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Generalized periods and mirror symmetry in dimensions $n > 3$ *

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Abstract

The predictions of the Mirror Symmetry are extended in dimensions $n > 3$ and are proven for projective complete intersections Calabi-Yau varieties. Precisely, we prove that the total collection of rational Gromov-Witten invariants of such variety can be expressed in terms of certain invariants of a new generalization of variation of Hodge structures attached to the dual variety.

To formulate the general principles of Mirror Symmetry in arbitrary dimension it is necessary to introduce the "extended moduli space of complex structures" $\mathcal{M}$. We show that the moduli space $\mathcal{M}$ is the base of generalized variation of Hodge structures. An analog $\mathcal{M} \to \oplus_k H^k(X^n, \mathbb{C})[n-k]$ of the classical period map is described and is shown to be a local isomorphism. The invariants of the generalized variations of Hodge structures are introduced. It is proven that their generating function satisfies the system of WDVV-equations exactly as in the case of Gromov-Witten invariants.

The basic technical tool utilized is the Deformation theory.

Contents

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1 Introduction

This work is devoted to the description of the collection of all rational Gromov-Witten invariants of Calabi-Yau varieties in arbitrary dimension via the invariants of a certain new generalization of variations of Hodge structures attached to the mirror dual varieties.

The first discovery in this direction was the striking prediction made by Candelas, de la Ossa, Green and Parkes [COGP] for the numbers of rational curves on quintic threefold in $\mathbb{P}^4$ in terms of the periods of some "dual" family of Calabi-Yau threefolds.

These numbers of rational curves on quintic are the simplest examples of rational Gromov-Witten invariants. According to the theory of Gromov-Witten invariants (see [KM]) the collection of all rational Gromov-Witten invariants of a projective algebraic manifold Y is encoded in the generating function

$$\text{(Potential)} \qquad \mathcal{F}(t),\ \ t\ \in\ H^*(Y,\mathbb{C}) \tag{1.1}$$

considered as series over the semigroup ring $\mathbb{Q}[B]$ where B is the semigroup of effective one-dimensional algebraic cycles modulo numerical equivalences. The total space of the cohomology groups $H^*(Y,\mathbb{C})$ is considered here as a complex supermanifold. It is convenient to choose a graded basis $\{\Delta_a\}$ in $H^*(Y,\mathbb{C})$. Let us denote by $\{t^a\}$ the dual set of linear coordinates and choose some generic representatives $\{\Gamma_a\}$ of the homology classes dual to $\{\Delta_a\}$. Intuitively, the Taylor coefficients $N(a_1,\ldots,a_n;\beta)$ in the series expansion

$$\mathcal{F}(t) = \sum_{n;a_1,\ldots,a_n;\beta} \frac{1}{n!} N(a_1,\ldots,a_n;\beta) q^\beta t^{a_1}\ldots t^{a_n} \tag{1.2}$$

count the numbers of algebraic maps $f: C \to Y$ where C is a rational curve with n marked points $\{x_1,\ldots,x_n\}$ such that $f(x_i) \in \Gamma_{a_i}$ and $f_*([C]) = \beta \in B$. However a lot of work is needed in order to give the precise definition for these numbers (see [BM]).

The conjectures of [COGP] were partially extended to higher dimensions in [BvS] (see also [GMP]) where the formulas describing hypothetically a subset of the Gromov-Witten invariants corresponding to the restriction of the third derivative of the potential to the subspace of the second cohomology group

$$\partial^3\mathcal{F}(t)\ |_{t\in H^2(Y,\mathbb{C})} \tag{1.3}$$

were proposed.

The next important achievement was made by A. Givental ([G]) who has established the conjectures from [COGP] and [BvS] which allow to express $\partial^3\mathcal{F}(t)|_{H^2(Y,\mathbb{C})}$ in terms of the classical periods associated with the mirror dual family.

We construct a generalization of the classical periods map in order to find the expression for the whole generating function $\mathcal{F}(t)$. In other words the aim of our work is to identify the total collection of rational Gromov-Witten invariants for the Calabi-Yau varieties of dimension $n > 3$ with certain invariants coming from a generalization of the theory of variations of Hodge structure on the dual varieties. We prove the coincidence of the two types of invariants for projective complete intersections Calabi-Yau varieties and its duals.

In section 3 the "extended moduli space of complex structures" $\mathcal{M}$ is introduced. Let X be a complex manifold with trivial canonical sheaf. Let us denote by J_X the corresponding complex structure on the underlying $C^\infty-$manifold X_{C^∞}. Recall that the tangent space to the classical moduli space of complex structures at a smooth point is identified canonically with $H^1(X,\mathcal{T}_X)$. According to Kodaira-Spencer theory a complex structure on X_{C^∞} close to J_X is described by an element $\gamma \in \Omega^{0,1}(X,\mathcal{T}_X)$ satisfying Maurer-Cartan equation

$$\bar\partial\gamma + \frac{1}{2}[\gamma,\gamma] = 0 \tag{1.4}$$

The correspondence: complex structure $J \to \gamma_J$ can be described as follows. A complex structure on X may be defined as a decomposition $T_\mathbb{R} \otimes \mathbb{C} = T \oplus \overline{T}$ of the complexified tangent space into the sum of complex conjugate subspaces which constitute formally integrable distributions (Newlander-Nirenberg theorem). A deformation of such decomposition corresponds to a graph of a linear map $\overline{T} \to T$, i.e. an element $\gamma \in \Omega^{0,1}(X, \mathcal{T}_X)$. The equation (1.4) is the condition of the formal integrability of $\overline{T}$. The elements γ, γ' describing the equivalent complex structures are related via the action of the group corresponding to the Lie algebra $\Omega^{0,0}(X, \mathcal{T}_X)$. The extended moduli space of complex structures $\mathcal{M}$ is described similarly by the elements [1]

$$\gamma \quad \in \quad \oplus_{p,q} \Omega^{0,q}(X, \Lambda^p \mathcal{T}_X)[p-q-1] \tag{1.5}$$

satisfying the eq. (1.4). The technic of the deformation theory which allows one to associate the moduli space $\mathcal{M}$ with the differential graded Lie algebra $\mathfrak{g} = \oplus_{q,p} \Omega^{0,q}(X, \Lambda^p T_X)[p-q-1]$ is recalled in §2. A trick from the rational homotopy theory (see [DGMS]) allows one to prove that the moduli space $\mathcal{M}$ is smooth with the tangent space at the base point $[X]$ canonically isomorphic to

$$\oplus_{p,q} H^q(X, \Lambda^p \mathcal{T}_X)[p-q] \tag{1.6}$$

We demonstrate in §3.3 using the formality theorem from [K1] that the super-moduli space $\mathcal{M}$ parametrizes the $A_\infty-$deformations of $D^bCoh(X)$.

The section §4 is devoted to the descrpition of the generalized period map. The condition $c_1(T_X) = 0$ implies that there exist nonvanishing holomorphic $n-$form Ω_{X_t}, $n = \dim_{\mathbb{C}} X$ for every $t \in \mathcal{M}^{classical}$. If one fixes a hyperplane $L \subset H^n(X, \mathbb{C})$ transversal to the last component of the Hodge filtration $F^{\geq n}$ at the base point $[X_0]$ then it allows one to define the classical period map $\mathcal{M}^{classical} \to H^n(X, \mathbb{C})$. It sends a point (t) of the moduli space of complex structures to the cohomology class of the holomorphic $n-$form $\Omega^L_{X_t}$ normalized so that $[\Omega^L_{X_t}] - [\Omega_{X_0}] \in L$. The theorem 1 and the proposition 4.2.4 describe a generalization of this map for the extended moduli space $\mathcal{M}$. Here W is an increasing filtration on the total sum of cohomology groups $H^*(X, \mathbb{C})$ complementary to the Hodge filtration. It turns out that our generalized period map

$$\Pi^W : \mathcal{M} \to \oplus_k H^k(X, \mathbb{C})[n-k], \;\; n = \dim_{\mathbb{C}} X \tag{1.7}$$

is locally an isomorphism.

The map Π^W arises from a structure which may be understood as certain generalization of the variation of Hodge structures (genaralized VHS) having the moduli space $\mathcal{M}$ as the base. This is explained in sections 5. We introduce also in this section the invariants of such generalized VHS. One of the important properties of these invariants is the fact that their generating function satisfies the system of WDVV-equations exactly as in the case of Gromov-Witten invariants.

In the section 6 we prove that the rational Gromov-Witten invariants of projective complete intersections Calabi-Yau manifolds coincide with the invariants of the generalized VHS introduced in section 5 which correspond to their mirror pairs. One reformulation of this result is the equality

$$C^{\gamma}_{\alpha\beta}(\tau) = \sum_i ((\partial\Pi)^{-1})^{\gamma}_i \partial_\alpha \partial_\beta \Pi^i$$

[1]For a graded object A we denote by $A[k]$ the tensor product of A with the trivial object concentrated in degree $(-k)$

where $\partial^3_{\alpha\beta\gamma}\mathcal{F} = \sum_\delta g_{\gamma\delta} C^\delta_{\alpha\beta}(\tau)$ is the third derivative of the generating function for rational Gromov-Witten invariants and $\Pi^i(\tau)$ is the vector of generalized periods depending on the point of the extended moduli space $\mathcal{M}$ associated with the mirror dual variety.

The importance of the problem of constructing a moduli space with the properties similar to $\mathcal{M}$ for understanding the Mirror Symmetry phenomena was anticipated in [W1]. It was conjectured in [K2] that such a moduli space is related with hypothetical moduli space of $A_\infty-$deformations of $D^bCoh(X)$. The definition of the moduli space $\mathcal{M}$ appeared for the first time in [BK]. The relations of our results with homological mirror symmetry conjecture are discussed in §3.3 and at the end of §5.4.

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All the moduli spaces which we consider are $\mathbb{Z}-$graded manifolds, in other words they are supermanifolds with additional $\mathbb{Z}-$grading on the structure sheaf compatible with the $\mathbb{Z}_2$-grading. To simplify notations we replace $\deg t^a$ by $\overline{a}$ in superscripts.

2 Basics of deformation theory

We recall here the basics of the Deformation theory which is the main technical tool used throughout the text. The material presented here is well-known to the specialists (see for example [SchSt],[GM],[K1]). The Deformation theory was developed in the work of a number of mathematicians (P.Deligne, V.Drinfeld, B.Feigin, W.Goldman and J.Millson, A.Grothendieck, M.Kontsevich, M.Schlessinger, J.Stasheff ...). Unfortunately many of their results remained unpublished for a long time.

We assume that we work over a field k of characteristic zero.

2.1 Moduli spaces via differential graded Lie algebras

The principal strategy of Deformation theory may be described as follows. Given some mathematical structure [2] A one can associate to A the differential graded Lie algebra[3] $\underline{Der}^*(A)$ defined canonically up to quasi-isomorphisms. Recall that the differential graded Lie algebra is a graded vector space equipped with

[2] for example A can be an associative algebra, complex manifold, vector bundle etc.
[3] sometimes it is more convenient to work with more general notion of $L_\infty-$algebra

differential and graded skew-symmetric bracket satisfying a list of axioms

$$\begin{gathered}\mathfrak{g}=\oplus_k\mathfrak{g}^k,\;\; d:\mathfrak{g}^k\to\mathfrak{g}^{k+1},\;\; d^2=0,\qquad [\cdot,\cdot]:\mathfrak{g}^k\otimes\mathfrak{g}^l\to\mathfrak{g}^{k+l}\\ d[\gamma_1,\gamma_2]=[d\gamma_1,\gamma_2]+(-1)^{\overline{\gamma_1}}[\gamma_1,d\gamma_2],\;\; [\gamma_2,\gamma_1]=-(-1)^{\overline{\gamma_1}\,\overline{\gamma_2}}[\gamma_1,\gamma_2]\\ [\gamma_1[\gamma_2,\gamma_3]]+(-1)^{\overline{\gamma_3}(\overline{\gamma_1}+\overline{\gamma_2})}[\gamma_3[\gamma_1,\gamma_2]]+(-1)^{\overline{\gamma_1}(\overline{\gamma_2}+\overline{\gamma_3})}[\gamma_2[\gamma_3,\gamma_1]]=0\end{gathered}\tag{2.1}$$

The correspondence $A\to\underline{Der}^*(A)$ may be viewed as a kind of "derived functor"(or rather "derived correspondence") with respect to the standard correspondence $A\to Der(A)$ which associates to A its Lie algebra of infinitesimal automorphisms. Then the equivalence classes of deformations of the structure A are described in terms of $\underline{Der}^*(A)$. In the standard approach of the deformation theory one considers inductevely the deformations up to the given order $1,2,\ldots,N,\ldots$. In other words, the algebras of functions on the standard parameter spaces of deformations are the Artin algebras with residue field k (in our context they will be $\mathbb{Z}-$graded generally). Recall that such an algebra $\mathfrak{A}$ is isomorphic to a direct sum $k\oplus\mathfrak{m}$ where k is a copy of the base field and $\mathfrak{m}$ is a finite-dimensional commutative nilpotent algebra ($\mathbb{Z}-$graded in general). Even more concretly, any Artin algebra with the residue field k is isomorphic to an algebra of the form $k[t_i]_{i\in S}/I$, where I is an ideal $I\supset t^Nk[t_i]$ and S is a finite set of (graded) generators. Then the deformations $\widetilde{A}_{/\mathfrak{A}}$ of the structure A over an Artin algebra $\mathfrak{A}$ are described by solutions to Maurer-Cartan equation

$$d\gamma+\frac{1}{2}[\gamma,\gamma]=0,\quad \gamma\in(\underline{Der}^*(A)\otimes\mathfrak{m})^1\tag{2.2}$$

The equivalent deformations $(\widetilde{A}_{/\mathfrak{A}})_1\simeq(\widetilde{A}_{/\mathfrak{A}})_2$ correspond to the solutions from the same orbit of the group associated with the nilpotent Lie algebra $(\underline{Der}^*(A)\otimes\mathfrak{m})^0$. This Lie algebra acts on the space $(\mathfrak{g}\otimes\mathfrak{m})^1$ by

$$\alpha\in(\mathfrak{g}\otimes\mathfrak{m})^0\to\quad\dot{\gamma}=d\alpha+[\alpha,\gamma]\tag{2.3}$$

It is convinient to introduce functor $\mathrm{Def}_{\mathfrak{g}}$ associated with a differential graded Lie algebra $\mathfrak{g}$

$$\mathrm{Def}_{\mathfrak{g}}(\mathfrak{A})=\{d\gamma+\frac{1}{2}[\gamma,\gamma]=0|\gamma\in(\mathfrak{g}\otimes\mathfrak{m})^1\}/\Gamma^0(\mathfrak{A})\tag{2.4}$$

which acts from the category of Artin algebras with residue field k to the category of sets. We will denote $\mathrm{Def}^0_{\mathfrak{g}}$ the corresponding functor in the more widely known case when only Artin algebras concentrated in degree 0 are involved. Sometimes we denote the more general functor on $\mathbb{Z}-$graded Artin algebras by $\mathrm{Def}^{\mathbb{Z}}_{\mathfrak{g}}$. The description above of the deformations of A in terms of $\underline{Der}^*(A)$ may be rephrased now by saying that the functor which associates to $\mathfrak{A}$ the set of equivalence classes of deformations $\widetilde{A}_{/\mathfrak{A}}$ is isomorphic to $\mathrm{Def}_{\underline{Der}^*(A)}$.

Furthermore, in the cases when the actual moduli space of deformations of A exists, the functor $\mathrm{Def}_{\mathfrak{g}}$ is equivalent to the functor $\mathrm{Hom}_{continuous}(\hat{\mathcal{O}},\cdot)$ where $\hat{\mathcal{O}}$ is the pro-Artin[4] algebra which is equal to the completion of the algebra of functions on the actual moduli space of deformations of A.

[4]=projective limit of Artin algebras

Given a differential graded Lie algebra $\mathfrak{g}$ such that the functor $\mathrm{Def}_{\mathfrak{g}}$ is equivalent to the functor represented by some pro-Artin algebra $\mathcal{O}_{\mathfrak{g}}$ one can define the formal moduli space $\mathcal{M}_{\mathfrak{g}}$ associated to $\mathfrak{g}$ by proclaiming $\mathcal{O}_{\mathfrak{g}}$ to be "the algebra of functions on $\mathcal{M}_{\mathfrak{g}}$".

The basic tool to deal with differential graded Lie algebras and formal moduli spaces associated to them is provided by the theorem on quasi-isomorphisms which is described in the next subsection.

2.2 Equivalence of Deformation functors

We need to recall first the following homotopy generalization of the notion of the morphism between two differential graded Lie algebras.

A sequence of linear maps

$$\begin{aligned} F_1 &: \mathfrak{g}_1 \to \mathfrak{g}_2 \\ F_2 &: \Lambda^2(\mathfrak{g}_1) \to \mathfrak{g}_2[-1] \\ F_3 &: \Lambda^3(\mathfrak{g}_1) \to \mathfrak{g}_2[-2] \\ &\dots \end{aligned} \tag{2.5}$$

defines an $L_\infty-$morphism of differential $\mathbb{Z}-$graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ if

$$\begin{aligned} dF_n(\gamma_1 \wedge \ldots \wedge \gamma_n) - \sum_i \pm F_n(\gamma_1 \wedge \ldots \wedge d\gamma_i \wedge \ldots \wedge \gamma_n) = \\ = \frac{1}{2} \sum_{k,l \geq 1,\ k+l=n} \frac{1}{k!l!} \sum_{\sigma \in S_n} \pm [F_k(\gamma_{\sigma(1)} \wedge \ldots \wedge \gamma_{\sigma(k)}), F_l(\gamma_{\sigma(k+1)} \wedge \ldots \wedge \gamma_{\sigma(k+l)})] + \\ + \sum_{i<j} \pm F_{n-1}([\gamma_i, \gamma_j] \wedge \gamma_1 \wedge \ldots \wedge \gamma_n) \end{aligned} \tag{2.6}$$

In particular, the first map is a morphism of complexes which respects the Lie brackets up to homotopy defined by the second map, which itself respects the Lie brackets up to higher homotopies and so on.

An $L_\infty-$map $F = \{F_n\}, F : \mathfrak{g}_1 \to \mathfrak{g}_2$ defines a natural transformations of the functors $F_* : \mathrm{Def}_{\mathfrak{g}_1} \to \mathrm{Def}_{\mathfrak{g}_2}$. If $\gamma \in (\mathfrak{g}_1 \otimes \mathfrak{m})^1$ is a solution to Maurer-Cartan equation then

$$F_*(\gamma) := \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\gamma \wedge \ldots \wedge \gamma) \tag{2.7}$$

is a solution to Maurer-Cartan equation in $(\mathfrak{g}_2 \otimes \mathfrak{m})^1$

Recall that an $L_\infty-$morphism $F = \{F_n\}, F : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called quasi-isomorphism if its linear part F_1 induces an isomorphism of cohomology of complexes $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$.

Basic Theorem of Deformation Theory. *If $F = \{F_n\} : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an $L_\infty-$morphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ which is a quasi-isomorphism then the natural transformation of deformation functors $F_* : Def_{\mathfrak{g}_1} \to Def_{\mathfrak{g}_2}$ is an isomorphism.*

2.3 Formal manifolds and odd vector fields

Here we recall the geometric picture of the theory presented above. In particular we give an interpretation of the notions of $L_\infty-$morphism and of the moduli space $\mathcal{M}_\mathfrak{g}$ described by differential graded Lie algebra $\mathfrak{g}$.

The set of maps $F_n : \Lambda^n\mathfrak{g}_1 \to \mathfrak{g}_2[1-n]$ can be thought of as the set of Taylor coefficients of a formal map $F : \mathfrak{g}_1[1] \to \mathfrak{g}_2[1]$ preserving the zero. Namely, the algebra of formal power series on a (super) vector space $\mathfrak{g}[1]$ can be identified with the dual to the free cocommutative coalgebra Symm($\mathfrak{g}[1]$) cogenerated by $\mathfrak{g}[1]$. Then all geometric objects associated with the $\mathbb{Z}-$graded (formal) manifold $\mathfrak{g}[1]$ may be described in terms of this coalgebra. In particular, a map of formal manifolds $\mathfrak{g}_1[1] \to \mathfrak{g}_2[1]$ corresponds to a coalgebra morphism Symm($\mathfrak{g}_1[1]$) $\to$ Symm($\mathfrak{g}_2[1]$). A map of free algebras is defined uniquely by its restriction to the set of generators. Similarly, a map of coalgebras Symm($\mathfrak{g}_1[1]$) $\to$ Symm($\mathfrak{g}_2[1]$) is defined uniquely by its components $F_n : S^n(\mathfrak{g}_1[1]) \to \mathfrak{g}_2[1]$, $n \geq 0$. Recall that in the category of $\mathbb{Z}-$graded vector spaces $S^n(V[1]) = \Lambda^n(V)[n]$. The set of maps (2.5) defines naturally a (formal power series) map of $\mathbb{Z}-$graded manifolds $\mathfrak{g}_1[1] \to \mathfrak{g}_2[1]$ preserving the origins, since the corresponding constant term is zero: $F_0 = 0$. The condition (2.6) can be translated into geometric terms as follows.

