solution of problem (4.10). □

5. Towards a global theory

Assume that $[a, \infty) \subset \mathbb{R}$ is given and consider functions $f(\phi, t)$ mapping $PC([-r, 0], \mathbb{R}^n) \times [a, \infty)$ to $\mathbb{R}^n$.

Assume further that a sequence (t_l) is given with $a \leq t_1 < t_2 < \ldots < t_l < \ldots$ and $t_l \to \infty$ as $l \to \infty$.

We will consider functions $y : [a - r, \infty) \to \mathbb{R}^n$ which are continuous from the left in their domain of definition, admit the right limits $y(t+)$ at every point and are such that $y(t+) \neq y(t)$ only for $t = t_l$, $l = 1, 2, \dots$ and $y|_{[a-r,a]} \in PC([a - r, a], \mathbb{R}^n)$. Denote this family of functions by $PC([a -$

$r, \infty)) = PC([a-r,\infty),\mathbb{R}^n)$. It is clear that for a function y having these properties, we have $y_t \in PC([-r,0],\mathbb{R}^n)$ for $t \in [a,\infty)$. Therefore, $f(y_t,t) : [a,\infty) \to \mathbb{R}^n$ is well defined for $t \in [a,\infty)$ and $y : [a-r,\infty) \to \mathbb{R}^n$ belongs to the class $PC([a-r,\infty))$ of functions presented above.

In accordance with the properties required in Section 3, we will assume the following: if $y \in PC([a-r,\infty))$, then the function $f(y_t,t) : [a,\infty) \to \mathbb{R}^n$ is Lebesgue integrable and moreover

(A*) there is a locally Lebesgue integrable function $M(t) : [a,\infty) \to \mathbb{R}$ such that for all $x \in PC_1$ and all $u_1, u_2 \in [a,+\infty)$,

$$\Big| \int_{u_1}^{u_2} f(x_s,s)\,ds \Big| \le \int_{u_1}^{u_2} M(s)\,ds;$$

(B*) there is a locally Lebesgue integrable function $L : [a,\infty) \to \mathbb{R}$ such that for all $x, y \in PC_1$ and all $u_1, u_2 \in [a,+\infty)$,

$$\Big| \int_{u_1}^{u_2} [f(x_s,s) - f(y_s,s)]\,ds \Big| \le \int_{u_1}^{u_2} L(s)\,\|x_s - y_s\|\,ds.$$

Concerning the impulse functions $I_l : \mathbb{R}^n \to \mathbb{R}^n$, $l = 1,2,\dots$, we assume the following conditions:

(A'*) there is a constant $K_1 > 0$ such that for all $l = 1,2,\dots$ and all $x \in \mathbb{R}^n$,

$$|I_l(x)| \le K_1;$$

(B'*) there is a constant $K_2 > 0$ such that for all $l = 1,2,\dots$ and all $x, y \in \mathbb{R}^n$,

$$|I_l(x) - I_l(y)| \le K_2|x-y|.$$

Let $PC_1 \subset PC([a-r,\infty))$ be an open set (in the topology of locally uniform convergence in $PC([a-r,\infty)))$ with the following property: if y is an element of PC_1 and $\bar{t} \in [a,\infty)$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & a-r \le t \le \bar{t} \\ y(\bar{t}_+), & \bar{t} < t \le \infty \end{cases}$$

is also an element of PC_1.

Similarly as in Section 3, define for $y \in PC([a-r,\infty))$

$$F(y,t)(\vartheta) = \begin{cases} 0, & a-r \le \vartheta \le a \text{ or } a-r \le t \le a, \\ \int_a^{\vartheta} f(y_s,s)\,ds, & a \le \vartheta \le t < \infty, \\ \int_a^{t} f(y_s,s)\,ds, & a \le t \le \vartheta < \infty \end{cases} \tag{5.1}$$

and

$$J(y,t)(\vartheta) = \sum_{l=1}^{\infty} H_{t_l}(t) H_{t_l}(\vartheta) I_l(y(t_l)) \tag{5.2}$$

for $\vartheta \in [a-r,\infty)$, $t \in [a-r,\infty)$ and an arbitrary $y \in PC([a-r,\infty))$. (H_{t_k} is the left continuous Heaviside function concentrated at t_k.)