The structure of the differential graded Lie algebra on a graded vector space $\mathfrak{g}$ is interpreted as a degree one vector field $Q_\mathfrak{g}, Q_\mathfrak{g}^2 = 0$ on the vector space $\mathfrak{g}[1]$ preserving the origin. Namely, the structure maps

$$d : \mathfrak{g} \to \mathfrak{g}[1], \quad [\cdot,\cdot] : S^2(\mathfrak{g}[1]) \to \mathfrak{g}[2] \tag{2.8}$$

are the components of uniquely defined degree one derivation $Q_\mathfrak{g}$ of the coalgebra Symm($\mathfrak{g}$). It corresponds to the vector field

$$Q(\gamma) = d\gamma + \frac{1}{2}[\gamma,\gamma] \tag{2.9}$$

The relations (2.1) which are satisfied by the structure maps (2.8) are exactly equivalent to the condition [5] $Q_\mathfrak{g}^2 = 0$.

The equations (2.6) are then reformulated as the single condition

$$F_*(Q_{\mathfrak{g}_1}) = Q_{\mathfrak{g}_2} \tag{2.10}$$

This picture implies that the moduli space described by a differential graded Lie algebra $\mathfrak{g}$ can be thought of as a "nonlinear cohomology" of the operator $Q_\mathfrak{g}$. The precise meaning of this may be described as follows. One has a subscheme

$$\text{"Ker}\, Q\text{"} := \{\gamma \in \mathfrak{g}[1] | Q(\gamma) = 0\} \tag{2.11}$$

of zeroes of the vector field $Q_\mathfrak{g}$. The vector fields of the form $[Q_\mathfrak{g}, \alpha]$ where α is an arbitrary constant vector field on $\mathfrak{g}[1]$ of degree -1 span a distribution "Im Q".

[5] The differential graded Lie algebras correspond to vector fields whose Taylor expansion contains only linear or quadratic terms. An arbitrary (formal) vector field $Q, Q^2 = 0, Q(0) = 0$ of degree one on $\mathfrak{g}[1]$ is equivalent by definition to the structure of $L_\infty-$algebra on $\mathfrak{g}$

This distribution is tangent to the subscheme "Ker Q" since $[Q,[Q,\alpha]]=0$. The moduli space $\mathcal{M}_{\mathfrak{g}}$ corresponds in geometric terms to the "quotent" of the submanifold "Ker Q" defined by the zeroes of the vector field Q by the distribution "Im Q". The functor $\mathrm{Def}_{\mathfrak{g}}$ describing the moduli space $\mathcal{M}_{\mathfrak{g}}$ is then identified with the natural functor describing the "quotent" "(Ker Q/Im Q)".

3 Extended moduli spaces of complex structures

We recall here following [BK] the definition of the extended moduli space of complex structures $\mathcal{M}$. We also recall the arguments showing that it is a smooth moduli space with the tangent space

$$T_{[X]} = \oplus p, q H^q(X, \Lambda^p \mathcal{T}_X)[p-q]$$

3.1 The sheaf of graded Lie algebras

Let X, $\dim_{\mathbb{C}} X = n$ be a smooth projective algebraic manifold such that $c_1(T_X) \in \mathrm{Pic}(X)$ is zero. We assume that X is defined over the complex numbers although most of our constructions are valid for X defined over an arbitrary algebraically closed field k of characteristic zero.

Consider the coherent sheaf of $\mathbb{Z}-$graded Lie algebras

$$\underline{\mathfrak{g}} = \oplus_k \underline{\mathfrak{g}}^k[k], \quad \underline{\mathfrak{g}}^k := \Lambda^{1-k}\mathcal{T}_X \tag{3.1}$$

endowed with the Schouten-Nijenhuys bracket. This bracket is uniquely defined by the following conditions:

(1) For $v_1, v_2 \in \mathcal{T}_X$ the commutator $[v_1, v_2]$ is the standard bracket on vector fields.

(2) For $v \in \mathcal{T}_X, f \in \mathcal{O}_X$ the commutator $[v, f] \in \mathcal{O}_X$ is the Lie derivative $Lie_v f$.

(3) The wedge product and the Lie bracket define the structure of an odd Poisson algebra(=Gerstenhaber algebra) on $\underline{\mathfrak{g}}[-1]$, in other words

$$[v_1, v_2 \wedge v_3] = [v_1, v_2] \wedge v_3 + (-1)^{(\overline{v}_1+1)\overline{v}_2} v_2 \wedge [v_1, v_3] \tag{3.2}$$

The technic of Deformation theory associates to the sheaf $\underline{\mathfrak{g}}$ a (formal) moduli space.

To describe this moduli space for $X_{/\mathbb{C}}$[6] let us take the Dolbeault resolution of $\underline{\mathfrak{g}}$ and consider the differential graded Lie algebra

$$\mathfrak{g} = \oplus_k \mathfrak{g}^k[k], \quad \mathfrak{g}^k := \oplus_{k=q-p+1} \Omega^{0,q}(X, \Lambda^p T_X) \tag{3.3}$$

[6]In the case of the manifold over arbitrary field of characteristic zero one should work with "simplicial Lie algebra" of Čech cochains.

endowed with the differential $\bar{\partial}$ and the extension of Schouten-Nijenhuys bracket by the cup-product of differential forms. Informally, the moduli space $\mathcal{M}$ associated with $\mathfrak{g}$ may be understood as the moduli space of solutions to Maurer-Cartan equation (1.4) in $\mathfrak{g}$ over $\mathbb{Z}-$graded bases modulo gauge equivalences.

We show in §3.3 that the moduli space associated to $\mathfrak{g}$ by the deformation theory parametrizes the $A_\infty-$deformations of $D^bCoh(X)$. On the other hand, because of the natural embedding $\mathcal{T} \subset \Lambda\mathcal{T}$ the deformations controlled by the sheaf $\Lambda\mathcal{T}$ generalize the deformations of complex structures on X.

3.2 The (formal) moduli space associated with $\Lambda\mathcal{T}$

To introduce the moduli space $\mathcal{M}$ we use the technic of deformation theory explained in §2. The moduli space is described by the deformation functor $\mathrm{Def}_{\mathfrak{g}}$ associated to $\mathfrak{g}$ (see §2.1). The algebra of functions on the moduli space $\mathcal{M}$ is by definition the algebra representing the functor $\mathrm{Def}_{\mathfrak{g}}$. First we need to introduce the odd "Laplacian" operator acting on the space $\mathfrak{g}$. The behaviour of this operator with respect to various algebraic structure on the graded vector space $\mathfrak{g}$ is described by the Batalin-Vilkovisky formalism (see for example [Schw]).

Odd Laplacian

It follows from the condition $c_1(T_X) = 0$ that there exists an everywhere nonvanishing holomorphic $n-$form $\Omega \in \Gamma(X, \Lambda^n T_X^*)$. It is defined canonically up to a multiplication by a constant. Let us fix a choice of Ω. It induces isomorphism of complexes $(\Omega^{0,*}(X, \Lambda^p T_X), \bar{\partial}) \simeq (\Omega^{0,*}(X, \Omega^{n-p}), \bar{\partial})$; $\gamma \mapsto \gamma \vdash \Omega$. One can define then the differential Δ on $\mathfrak{g}$ by the formula

$$(\Delta\gamma) \vdash \Omega = \partial(\gamma \vdash \Omega) \tag{3.4}$$

The Lie bracket on $\mathfrak{g}$ satisfies the following identity (Tian-Todorov lemma)

$$[\gamma_1, \gamma_2] = (-1)^{\deg\gamma_1+1}(\Delta(\gamma_1 \wedge \gamma_2) - (\Delta\gamma_1) \wedge \gamma_2 - (-1)^{\deg\gamma_1+1}\gamma_1 \wedge \Delta\gamma_2) \tag{3.5}$$

where $\deg\gamma = q - p + 1$ for $\gamma \in \Omega^{0,q}(X, \Lambda^p T_X)$. In particular Δ is a derivation of the differential graded Lie algebra structure.

Diagram of quasi-isomorphisms

Denote by $\mathbf{H}$ the graded vector space

$$\mathbf{H} = \oplus_k \mathbf{H}^k, \ \dim \mathbf{H}^k = \sum_{q-p=k} \dim H^q(X, \Lambda^p T_X) \tag{3.6}$$

Denote by $\mathbb{C}[[t_{\mathbf{H}}]]$ the graded algebra of formal power series on $\mathbf{H}$. We recall here the proof from [BK] of the nonobstructedness of the $\mathbb{Z}-$graded moduli space associated to $\mathfrak{g}$. Analogous arguments will be applied later for similar differential graded Lie algebras. It is convinient to fix some choice of a set $\{t^a\}$ of linear coordinates on $\mathbf{H}$.

Proposition 3.2.1. *The deformation functor associated to $\mathfrak{g}$ is isomorphic to the functor represented by the algebra $\mathbb{C}[[t_{\boldsymbol{H}}]]$. Equivalently, there exists a versal solution*[7] *to the Maurer-Cartan equation*

$$\bar{\partial}\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0 \tag{3.7}$$

in formal power series with values in $\mathfrak{g}$

$$\gamma(t) = \sum_a \gamma_a t^a + \frac{1}{2!}\sum_{a_1,a_2} \gamma_{a_1 a_2} t^{a_1} t^{a_2} + \ldots \in (\mathfrak{g}\,\widehat{\otimes}\mathbb{C}[[t_{\boldsymbol{H}}]])^1 \tag{3.8}$$

Remark 3.2.2. *The solution of the form (3.8) is versal iff the cohomology classes $[\gamma_a]$ form a basis of cohomology of the complex $(\mathfrak{g}, \bar{\partial})$.*

Proof. The idea is to use the well-known trick from rational homotopy theory of Kähler manifolds (see [DGMS]) and the theorem from §2.2. Notice it follows from the equation (3.5) that in the following diagram all arrows are the morphisms of differential graded Lie algebras

$$(\mathfrak{g}, \bar{\partial}) \leftarrow (\mathrm{Ker}\Delta, \bar{\partial}) \rightarrow (\mathrm{Ker}\Delta/\mathrm{Im}\Delta, d := 0) \tag{3.9}$$

The $\partial\bar{\partial}-$lemma (see [GH]) implies that both arrows are in fact quasi-isomorphisms. □

Corollary 3.1. *One can associate to $\mathfrak{g}$ the smooth (formal)*[8] *moduli space $\mathcal{M}$, $\widehat{\mathcal{O}}_{\mathcal{M}} \simeq \mathbb{C}[[t_{\boldsymbol{H}}]]$. The tangent space to $\mathcal{M}$ at the base point $[X]$ is canonically isomorphic to the $\mathbb{Z}-$graded vector space $\oplus_{q,p} H^q(X, \Lambda^p T_X)[p-q]$.*

Remark 3.2.3. *The differential Lie algebra $\mathfrak{g}$ and the associated moduli space were first introduced in [BK] where $\mathfrak{g}$ was considered with a slightly different grading. The $k-$th graded component of the similar differential graded Lie algebra $\mathfrak{g}^{\tilde{*}}$ defined in [BK] is $\mathfrak{g}^{\tilde{k}} = \oplus_{q+p-1=k}\Omega^{0,q}(X, \Lambda^p T_X)$. The corresponding moduli spaces considered as $\mathbb{Z}_2-$graded (formal) manifolds are canonically isomorphic since the two gradings agree mod $\mathbb{Z}_2$.*

Remark 3.2.4. *The classical moduli space of complex structures $\mathcal{M}^{classical}$ is naturally a subspace of $\mathcal{M}$. The classical deformations are parametrized by $t \in \mathcal{M}$ such that $\gamma(t)$ is equivalent to a solution with values in $\Omega^{0,1}(X, T_X)$.*

[7] This means that any other solution over arbitrary Artin algebra (or projective limit of Artin algebras) is equivalent to a solution obtained from this solution via a base change. It is also a "minimal" solution having these properties. It is common to use the adjective "versal" in such situations.

[8] For our purposes it is sufficient to work on the level of formal manifolds. In fact the standard technic of Kuranishi spaces may be used to show that all our moduli spaces exist on the level of analytic manifolds, i.e. all the power series representing the versal solutions may be chosen to be convergent

3.3 Moduli space of A_∞−deformations of $D^bCoh(X)$.

Here we sketch demonstration of the fact that the supermoduli space $\mathcal{M}$ parametrizes the A_∞−deformations of $D^b(Coh(X)$. The notion of A_∞−algebra was introduced by J.Stasheff. The A_∞−categories in the context of mirror symmetry were considered for the first time in [K2]. The main result which is used in this subsection is the formality theorem from [K1].

Recall (see [St]) that the structure of the A_∞−algebra on a graded vector space A is defined by odd degree one derivation M of the free coassociative coalgebra generated by $A[1]$

$$\mathcal{C}(A[1]) := A[1] \oplus (A[1])^{\otimes 2} \oplus \dots$$

such that $[M, M] = 0$. Degree one derivations of the free coalgebra are in one-to-one correspondence with collections of their components $m_k : A^{\otimes k} \to A[2-k]$ for all $k \in \mathbb{N}$. The condition $[M, M] = 0$ is translated into the infinite number of identities:

$$\sum_{k+l=n+1} \sum_{i=0}^{k-1} \pm m_k(x_1, \dots, x_i, m_l(x_{i+1}, \dots, x_{i+l}), x_{i+l+1}, \dots, x_n) = 0 \qquad (3.10)$$

The first identity means that m_1 is a differential on the graded vector space A. The next identity says that m_1 is a derivation with respect to the product defined by m_2. The third identity is the associativity of the product m_2 up to the homotopy defined by m_3. The next identity is the compatibility of m_3 with the product m_2 up to higher homotopies and so on.

There exists a homotopy generalization of the notion of module over the associative algebra.

Definition 3.3.1. *The structure of module over an A_∞−algebra A on graded vector space E is a degree one connection M^E on the free comodule generated by E*

$$\mathcal{C}(A[1], E) := E \oplus (E \otimes A[1]) \oplus \dots \oplus (E \otimes (A[1])^{\otimes k}) \oplus \dots \qquad (3.11)$$

such that $[M^E, M^E] = 0$.

Recall that a connection on a module $\mathfrak{M}$ over differential coalgebra $(\mathcal{C}, d)$ is a linear operator $\nabla : \mathfrak{M} \to \mathfrak{M}$ compatible with the coalgebra module structure: $\Delta^{\mathfrak{M}} \circ \nabla = (\nabla \otimes Id \pm Id \otimes d) \circ \Delta^{\mathfrak{M}}$. Again such connections acting on free modules are in one-to-one correspondence with collections of linear maps $m_i^E : A^{\otimes i} \otimes E \to E[1-i]$ satisfying an infinite number of identities similar to (3.10). For example, if A is in fact a differential graded associative algebra and $m_i^E = 0$ for $i \geq 2$, then E is a differential graded module over A.

Recall that an A_∞−category $\mathcal{X}$ is a collection of objects $Ob\,\mathcal{X}$ together with graded vector spaces of morphisms $\mathrm{Hom}^*(E_1, E_2)$ for any pair $E_1, E_2 \in \mathcal{C}$ and "higher compositions"

$$m_k(E_0, \dots, E_k) : \mathrm{Hom}^*(E_0, E_1) \otimes \dots \otimes \mathrm{Hom}^*(E_{k-1}, E_k) \to \mathrm{Hom}^*(E_1, E_k)[2-k]$$

defined for any $k \geq 0$ and $E_0, \ldots, E_k \in Ob(\mathcal{X})$. The maps $m_k(E_0, \ldots E_k)$ should satisfy an infinite number of "associativity up to homotopy" constraints similar to (3.10) which can be formulated by saying that $\oplus_{i<j}\mathrm{Hom}^*(E_i, E_j)$ should form an $A_\infty-$algebra. In particular $m_1(E)$ is a differential on $\mathrm{Hom}^*(E, E)$. One assumes usually also that the identity morphism $1_E \in \mathrm{Hom}^0(E, E)$ is singled out and satisfies $m_2(1_E, f) = f, m_k(f_1, \ldots, 1_E, \ldots, f_{k-1}) = 0$ for any $k \neq 2$.

One can show that for an $A_\infty-$algebra A the collection of all $A_\infty-$modules forms an $A_\infty-$category which we will denote by $Mod_\infty(A)$. For example the space $\mathrm{Hom}^*(E_1, E_2)$ is defined as the space of all comodule morphisms $\mathcal{C}(A[1], E_1) \to \mathcal{C}(A[1], E_2)$. This is the same as arbitrary collection of linear maps $f_k : A^{\otimes k} \otimes E_1 \to E_2[-k]$ for $k \geq 0$. The differential acting on $\mathrm{Hom}^*(E_1, E_2)$ is given by $m_1(\phi) = M^{E_2} \circ \phi \pm \phi \circ M^{E_1}$.

The deformations of the $A_\infty-$category $Mod_\infty(A)$ are described by the functor $\mathrm{Def}^{\mathbb{Z}}$ associated with the differential graded Lie algebra which consists of derivations of free coassociative coalgebra generated by $A[1]$ equipped with counit

$$Der^*(\mathcal{C}_1(A[1])),\ \ \mathcal{C}_1(A[1]) = k \cdot 1 \oplus A[1] \oplus \ldots \oplus (A[1])^{\otimes n} \oplus \ldots \tag{3.12}$$

The differential is the commutator with $M \in Der^1(\mathcal{C}(A[1])) \subset Der^1(\mathcal{C}_1(A[1]))$. This is the differential graded Lie algebra of Hochschild cochains $C^*(A, A)[1]$. Let us suppose that we are given a solution to Maurer-Cartan equation $\Gamma(t) \in (C(A, A)[1]\widehat{\otimes}\mathfrak{M}_R)^1$ where $R = k \oplus \mathfrak{M}_R$ is a (pro)-Artin algebra assumed to be $\mathbb{Z}-$graded. Then the objects of the deformed category $(\widetilde{Mod_\infty})_{/R}$ can be described as pairs

$$(E, \widetilde{M}^E),\ E \in Ob(Mod_\infty),\ \widetilde{M}^E = M^E + \Gamma^E, \tag{3.13}$$
$$\Gamma^E \in (Connec^*_\Gamma(\mathcal{C}(A[1], E))\widehat{\otimes}\mathfrak{M}_R)^1, [M^E + \Gamma^E, M^E + \Gamma^E] = 0$$

Here $\widetilde{M}^E = M^E + \Gamma^E$ is a connection depending on the parameters $t \in R$ on the free comodule $\mathcal{C}(A[1], E)$ which lifts the derivation $M + \Gamma$ acting on the coalgebra $\mathcal{C}_1(A[1])$.

One of the ways to describe the $A_\infty-$category structure on $D^bCoh(X)$ is to identify it with the category which is obtained from the differential graded category of finitely generated projective modules over the algebra $\mathcal{A}_X = (\Omega^{0,*}(X), \bar{\partial})$ by taking the same collection of objects and by taking H^0 of the complex $Hom^*(\mathcal{E}_1^*, \mathcal{E}_2^*)$ as the linear space of morphisms between two objects $\mathcal{E}_1, \mathcal{E}_2$. Then one can use the homotopy technic (see for example [M,P]) to construct the $A_\infty-$category structure on $D^bCoh(X)$. The $A_\infty-$category $D^bCoh(X)$ is $A_\infty-$equivalent to the category of finitely generated projective modules over $\mathcal{A}_X$.

Let $\mathfrak{g}^{\tilde{*}}$ denotes our differential graded Lie algebra of Dolbeault forms with coefficients in polyvector fields which is considered with slightly different grading $(\mathfrak{g}^{\tilde{*}})^k = \oplus_{q+p-1=k}\Omega^{0,q}(X, \Lambda^p\mathcal{T})$. Since the two gradings agree $mod\, 2$ the corresponding supermoduli spaces are the same. It is possible to prove using the formality theorem from [K1] that the differential graded Lie algebra $\mathfrak{g}^{\tilde{*}}$ is quasi-isomorphic to the Lie algebra of local Hochschild cochains on differential graded algebra $\mathcal{A}_X$. Therefore equivalence classes of solutions to Maurer-Cartan equation in $\mathfrak{g}$ parametrize $A_\infty-$deformations of $D^bCoh(X)$. Given a solution to Maurer-Cartan equation $\gamma(t) \in (\mathfrak{g}^{\tilde{*}}\widehat{\otimes}\mathfrak{M}_R)^1$ the quasi-isomorphism referred to above gives similar solution $\Gamma(t)$ in the Lie algebra of local Hochschild cochains on $\mathcal{A}_X$. The objects of deformed category $D^b\widetilde{Coh(X)}_{/R}$ can be identified with pairs $(\mathcal{E}, \widetilde{\nabla}^{\mathcal{E}})$. where $\mathcal{E} = (\mathcal{E}^*, \delta^{\mathcal{E}})$ is a bounded complex of locally free sheaves and $\widetilde{\nabla}^{\mathcal{E}} = (\bar{\partial} + \delta^{\mathcal{E}} + m_2^{\mathcal{E}}) + \Gamma^{\mathcal{E}}$ is a deformed conection on free comodule over $\mathcal{C}(\mathcal{A}_X[1])$ generated by $\Omega^{0,*}(X, \mathcal{E}^*)$ which depends on $t \in R$ and satisfies $[\widetilde{\nabla}^E, \widetilde{\nabla}^E] = 0$.

4 Generalized periods.

To a point of classical moduli space $[X] \in \mathcal{M}^{classical}$ one associates a vector of periods of the holomorphic $n-$form defined up to multiplication by a constant. We introduce in this section the periods associated to the generalized deformations of complex structure parametrized by $\mathcal{M}$. The properties of these periods are investigated in §5.

4.1 Classical periods.

We recall first the notion of periods associated with classical complex structures on X_{C^∞}. Let us denote by J_X the complex structure on the underlying C^∞ manifold X_{C^∞} corresponding to a point $[X] \in \mathcal{M}^{classical}$. The condition $c_1(T_X) = 0$ (as an element of $Pic(X)$) implies that there exists holomorphic nowhere vanishing $n-$form $\Omega_X \in \Gamma(X, \Omega_X^n)$ (holomorphic volume element). Such form is defined canonically up to multiplication by a constant. The deformations of the pair (J_X, Ω_X) are described by the deformation functor $\mathrm{Def}^0_{\underline{Der}^*(J_X,\Omega_X)}$ associated with the differential graded Lie algebra

$$\begin{gathered}\underline{Der}^*(J_X, \Omega_X) = \oplus_q \Omega^{0,q}(X, \mathcal{T}_X)[-q] \oplus_{q'} \Omega^{0,q'}(X, \mathcal{O}_X)[-q'-1] \\ d := \bar{\partial} + \Delta, \ \ [\ ,\ ] := \text{Schouten-Nijenhuys bracket} \times \text{cup-product}\end{gathered} \tag{4.1}$$

Proposition 4.1.1. *The classes of gauge equivalences of solution to Maurer-Cartan equation*

$$(\rho_J, f) \in \Omega^1(X, \mathcal{T}_X) \oplus \Omega^0(X, \mathcal{O}_X) \tag{4.2}$$

define deformations of the pair (J_X, Ω_X) consisting of the complex structure together with the choice of holomorphic volume form. The deformations of the complex structures are described by ρ_J. The deformed holomorphic $n-$form is

$$exp(f + \rho_J) \vdash \Omega_X \tag{4.3}$$

Proof. The Maurer-Cartan equation for (ρ_J, f) can be written as

$$\bar{\partial}\rho_J + \frac{1}{2}[\rho_J, \rho_J] = 0 \tag{4.4}$$

$$\Delta\rho_J + \bar{\partial} f + [\rho_J, f] = 0 \tag{4.5}$$

The first equation means that ρ_J defines a deformation of the complex structure. If one chooses a local holomorphic coordinates $(z^1, \ldots, z^n)$ on X then the differentials of the deformed set of holomorphic coordinates are given by

$$d\tilde{z}^i = dz^i + \sum_{\bar{j}} (\rho_J)^i_{\bar{j}} d\bar{z}^{\bar{j}} \tag{4.6}$$

where $\rho_J = \sum_{i,\bar{j}} (\rho_J)^i_{\bar{j}} d\bar{z}^{\bar{j}} \frac{\partial}{\partial z^i}$. One has $\Omega_X = a(z) \prod_i dz^i$ in the local coordinates. The second equation implies now that the $n-$form

$$\exp(f + \rho_J) \vdash \Omega_X = e^f a(z) \prod_i d\tilde{z}^i \tag{4.7}$$

is closed. One can show similarly that the gauge transformations preserve the class of equivalenses of the complex structure defined by ρ_J and the cohomology class of (4.3) □

Remark 4.1.2. *The moduli space of pairs (J_X, Ω_X) form naturally a linear bundle $\mathcal{L} \to \mathcal{M}^{classical}$ over the moduli space of complex structures. Analogously one can prove that the formal completion at the point (J_X, Ω_X) of the algebra of functions on the total space of the linear bundle $\mathcal{L}$ is the algebra representing the deformation functor $Def^0_{\underline{Der}^*(J_X, \Omega_X)}$.*

The correspondence $[X] \to \mathbb{C}{\cdot}\Omega_X$ defines the (projectivization of the) periods map $\mathcal{M}^{classical} \to \mathbb{P}H^n(X, \mathbb{C})$, $\dim_{\mathbb{C}} = n$. If one chooses a value of Ω_{X_0} at some point $[X_0] \in \mathcal{M}^{classical}$ and a hyperplane $L \subset H^n(X, \mathbb{C})$ transversal to the line $\mathbb{C} \cdot \Omega_{X_0}$ then one can define the periods map with values in $H^n(X, \mathbb{C})$:

$$\Pi^L([X]) = \Omega^L_X, \;\; \Omega^L_X - \Omega_{X_0} \in L \tag{4.8}$$

Given a set of elements $\{G^i_{(n)}\} \in H_n(X, \mathbb{C})$ which form a locally constant frame the map Π^L is described by the vector of periods $(\int_{G^i_{(n)}} \Omega^L)$.

4.2 The generalized periods map $\mathcal{M} \to \oplus_k H^k(X, \mathbb{C})[n-k]$

We introduce in this section the generalized periods map $\Pi^W : \mathcal{M} \to \oplus_k H^k(X, \mathbb{C})[n-k]$ where W is a filtration on $\oplus_k H^k(X, \mathbb{C})[-k]$ complementary to the Hodge filtration (eq. 4.21). In applications to mirror symmetry W will be the limiting weight filtration arising from degenerating family of Hodge structures. It is shown in proposition 4.2.4 that Π^W is a local isomorphism.

Family of differential graded Lie algebras.

Let us consider the family of differential graded Lie algebras [9]

$$\mathfrak{g}_\hbar = \oplus_k {\mathfrak{g}_\hbar}^k, \quad {\mathfrak{g}_\hbar}^k := \oplus_{k=q-p+1} \Omega^{0,q}(X, \Lambda^p T_X)$$
$$d_\hbar := \bar{\partial} + \hbar\Delta, \ [\ ,\ ] := (\text{Schouten-Nijenhuys bracket}) \times (\text{cup-product}) \quad (4.9)$$

Here we would like to consider $\mathfrak{g}_\hbar$ as a differential graded Lie algebra over $\mathbb{C}[[\hbar]]$. Notice that it follows from the $\partial\bar{\partial}-$lemma (see [GH]) that the cohomology of the operator $\bar{\partial} + \hbar\Delta$ acting on $\oplus_{p,q}\Omega^{0,q}(X, \Lambda^p T_X)[p-q-1]\widehat{\otimes}\mathbb{C}[[\hbar]]$ is a free $\mathbb{C}[[\hbar]]-$module. The same diagram (3.9) of quasi-isomorphisms proves the statement similar to proposition 3.2.1 in the case of the family of differential graded Lie algebras $\mathfrak{g}_\hbar$.

Proposition 4.2.1. *There exists a solution to Maurer-Cartan equation in* $\mathfrak{g}_\hbar$ *depending formally on* $\hbar$

$$\bar{\partial}\widetilde{\gamma}_\hbar(t) + \hbar\Delta\widetilde{\gamma}_\hbar(t) + \frac{1}{2}[\widetilde{\gamma}_\hbar(t), \widetilde{\gamma}_\hbar(t)] = 0$$
$$\widetilde{\gamma}_\hbar(t) = \sum_a (\widetilde{\gamma}_\hbar)_a t^a + \frac{1}{2!}\sum_{a_1,a_2} (\widetilde{\gamma}_\hbar)_{a_1 a_2} t^{a_1} t^{a_2} + \dots, \ \ (\widetilde{\gamma}_\hbar)_{a_1\dots a_k} \in \mathfrak{g}\widehat{\otimes}\mathbb{C}[[\hbar]] \tag{4.10}$$

such that $(\widetilde{\gamma}_\hbar)_a$ *form a basis of the* $\mathbb{C}[[\hbar]]-$*module of the cohomology of operator* $\bar{\partial} + \hbar\Delta$ *acting on* $\mathfrak{g}\widehat{\otimes}\mathbb{C}[[\hbar]]$.

□

We will call solution satisfying this condition "fibered versal solution to Maurer-Cartan equation in $\mathfrak{g}_\hbar$".

If $\widetilde{\gamma}$ is such a solution then for arbitrary family of coordinates change depending formally on $\hbar$

$$t'(\hbar) = t_{(0)} + \hbar t_{(1)} + \dots$$

the formula

$$\widetilde{\gamma}'(t, \hbar) = \widetilde{\gamma}(t(\hbar), \hbar) \tag{4.11}$$

gives again a versal fibered solution to Maurer-Cartan equation in $\mathfrak{g}_\hbar$. We will see that an arbitrary filtration on $\oplus_k H^k(X, \mathbb{C})[-k]$ complementary to the Hodge filtration defines a way to single out a canonical choice of such solution.

Introduce the rescalling operator

$$l_\hbar : \sum \gamma^{p,q} \to \sum \hbar^{\frac{p+q-2}{2}} \gamma^{p,q} \text{ where } \gamma^{p,q} \in \Omega^{0,q}(X, \Lambda^p T_X) \tag{4.12}$$

If $\gamma \in (\mathfrak{g}_\hbar \otimes \mathfrak{m})^1, \ \hbar \neq 0$ is a solution to the Maurer-Cartan equation depending formally on $\hbar$ then $l_\hbar(\gamma)$ gives a formal family of solutions to Maurer-Cartan equation in $\mathfrak{g}_{\hbar=1}$.

[9] From here the differential graded Lie algebra $\mathfrak{g}$ and the corresponding moduli space $\mathcal{M}$ will be sometimes referred to also as $\mathfrak{g}_0$ and $\mathcal{M}_0$

Exponential map.

Proposition 4.2.2. *For a solution to Maurer-Cartan equation* $\gamma \in (\mathfrak{g}_{\hbar=1} \otimes \mathfrak{m})^1$ *one has*

$$exp(\gamma) = 1 + \gamma + \frac{1}{2}\gamma \wedge \gamma + \dots \ \in Ker \bar{\partial} + \Delta \tag{4.13}$$

The $(\bar{\partial} + \Delta)-$*cohomology class of* e^{γ} *is the same for gauge equivalent solutions.*

Proof. Let us denote by m_{γ} the operator of the contraction by $\gamma \in (\mathfrak{g} \otimes \mathfrak{m})^{odd}$ acting on $\Omega^* \otimes \mathfrak{A}$. The standard formulas

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \dots + \tag{4.14}$$

$$[[\partial, m_{\gamma_1}], m_{\gamma_2}] = m_{[\gamma_1, \gamma_2]} \tag{4.15}$$

imply that

$$e^{-m_{\gamma}} \, \partial \, e^{m_{\gamma}} = \partial + [\partial, m_{\gamma}] + \frac{1}{2} m_{[\gamma,\gamma]} \tag{4.16}$$

$$e^{-m_{\gamma}} \, \bar{\partial} \, e^{m_{\gamma}} = \bar{\partial} + m_{\bar{\partial}\gamma} \tag{4.17}$$

Applying the sum of eq. (4.16) and eq. (4.17) to the holomorphic $n-$form Ω_{X_0} and multiplying both sides by e^{γ} one gets

$$(\Delta + \bar{\partial})e^{\gamma} = e^{\gamma} \wedge (\Delta\gamma + \bar{\partial}\gamma + \frac{1}{2}[\gamma, \gamma]) \tag{4.18}$$

The independence of the cohomology class on the choice of solutions in the same class of equivalence follows from (4.18) if one puts $\gamma' = \gamma + \epsilon\alpha$, $\deg(\epsilon) = 1$. □

We will denote the cohomology of the complex

$$C^k = \oplus_{q-p=k} \Omega^{0,q}(X, \Lambda^p T_X)[p - q], \ \ \bar{\partial} + \Delta : C^k \to C^{k+1} \tag{4.19}$$

by H. Of course, H is isomorphic to the graded vector space $\mathbf{H}$, but we prefer to reserve the special notation for the former vector space since it is defined canonically and the vector space $\mathbf{H}$ is only defined up to arbitrary linear isomorphism.

Normalized solution to Maurer-Cartan equation.

Let F be the Hodge filtration which we would like to view as the following filtration $F^{\geq n} \subset F^{\geq n - \frac{1}{2}} \subset \dots \subset F^{\geq 0}$ on the direct sum of all cohomology groups:[10]

$$F^{\geq r} := \oplus_{p-q \geq 2r-n} H^{(p,q)} \tag{4.20}$$

$$H^{(p,q)} := \{[\phi] \in H^{p+q}(X, \mathbb{C})[-p - q] | \phi \in \Omega^{p,q}\}, \ r \in \frac{1}{2}\mathbb{Z}$$

[10] The reason for such a choice of indexes on the subspaces of the filtration will become clear from what follows

Let

$$W_{\leq 0} \subset W_{\leq \frac{1}{2}} \subset \ldots \subset W_{\leq n}$$

be an increasing filtration which is complementary to the Hodge filtration in the following sense

$$\forall\ r \quad H^*(X,\mathbb{C}) = F^{\geq r} \oplus W_{\leq r-\frac{1}{2}} \tag{4.21}$$

Here a filtration on $H^*(X,\mathbb{C})$ is by definition a filtration in the category of $\mathbb{Z}-$graded vector spaces, i.e. it is the same as a set of filtrations on every graded component of $H^*(X,\mathbb{C})$. As a side remark let us notice that the whole construction works analogously if W is understood as a filtration in the category of $\mathbb{Z}_2-$graded vector spaces.

Let us assume that the choice of the holomorphic volume form Ω_{X_0} corresponding to the complex structure X_0 is fixed.

Theorem 1. *There exists fibered versal solution $\widetilde{\gamma}_\hbar^W$ to Maurer-Cartan equation in $\mathfrak{g}_\hbar$ which depends formally on $\hbar$*

$$\bar{\partial}\widetilde{\gamma}_\hbar^W(t) + \hbar\Delta\widetilde{\gamma}_\hbar^W(t) + \frac{1}{2}[\widetilde{\gamma}_\hbar^W(t), \widetilde{\gamma}_\hbar^W(t)] = 0$$

such that [11]

$$\begin{gathered}\left[exp\,(l_\hbar\widetilde{\gamma}_\hbar^W) - 1\right] \vdash \Omega_{X_0} \in \oplus_r W_{\leq r}\hbar^{s(r)}\mathbb{C}[[\hbar^{-1}]]\widehat{\otimes}\mathbb{C}[[t_{\boldsymbol{H}}]] \\ where\ \ s(r) = -r + n - 1\end{gathered} \tag{4.22}$$

Proof. The condition (4.21) implies that one has the direct sum decomposition

$$H^*(X,\mathbb{C}) = \oplus_r F^{\geq r} \cap W_{\leq r} \tag{4.23}$$

Let $\{\Delta_a\}$ be a basis in the vector space H of the cohomology of complex (4.19) compatible with the direct sum decomposition (4.23) so that

$$\Delta_a \vdash \Omega_{X_0} \in F^{\geq r_a} \cap W_{\leq r_a} \tag{4.24}$$

The cohomology of the complex (4.19) are isomorphic to the graded vector space $\mathbf{H}$ introduced in §3.2. Let $\{t^a\}$ be a set of linear coordinates on $\mathbf{H}$ which form the basis dual to $\{\Delta_a\}$. Let

$$\widetilde{\gamma}_\hbar = \sum_a (\widetilde{\gamma}_\hbar)_a t^a + \frac{1}{2}\sum_{a_1 a_2}(\widetilde{\gamma}_\hbar)_{a_1 a_2} t^{a_1} t^{a_2} + \ldots$$

denotes a fibered versal solution to Maurer-Cartan equation in $\mathfrak{g}_\hbar$. The linear in $t_{\mathbf{H}}$ terms $(\widetilde{\gamma}_\hbar)_a$ are in $\mathrm{Ker}\,\bar{\partial} + \hbar\Delta$. One can assume without loss of generality that $[l_\hbar(\widetilde{\gamma}_\hbar)_a] = \hbar^{s(r_a)}\Delta_a$. This can be achieved by a substitution

$$t'(\hbar) = a(\hbar)t,\ a(\hbar) \in GL(\mathbf{H})[[\hbar]]$$

[11] A geometric interpretation of this condition will be given in §5.6

which is linear in $t_{\mathbf{H}}$. Let us consider now the effect of an arbitrary substitution depending formally on $\hbar$ with linear term equal to identity

$$(t')^a = t^a_{(0)} + \sum_{k\geq 1} t^a_{(k)}\hbar^k,\ \ t^a_{(0)} = t^a$$

Notice that if one puts

$$\left[\exp\left(l_\hbar \widetilde{\gamma}^W_\hbar\right) - 1\right] = \sum_a \Delta^a \hbar^{s(r_a)} \sum_{k=-\infty}^{k=+\infty} \Phi^a_{(k)}\hbar^k$$

then

$$\Phi^a_{(k)} = t^a_{(k)} + O(t^2),\ \ \text{for}\, k \geq 0 \tag{4.25}$$

The condition (4.22) is equivalent to the system of equations

$$\Phi^a_{(k)} = 0,,\ \ \text{for}\, k > 0 \tag{4.26}$$

We see from expression (4.25) that this system has unique solution in formal power series in $t^a_{(0)}$:

$$t^a_{(k)} = f^a_{(k)}(t^a_{(0)}),\ \ k > 0$$

The power series

$$\widetilde{\gamma}^W_\hbar(t) = \widetilde{\gamma}_\hbar(t + \sum_{k=1}^{\infty} f_{(k)}(t)\hbar^k, \hbar) \tag{4.27}$$

satisfies all the conditions of the theorem. □

Proposition 4.2.3. *The formal power series map*

$$\Phi^W = \left[exp\left(l_\hbar \widetilde{\gamma}^W_\hbar\right) - 1\right] \vdash \Omega_{X_0} \tag{4.28}$$
$$\Phi^W : \mathcal{M} \to \oplus_r W_{\leq r}\hbar^{s(r)}\mathbb{C}[[\hbar^{-1}]]$$

does not depend on the choice of the solution $\widetilde{\gamma}^W_\hbar$ satisfying the condition of the theorem 1 and depends only on the choice of the filtration W and the value of Ω_{X_0}.

□

The map Π^W.

Proposition 4.2.4. *The power series*

$$\Pi^W = \left[exp\left(l_\hbar \widetilde{\gamma}^W_\hbar\right)|_{\hbar=1} \vdash \Omega_{X_0}\right] \in \oplus_k H^k(X,\mathbb{C})[n-k]\widehat{\otimes}\hat{\mathcal{O}}_{\mathcal{M}} \tag{4.29}$$

is well-defined and induces isomorphism of germs of (formal) graded manifolds

$$(\mathcal{M}, [X_0]) \to (\oplus_k H^k(X,\mathbb{C})[n-k], \Omega_{X_0}) \tag{4.30}$$

Proof. If one puts

$$\big[\exp\big(l_\hbar \widetilde{\gamma}^W_\hbar\big)\big] = 1 + \sum_a \Delta^a \hbar^{s(r_a)} (\Phi^W)^a_{(0)} + (\Phi^W)^a_{(-1)} \hbar^{-1} + \dots \tag{4.31}$$

then $(\Phi^W)^a_{(-i)}(t) \in \mathfrak{M}^{i+1}_{\mathbb{C}[[t_{\mathbf{H}}]]}$, $(\Phi^W)^a_{(0)} = t^a + O(t^2)$ □

Remark 4.2.5. *The proposition 4.2.3 implies that the map $\Pi^W : \mathcal{M} \to H^*(X,\mathbb{C})[n]$ defined by the series (4.29) depends only on the choice of the filtration W and the value of Ω_{X_0}.*

Proposition 4.2.6. *The restriction of the generalized periods map $\Pi^W(t)|_{\mathcal{M}^{classical}}$ coincides with the classical periods $[\Omega^L_{[X_t]}] \in H^n(X,\mathbb{C})[n]$ normalized according to (4.8) using the hyperplane $L = W_{\leq n-1} \cap H^n(X,\mathbb{C})$.*

Proof. Let (ρ, f_ρ) be a solution to Maurer-Cartan equation in $\mathfrak{g}_{\hbar=1}$ corresponding to the deformations $(X_\rho, \Omega^L_{X_\rho})$ of the pair which consists of complex structure and holomorphic n-form. The $n-$form $\Omega^L_{X_\rho}$ is assumed to be normalized so that $\Omega^L_{X_\rho} - \Omega_{X_0} \in L$. Notice that $\gamma_\hbar = \rho + \hbar f_\rho$ is the solution to the Maurer-Cartan equation in $\mathfrak{g}_\hbar$ and

$$\big[\exp(l_\hbar \gamma_\hbar) - 1\big] \vdash \Omega_{X_0} = \big[\exp(\rho + f_\rho) - 1\big] \vdash \Omega_{X_0} \in W_{\leq n-1}\hbar^0$$

Therefore it follows from (4.25) and (4.26) that one can assume without loss of generality that

$$\widetilde{\gamma}^W(t,\hbar)|_{t\in\mathcal{M}^{classical}} = \rho(t) + \hbar f_{\rho(t)} \tag{4.32}$$

Hence $\Pi^W(t) = [\Omega^L_{X_t}]$, for $t \in \mathcal{M}^{classical}$. □

The map $\mathrm{Gr}\,\Phi^W$.