Taking $F(y,t)$ from (5.1) and $J(y,t)$ from (5.2), let

$$G(y,t)(\vartheta) = F(y,t)(\vartheta) + J(y,t)(\vartheta) \tag{5.3}$$

for $y \in PC([a-r,\infty))$, $t \in [a-r,\infty)$ and $\vartheta \in [a-r,\infty)$. The values of the function $G(y,t)$ belongs clearly to $PC([a-r,\infty))$; that is,

$$G : PC_1 \times [a-r,\infty) \to PC([a-r,\infty)).$$

Analogously as in Section 3, it can be checked out that for $s_1, s_2 \in [a,\infty)$ and $x, y \in PC_1$ we have

$$\|G(x,s_2) - G(x,s_1)\|_{loc} \le h(s_2) - h(s_1) \tag{5.4}$$

and

$$\|G(x,s_2) - G(x,s_1) - G(y,s_2) + G(y,s_1)\|_{loc} \le \|x-y\|(h(s_2)-h(s_1)), \tag{5.5}$$

where

$$h(t) = \int_a^t [M(s) + L(s)]ds + \max(K_1,K_2) \sum_{k=1}^{\infty} H_{t_k}(t), \quad t \in [a,\infty)$$

is a nondecreasing real function which is continuous from the left at every point, continuous for all $t \ne t_l$ and $h(t_l+)$ exists for every l and $\|\cdot\|_{loc}$ is any local norm of elements in $PC([a-r,\infty))$; i.e., if $z \in PC([a-r,\infty))$ then $\|z\|_{loc} = \sup_{\vartheta\in[\alpha,\beta]} |z(\vartheta)|$ for an arbitrary compact interval $[\alpha,\beta] \subset [a-r,\infty)$.

According to (5.4) and (5.5), it can be easily seen that the function G defined by (5.3) belongs to the class $\mathcal{F}(\Omega,h)$, where $\Omega = PC_1 \times [c,d]$ and $[c,d]$ is any compact subinterval of $[a,\infty)$.

Consider the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG\,(x,t)\,. \tag{5.6}$$

Assume that $t_0 \in [a,\infty)$ and $\phi \in PC\,([-r,0],\mathbb{R}^n)$ are given. Define a function $\widetilde{x} \in PC([a-r,\infty))$ by

$$\widetilde{x}(\vartheta) = \begin{cases} \phi(\vartheta - t_0) \text{ for } \vartheta \in [t_0-r,t_0], \\ \widetilde{x}(t_0-r) = \phi(-r) \text{ for } \vartheta \in [a,t_0-r], \\ \widetilde{x}(t_0) = \phi(0) \text{ for } \vartheta \in [t_0,\infty). \end{cases} \tag{5.7}$$

Looking at the initial-value problem for (5.6), with $x(t_0) = \widetilde{x}$, the local existence and uniqueness Theorem 2.15, together with the equivalence result given in Theorem 3.5, can be used to obtain the following.

Theorem 5.1. *If the conditions* (A*), (B*), (A'*) *and* (B'*) *are fulfilled and if* $\widetilde{x} \in PC_1$ *from* (5.7) *is such that*

$$\widetilde{x}(\vartheta) + H_{t_l}(\vartheta) I_l(\widetilde{x}(t_0)) \in PC_1 \tag{5.8}$$

when $t_0 = t_l$ *for some* $l = 1, 2, \ldots,$ *then there is a* $\Delta > 0$ *such that on the interval* $[t_0, t_0 + \Delta]$ *there exists a unique solution* $y : [t_0, t_0 + \Delta] \to \mathbb{R}^n$ *of the problem* (3.1) *for which* $y_{t_0} = \phi$.

By Theorem 2.15, for $\widetilde{x} \in PC_1$ the relation

$$\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0) \in PC_1$$

is needed. This condition assures that the solution of the initial-value problem for the generalized ordinary differential equation (5.6) does not jump off the set PC_1 immediately at the moment t_0. Note that in our situation of the function G given by (5.3), we have $G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0) = 0$ if $t_0 \neq t_l$, $l = 1, 2, \ldots$ and $[G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)](\vartheta) = H_{t_l}(\vartheta) I_l(\widetilde{x}(t_0))$ if $t_0 = t_l$ for some $l = 1, 2, \ldots$. This gives then the condition (5.8) from Theorem 5.1.

By Theorem 3.4 we have also the opposite; i.e., we have a one-to-one correspondence between the solutions of the problem (3.1) and the solutions of the initial-value problem for (5.6) with $x(t_0) = \widetilde{x}$.

Having the result of Theorem 5.1, the concept of a maximal solution of the problem (3.1) can be introduced by taking $\sigma = \sup \Delta > 0$, where $\Delta > 0$ is such that there is a unique solution $y : [t_0, t_0 + \Delta] \to \mathbb{R}^n$ of the problem (3.1) for which $y_{t_0} = \phi$. Hence we have a function $y : [t_0, t_0 + \sigma) \to \mathbb{R}^n$ which is the maximal solution of the problem (3.1) on every closed interval $[t_0, t_0 + \Delta]$ with $0 < \Delta < \sigma$, but there is no solution of the problem (3.1) on closed intervals $[t_0, t_0 + \Delta]$ with $\sigma \leq \Delta$.

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