Let us consider the map $\mathrm{Gr}\,\Phi^W : \mathcal{M} \to \mathrm{Gr}\,W$ defined using the expansion (4.31) as

$$\mathrm{Gr}\,\Phi^W(t) = \sum_a (\Phi^W)^a_{(0)} \Delta_a \vdash \Omega_{X_0} \tag{4.33}$$

It follows from the equation (4.25) that it is a local isomorphism as well. In applications to mirror symmetry this map will be used in order to provide a natural set of coordinates on the moduli space $\mathcal{M}$. In the sequel it will be convinient to distinguish this set of coordinates associated with the filtration W and denote them by $\{\tau_W\}$. Let us notice that the elements $\Delta_a \vdash \Omega_{X_0} \in F^{\geq r_a} \cap W_{\leq r_a}$ project naturally to a basis in $\mathrm{Gr}\,W$. Let $\{\tau^a_W\}$ denote the corresponding dual basis. The set of coordinates $\{\tau_W\}$ is characterized by the property

$$(\Phi^W)^a_{(0)} = \tau^a_W \tag{4.34}$$

It is easy to calculate the restriction of $\mathrm{Gr}\,\Phi^W$ on $\mathcal{M}^{classical}$. The restriction of the solution $\widetilde{\gamma}_\hbar^W$ to $\mathcal{M}^{classical}$ can be assumed to be of the form (4.32). Hence the map $\mathrm{Gr}\,\Phi^W$ sends the complex structure $[X_\rho]$ to the class

$$\mathrm{Gr}\,\Phi^W|_{\mathcal{M}^{classical}} : [X_\rho] \to [\Omega^L_{X_\rho} - \Omega_{X_0}] \in W_{\leq n-1}/W_{\leq n-\frac{3}{2}} \tag{4.35}$$

Remark 4.2.7. *Let $\Gamma_0 \in H_n(X, \mathbb{C})$ be a homology class orthogonal to the hyperplane $L = W_{\leq n-1} \cap H^n(X, \mathbb{C})$. Then the condition (4.8) may be written as*

$$\int_{\Gamma_0} \Omega^L_{X_\rho} = const \tag{4.36}$$

Let $\{\Gamma^i_1\} \subset H_n(X, \mathbb{C})$ is a set of elements which defines together with Γ_0 a basis of the subspace of $H_n(X, \mathbb{C})$ annihilating $W_{\leq n-\frac{3}{2}}$. The restriction of the map $Gr\,\Phi^W$ on $\mathcal{M}^{classical}$ associated with the base point $\rho = 0$ and the filtration W may be written as the vector

$$Gr\,\Phi^W|_{\mathcal{M}^{classical}} = (\int_{\Gamma^i_1} \Omega^L_{X_\rho} - \int_{\Gamma^i_1} \Omega_{X_0}) \tag{4.37}$$

It follows from the condition (4.21) that

$$dim\ (W_{\leq n-1} \cap H^n)/(W_{\leq n-\frac{3}{2}} \cap H^n) = dim\ H^{n-1,1} \tag{4.38}$$

Therefore the map (4.35) defines a set of coordinates on $\mathcal{M}^{classical}$.

5 Generalized variations of Hodge structures and its invariants

We have seen in the previous chapter that one can define a generalization of the classical periods map on the extended moduli space $\mathcal{M}$. We investigate the properties of this map in this chapter. Namely we show that the moduli space $\mathcal{M}$ is the base of generalized variations of Hodge structures on $H^*(X, \mathbb{C})$. We introduce the invariants of this generalized VHS in §5.8 This is a collection of polylinear S_k-symmetric maps of graded vector spaces

$$A^{(k)}_{[X]}:\ \mathcal{H}^{\otimes k}\ \to\ \mathcal{H}, \quad \mathcal{H} = \oplus_{p,q} H^q(X, \Lambda^p T_X)[p-q] \tag{5.1}$$

associated with filtration W on $H^*(X, \mathbb{C})$ complementary to the Hodge filtration. It is shown in §5.13 that if W is isotropic with respect to the Poincare pairing then these invariants define a structure of the Frobenius manifold on $\mathcal{M}$ (in another terminology they define a solution to Witten-Dijkgraaf-Verlinde-Verlinde equation). We prove in chapter 6 that the rational Gromov-Witten invariants of the projective complete intersections Calabi-Yau varieties coincide with the invariants of generalized VHS which are attached to the mirror varieties.

5.1 Nonlinear limits of families of affine spaces

Let $p : E \to \mathbb{A}^1_k$ be a regular morphism of (formal) varieties over algebraically closed field k, $\operatorname{char} k = 0$. Denote by $F_\hbar$ the fiber lying over the point $\hbar \in \mathbb{A}^1_k$. Let the fibers $F_\hbar, \hbar \neq 0$ are equipped with affine structure[12] such that the corresponding family of affine connections has pole of order one at $\hbar = 0$

$$\nabla(\hbar) : \mathcal{T}_{E/\mathbb{A}^1_k} \to \mathcal{T}_{E/\mathbb{A}^1_k} \otimes \Omega^1_{E/\mathbb{A}^1_k}[F_0] \tag{5.2}$$

In other words the connection $\nabla(\hbar)$ is represented as

$$d + \sum_{p=-1}^{\infty} \Gamma_{ij}^{k\ (p)} \hbar^p, \ \ \Gamma_{ij}^{k\ (-1)} \neq 0 \tag{5.3}$$

for any choice of (formal) trivialization $E \sim F_0 \times \mathbb{A}^1_k$ and set of (formal) coordinates on F_0.

Proposition 5.1.1. *The term $\Gamma_{ij}^{k\ (-1)}$ with the highest order of pole defines canonical structure of commutative associative algebra on TF_0.*

Proof. Note that the expression $\Gamma_{ij}^k(\hbar)$ transforms under a fiberwise change of coordinates $x = f(\tilde{x}, \hbar), f = f_0(\tilde{x}) + f_1(\tilde{x})\hbar + \dots$ into[13]

$$\sum_l -(J^{-1})^k_l \frac{\partial J^l_j}{\partial \tilde{x}^i} + \sum_{m,n,l} (J^{(-1)})^k_m J^l_i J^n_j \Gamma^m_{ln} \tag{5.4}$$

where $J^k_l(\tilde{x}, \hbar) = \partial f^k / \partial \tilde{x}^l$, and $(J^{(-1)})^k_l$ is the inverse matrix to J^i_j. Therefore the term $\Gamma_{ij}^{k\ (-1)}(x)$ having the highest order of pole is a well-defined tensor on F_0. The connection $\nabla(\hbar)$ is torsionless: $T^k_{ij}(\hbar) = \Gamma^k_{ij}(\hbar) - \Gamma^k_{ji}(\hbar) = 0$. Hence,

$$\Gamma_{ij}^{k\ (-1)} = \Gamma_{ji}^{k\ (-1)} \tag{5.5}$$

The connection $\nabla(\hbar)$ is also flat for all $\hbar \neq 0$:

$$d\Gamma(\hbar) + \frac{1}{2}[\Gamma(\hbar), \Gamma(\hbar)] = 0 \tag{5.6}$$

Hence, $[\Gamma^{(-1)}, \Gamma^{(-1)}] = 0$. This can be rewritten as

$$\sum_n \Gamma_{in}^{k\ \ (-1)} \Gamma_{mj}^{n\ (-1)} = \sum_n \Gamma_{mn}^{k\ \ \ (-1)} \Gamma_{ij}^{n\ (-1)} \tag{5.7}$$

The components of the tensor $\Gamma_{ij}^{k\ (-1)}$ are the structure constants of the commutative associative multiplication on (the fibers of) TF_0. ☐

[12]Remind that affine structure in the analytic setting is an atlas having affine transformations as transition functions.

[13]We assume here for simplicity that the fibers are purely even varieties.

5.2 Tangent sheaf $\mathcal{T}_{\mathcal{M}}$.

We describe here the tangent sheaf of $\mathcal{M}$ and recall following [BK] how to construct canonical algebra structure on it.

Consider the space T_γ of first-order deformations (over $\mathbb{Z}-$graded bases) of a given versal solution to the Maurer-Cartan equation $\gamma(t) \in (\mathfrak{g}\,\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]])^1$. The $\mathbb{C}[[t_{\mathbf{H}}]]-$module T_γ is identified naturally with the cohomology of the complex $(\mathfrak{g}\,\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]][1], \bar{\partial} + [\gamma(t),\,\cdot\,])$. It is a $\mathbb{Z}-$graded module over the algebra of functions on the moduli space. In geometrical language it corresponds to the tangent sheaf of $\mathcal{M}$.

The operator $\bar{\partial}$ as well as the operators ad_γ, $\gamma \in \mathfrak{g}$ are differentiations with respect to the natural algebra structure on $\mathfrak{g}[1]$ defined by the wedge product. Hence the $\mathbb{C}[[t_{\mathbf{H}}]]-$linear extension of the wedge product defines the natural $\mathcal{O}_{\mathcal{M}}-$linear algebra structure on T_γ. This algebra structure is functorial with respect to the isomorphisms $T_{\gamma_1} \sim T_{\gamma_2}$ induced by the gauge equivalences $\gamma_1 \sim \gamma_2$

The 3-tensor of structure constants of the multiplication on T_γ can be written explicitly as follows. Recall that we have fixed for convinience a choice $\{t^\alpha\}$ of linear coordinates on $\mathbf{H}$. The (uni)versality property of $\gamma(t)$ implies that the classes of partial derivatives $[\partial_\alpha\gamma]$ generate freely the $\mathbb{C}[[t_{\mathbf{H}}]]-$module T_γ.

Proposition 5.2.1. *The equation*

$$\partial_\alpha\gamma(t) \wedge \partial_\beta\gamma(t) = \sum_\delta A^\delta_{\alpha\beta}(t)\,\partial_\delta\gamma(t) \quad mod\ Im\ \bar{\partial}_{\gamma(t)} \tag{5.8}$$

uniquely determines the formal power series 3-tensor $A^\delta_{\alpha\beta}(t) \in \mathbb{C}[[t_{\boldsymbol{H}}]]$. The components of the tensor are the structure constants of commutative associative $\hat{\mathcal{O}}_{\mathcal{M}}-$algebra structure.

We will denote the product of two elements $v_1, v_2 \in \mathcal{T}_{\mathcal{M}}$ by $v_1 \circ v_2$.

Recall that the third derivative of the generating function for Gromov-Witten invariants of some projective variety Y defines the commutative algebra structure on the tangent sheaf to $H^*(Y,\mathbb{C})$ considered as a supermanifold. We have found similar structure on $\mathcal{M}$. Although, such a structure alone is relatively noninteresting (for example in many cases one can choose locally some coordinates $\{u_i\}$ on the underlying supermanifold $H^*(Y,\mathbb{C})$ such that the multiplication is given simply by $\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij}\frac{\partial}{\partial u^i}$), it indicates the existence of canonical nonlinear family of affine spaces having the former supermanifold as the central fiber.

5.3 Family of moduli spaces

Our aim now is to demonstarte that the extended moduli space of Calabi-Yau manifolds introduced in chapter 3 is naturally the limiting fiber of the family of affine spaces with the property (5.2). We describe first the canonical family of the moduli spaces $\mathcal{M}_\hbar$.

Extended moduli spaces of complex manifolds with holomorphic volume element

In this subsection the moduli space $\mathcal{M}$ is embedded into a natural $1-$parameter family of moduli spaces $\mathcal{M}_\hbar$, $\mathcal{M}_0 = \mathcal{M}$.

Let us consider a family of sheaves of differential graded Lie algebras $(\underline{\mathfrak{g}}, \hbar\Delta)$. The deformation theory associates to $(\underline{\mathfrak{g}}, \hbar\Delta)$ certain moduli space $\mathcal{M}_\hbar$. It is convinient to describe this moduli space with the help of Dolbeult resolution of $(\underline{\mathfrak{g}}, \hbar\Delta)$. This is the family of differential graded Lie algebras $\mathfrak{g}_\hbar$ introduced in §4.2. Same diagram (3.9) of quasiisomorphisms proves that the deformation functor associated with the differential graded Lie algebra $\mathfrak{g}_\hbar$ for arbitrary given $\hbar$ is equivalent to the functor represented by the algebra $\mathbb{C}[[t_{\mathbf{H}}]]$. This can be reformulated as usual as the existence of a versal solution to the Maurer-Cartan equation $\gamma_\hbar(t) \in (\mathfrak{g}_\hbar \widehat{\otimes} \mathbb{C}[[t_{\mathbf{H}}]])$ whose linear term gives a basis of the graded vector space equal to the cohomology of the complex $(\mathfrak{g}_\hbar, \bar{\partial} + \hbar\Delta)$.

According to the proposition 4.1.1 the subspace of the moduli space $\mathcal{M}_{\hbar=1}$ corresponding to solutions with values in $\Omega^{0,1}(X, T_X) \oplus \Omega^{0,0}(X, \mathcal{O}_X) \subset \mathfrak{g}_{\hbar=1}$ is the moduli space of deformations of pair which consists of complex structure and holomorphic $n-$form.

The total space of the family $\mathcal{M}_\hbar$

The family $\mathcal{M}_\hbar, \hbar \in \mathbb{A}^1_{\mathbb{C}}$ of moduli spaces form the bundle $p : \widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$. The total space of the bundle can be described similarly as the moduli space associated with the differential graded Lie algebra $\widetilde{\mathfrak{g}}$. This is the algebra $\mathfrak{g}$ with one added element D of degree 1 such that $[D, \gamma] := \Delta\gamma$. Again, the arguments involving the use of the diagram (3.9) prove that $\widetilde{\mathfrak{g}}$ is quasi-isomorphic to an an abelian Lie algebra$\widetilde{\mathbf{H}} := \mathbf{H} \oplus \mathbf{D}$, where $\mathbf{D}$ is a one-dimensional vector space in degree 0. Denote by $\hbar$ thecorresponding linear coordinate along $\mathbf{D}$, $\deg \hbar = 0$

Proposition 5.3.1. *The deformation functor associated with $\widetilde{\mathfrak{g}}$ is isomorphic to the functor represented by the algebra $\mathbb{C}[[t_{\widetilde{\boldsymbol{H}}}]]$. Equivalently, there exists a versal solution to Maurer-Cartan equation in $(\widetilde{\mathfrak{g}} \widehat{\otimes} t_{\widetilde{\boldsymbol{H}}} \mathbb{C}[[t_{\widetilde{\boldsymbol{H}}}]])^1$. This solution can be taken to be of the form $\gamma_{\widetilde{\mathcal{M}}} = \widetilde{\gamma}_\hbar + \hbar D$ where*

$$\bar{\partial}\widetilde{\gamma}_\hbar(t) + \hbar\Delta\widetilde{\gamma}_\hbar(t) + \frac{1}{2}[\widetilde{\gamma}_\hbar(t), \widetilde{\gamma}_\hbar(t)] = 0$$

$$\widetilde{\gamma}_\hbar = \sum_a (\widetilde{\gamma}_\hbar)_a t^a + \frac{1}{2!} \sum_{a_1, a_2} (\widetilde{\gamma}_\hbar)_{a_1 a_2} t^{a_1} t^{a_2} + \ldots \tag{5.9}$$

$$\widetilde{\gamma}_\hbar \in (\mathfrak{g} \widehat{\otimes} \mathbb{C}[[t_{\widetilde{\boldsymbol{H}}}]])^1, \;\; (\widetilde{\gamma}_\hbar)_{a_1 \ldots a_k} \in \mathfrak{g} \widehat{\otimes} \mathbb{C}[[\hbar]]$$

Proof. Define the differential graded Lie subalgebra $\widetilde{\mathrm{Ker}} := \mathrm{Ker}[D, \;] \subset \widetilde{\mathfrak{g}}$. It is a direct sum of two differential graded Lie subalgebras $\widetilde{\mathrm{Ker}} = D \oplus \mathrm{Ker}\Delta, \mathrm{Ker}\Delta \subset \mathfrak{g}$. The $\partial\bar{\partial}-$lemma implies again that the natural embedding $\widetilde{\mathrm{Ker}} \subset \widetilde{\mathfrak{g}}$ and the projection $\widetilde{\mathrm{Ker}} \to \widetilde{\mathbf{H}}$ are quasi-isomorphisms. The versal solution can be taken to be $\gamma_0(t) + \hbar D$ where $\gamma_0(t) = \sum_n \frac{1}{n!} f_n(t)$ and $f_n : S^n\mathbf{H} \to \mathrm{Ker}\Delta$ are the components of a quasi-isomorphism $\mathbf{H} \to \mathrm{Ker}\Delta$ homotopy inverse to the natural projection $\mathrm{Ker}\Delta \to \mathbf{H}$. ☐

Remark 5.3.2. *A solution of the form (5.9) is versal iff the cohomology classes* $[(\widetilde{\gamma}_\hbar)_a]$ *freely generate the* $\mathbb{C}[[\hbar]]-$*module of cohomology of the complex* $(\mathfrak{g}\widehat{\otimes}\mathbb{C}[[\hbar]], \bar{\partial} + \hbar\Delta)$

We see that the "fibered versal solutions" introduced in §4.2. describe the total space of the bundle $\widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$. The choice of class of gauge equivalence of such solutions corresponds to the fiberwise choice of coordinates $\widetilde{\mathcal{M}} \to \mathbf{H} \times \mathbb{A}^1_{\mathbb{C}}$.

5.4 Affine structure on $\mathcal{M}_\hbar, \hbar \neq 0$

In this subsection we show that the moduli spaces $\mathcal{M}_\hbar$ for $\hbar \neq 0$ have canonical affine structure. It is induced by the exponential map from §9.

Let

$$\mathfrak{a} = \oplus_k \mathfrak{a}^k, \ \ \mathfrak{a}^k := \oplus_{k=q-p+1}\Omega^{0,q}(X, \Lambda^p T_X) \tag{5.10}$$

denotes the differential $\mathbb{Z}-$graded abelian (i.e. with zero bracket) Lie algebra equipped with the differential $\bar{\partial} + \Delta$. Since $\mathfrak{a}$ is abelian, the deformation functor associated to $\mathfrak{a}$ is representable and the moduli space associated with $\mathfrak{a}$ is canonically isomorphic to (formal neighborhood of zero in) the cohomology of the complex $(\mathfrak{a}, \bar{\partial} + \Delta)[1]$. Therefore it has canonical structure of affine space.

Proposition 5.4.1. *The differential graded Lie algebra* $\mathfrak{g}_\hbar$ *with* $\hbar \neq 0$ *is quasi-isomorphic to the abelian Lie algebra* $\mathfrak{a}$.

Proof. The differential graded Lie algebras $\mathfrak{g}_\hbar$ are all quasi-isomorphic for $\hbar \neq 0$. The quasi-isomorphism is given by the map

$$i_\hbar : \mathfrak{g}_\hbar \to \mathfrak{g}_{\hbar=1}, \ \ i_\hbar : \phi^{p,q} \to \hbar^{p-1}\phi^{p,q}, \ \text{for} \ \phi^{p,q} \in \Omega^{0,q}(X, \Lambda^p T_X) \tag{5.11}$$

Therefore it is enough to consider the differential graded Lie algebra $\mathfrak{g}_{\hbar=1}$. It turns out that one can continue the identity map $\phi_{(1)} : \mathfrak{g}_{\hbar=1} \to \mathfrak{a}$ of complexes of vector spaces to an $L_\infty-$morphism of differential graded Lie algebras.

Define the higher components of the $L_\infty-$map $\phi = \{\phi_{(n)}\} : \mathfrak{g}_{\hbar=1} \to \mathfrak{a}$ as follows

$$\phi_{(2)} : \gamma_1 \wedge \gamma_2 \to \gamma_1 \cdot \gamma_2 \tag{5.12}$$

$$\phi_{(3)} : \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \to \gamma_1 \cdot \gamma_2 \cdot \gamma_3 \tag{5.13}$$

$$\dots \tag{5.14}$$

where the symbol "$\cdot$" denotes the natural wedge product on $\mathfrak{a}[1]$. Let $\gamma_1, \dots, \gamma_n \in \mathfrak{g}_{\hbar=1}$. Consider the Artin algebra $\mathfrak{A} := \otimes_{i=1}^n \{\mathbb{C}[\epsilon_i]/\epsilon_i^2 = 0\}$ where $\mathrm{deg}\epsilon_i + \mathrm{deg}\gamma_i - 1 = 0$. Let us look at the equation (4.18) for $\gamma = \sum_i \gamma_i \epsilon_i$. Considering the coefficients in front of the term $\Pi_i \epsilon_i$ in the equation (4.18) we see that $\{\phi_{(n)}\}$ has the required property (2.6) ☐

Remark 5.4.2. *The equation (4.18) can be directly interpreted as the required compatibility of the map $\{\phi_n\}$ of formal manifolds with the action of vector fields $Q_{\mathfrak{g}_{\hbar=1}}$ and $Q_{\mathfrak{a}}$ (see equation (2.10)). The left hand side is the value of the vector field $Q_{\mathfrak{a}}$ at the point $\phi(\gamma)$ and the right hand side is the vector field $\phi_*(Q_{\mathfrak{g}_{\hbar=1}})$.*

Corollary 5.1. *The moduli spaces $\mathcal{M}_\hbar$, $\hbar \neq 0$ have canonical affine structure. A family of sets of affine coordinates is given by (compare with formula (4.3))*

$$\int_{G_i} e^{l_\hbar \widetilde{\gamma}_\hbar(t)} \vdash \Omega_{X_0} \tag{5.15}$$

where $\{G_i\}$ is a basis in $H_*(X, \mathbb{C})$ and Ω_{X_0} is the holomorphic nowhere vanishing $n-$form on X_0.

Completed tensor products. Let $\mathcal{T}_1, \mathcal{T}_2$ are finetely generated modules over the topological(=projective limit of Artin) algebras $\mathcal{O}_1, \mathcal{O}_2$. We denote as usual by $\mathcal{T}_1 \widehat{\otimes} \mathcal{T}_2$ the completion of the tensor product with respect to the filtration by $\mathcal{O}_1 \otimes \mathcal{O}_2-$submodules $\mathcal{F}_i = \oplus_{i+j=k} \mathfrak{M}^i_{\mathcal{O}_1} \otimes \mathfrak{M}^j_{\mathcal{O}_2}$. $\mathcal{T}_1 \widehat{\otimes} \mathcal{T}_2$ is naturally a $\mathcal{O}_1 \widehat{\otimes} \mathcal{O}_2-$module. Let $\mathcal{N}_\hbar$ be a finitely generated $\mathbb{C}(\hbar)-$module (or $\mathbb{C}((\hbar))-$module) and $\mathcal{T}$ be a finetely generated $\mathcal{O}-$module where $\mathcal{O}$ is a pro-Artin algebra with the maximal ideal $\mathfrak{M}_{\mathcal{O}}$. The localization at $\hbar = 0$ of the $\mathbb{C}[[\hbar]] \widehat{\otimes} \mathcal{O}-$module $\mathcal{N}_\hbar \widehat{\otimes} \mathcal{T}$ consists of the elements of the form

$$\sum_{k=1}^{k=N} \hbar^{-k} \Delta_k, \ \ \Delta_k \in \mathcal{N}_\hbar \widehat{\otimes} \mathcal{T}$$

We denote via $\mathcal{N}_\hbar \widehat{\otimes}_{(0)} \mathcal{O}$ the minimal completion of the localization at $\hbar = 0$ of $\mathcal{N}_\hbar \widehat{\otimes} \mathcal{T}$ which contains all the elements of the form

$$\sum_{k=1}^{k=+\infty} \hbar^{-k} f_k(t)\, \Delta, \ \ f_k(t) \in \mathfrak{M}^k_{\mathcal{O}}, \Delta \in \mathcal{N}_\hbar \widehat{\otimes} \mathcal{T}$$

In particular $\mathcal{N}_\hbar \widehat{\otimes}_{(0)} \mathcal{T}$ is a $\mathbb{C}[[\hbar]] \widehat{\otimes} \mathcal{O}-$module.

Remark 5.4.3. *The affine structure on $\mathcal{M}_\hbar$ defines the canonical affine map $\mathcal{M}_\hbar \to T_{\pi_\hbar}$. It is distinguished from other affine maps by the condition that it induces an identity map on $T_{\pi_\hbar}$. In terms of a versal fibered solution $\widetilde{\gamma}_\hbar$ it can be written as*

$$\Phi^H(t, \hbar) = \left[\hbar exp\left(\frac{1}{\hbar}\widetilde{\gamma}_\hbar\right) - 1\right] \in H_\hbar \widehat{\otimes}_{(0)} \mathbb{C}[[t_{\boldsymbol{H}}]] \tag{5.16}$$

where $H_\hbar$ is the graded $\mathbb{C}((\hbar))-$module equal to the cohomology of the complex

$$C^*_\hbar = \oplus_k C^k_\hbar, \ \ C^k_\hbar = \oplus_{q-p=k} \Omega^{0,q}(X, \Lambda^p T_X) \widehat{\otimes} \mathbb{C}(\hbar), \ \ \bar{\partial} + \hbar\Delta : C^k_\hbar \to C^{k+1}_\hbar \tag{5.17}$$

Singularity at $\hbar = 0$ of the family of affine structures on $\mathcal{M}_\hbar$

In this subsection we show that the family of affine structures on $\mathcal{M}_\hbar$ have the singularity (5.2) at $\hbar = 0$.

Proposition 5.4.4. *The family of affine structures on $\mathcal{M}_\hbar, \hbar \neq 0$ has pole of order one at $\hbar = 0$*

Proof. A choice of (a class of gauge equivalence of) versal fibered solution to Maurer-Cartan equation $\widetilde{\gamma}_\hbar \in (\mathfrak{g}\widehat{\otimes} t_{\mathbf{H}}\mathbb{C}[[t_{\widehat{\mathbf{H}}}]])^1$ corresponds geometrically to a fiberwise choice of coordinates $\rho : \widetilde{\mathcal{M}} \to \mathbf{H} \times \mathbb{A}^1_{\mathbb{C}}$. The affine structure on the fiber $\mathcal{M}_\hbar, \hbar \neq 0$ is induced via period mapping to the cohomology of the complex (4.19)

$$(t, \hbar) \ \to \big[\exp(i_\hbar(\widetilde{\gamma}_\hbar))\big] \in H \otimes (\mathbb{C}(\hbar)\widehat{\otimes}_{(0)}\mathbb{C}[[t_{\mathbf{H}}]]) \tag{5.18}$$

The relative tangent sheaf of the total space of the family $p : \widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$ is identified naturally with the cohomology of the complex $(\mathfrak{g}\widehat{\otimes}\mathbb{C}[[t_{\widetilde{\mathbf{H}}}]][1], \bar{\partial} + \hbar\Delta + [\widetilde{\gamma}_\hbar, \cdot])$. Denote this $\mathbb{C}[[t_{\widetilde{\mathbf{H}}}]]-$module by $\mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}}$. The Jacobian of map (5.18) is equal to

$$\delta\gamma \in \mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}} \ \to \big[i_\hbar(\delta\gamma) \wedge \exp(i_\hbar\widetilde{\gamma}_\hbar)\big] \in H \otimes (\mathbb{C}(\hbar)\widehat{\otimes}_{(0)}\mathbb{C}[[t_{\mathbf{H}}]]) \tag{5.19}$$

Let us calculate the affine connection

$$\Gamma^k_{ij}(\hbar) = \sum_\alpha (\frac{\partial a^{-1}}{\partial t})^k_\alpha \partial_i \partial_j a^\alpha \tag{5.20}$$

where $\{a^\alpha\}$ is a set of linear coordinates on H. Let $\partial_i\widetilde{\gamma}(t, \hbar),\ \partial_j\widetilde{\gamma}(t, \hbar) \in \mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}}$ are the cohomology classes corresponding to the coordinate vector fields $\partial_i,\ \partial_j$ tangent to the fibers. Differentiating twice the expression (5.18) and taking the inverse to the map (5.19) we see that

$$\nabla(\hbar)_{\partial_i}\partial_j = \big[\frac{1}{\hbar}\partial_i\widetilde{\gamma}_\hbar \wedge \partial_j\widetilde{\gamma}_\hbar + \partial_i\partial_j\widetilde{\gamma}_\hbar\big] \in \mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}} \otimes \mathbb{C}[\hbar^{-1}] \tag{5.21}$$

Family of vector fields $[\partial_i\widetilde{\gamma}_\hbar(t, \hbar) \wedge \partial_j\widetilde{\gamma}_\hbar(t, \hbar) + \hbar\partial_i\partial_j\widetilde{\gamma}_\hbar(t, \hbar)]$ is regular at $\hbar = 0$ and does not vanish at $\hbar = 0$. Therefore the order of pole at $\hbar = 0$ of the connection $\Gamma^k_{ij}(\hbar)$ is equal to one. □

Proposition 5.4.5. *The algebra structure defined on $T_{\mathcal{M}}$ by the "residue" of the family of affine connections on $\mathcal{M}_\hbar$ coincides with the algebra structure defined in §5.2*

Proof. It follows from the equations (5.21) and (5.8) □

Conjecturally the family of moduli spaces $\mathcal{M}_\hbar$ should be possible to describe entirely in terms of the $A_\infty-$ category $D^bCoh(X)$. The fibers $\mathcal{M}_\hbar$ for $\hbar \neq 0$ equipped with their affine structure should be identified with open subsets in periodic cyclic cohomology of $D^bCoh(X)$. The cyclic cohomology corresponds to infinitesimal deformations of a section of $\widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$.

5.5 Flags and flat connections over $\mathbb{A}^1_{\mathbb{C}}$

In this subsection we recall the correspondence due to Rees (see also [S]) between filtrations on a $\mathbb{C}-$vector space and $\mathbb{G}_m-$equivariant vector bundles on $\mathbb{A}^1_{\mathbb{C}}$. (Recall that $\mathbb{G}_m$ denotes here the algebraic group whose set of $\mathbb{C}-$points coincide with $\mathbb{C} \setminus 0$)

Proposition 5.5.1. *The category of finite-dimensional $\mathbb{C}-$vector spaces equipped with decreasing filtration $(F^{\geq k})_{k\in\mathbb{Z}}$ is canonically equivalent to the category of coherent locally free $\mathbb{G}_m-$equivariant sheaves over $\mathbb{A}^1_{\mathbb{C}}$.*

Proof. Let $\mathcal{V} \to \mathbb{A}^1_{\mathbb{C}}$ be a coherent locally free sheaf equipped with an action of $\mathbb{G}_m$ which covers the natural action of $\mathbb{G}_m$ on the affine line $\mathbb{A}^1_{\mathbb{C}}$. In particular $\mathbb{G}_m$ acts on the fiber $\mathcal{V}_0$. This action defines canonical grading $\mathcal{V}_0 = \oplus_{i\in\mathbb{Z}}\mathcal{V}_0^i$ where $\mathbb{G}_m$ acts on the graded component $\mathcal{V}_0^i$ via

$$v \to \lambda^i v \ \text{ for } \ \lambda \in \mathbb{C}^*,\, v \in \mathcal{V}_0^i \tag{5.22}$$

Similarly, any fiber $\mathcal{V}_{\hbar_0}$, $\hbar_0 \neq 0$ receives canonical decreasing filtration

$$F^{\geq r} \subset F^{\geq r-1} \cdots \subset F^{\geq s} = \mathcal{V}_{\hbar_0},\, r, s \in \mathbb{Z} \tag{5.23}$$

This is the filtration by the orders of growth at $0 \in \mathbb{A}^1_{\mathbb{C}}$ of $\mathbb{G}_m-$invariant sections. Given a choice of isomorphism of $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}-$modules $\mathcal{V} \simeq \mathcal{V}_0 \otimes \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}$ a $\mathbb{G}_m-$invariant section $v(\hbar)$ can be written

$$v(\hbar) = \hbar^{-p}v^{(-p)} + \hbar^{-p+1}v^{(-p+1)} + \ldots,\ v^{(i)} \in \mathcal{V}_0 \tag{5.24}$$

where the order of growth $\hbar^{-p}$ and the "residue" $v^{(-p)} \in \mathcal{V}_0^{-p}$ does not depend on the choice of the isomorphism $\mathcal{V} \simeq \mathcal{V}_0 \otimes \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}$. Put $v(\hbar_0) \in F^{-p} \subset \mathcal{V}_{\hbar_0}$ if the order at $\hbar = 0$ of the $\mathbb{G}_m-$invariant section $v(\hbar)$ is less or equal to $\hbar^{-p}$. The action of $\mathbb{G}_m$ identifies all the fibers $\mathcal{V}_\hbar$ for $\hbar \neq 0$. This identification respects the filtrations. The corresponding associated graded quotent is canonically isomorphic to $\mathcal{V}_0$.

Conversely, given a vector space $\mathcal{V}_1$ equipped with filtration $(F^{\geq k})_{k\in\mathbb{Z}}$ define $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}-$module to be the $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}-$submodule of $\mathcal{V}_1 \otimes \mathbb{C}[\hbar^{-1}, \hbar]$generated by the elements of the form $\hbar^p v$ where $v \in F^{-p},\ p \in \mathbb{Z}$. This is a free finitely generated $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}-$module. It has a natural $\mathbb{G}_m-$action

$$\hbar^i v \to \lambda^i \hbar^i v,\ \text{for}\, \lambda \in \mathbb{C}^* \tag{5.25}$$

□

More generally, a vector space equipped with filtration having arbitrary rational indices $\alpha_i \in \mathbb{Q}$ corresponds to a vector bundle over $\mathbb{A}^1_{\mathbb{C}}$ endowed with a connection over $\mathbb{A}^1_{\mathbb{C}} \setminus 0$ with regular singularities at zero and at infinity, which has monodromy of finite order.

A similar correspondence exists between increasing filtrations and local systems over $\mathbb{CP}^1 \setminus 0$ having a pole of the first order at infinity. The subspace having an index α corresponds to the $\mathbb{G}_m-$invariant sections of order $\leq \hbar^\alpha$ at infinity.

For projective complex manifold X^n one has the associated Hodge filtration (4.20). It corresponds to vector bundle of graded vector spaces $\mathcal{L}^{Hodge}$ over $\mathbb{A}^1_{\mathbb{C}}$ equipped with connection having regular singularity at $\hbar = 0$. The fiber $\mathcal{L}^{Hodge}_\hbar$ over $\hbar$ is the cohomology of the operator $\bar{\partial} + \hbar\partial$ acting on the graded vector space of differential forms. The flat multivalued sections of $\mathcal{L}^{Hodge}$ are

$$[\phi^{r,q}\hbar^{\frac{r-q+n}{2}}],\ \text{for}\, \phi^{r,q} \in \Omega^{r,q}(X) \tag{5.26}$$

5.6 The bundle $\widehat{\mathcal{M}^W} \to \mathbb{CP}^1$ and the map Φ^W.

In this subsection we give a geometric interpretation for the results concerning the generalized period map from §4.

An increasing filtration W on $\oplus_i H^i(X, \mathbb{C})[-i]$ introduced in §4.2 defines an extension of $\mathcal{L}^{Hodge}$ to a vector bundle over $\mathbb{CP}^1$ as we saw it in §5.5.

Proposition 5.6.1. *A filtration W complementary to the Hodge filtration in the sense of eq. (4.21) defines a trivial sheaf over $\mathbb{CP}^1$.*

Proof. It follows from (4.21) that $\oplus_i H^i(X, \mathbb{C})[-i] = \oplus_s (F^{\geq s} \cap W_{\leq s})$ The trivialization of the extension of the sheaf $\mathcal{L}^{Hodge}$ is given by the sections $\hbar^{-s} v(\hbar)$ where $v(\hbar) \in F^{\geq s} \cap W_{\leq s}$ is a flat (multivalued) section. ☐

The extension of the bundle $\widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$.

We explain here that the filtration W defines canonical extension $\widehat{\mathcal{M}^W} \to \mathbb{CP}^1$ of the bundle $\widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$.

Let $\pi_\hbar$ denotes the base point of the formal moduli space $\mathcal{M}_\hbar$. The $\mathbb{Z}-$graded tangent space $T_{\pi_\hbar}\mathcal{M}_\hbar$ at the base point is identified naturally with the cohomology of the complex $(\mathfrak{g}, \bar{\partial} + \hbar\Delta)[1]$. A choice of nowhere vanishing holomorphic $n-$form induces an isomorphism of $\mathbb{Z}-$graded vector spaces $T_{\pi_\hbar}\mathcal{M}_\hbar \simeq \mathcal{L}^{Hodge}_\hbar[n]$. The isomorphism is defined canonically up to a multiplication by nonzero constant. Therefore we get the structure of a local system with regular singularity at $\hbar = 0$ on the bundle $T_{\pi_\hbar} \to \mathbb{A}^1_{\mathbb{C}}$. Let us denote by $\mathcal{D}^0_{\partial/\partial\hbar}$ the corresponding connection. The flat multivalued sections of the local system can be written as

$$[\sum_{p,q} \hbar^{-\frac{p+q}{2}+n}\gamma^{p,q}],\ \text{for}\ \sum_{p,q}\gamma^{p,q} \in \operatorname{Ker}\bar{\partial} + \Delta,\ \gamma^{p,q} \in \Omega^{0,q}(X, \Lambda^p T_X) \tag{5.27}$$

The filtration F maps to the filtration

$$(F^H)^{\geq r} = \{[\gamma] | \gamma \in \oplus_{p+q \leq 2(n-r)}\Omega^{0,q}(X, \Lambda^p T_X)\}$$

on the graded vector space $H = T_{\pi_1}\mathcal{M}_1$. The filtration W maps similarly to a filtration W^H and defines an extension $\widehat{T^W_{\pi_\hbar}} \to \mathbb{CP}^1$ of the bundle $T_{\pi_\hbar} \to \mathbb{A}^1_{\mathbb{C}}$. The affine structure on the fibers $\mathcal{M}_\hbar$, $\hbar \neq 0$ gives an identification of $\mathcal{M}_\hbar$, for $\hbar \neq 0$ with formal neighborhood of zero in $T_{\pi_\hbar}\mathcal{M}_\hbar$. Therefore the filtration W induces extension $\widehat{\mathcal{M}^W} \to \mathbb{CP}^1$ of the bundle of formal manifolds $\widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$. Notice that the family of affine structures on $\mathcal{M}_\hbar$ continues analytically at $\hbar = \infty \in \mathbb{CP}^1$.

Suppose that we are given a complex analytic bundle over $\mathbb{CP}^1$ and a section with analytically trivial normal bundle. Then it is easy to see that the formal neighborhood of this section is trivial as a nonlinear bundle and the trivialization is unique. It follows from the fact that $H^1(\mathbb{CP}^1, G)$=0 where G is the projective limit of nilpotent algebraic groups of jets of formal diffeomorphisms of a superspace with the first jet the same as of the identity diffeomorphism. The fact about H^1 follows from $H^1(\mathbb{CP}^1, \mathcal{O}) = H^2(\mathbb{CP}^1, \mathcal{O}) = 0$.

The theorem 1 and the proposition 4.2.3 describe the analog of this phenomena in the category of formal bundles over $\mathbb{CP}^1$. We saw above that a choice of fibered versal solution $\widetilde{\gamma}_\hbar$ corresponds to a fiberwise choice of coordinates on $\widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$ and the formal power series $\exp(l_\hbar \widetilde{\gamma}_\hbar)$ represents the map to the affine space associated with the affine structure on the fibers $\mathcal{M}_\hbar$. The theorem 1 is interpreted then as the fact that the nonlinear formal bundle $\widehat{\mathcal{M}^W}$ is trivial.

5.7 Canonical connection on $\widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$ and symmetry vector field.

In this subsection we will see that there exists a natural symmetry vector field acting on all the structures on $\mathcal{M}$. It is given by the "residue" of the canonical connection on the family $\mathcal{M}_\hbar$.

Let us consider the following connection on $T_{\pi_\hbar}$:

$$\mathcal{D}^T_{\partial/\partial\hbar} = \mathcal{D}^0_{\partial/\partial\hbar} + \frac{(n-1)}{\hbar} \tag{5.28}$$

The flat multivalued sections of this local system are

$$[\sum_{p,q} \hbar^{-\frac{p+q}{2}+1}\gamma^{p,q}], \text{ for } \sum_{p,q}\gamma^{p,q} \in \operatorname{Ker}\bar{\partial} + \Delta, \ \gamma^{p,q} \in \Omega^{0,q}(X, \Lambda^p T_X) \tag{5.29}$$

This local system corresponds to the same filtration F^H having all the indexes shifted by $(1-n)$:

$$(F^H)^{\geq 1} \subset (F^H)^{\geq \frac{1}{2}} \subset \cdots \subset (F^H)^{\geq 1-n} \tag{5.30}$$

Let $\mathcal{D}_{\partial/\partial\hbar}$ denotes the connection on the bundle $p : \widetilde{\mathcal{M}} \to \mathbb{A}^1_{\mathbb{C}}$ induced via the canonical map $\mathcal{M}_\hbar \to T_{\pi_\hbar}$ from the connection $\mathcal{D}^T_{\partial/\partial\hbar}$. In other words by definition the covariant derivative along $\partial/\partial\hbar$ is the following formal series

$$\mathcal{D}_{\partial/\partial\hbar} := \frac{\partial}{\partial\hbar} - \sum_{a,b} \frac{\partial(\Phi^H)^a}{\partial\hbar}\Big(\frac{\partial\Phi^{-1}}{\partial\tau}\Big)^b_a \frac{\partial}{\tau^b} \tag{5.31}$$

where $\Phi^H(\tau,\hbar) = [l_\hbar(\hbar\exp(\frac{1}{\hbar}\widetilde{\gamma}_\hbar) - \hbar)] = [\exp(l_\hbar\widetilde{\gamma}_\hbar) - 1]$ is the map $\mathcal{M}_\hbar \to T_{\pi_\hbar}$ written in the coordinates corresponding to a fibered versal solution $\widetilde{\gamma}_\hbar$ and the frame on $T_{\pi_\hbar}$ which is covariantly constant with respect to $\mathcal{D}^T_{\partial/\partial\hbar}$.

Proposition 5.7.1. *The connection $\mathcal{D}_{\partial/\partial\hbar}$ has regular singularity at $\hbar = 0$, in other words*

$$\mathcal{D}_{\partial/\partial\hbar} = \frac{\partial}{\partial\hbar} + \hbar^{-1}E_{-1}(\tau)^\beta\frac{\partial}{\tau^\beta} + E_0^\beta(\tau)\frac{\partial}{\tau^\beta} + \hbar E_1 + \dots \tag{5.32}$$

Proof. The proof is parallel to the proof of the prop. 5.4.4. The tangent space map induced by Φ is

$$\frac{\partial\Phi}{\partial\tau} : \delta\widetilde{\gamma}_\hbar \to l_\hbar\delta\widetilde{\gamma}_\hbar\, e^{l_\hbar\widetilde{\gamma}_\hbar} \tag{5.33}$$

The image of $\partial\Phi/\partial\hbar$ under the inverse map is

$$\frac{\partial\Phi}{\partial\tau}^{-1}\frac{\partial\Phi}{\partial\hbar} = l_\hbar^{-1}\partial_\hbar(l_\hbar)\widetilde{\gamma}_\hbar + \partial_\hbar\widetilde{\gamma}_\hbar \in \mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}} \otimes \mathbb{C}[\hbar^{-1}] \tag{5.34}$$

Notice that

$$l_\hbar^{-1}\partial_\hbar(l_\hbar):\ \ [\sum_{p,q}\gamma^{p,q}] \to \sum_{p,q}\frac{p+q-2}{2\hbar}[\gamma^{p,q}] \tag{5.35}$$

Therefore the connection $\mathcal{D}_{\partial/\partial\hbar}$ takes the form (5.32) with

$$E_{-1}(\tau) = [\sum_{p,q}\frac{2-p-q}{2}\widetilde{\gamma}_\hbar^{p,q}|_{\hbar=0}(\tau)] \in \mathcal{T}_\mathcal{M} \tag{5.36}$$

where $\mathcal{T}_\mathcal{M}$ is identified with the cohomology of the complex $(\mathfrak{g}\widehat{\otimes}\mathbb{C}[[\tau_W]][1], \bar\partial + [\widetilde{\gamma}|_{\hbar=0},\cdot])$. □

It follows easily from the transformation law of the connection $\mathcal{D}_{\partial/\partial\hbar}$ under the fiberwise coordinate changes that the "residue" part $E_{-1}(\tau)$ is a well defined vector field on $\mathcal{M}$. In the sequel we will often denote this vector field simply by $E(\tau)$.

Proposition 5.7.2. *The vector field $E(\tau)$ acts as a conformal symmetry on the algebra structure on $\mathcal{T}_\mathcal{M}$: $Lie_E(\circ) = \circ$, that is, for any vector fields $u, v \in \mathcal{T}_\mathcal{M}$*

$$[E, u\circ v] - [E,u]\circ v - u\circ[E,v] = u\circ v \tag{5.37}$$

Proof. The connection $\mathcal{D}_{\partial/\partial\hbar}$ respects the affine structure on the fibers $\mathcal{M}_\hbar$. The compatibility of $\mathcal{D}_{\partial/\partial\hbar}$ with the affine structure can be written as

$$\mathcal{D}_{\partial/\partial\hbar}(\nabla_{u_\hbar}(v_\hbar)) = \nabla_{\mathcal{D}_{\partial/\partial\hbar}(u_\hbar)}(v_\hbar) + \nabla_{u_\hbar}(\mathcal{D}_{\partial/\partial\hbar}(v_\hbar)) \tag{5.38}$$

where $u_\hbar = u + u_{(1)}\hbar + \dots,\quad v_\hbar = v + v_{(1)}\hbar + \dots\quad \in \Gamma(\mathbb{A}^1_\mathbb{C}, p_*\mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_\mathbb{C}})$. Taking the terms of order $\hbar^{-2}$ at both sides of the formula (5.38) one obtains the equation (5.37). □

Proposition 5.7.3. *Assuming that $\widetilde{\gamma}_\hbar^W$ is the solution which satisfies the conditions of the theorem 1 one has*

$$\mathcal{D}_{\partial/\partial\hbar} = \frac{\partial}{\partial\hbar} - \hbar^{-1}s(r_a)\tau^a\frac{\partial}{\partial\tau^a} \tag{5.39}$$

where $s(r_a) = -r_a + n - 1$ for $\tau^a \in (W_{\leq r_a}/W_{\leq r_a - \frac{1}{2}})^{dual}$. In particular, one has $E_0 = E_1 = \cdots = 0$ in the coordinates corresponding to $\widetilde{\gamma}_\hbar^W$.

Proof. Let us notice that

$$\Big(\frac{\partial\Phi}{\partial\tau}^{-1}\Big)^a_b \in \hbar^{-s(r_b)}\mathbb{C}[\hbar^{-1}]\widehat{\otimes}\mathbb{C}[[\tau_W]]$$
$$\frac{\partial\Phi^b}{\partial\hbar} \in \hbar^{s(r_b)-1}\mathbb{C}[\hbar^{-1}]\widehat{\otimes}\mathbb{C}[[\tau_W]]$$

Then the proposition 5.7.1 implies that

$$E(\hbar) = -\sum_{a,b}\Big(\frac{\partial(\Phi^b_{(0)}\hbar^{s(r_b)})}{\partial\tau^a}\Big)^{-1}\frac{\partial(\Phi^b_{(0)}\hbar^{s(r_b)})}{\partial\hbar}\frac{\partial}{\partial\tau^a} =$$
$$= -\hbar^{-1}\sum_a s(r_a)\tau^a\frac{\partial}{\partial\tau^a} \tag{5.40}$$

□

5.8 Invariants of the generalized VHS.

Fix a point $[X_z]$ in the classical moduli space of complex structures on X_{C^∞} and an increasing filtration W on $\oplus_k H^k(X_z,\mathbb{C})[-k]$ which is complementary to the Hodge filtration. Then the coefficients in the Taylor expansion of the power series representing the $3-$tensor $A^a_{bc}(\tau_W)$ written in coordinates $\{\tau_W\}$ form collection of polylinear S_k-symmetric maps

$$A^{(k)}_z: \ \mathcal{H}^{\otimes k} \ \rightarrow \ \mathcal{H}, \quad \mathcal{H} = \oplus_{p,r}H^r(X,\Lambda^p T_X)[p-r] \tag{5.41}$$

where we have used the natural identification of graded vector spaces $\mathcal{H} \simeq \operatorname{Gr} W^H \overset{\vdash\Omega_X}{\simeq} \operatorname{Gr} W$

In the view of the proposition 5.5.1 it is natural to consider the family of moduli spaces $\mathcal{M}_\hbar$ equipped with the canonical connection $\mathcal{D}_{\partial/\partial\hbar}$ as a nonlinear analog of the Hodge filtration. On the other hand as we will explain below such families of the moduli spaces can be canonically identified when the base point $\pi \in \mathcal{M}^{classical}$ changes. It is natural to consider the family of maps (5.41) as invariants of the generalized variations of Hodge structure. We will see that these invariants form exactly the same structure as the Gromov-Witten invariants.

Potentiality of $A_{ab}^c(\tau_W)$

Here we start to study the properties of the invariants of generalized VHS. As one of the consequences we show that the tensor of algebra structure constants A_{ab}^c written using the coordinates $\{\tau_W\}$ on $\mathcal{M}$ satisfies

$$\forall \ \ a,b,c,d \quad \partial_a A_{cd}^b = (-1)^{\bar{a}\bar{c}} \partial_c A_{ad}^b \tag{5.42}$$

Theorem 2. *The power series representing the family of affine connections on the moduli spaces $\mathcal{M}_\hbar$ is written using the fiberwise coordinates corresponding to $\widetilde{\gamma}^W(\tau,\hbar)$ as*

$$\nabla(\hbar) = d + \hbar^{-1} A_{ab}^c(\tau) \tag{5.43}$$

Proof. Let

$$\begin{gathered}(\Phi^W)^a(\tau_W) = \big[e^{l_\hbar \widetilde{\gamma}^W(\tau,\hbar)} - 1\big]^a = \\ \Phi_{(0)}^a(\tau)\hbar^{s(r_a)} + \Phi_{(-1)}^a(\tau)\hbar^{s(r_a)-1} + \dots \end{gathered} \tag{5.44}$$

$$\text{where} \ \ (\Phi^W)_{(-i)}^a \in \mathfrak{M}_{\mathbb{C}[[\tau_W]]}^{i+1} \ \ \text{and} \ \ (\Phi^W)_{(0)}^a(\tau) = \tau^a$$

are the components of the map (4.28). In particular

$$\frac{\partial}{\tau^b}\frac{\partial}{\tau^c}\Phi_{(0)}^a(\tau) = 0 \tag{5.45}$$

for any a, b, c. It follows from the equation (5.44) that

$$\Big(\frac{\partial \Phi}{\partial \tau}^{-1}\Big)_a^b \in \hbar^{-s(r_a)}\mathbb{C}[[\hbar^{-1}]] \tag{5.46}$$

Therefore

$$\Gamma_{bc}^e(\hbar) = \sum_a \partial_b \partial_c \Phi^a \Big(\frac{\partial \Phi}{\partial \tau}^{-1}\Big)_a^e \in \hbar^{-1}\mathbb{C}[[\hbar^{-1}]]\widehat{\otimes}\mathbb{C}[[\tau_W]] \tag{5.47}$$

On the other hand according to the proposition 5.4.4

$$\Gamma_{bc}^e(\hbar) \in \hbar^{-1}\mathbb{C}[[\hbar]]\widehat{\otimes}\mathbb{C}[[\tau_W]] \tag{5.48}$$

□

Corollary 5.2. *The tensor of algebra structure constants $A_{ab}^c(\tau_W)$ written in the coordinates on $\mathcal{M}$ corresponding to the solution $\widetilde{\gamma}^W|_{\hbar=0}$ has the property (5.42).*

Proof. The tensor $A_{ab}^c(\tau)$ can be interpreted as the "residue" of the affine connection on $\mathcal{M}_\hbar$ (see 5.4.5). Consider the equation (5.6) written in the fiberwise coordinates corresponding to the solution $\widetilde{\gamma}^W(\tau,\hbar)$. This equation expresses the fact that $\nabla(\hbar)$ is flat. The term of order $\hbar^{-1}$ gives

$$d\Big(\sum_a A_{ab}^c d\tau^a\Big) = 0 \tag{5.49}$$

which is exactly the required property (5.42). □

5.9 Flat metric on $\mathcal{M}$

Here we explain that if the filtration W on $H^*(X,\mathbb{C})$ is isotropic with respect to the Poincare pairing then a (holomorphic) flat metric which is compatible with the algebra structure (equation (5.67)) and the affine structure $\{\tau_W\}$ (equation (5.72)) is induced on $\mathcal{M}$.

Assume that the filtration $W_{\leq 0} \subset W_{\leq \frac{1}{2}} \subset \cdots \subset W_{\leq n} = \oplus_k H^k(X,\mathbb{C})[-k]$ introduced in §9 is isotropic with respect to the Poincare pairing $<,>$ in the following sense:

$$< \phi_1, \phi_2 >= 0, \text{ for } \phi_1 \in W_{\leq r},\ \phi_2 \in W_{\leq n-r-\frac{1}{2}} \tag{5.50}$$

This property together with the corresponding property

$$< \phi_1, \phi_2 >= 0, \text{ for } \phi_1 \in F^{\geq r},\ \phi_2 \in F^{\geq n-r+\frac{1}{2}} \tag{5.51}$$

of the Hodge filtration implies

$$< \phi_s, \phi_{s'} >\neq 0, \text{ for } \phi_r \in F^{\geq r} \cap W_{\leq r},\ \phi_{r'} \in F^{\geq r'} \cap W_{\leq r'} \Longrightarrow r + r' = n \tag{5.52}$$

Assume that a choice of nowhere vanishing holomorphic $n-$form Ω_X is fixed. Define the pairing

$$(\gamma_1, \gamma_2) := \int_X (\gamma_1 \wedge \gamma_2) \vdash \Omega_X \wedge \Omega_X, \text{ for } \gamma_1,\ \gamma_2 \in \oplus_{p,q}\Omega^{0,q}(X, \Lambda^p T_X) \tag{5.53}$$

Proposition 5.9.1.

$$(\bar{\partial}\gamma_1, \gamma_2) = (-1)^{p_1+q_1+1}(\gamma_1, \bar{\partial}\gamma_2) \tag{5.54}$$

$$(\Delta\gamma_1, \gamma_2) = (-1)^{p_1+q_1}(\gamma_1, \Delta\gamma_2) \tag{5.55}$$

Proof. The proof follows from the integration by parts formulas. □

Proposition 5.9.2. *The pairing (5.53) induces a natural pairing on the opposite fibers of the local system $T_{\pi_\hbar} \otimes T_{\pi_{-\hbar}} \to \mathbb{C}$*

Proof. Recall that the fibers of the local system $T_{\pi_\hbar} \to \mathbb{A}^1_{\mathbb{C}}$ are naturally identified with the cohomology of the complex $(\mathfrak{g}, \bar{\partial} + \hbar\Delta)[1]$. □

Proposition 5.9.3. *The pairing $\hbar^{n-2}(,) : T_{\pi_\hbar} \otimes T_{\pi_{-\hbar}} \to \mathbb{C}$ is locally constant:*

$$\partial_\hbar \hbar^{n-2}(u_\hbar, v_{-\hbar}) = \hbar^{n-2}(\mathcal{D}^T_{\partial/\partial\hbar} u_\hbar, v_{-\hbar}) + \hbar^{n-2}(u_\hbar, \mathcal{D}^T_{-\partial/\partial\hbar} v_{-\hbar}) \tag{5.56}$$

where $u_\hbar = u + u_{(1)}\hbar + \dots,\quad v_\hbar = v + v_{(1)}\hbar + \dots \quad \in \Gamma(\mathbb{A}^1_{\mathbb{C}}, \pi^*_\hbar T_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}})$.

Proof. The locally constant elements of $\Gamma(\mathbb{A}^1_{\mathbb{C}} \setminus 0, \pi^*_\hbar T_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}})$ are of the form (5.29). □

Consider the same pairing written in the locally constant frame on $T_{\pi_\hbar} \to \mathbb{A}^1_{\mathbb{C}}$.

Proposition 5.9.4. *Assuming that the filtration W is isotropic one has*

$$\forall \; u,v \;\; \in \oplus_r W^H_{\leq r} \hbar^{s(r)} \mathbb{C}[[\hbar^{-1}]] \qquad (u(\hbar), v(-\hbar)) \in \mathbb{C}[[\hbar^{-1}]] \tag{5.57}$$

In other words the pairing $(\,,\,) : T_{\pi_\hbar} \otimes T_{\pi_{-\hbar}} \to \mathbb{C}$ extends over $\hbar = \infty$.

Proof. For $u \in H$ the corresponding element of $(\bar{\partial} + \hbar\Delta)-$cohomology class can be written as $l_\hbar^{-1} u$, where $l_\hbar^{-1} : [\sum \gamma^{p,q}] \to [\sum \hbar^{-\frac{p+q-2}{2}} \gamma^{p,q}]$. Notice that for any $u, v \in H$

$$\begin{aligned}
&\int_X (l_\hbar^{-1} u \wedge l_{-\hbar}^{-1} v) \vdash \Omega \wedge \Omega = \\
&\qquad = (-1)^{1+\overline{v}(n-\frac{1}{2})} \int_X (l_\hbar^{-1} u \vdash \Omega) \wedge (l_\hbar^{-1} v \vdash \Omega) \\
&\qquad\qquad = (-1)^{1+\overline{v}(n-\frac{1}{2})} \hbar^{2-n} \int_X (u \vdash \Omega) \wedge (v \vdash \Omega)
\end{aligned} \tag{5.58}$$

where $\deg v^{p,q} = q - p$ for $v^{p,q} \in \Omega^{0,q}(X, \Lambda^p T_X)$. To complete the proof it is enough to notice that by the property (5.52) for $u \in W_{\leq r}, v \in W_{\leq r'}$

$$\int_X (u \vdash \Omega) \wedge (v \vdash \Omega) \neq 0 \Longrightarrow s(r) + s(r') \leq n - 2 \tag{5.59}$$

□

Let us denote by

$$\Phi^{W,H}(\tau_W, \hbar) = \left[\hbar \exp \frac{1}{\hbar} \widetilde{\gamma}^W(\tau, \hbar) - \hbar\right] \in H_{(\hbar)} \widehat{\otimes}_{(0)} \mathbb{C}[[\tau_W]] \tag{5.60}$$

the formal power series representing the canonical map $\mathcal{M}_\hbar \to T_{\pi_\hbar}$ written using the fiberwise coordinates corresponding to $\widetilde{\gamma}^W_\hbar$.

Consider the pairing induced on the tangent sheaf of $\mathcal{M}$ via this map from the pairing $T_{\pi_\hbar} \otimes T_{\pi_{-\hbar}} \to \mathbb{C}$

$$(\Phi^W)^*(u,v) := (\partial_u \Phi^{W,H}(\tau_W, \hbar), \partial_v \Phi^{W,H}(\tau_W, -\hbar)) \tag{5.61}$$

Theorem 3. *Assuming that the filtration W is isotropic the induced pairing does not depend on $\hbar$:*

$$\forall \; u, v \in \mathcal{T}_{\mathcal{M}_0} \quad (\Phi^W)^*(u,v) \in \hbar^0 \mathbb{C}[[\tau_W]] \tag{5.62}$$

Proof. Notice that

$$(\Phi^W)^*(u,v) = \int_X \big(\partial_u \widetilde{\gamma}^W(\tau,\hbar) e^{\frac{1}{\hbar}\widetilde{\gamma}^W(\tau,\hbar)} \wedge \partial_v \widetilde{\gamma}^W(\tau,-\hbar) e^{-\frac{1}{\hbar}\widetilde{\gamma}^W(\tau,-\hbar)}\big) \vdash \Omega \wedge \Omega = \tag{5.63}$$

$$\int_X \big(\partial_u \widetilde{\gamma}^W(\tau,\hbar)\partial_v \widetilde{\gamma}^W(\tau,-\hbar) e^{\frac{1}{\hbar}((\widetilde{\gamma}^W_{(0)}(\tau)+\hbar\widetilde{\gamma}^W_{(1)}(\tau)+\dots)-(\widetilde{\gamma}^W_{(0)}(\tau)-\hbar\widetilde{\gamma}^W_{(1)}(\tau)+\dots))}\big) \vdash \Omega \wedge \Omega \tag{5.64}$$

Hence $(\Phi^W)^*(u,v) \in \mathbb{C}[[\hbar]]\widehat{\otimes}\mathbb{C}[[\tau_W]]$

On the other hand for any $u \in \mathcal{T}_{\mathcal{M}_0}$ its image under the map $(\Phi^{W,H})_*$ written in the locally constant frame on $T_{\pi_\hbar} \to \mathbb{A}^1_{\mathbb{C}}$ lies in

$$\Phi^W_* u \in \oplus_r W^H_r \hbar^{s(r)} \mathbb{C}[[\hbar^{-1}]]\widehat{\otimes}\mathbb{C}[[\tau_W]] \tag{5.65}$$

Therefore by the proposition 5.9.4

$$\forall u,v \in \mathcal{T}_{\mathcal{M}_0} \quad (\Phi^W)^*(u,v) \in \mathbb{C}[[\hbar^{-1}]]\widehat{\otimes}\mathbb{C}[[\tau_W]] \tag{5.66}$$

□

Proposition 5.9.5. *The pairing* $(\Phi^W)^* : T^{\otimes 2}_{\mathcal{M}_0} \to \mathbb{C}$ *is compatible with the algebra structure on* $T_{\mathcal{M}_0}$ *in the following sense:*

$$\forall \; u,v,w \in \mathcal{T}_{\mathcal{M}_0} \qquad (\Phi^W)^*(u \circ w, v) = (\Phi^W)^*(u, w \circ v) \tag{5.67}$$

Proof. The pairing $T_{\pi_\hbar} \otimes T_{\pi_{-\hbar}} \to \mathbb{C}$ induces via the canonical map $\mathcal{M}_\hbar \to T_{\pi_\hbar}$ the pairing

$$\Phi^*_\hbar : \mathcal{T}_{\mathcal{M}_\hbar} \otimes \mathcal{T}_{\mathcal{M}_{-\hbar}} \to \mathcal{O}_{\mathcal{M}_\hbar \times \mathcal{M}_{-\hbar}} \tag{5.68}$$

One can regard the pairing $(\Phi^W)^* : \mathcal{T}^{\otimes 2}_{\mathcal{M}_0} \to \mathcal{O}_{\mathcal{M}_0}$ as being induced from the pairing $\Phi^*_\hbar$ via the identification of the fibers $\mathcal{M}_\hbar \sim \mathcal{M}_0$ associated with the fiberwise coordinate choice corresponding to $\widetilde{\gamma}^W_\hbar$. The pairing $\Phi^*_\hbar(\, , \,)$ is compatible with the affine connections on the fibers $\mathcal{M}_\hbar, \mathcal{M}_{-\hbar}$:

$$\begin{aligned} &\forall \;\; u,v,w \in \;\; \Gamma(\widetilde{\mathcal{M}}, \mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}}) \\ &\quad Lie_{(w(\hbar),w(-\hbar))}\Phi^*_\hbar(u,v) = \Phi^*_\hbar(\nabla(\hbar)_w u(\hbar), v(-\hbar)) + \Phi^*_\hbar(u, \nabla(-\hbar)_w v(-\hbar)) \end{aligned} \tag{5.69}$$

To prove this equality it is enough to notice that it becomes trivial if one uses affine coordinates on the fibers $\mathcal{M}_\hbar, \mathcal{M}_{-\hbar}$. For $u,v,w \in \Gamma(\mathcal{M}_0, \mathcal{T}_{\mathcal{M}_0})$ denote by $u(\hbar), v(\hbar), w(\hbar) \in \Gamma(\widetilde{\mathcal{M}}, \mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}})$ the relative vector fields such that $u(0) = u, v(0) = v, w(0) = w$ and $u(\hbar), v(\hbar), w(\hbar)$ are constant in the trivialization of p defined by $\widetilde{\gamma}^W_\hbar$. The equation (5.67) is obtained from (5.69) applied to $u(\hbar), v(\hbar), w(\hbar)$ by looking at the terms of order $\hbar^{-1}$. □

Second proof. Taking the terms of zero order in $\hbar$ in (5.64) we see that

$$(\Phi^W)^*(u,v) = \int_X (\partial_u \widetilde{\gamma}^W_{(0)} \wedge \partial_v \widetilde{\gamma}^W_{(0)} \wedge e^{2\widetilde{\gamma}^W_{(1)}(\tau)}) \vdash \Omega \wedge \Omega \tag{5.70}$$

Therefore

$$\forall\ u,v,w\ \in \mathcal{T}_{\mathcal{M}_0}\ \ (\Phi^W)^*(u \circ w, v) = \int_X (\partial_u \widetilde{\gamma}^W_{(0)} \wedge \partial_v \widetilde{\gamma}^W_{(0)} \wedge \partial_w \widetilde{\gamma}^W_{(0)} \wedge e^{2\widetilde{\gamma}^W_{(1)}(\tau)}) \vdash \Omega \wedge \Omega = (\Phi^W)^*(u, w \circ v) \tag{5.71}$$

□

Recall that $\{\Delta_a\}$ denotes a basis in the graded vector space H compatible with the direct sum decomposition $H^*(X,\mathbb{C}) = \oplus_r F^{\geq r} \cap W_{\leq r}$. We have also denoted via $\{\tau^a_W\}$ the natural coordinates on $\mathcal{M}$ assocated with the formal power series $\widetilde{\gamma}^W|_{\hbar=0}(\tau)$. Since the filtration W is isotropic the Poincare pairing induces the pairing on $\mathrm{Gr}\, W_*$ which we will denote by the same symbol $<,>$.

Proposition 5.9.6.

$$(\Phi^W)^*(\partial_a, \partial_b) = (-1)^{\bar{b}(n-\frac{1}{2})-r_b+n} < \Delta_a \vdash \Omega, \Delta_b \vdash \Omega > \tag{5.72}$$

where $\Delta_b \in W^H_{\leq r_b} \cap (F^H)^{\geq r_b}$ *and* $\bar{b} = k$ *for* $\Delta_b \in H^k$. *In particular, the pairing* $(\Phi^W)^* : (\mathcal{T}_{\mathcal{M}_0})^{\otimes 2} \to \mathcal{O}_{\mathcal{M}_0}$ *is constant in the coordinates* $\{\tau_W\}$.

Proof. It follows from the theorem 3 that one should take into account only the following components of Φ^W in (5.61):

$$(\Phi^W)^*(\partial_a, \partial_b) = \sum_{c,d} \hbar^{s(r_c)}(-\hbar)^{s(r_d)}([\partial_a \Phi^c_{(0)} \Delta_c], [\partial_b \Phi^d_0 \Delta_d]) = (-1)^{\bar{b}(n-\frac{1}{2})-r_b+n} < \Delta_a \vdash \Omega, \Delta_b \vdash \Omega > \tag{5.73}$$

where in the last step we have used the property (4.34) and the calculation (5.58). □

Corollary 5.3. *The pairing* $(\Phi^W)^*(\,,\,)$ *is symmetric and nondegenerate.*

Proof. These properties follow from the equation (5.67) and (5.72) correspondingly.

Proposition 5.9.7. *The vector field* $E \in \mathcal{T}_{\mathcal{M}_0}$ *is conformal with respect to the metric induced by* Φ^W *:* $Lie_E(\Phi^W)^*(\,,\,) = (2-n)(\Phi^W)^*(\,,\,)$*, that is*

$$\forall\ u,v \in\ \mathcal{T}_{\mathcal{M}_0}\quad E(\Phi^W)^*(u,v) - (\Phi^W)^*([E,u],v) - (\Phi^W)^*(u,[E,v]) = (2-n)(\Phi^W)^*(u,v) \tag{5.74}$$

Proof. As in the proof of proposition 5.9.5 let us regard the pairing $(\Phi^W)^* : \mathcal{T}_{\mathcal{M}}^{\otimes 2} \to \mathcal{O}_{\mathcal{M}}$ as being induced from the pairing $\Phi^*_\hbar$ defined in (5.68). The connection $\mathcal{D}_{\partial/\partial\hbar}$ respects the pairing $\hbar^{n-2}\Phi^*_\hbar$:

$$\mathcal{D}_{\partial/\partial\hbar}\hbar^{n-2}\Phi^*_\hbar(u(x,\hbar), v(y,-\hbar)) = \\ \hbar^{n-2}\Phi^*_\hbar(\mathcal{D}_{\partial/\partial\hbar}u(x,\hbar), v(y,-\hbar)) + \hbar^{n-2}\Phi^*_\hbar(u(x,\hbar), \mathcal{D}_{-\partial/\partial\hbar}v(y,-\hbar)) \quad (5.75)$$

Let us apply this equality to $u(\hbar), v(\hbar) \in \Gamma(\mathbb{A}^1_{\mathbb{C}}, p_*\mathcal{T}_{\widetilde{\mathcal{M}}/\mathbb{A}^1_{\mathbb{C}}})$ which are constant in the trivialization corresponding to $\widetilde{\gamma}^W_\hbar$. Notice that for such vector fields

$$(\Phi^W)^*(u(\tau,0), v(\tau,0)) = \Phi^*_\hbar(u(\tau,\hbar), v(\tau,-\hbar)) \quad (5.76)$$

Let us restrict both sides of the equation (5.75) to $x = y$ and pick up the terms of order $\hbar^{n-3}$. A short calculation gives the equation (5.74). □

5.10 Flat identity

Consider the coordinates on $\mathcal{M}$ associated with the power series $\widetilde{\gamma}^W_{(0)} = \widetilde{\gamma}^W|_{\hbar=0}(\tau_W)$.

Proposition 5.10.1. *The coordinate vector field ∂_0 corresponding to the element $\Delta_0 = [1] \in W^H_{\leq n}/W^H_{\leq n-\frac{1}{2}}$ is the identity with respect to the algebra structure on $\mathcal{T}_{\mathcal{M}}$.*

Proof. We need to prove that

$$\partial_0\widetilde{\gamma}^W_{(0)} = [1] \ \in \ \mathrm{Ker}\,\bar\partial + [\widetilde{\gamma}^W_{(0)}, \,]/\mathrm{Im}\,\bar\partial + [\widetilde{\gamma}^W_{(0)}, \,] \quad (5.77)$$

The relative vector field corresponding to

$$\delta\widetilde{\gamma} = \hbar[1] \in \mathrm{Ker}\,\bar\partial + \hbar\Delta + [\widetilde{\gamma}^W, \,]/\mathrm{Im}\,\bar\partial + \hbar\Delta + [\widetilde{\gamma}^W, \,] \quad (5.78)$$

is sent by the map

$$[\exp(l_\hbar\widetilde{\gamma}^W)]_* : \mathcal{T}\mathcal{M}_\hbar \to \mathcal{T}T_{\pi_\hbar} \quad (5.79)$$

to the family of eulerian vector fields

$$\Big[\frac{\delta\exp(l_\hbar\widetilde{\gamma}^W)}{\delta\widetilde{\gamma}}\Big] = [\exp(l_\hbar\widetilde{\gamma}^W)] = x_\hbar\frac{\partial}{\partial x_\hbar} \quad (5.80)$$

in the fibers of the local system $T_{\pi_\hbar}$. On the other hand, since the series $\widetilde{\gamma}^W$ satisfies the conditions of the theorem 1 the image of the family of eulerian vector fields under the inverse to the map (5.79) is the family written in the coordinates τ_W as

$$\sum_{a,b}(1 + \Phi^{W,a})\Big(\frac{\partial\Phi^W}{\partial\tau}^{-1}\Big)^b_a\frac{\partial}{\partial\tau^b} = \hbar\frac{\partial}{\partial\tau^0} + \hbar^0 v_0(\tau) + \hbar^{-1}v_{-1}(\tau) + \ldots \quad (5.81)$$

Therefore

$$\frac{\partial}{\partial\tau^0} = [1] \in \mathrm{Ker}\,\bar\partial + \hbar\Delta + [\widetilde{\gamma}^W, \,]/\mathrm{Im}\,\bar\partial + \hbar\Delta + [\widetilde{\gamma}^W, \,] \quad (5.82)$$

□

5.11 Dependence on the base point $\pi \in \mathcal{M}^{classical}$

In this subsection we study the variation of the structure introduced above with respect to the deformations of the base point π.

Let us denote by ϕ the coordinates defined by the map (4.35) on a neighborhood of the point $[X_0]$ of the moduli space $\mathcal{M}^{classical}$ of complex structures on X_{C^∞}.

Let $W(\phi), W(0) = W$ be the family of filtrations on $\oplus_i H^i(X_\phi, \mathbb{C})$ locally constant with respect to the Gauss-Manin connection. If $W(0)$ is complementary to the Hodge filtration at $\phi = 0$ the filtration $W(\phi)$ remains complementary to the Hodge filtration for ϕ sufficiently small. Let Ω_ϕ be the family of $n-$forms which are holomorphic in the complex structure corresponding to ϕ and which are normalized according to the formula (4.8).

We see that we get a family of (formal) moduli spaces $\mathcal{M}(\phi)$ equipped with coordinate systems $\operatorname{Gr} W(\phi) \to \mathcal{M}(\phi)$, multiplications on $T_{\mathcal{M}(\phi)}$, symmetry vector fields, and the pairings induced by generalized period mappings from the pairings defined by Ω_ϕ. We claim that this family of structures essentially does not depend on the base point $\phi \in \mathcal{M}^{clasical}$. The graded vector spaces $\operatorname{Gr} W(\phi)$ are identified by the Gauss-Manin connection. Consequently one gets the identification of the moduli spaces $\mathcal{M}(\phi)$ for sufficiently small ϕ. Denote by τ^ϕ the linear coordinate on $W_{\leq n-1}/W_{\leq n-\frac{3}{2}}$ corresponding to ϕ.

Proposition 5.11.1.

$$\frac{\partial}{\partial \phi} A^c_{ab}(\tau_W, \phi)|_{\phi=0} = \frac{\partial}{\partial \tau^\phi} A^c_{ab}(\tau_W, 0) \tag{5.83}$$

Proof. Let $\mathfrak{g}(\phi), \mathfrak{g}_{\hbar=1}(\phi)$ denote the differential graded Lie algebras associated with the base point X_ϕ. Let

$$(\rho_\phi, f_\phi) \in \Omega^{0,1}(X(0), \mathcal{T}) \oplus \Omega^{0,0}(X(0), \mathcal{O})$$

denote the elements describing according to the proposition 4.1.1 the deformations of the complex structure corresponding to X_ϕ and the holomorphic volume element Ω_ϕ The proof follows from the standard quasi-isomorphisms $\mathfrak{g}(\phi) \sim (\mathfrak{g}(0), d := \bar{\partial} + [\rho_\phi, \cdot])$, $\mathfrak{g}_{\hbar=1}(\phi) \sim (\mathfrak{g}_{\hbar=1}(0), d := \bar{\partial} + \Delta + [\rho_\phi + f_\phi, \cdot])$. $\square$

It was shown in the proposition 4.2.6 that

$$\Omega(\phi) = \exp(\widetilde{\gamma}^W(\tau^\phi, \hbar = 1)) \vdash \Omega(0) \tag{5.84}$$

From this fact it can be deduced analogously that

$$\frac{\partial}{\partial \phi} (\Phi^W)^*(\partial_a, \partial_b) = 0 \tag{5.85}$$

5.12 Frobenius manifolds

Let us recall the definition of formal Frobenius (super) manifold as given in [KM], [Du]. Let $\mathcal{H}$ be a finite-dimensional $\mathbb{Z}_2$-graded affine space over[14] $\mathbb{C}$. It is convenient to choose some set of coordinates $x_{\mathcal{H}} = \{x^a\}$ which defines the basis $\{\partial_a := \partial/\partial x^a\}$ of vector fields which are generators of affine transformations. Let one of the coordinates is distinguished and denoted by x_0. Let $A^c_{ab} \in \mathbb{C}[[x_{\mathcal{H}}]]$ be a formal power series representing 3-tensor field, g_{ab} be a nondegenerate symmetric pairing on $\mathcal{H}$.

One can use the A^c_{ab} in order to define a structure of $\mathbb{C}[[x_{\mathcal{H}}]]$-algebra on $\mathcal{H}\otimes\mathbb{C}[[x_{\mathcal{H}}]]$, the (super)space of all continious derivations of $\mathbb{C}[[x_{\mathcal{H}}]]$, with multiplication denoted by $\circ$:

$$\partial_a \circ \partial_b := \sum_c A^c_{ab}\partial_c \tag{5.86}$$

One can use g_{ab} to define the symmetric $\mathbb{C}[[x_{\mathcal{H}}]]$-pairing on $\mathcal{H}\otimes\mathbb{C}[[x_{\mathcal{H}}]]$:

$$\langle \partial_a, \partial_b\rangle := g_{ab} \tag{5.87}$$

These data define the structure of formal Frobenius manifold on $\mathcal{H}$ iff the following equations hold:

(1) (Associativity/Commutativity/Identity)

$$\forall\, a,b,c,d \quad \sum_e A^e_{ab}A^d_{ec} = (-1)^{\bar{a}(\bar{b}+\bar{c})}\sum_e A^e_{bc}A^d_{ea} \tag{5.88}$$

$$\forall\, a,b,c \quad A^c_{ba} = (-1)^{\bar{a}\bar{b}}A^c_{ab} \tag{5.89}$$

equivalently, A^c_{ab} are structure constants of a supercommutative associative
$\mathbb{C}[[x_{\mathcal{H}}]]$-algebra. It is required also that ∂_0 is an identity element of this algebra.

(2) (Invariance)

$$\forall\, a,b,c \quad \langle a\circ b, c\rangle = \langle a, b\circ c\rangle, \tag{5.90}$$

equivalently, the pairing g_{ab} is invariant with respect to the multiplication $\circ$.

(3) (Potential)

$$\forall\, a,b,c,d \quad \partial_d A^c_{ab} = (-1)^{\bar{a}\bar{d}}\partial_a A^c_{db}\ \ , \tag{5.91}$$

which implies, assuming (5.89) and (5.90), that the series A_{abc} are the third derivatives of a single power series $\mathbb{C}[[x_{\mathcal{H}}]]$

$$A_{abc} = \partial_a\partial_b\partial_c\Phi \tag{5.92}$$

[14]One can work over an arbitrary field of characteristic zero

(4) (Euler vector field) There exists an even vector field E acting as a conformal symmetry with respect to the multiplication and the metric on $\mathcal{T}_{\mathcal{H}}$ in the following sense:

$$\mathrm{Lie}_E(g) = Dg, \;\; D \in \mathbb{Q} \tag{5.93}$$

$$\mathrm{Lie}_E(\circ) = \circ \tag{5.94}$$

5.13 Family of Frobenius manifold structures on $\mathcal{M}$

Recall that the set of rational Gromov-Witten invariants of a projective manifold Y defines the Frobenius manifold structure on $\oplus_{p,q} H^q(Y, \Omega^p)[-q-p]$ considered as a (super)affine space. Analogously, given a filtration W complementary to the Hodge filtration the set of the invariants of generalized variations of Hodge structures defines the Frobenius manifold structure on $\mathcal{M}$ identified with (super)affine space $\oplus_{p,q} H^q(X, \Lambda^p \mathcal{T}_X)[-q-p]$ with the help of the map $\mathrm{Gr}\,\Phi^W$ (see §5.8).

Theorem 4. *Assume that we are given an isotropic filtration W on the graded vector space $H^*(X, \mathbb{C})$ which is complementary to the Hodge filtration. Then the affine structure on $\mathcal{M}$ corresponding to the series $\widetilde{\gamma}^W|_{\hbar=0}(\tau_W)$, the algebra structure on $T_{\mathcal{M}}$, the pairing $(\Phi^W)^*(\,,\,)$ induced by the generalized period mapping and the symmetry vector field E define Frobenius manifold structure on $\mathcal{M}$.*

Proof. We have checked all the required properties above in this section. □

Remark 5.13.1. *Frobenius manifold structure on $\mathcal{M}$ described in [BK] is obtained if one takes as the filtration W the complex conjugate to the Hodge filtration $\overline{F}$.*

Remark 5.13.2. *The expression (5.71) suggests that there might exists a formula of Chern-Simons type (see [BK],[BCOV]) for the function $\mathcal{F}^W$ whose third derivative equals $A_{abc}(\tau_W)$*

6 Mirror Symmetry in dimensions $n > 3$

Let (X, Y) be a pair of mirror dual Calabi-Yau manifolds. Hypothetically there exists singular point on the compactification of the moduli space of deformations of X such that the associated limiting weight filtration W is complementary to the Hodge filtration on $\oplus_k H^k(X, \mathbb{C})[-k]$ and its associated quotent $\mathrm{Gr}\, W$ is naturally isomorphic to $\oplus_k H^k(Y, \mathbb{C})[-k]$. Let us consider the invariants of generalized variations of Hodge structure associated with X which are written using the filtration W at the singular point.

Mirror Symmetry conjecture in higher dimensions. *The rational Gromov-Witten invariants of Y coincides with the invariants (5.1) of generalized variations of Hodge structure associated with X.*

In this section we prove this conjecture in the case of projective complete intersections.

6.1 Projective hypersurfaces

Let $Y^n \subset \mathbb{P}^{n+1}$ denotes a smooth variety defined by the equation

$$P(x_1, \dots, x_{n+2}) = 0$$

of degree $(n+2)$ in $(n+2)$ homogenious coordinates $(x_1 : \dots : x_{n+2})$. The canonical class of Y^n is trivial. A nonzero section of the sheaf of holomorphic $n-$forms can be written as

$$\sum_{i=1}^{i=n+2} (-1)^i \frac{x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{n+2}}{dP} \tag{6.1}$$

Let X^n_z denotes the family of Calabi-Yau varieties obtained by the resolution of singularities from the varieties:

$$\big\{(x_1 : \ldots : x_{n+2}) | x_1^{n+2} + \ldots + x_{n+2}^{n+2} = z \cdot x_1 x_2 \ldots x_{n+2}\big\} / \big(\mathbb{Z}/(n+2)\mathbb{Z}\big)^n \tag{6.2}$$

where $\big(\mathbb{Z}/(n+2)\mathbb{Z}\big)^n$ is the group of diagonal matrices acting on the zero locus of the above equation

$$\big\{\mathrm{diag}(\zeta_1, \dots, \zeta_{n+2}) | \zeta_i^{n+2} = 1, \prod_{i=1}^{i=n+2} \zeta_i = 1\big\} / \big\{\mathrm{diag}(\zeta, \dots, \zeta) | \zeta^{n+2} = 1\big\} \tag{6.3}$$

The $1-$parameter family of varieties X^n_z is mirror dual to the universal family of Calabi-Yau hypersurfaces $Y^n \subset \mathbb{P}^n$ (see[BvS],[GMP]). The varieties X^n and Y^n are known to satisfy (see [BB])

$$\dim \ H^q(Y, \Omega^p) = \dim \ H^q(Y, \Omega^{n-p}) \tag{6.4}$$

6.2 Complete intersections

More generally, let $Y_n(l_i)$ be a smooth variety defined as the intersection of the hypersurfaces $\mathcal{L}_1, \dots, \mathcal{L}_r \subset \mathbb{P}^{n+r}$ of degrees $l_1, \dots, l_r,\ l_1 + \ldots + l_r = n + r + 1$. A construction of the mirror dual family of Calabi-Yau manifolds $X^n_z(l_i)$ was proposed in [BvS]. The smooth varieties $X^n_z(l_i)$ are the Calabi-Yau compactifications of the complete intersection of r hypersurfaces

$$\begin{cases} u_1 + & \ldots & + u_{l_1} = 1 \\ u_{l_1+1} + & \ldots & + u_{l_1+l_2} = 1 \\ & \ldots & \\ u_{l_1+\cdots+l_{r-1}} + & \ldots & + u_{l_1+\cdots+l_r} = 1 \end{cases} \tag{6.5}$$

inside the algebraic tori $\{u_1 \cdot u_2 \cdots u_{n+r+1} = z | z \neq 0\} \subset \mathbb{A}^{n+r+1}$. The pairs of varieties $Y_n(l_i), X_n(l_i)$ have Hodge numbers satisfying (6.4) according to [BB].

6.3 Weight filtration

Let us denote by $\nu : \mathcal{X}(l_i) \to S$ the total space of the family $X_z(l_i)$. Let W'_* denotes the standard limiting weight filtration on $\oplus_k H^k(X^n_z, \mathbb{C})[-k]$ associated with the singular point $z = 0$ in the family X^n_z. Recall that there exists a unique increasing filtration $W'_{\leq 0} \subset W'_{\leq 1} \subset \ldots \subset W'_{\leq 2n}$ associated with a nilpotent operator N acting on a vector space V with the properties

- $N : W'_{\leq l} \subset W'_{\leq l-2}$
- $N^l : \mathrm{Gr}_{n+l} W' \to \mathrm{Gr}_{n-l} W'$ is an isomorphism

The limiting weight filtration on $\oplus_k H^k(X_z, \mathbb{C})$ is the filtration defined by the nilpotent operator $N = \log(T)$ where T is the unipotent part of the monodromy around $z = 0$. Denote by $W_{\leq r} = W'_{\leq 2r}$ the same filtration with all the indexes divided by two:

$$W_{\leq 0} \subset W_{\leq \frac{1}{2}} \subset \ldots \subset W_{\leq n} = H^*(X, \mathbb{Q}) \tag{6.6}$$

It can be shown using the technic from [BB] and [DL] that W_* is complementary to the Hodge filtration $F^*(X_z)$ in the sense of (4.21) for sufficiently small z and that the monodromy is in fact unipotent (this is an extension of the condition of the (strong) maximal degeneracy of the point $z = 0$, see for example [D],[M]).

Let us introduce the following bigrading on the space $\mathrm{Gr}\, W = \oplus_r W_{\leq r}/W_{\leq r-\frac{1}{2}}$:

$$\begin{gathered}\mathrm{Gr}\, W = \oplus_{i,j \in 0 \ldots n} (\mathrm{Gr}\, W)^{ij}, \\ (\mathrm{Gr}\, W)^{ij} = \big(W_{\leq n-(i+j)/2} \cap H^{j-i+n}(X, \mathbb{C})\big) / \big(W_{\leq n-(i+j+1)/2} \cup H^{j-i+n}(X, \mathbb{C})\big)\end{gathered} \tag{6.7}$$

Since W is complementary to the Hodge filtration F one has $\mathrm{Gr}\, W \simeq \mathrm{Gr}\, F$ and

$$(\mathrm{Gr}\, W)^{ij} \simeq H^j(X, \Omega^{n-i}_X) \tag{6.8}$$

A choice of holomorphic $n-$form Ω_X defines a pairing (5.53) on the space $\mathrm{Gr}\, F^H = \oplus_{i,j} H^j(X, \Lambda^i T_X)[-i-j]$. It induces a natural pairing on the the space $\mathrm{Gr}\, F = \oplus_{i,j} H^j(X, \Omega^{n-i}_X)[-i-j]$ via the identification $\mathrm{Gr}\, F^H \overset{\vdash \Omega_X}{=} \mathrm{Gr}\, F$ where the same choice of the holomorphic $n-$form Ω_X is used. When considered as a pairing on $\mathrm{Gr}\, W \simeq \mathrm{Gr}\, F$ this pairing does not depend either on the deformations of complex structure on X nor on the choice of Ω. This pairing is symmetric with respect to the total grading (6.7) on $\mathrm{Gr}\, W$. It differs from the pairing induced from the Poincare pairing by certain signs depending on the given bigraded component (see eq. (5.72)).

It is known that the operator N acting on $\mathrm{Gr}\, W$ has properties similar to that of the operator of multiplication by the Kähler 2-form acting on the cohomologies of a projective manifold. In particular, the operator N defines the "Lefschetz decomposition" on $\mathrm{Gr}\, W$. The restriction of the pairing to the corresponding primitive subspaces satisfies the analogs of the Hodge-Riemann bilinear identities (see [D] and references therein).

Recall that the only nonzero Hodge numbers of Y are $h^{p,q}$ for $p+q = \dim_{\mathbb{C}} Y$, and $h^{p,p} = 1$ for $p \neq \dim_{\mathbb{C}} Y/2$. Let us denote by L the operator of the multiplication by the Kähler 2-form acting on $H^*(Y,\mathbb{C})$. The Lefschetz decomposition on $H^*(Y,\mathbb{C})$ has the following structure

$$H^*(Y,\mathbb{C}) = <\Delta_0, \dots, \Delta_n> \oplus (\oplus_{k=1 \dots \dim P^n(Y,\mathbb{C})} < \Theta_k >) \tag{6.9}$$
$$\Delta_0 = 1, \;\; L\Delta_i = \Delta_{i+1}, \;\; L\Delta_n = 0, \;\; \Delta_i \in H^{2i}(Y,\mathbb{C}), \;\;\; n = \dim_{\mathbb{C}} Y$$
$$L\Theta_k = 0, \;\;\; \Theta_k \in H^n(Y,\mathbb{C})$$

It follows from the property (4.21) that the "Lefschetz decomposition" induced by the operator N on $\mathrm{Gr}\, W$ has similar structure.

Proposition 6.3.1. *There exists an isomorphism of the $\mathbb{Z}-$graded* [15] *vector spaces*

$$f : H^*(Y,\mathbb{C}) \to Gr\, W \tag{6.10}$$

preserving the pairing and such that $f^{-1}Nf = L$

Proof. It follows from the coincidence of the dimensions of the components of the Lefschetz decompositions. □

Let us denote

$$\widetilde{\Delta}_i = f(\Delta_i), \quad \widetilde{\Theta}_k = f(\Theta_k) \tag{6.11}$$

We will use the linear map f in §6.5 to compare the Gromov-Witten invariants of $Y^n(l_i)$ with the invariants of the generalized variations of Hodge structures attached to $X^n(l_i)$

6.4 q-expansion

Let

$$\Gamma_i \in H_n(X_z,\mathbb{C}) \cap W^{\perp}_{\leq -i}, \quad N\Gamma_i = \Gamma_{i-1}$$
$$\Xi_k \in W^{\perp}_{\leq -n/2}, \;\; N\Xi_k = 0 \tag{6.12}$$

the elements which form a frame in $H_*(X_z,\mathbb{C})$ which projects to the frame in $\mathrm{Hom}(\mathrm{Gr}\, W,\mathbb{C})$ dual to $\{\widetilde{\Delta}_i, \widetilde{\Theta}_k\}$. It follows that the monodromy transforms Γ_0, Γ_1 into

$$T(\Gamma_0) = \Gamma_0, \;\; T(\Gamma_1) = \Gamma_1 + \Gamma_0 \tag{6.13}$$

Let $\Omega(z)$ denotes the holomorphic $n-$form on X_z normalized so that

$$\int_{\Gamma_0} \Omega(z) = 1 \tag{6.14}$$

[15] When we speak about the $\mathbb{Z}-$grading on $\mathrm{Gr}\, W$ we mean the total sum of gradings (6.7)

Recall that we have associated the formal power series $A_{abc}(\tau, z)$ to the choice of the base point z of the classical moduli space of complex structures, of the isotropic filtration W complementary to the Hodge filtration, and of the normalization of the holomorphic $n-$form $\Omega(z)$.

According to the formula (4.35) affine coordinate induced on the base of the family X_z^n via the map $\mathrm{Gr}\,\Phi^W|_{\mathcal{M}^{classical}}$ can be identified with the period

$$t^{base}(z) = \int_{\Gamma_1} \Omega(z) \tag{6.15}$$

It follows from the formula (6.13) that the family of formal power series $A_{abc}(\tau; t^{base}(z))$ depends on $t^{base}(z)$ only through $q = \exp((2\pi i)t^{base}(z))$. Denote by τ^{Γ_1} the coordinate on the formal moduli space $\mathcal{M}(q)$ corresponding to the element $[\Gamma_1] \in (W_{\leq n-1}/W_{\leq n-\frac{3}{2}})^{dual}$.

Proposition 6.4.1.

$$A_{abc}(\tau, \tau^{\Gamma_1}; q) = A_{abc}(\tau, 0; \exp(2\pi i\, \tau^{\Gamma_1})q) \tag{6.16}$$

considered as power series in (τ, q)

Proof. It follows immediately from the prop. 5.11.1. □

One can assume without loss of generality that the element Γ_1 defined up to an addition of an element proportional to Γ_0 is chosen so that the coordinates q on the base of the family $X(l_i)$ coincides with the similar coordinate from [BvS],[G].

6.5 Invariants of the generalized VHS and Gromov-Witten invariants

Let $C_{abc}(\tau', q)$, $\tau' \in H^*(Y^n, \mathbb{C})$ is the generating function encoding the whole set of rational Gromov-Witten invariants (for the definition of $C_{abc}(\tau', q)$ see [KM, BM]) of the smooth projective Calabi-Yau variety $Y^n(l_i)$ which is the intersection of the hypersurfaces of degrees $l_1, \ldots, l_r$. We would like to identify $C_{abc}(\tau', q)$ with the generating function $A_{abc}(\tau, q)$ of the invariants introduced in section 5 which are attached to the dual family $X^n(l_i)$.

Picard-Fuchs equations

Here we explain how to write down the equations satisfied by the periods of the holomorphic $n-$form $\Omega(q)$:

$$\int_{\Gamma_i} \Omega(q) \tag{6.17}$$

where $N\Gamma_i = \Gamma_{i-1}$ are the locally constant elements defined in (6.12).

Let us denote

$$\widetilde{\Delta}_i^q \in W_{\leq n-i} \cap F^{\geq n-i} \cap H^n(X_q, \mathbb{C}), \quad <\widetilde{\Delta}_i^q, \dots, \widetilde{\Delta}_n^q> = \operatorname{Im} N^i,$$
$$\widetilde{\Theta}_k^q \in W_{\leq n/2} \cap F^{\geq n/2} \tag{6.18}$$

the sections of $R^*\nu_*(\mathbb{C}_{\mathcal{X}})$ whose images in $\operatorname{Gr} W$ coincide with $\widetilde{\Delta}_i, \widetilde{\Theta}_k$.

Proposition 6.5.1. *The covariant derivatives of $\widetilde{\Delta}_i^q$ with respect to the Gauss-Manin connection have the following form:*

$$\mathcal{D}_q \widetilde{\Delta}_{i-1}^q = \frac{1}{q} a_i(q) \widetilde{\Delta}_i^q \;, \;\; \textit{for } i \in 1 \dots n-1 \tag{6.19}$$

$$a_i(q) = 1 + o(q) \tag{6.20}$$

Proof. The operator N is locally constant. It follows that the subspace generated by $\widetilde{\Delta}_i^q, \dots, \widetilde{\Delta}_n^q$ is preserved by the Gauss-Manin connection. The equation (6.19) follows now from the Griffiths transversality condition $D_q F^{\geq s} \subset F^{\geq s-1}$. The form (6.20) of the coefficients $a_i(q)$ follows from the nilpotent orbit theorem (see [D] and references therein). □

One can check that $\Omega(q) = \widetilde{\Delta}_0^q$. The conditions (6.14) and (6.15) imply also that [16] $\mathcal{D}_t \Omega(q) = \widetilde{\Delta}_1^q$.

Corollary 6.1. *The periods of the holomorphic $n-$form $\Omega(q)$ satisfy*

$$\partial_t(\frac{1}{a_n(e^{2\pi i t})}\partial_t(\dots \partial_t(\frac{1}{a_1(e^{2\pi i t})}\partial_t \int_{\Gamma_i} \Omega(e^{2\pi i t}) \;) \dots) = 0 \tag{6.21}$$

□

Let

$$\widetilde{\Delta}_i^H, \widetilde{\Theta}_k^H \in \oplus_{r,s} H^r(X_q, \Lambda^s T_{X_q})[s-r]$$

correspond to $\widetilde{\Delta}_i, \widetilde{\Theta}_k \in \operatorname{Gr} W$ via the composition of isomorphisms

$$\operatorname{Gr} W \simeq \operatorname{Gr} F \overset{\vdash \Omega(q)}{\simeq} \oplus_{r,s} H^r(X_q, \Lambda^s T_{X_q})[s-r]$$

In particular, $\widetilde{\Delta}_0^H = 1$ and $\widetilde{\Delta}_1^H = [\frac{\partial}{\partial t}]$.

Proposition 6.5.2. *The following identities hold*

$$\widetilde{\Delta}_1^H \cdot \widetilde{\Delta}_{i-1}^H = a_i(q) \widetilde{\Delta}_i^H, \;\; \textit{for } i \in 1 \dots n-1 \tag{6.22}$$

in the algebra $\oplus_{r,s} H^r(X_q, \Lambda^s T_{X_q})[s-r]$

[16] In the sequel we will denote the coordinate t^{base} simply by t when it does not seem to lead to a confusion

Proof. The covariant derivative of the flag F_t is a linear map

$$\mathrm{Gr}\, F_t^{\geq *} \to \mathrm{Gr}\, F_t^{\geq *-1}$$

It coincides as an element of

$$\oplus_{r,s} \mathrm{Hom}\,(H^r(X_t, \Omega_X^s), H^{r+1}(X, \Omega_X^{s-1}))$$

with the cup multiplication by $\partial/\partial t$ □

Small quantum cohomology differential equations

Here we decribe the analogous differential equation involving the structure constants of the small quantum multiplication on $H^*(Y, \mathbb{C})$.

Proposition 6.5.3. *The following identities hold in the small quantum cohomology algebra of* $Y(l_i)$

$$\Delta_1 \cdot \Delta_{i-1} = c_i(q)\Delta_i, \ \ \textit{for}\ \ i \in 1 \dots n-1 \tag{6.23}$$

$$c_i(q) = 1 + o(q) \tag{6.24}$$

Proof. It follows almost immediately from the definitions. The only thing to check which is not immediately obvious is the fact that $\Delta_1 \cdot \Delta_{\frac{n}{2}-1} \in \ < \Delta_{\frac{n}{2}} >$ in the case $\dim Y$ is even. This follows from the fact that the calculation of the small quantum multiplication $\Delta_1 \cdot \Delta_{\frac{n}{2}-1}$ can be reduced to the calculation of certain intersection numbers on the space of stable maps to the projective space $\mathbb{CP}^{n+r+1} \supset Y(l_i)$. □

Consider the differential equation of the $n+1$-st order

$$\partial_t(\frac{1}{c_n(e^{2\pi i t})}\partial_t(\dots \partial_t(\frac{1}{c_1(e^{2\pi i t})}\partial_t\ \psi(t)\ \ )\dots) = 0 \tag{6.25}$$

Proposition 6.5.4. *For the pair of mirror families* $X(l_i), Y(l_i)$ *one has*

$$a_i^X(q) = c_i^Y(q) \tag{6.26}$$

Proof. The theorem 11.8 from [G] implies that the two differential equations (6.21) and (6.25) have the same space of solutions. This identifies a_i^X and c_i^Y up to a multiplication by a constant. The latter ambiguity is fixed by the conditions (6.20) and (6.24). □

The two generating functions

Here we prove the theorem establishing the coincidence of the Gromov-Witten invariants of the projective complete intersection Calabi-Yau manifolds and the invariants introduced in the section 5 associated with their mirror duals. Recall that we use the map $f : H^*(Y, \mathbb{C}) \to \mathrm{Gr}\, W$ defined in the proposition 6.3.1 to compare the generating power series $A(\tau, q)$ and $C(\tau, q)$.

Theorem 5.

$$C_{abc}(\tau, q) = A_{abc}(\tau, q) \tag{6.27}$$

in other words the rational Gromov-Witten invariants of $Y^n(l_i)$ coincide with the invariants of generalized variations of Hodge structures associated with $X^n(l_i)$

Proof. The idea of the proof is to use the constraints on the series $A_{abc}(\tau, q)$, $C_{abc}(\tau, q)$ arising from the equations defining the Frobenius manifold structure and the proposition 6.5.4. Let us consider the Taylor expansions

$$C_{abc}(\tau, q) = \sum_{m \geq 0, d_1 \dots d_k} \frac{1}{k!} \varepsilon(a, b, c, d_1 \dots d_k) < e_a, e_b, e_c, e_{d_1}, \dots, e_{d_k} >_m^C q^m \tau^{d_1} \dots \tau^{d_k} \tag{6.28}$$

$$A_{abc}(\tau, q) = \sum_{m \geq 0, d_1 \dots d_k} \frac{1}{k!} \varepsilon(a, b, c, d_1 \dots d_k) < e_a, e_b, e_c, e_{d_1}, \dots, e_{d_k} >_m^A q^m \tau^{d_1} \dots \tau^{d_k}$$

where $\{e_i\}$ denotes the basis (6.9), (6.11) in $H^*(Y, \mathbb{C}) \overset{f}{\simeq} \operatorname{Gr} W$ and ε is the standard sign depending on the parity of the elements e_{d_i}. Recall that the conformal symmetry vector field acts on both power series (see eq. (5.37) for the case of A_{abc})

$$E(\tau) = \sum_a \frac{1}{2} (\deg \tau^a + 2) \tau^a \frac{\partial}{\partial \tau^a} \tag{6.29}$$

where $\deg \tau^a = -k$ for $\tau^a \in (W_{\leq (n-k)/2} / W_{\leq (n-k-1)/2})^{dual} \overset{f^*}{\simeq} (H^k(Y, \mathbb{C}))^{dual}$. It follows that

$$< e_{a_1}, \dots, e_{a_k} >_m^{A(C)} \neq 0 \Rightarrow \sum_i \deg e_{a_i} = 2 \mathrm{dim}_{\mathbb{C}} X + 2(k-3) \tag{6.30}$$

where $\deg e_a = k$ for $e_a \in H^k(Y, \mathbb{C}) \overset{f}{\simeq} W_{\leq (n-k)/2} / W_{\leq (n-k-1)/2}$. It follows from the proposition 5.10.1 that $A_{ab0}(\tau, q) = \eta_{ab}$ where η_{ab} is the 2−tensor of the metrics. The proposition 6.5.4 together with the grading conditions defined by the symmetry vector field $E(\tau)$ imply that $C_{abc}(0, q) = A_{abc}(0, q)$ whenever one of the indexes a, b, c corresponds to $H^2(Y, \mathbb{C}) \simeq (\operatorname{Gr} W)^2$. The proposition 6.4.1 implies that the following analog of the "Divisor axiom" holds for the series $A_{abc}(\tau, q)$

$\forall \;\; a, b, c, d_1 \dots d_k; m$

$$< e_a, e_b, e_c, e_{d_1}, \dots, e_{d_k}, \widetilde{\Delta}_1 >_m^A = < e_a, e_b, e_c, e_{d_1}, \dots, e_{d_k} >_m^A \tag{6.31}$$

where $\widetilde{\Delta}_1$ is the basis element corresponding to $\partial / \partial t^{base}$. Let us consider the associativity equaion

$$\forall \; a, b, c, d, \; \sum_{f,g} A_{abf} \eta^{fg} A_{gcd} = (-1)^{\bar{a}(\bar{b}+\bar{c}} \sum_{f,g} A_{bcf} \eta^{fg} A_{gad} \tag{6.32}$$

and similarly for C_{abc}. Notice it follows from the grading condition (6.30) that any nonzero expression $< e_{i_1}, \ldots, e_{i_k} >_m^C$ contains no more then two elements from the "nonalgebraic" subspace generated by Θ_i. Analogously the same is true for the Taylor coefficients of the series $A_{abc}(\tau, q)$ and the elements from the subspace generated by $\widetilde{\Theta}_i$. Using the equation (6.32) as in the proof of the theorem 3.1 from [KM] all the Taylor coefficients of the series A_{abc} and C_{abc} can now be identified inductively. □

Let $\{\mathcal{G}_i\}$ denotes a locally constant frame in $H_*(X_q, \mathbb{C})$.

Corollary 6.2. *The Gromov-Witten invariants of $Y(l_i)$ are expressed in terms of the generalized periods $\int_{\mathcal{G}_i} \Pi^W(\tau, q)$ (see formula (4.29)) associated with the dual family $X(l_i)$*

$$C_{ab}^c(\tau, q) = \sum_i ((\partial\Pi)^{-1})_i^c \partial_a \partial_b \Pi^i \tag{6.33}$$

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