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# GENERALIZED POLAR COORDINATE TRANSFORMATIONS FOR DIFFERENTIAL SYSTEMS [1]

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1. **Introduction.** If $m(t)$ and $k(t)$ are real-valued, continuous functions on an interval $I$ on the real line, and $m(t) > 0$ for $t \in I$, then it is well known that under the polar coordinate transformation

$$u(t) = \rho(t) \sin \theta(t), \qquad m(t)u'(t) = \rho(t) \cos \theta(t), \tag{1.1}$$

the differential equation

$$\mathfrak{L}[u](t) \equiv [m(t)u'(t)]' - k(t)u(t) = 0, \qquad t \in I, \tag{1.2}$$

is equivalent to the nonlinear differential system

$$\begin{aligned} &\text{(a)} \quad \theta'(t) = q(t; \sin \theta(t), \cos \theta(t)), \quad \text{where} \\ &\qquad q(t; s,c) = \frac{c^2}{m(t)} - k(t)s^2, \\ &\text{(b)} \quad \rho'(t) = \left\{ \left[ \frac{1}{m(t)} + k(t) \right] \sin \theta(t) \cos \theta(t) \right\} \rho(t). \end{aligned} \tag{1.3}$$

To the present author it appears impossible to ascribe the introduction of the transformation (1.1) to any specific person, for the use of polar coordinates in the study of differential systems is of long standing, appearing in particular in the perturbation theory of two-dimensional real autonomous dynamical systems. The first published use of this substitution in the derivation of certain results of the Sturmian theory for a linear homogeneous differential equation (1.2) appears to be that of Prüfer [11], however, and in the literature this substitution is widely known as the Prüfer transformation of (1.2).

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It is to be remarked, however, that the value of this method for the study of the Sturmian theory was independently promoted by H. J. Ettlinger. The present author was introduced to the method as a beginning graduate student in Professor Ettlinger's class in the academic year 1926-1927. This method appears in the paper [18] of W. M. Whyburn, who credits Ettlinger with having suggested the method to him. In the unpublished address [12] of the present author, presented to the Illinois Section of the Mathematical Association of America in May 1932, the method was employed to obtain oscillation, separation and comparison theorems of the Sturmian theory which extended results that had been presented in Professor Ettlinger's classes. Subsequently, a more comprehensive treatment of such results by this method was published by Kamke [6], [7], [8].

The basic idea that is fundamental for the development of an analogue of the polar coordinate transformation (1.1) for selfadjoint matrix differential systems was established by Barrett [2]. Specifically, Barrett considered a real selfadjoint matrix differential equation of the second order,

$$(1.4) \qquad [M(t)U'(t)]' - K(t)U(t) = 0, \qquad t \in I,$$

where $M(t)$, $K(t)$ are $n \times n$ real, symmetric, continuous matrix functions on $I$, with $M(t)$ positive definite on this interval. Shortly thereafter the present author [15] established similar results of somewhat more general character for a differential system that may be described as the complex form of the canonical accessory differential equations for a variational problem of Bolza type.

The present paper is concerned with extensions of these earlier results on generalized polar coordinate transformations, in the following specific categories:

(i) The presentation of the existence of such transformations in a context that is a direct generalization of that used in the scalar case, in contrast to the "constructive methods" of Barrett [2] and Reid [15].

(ii) The employment of such transformations in the treatment of disconjugacy criteria for selfadjoint differential systems which are not required to satisfy the conditions of normality present in the earlier treatments of Barrett [2], Reid [15], and Etgen [3], [4].

(iii) The establishment of certain oscillation theorems of general Sturmian type for selfadjoint differential systems, which go beyond those obtained previously by other methods.

(iv) The derivation of a type of "coupled polar coordinate transformation" for the simultaneous representation of solutions of a given matrix differential system and solutions of the adjoint matrix differential system.

Matrix notation is used throughout; in particular, matrices of one column are termed vectors, and for a vector $y = (y_\alpha)$, $(\alpha = 1, \cdots, n)$, the norm $|y|$ is given by $(|y_1|^2 + \cdots + |y_n|^2)^{1/2}$; the linear vector space of ordered $n$-tuples of complex numbers, with complex scalars, is denoted by $\mathfrak{C}_n$. The $n \times n$ identity matrix is denoted by $E_n$, or by merely $E$ when there is no ambiguity, while 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix $M$ is denoted by $M^*$. If $M$ is an $n \times n$ matrix the symbol $\nu[M]$ is used for the maximum of $|My|$ on the unit ball $\{y \,|\, |y| \leqq 1\}$ in $\mathfrak{C}_n$. The notation $M \geqq N$, $\{M > N\}$, is used to signify that $M$ and $N$ are hermitian matrices of the same dimensions and $M - N$ is a nonnegative, {positive}, definite hermitian matrix. If the elements of a matrix $M(t)$ are a.c. (absolutely continuous) on an interval $[a, b]$, then $M'(t)$ signifies the matrix of derivatives at values for which these derivatives exist and the zero matrix elsewhere; correspondingly, if the elements of $M(t)$ are (Lebesgue) integrable on $[a, b]$ then $\int_a^b M(t)dt$ denotes the matrix of integrals of respective elements of $M(t)$. If $M(t)$ and $N(t)$ are equal a.e. (almost everywhere) on their domain of definition we write simply $M(t) = N(t)$. Also, for brevity a matrix function $M(t)$ is called continuous, integrable, etc., when each element of the matrix possesses the specified property. If $M = [M_{\alpha j}]$, $N = [N_{\alpha j}]$, $(\alpha = 1, \cdots, n; j = 1, \cdots, r)$, are $n \times r$ matrices, for typographical simplicity the symbol $(M; N)$ is used to denote the $2n \times m$ matrix whose $j$th column has elements $M_{1j}, \cdots, M_{nj}, N_{1j}, \cdots, N_{nj}$.

For a given compact interval $[a, b]$ on the real line the symbols $\mathcal{L}_{nr}[a, b]$, $\mathcal{L}^p_{nr}[a, b]$, $\mathcal{L}^\infty_{nr}[a, b]$ are used to denote the classes of $n \times r$ matrix functions $M(t) = [M_{\alpha\beta}(t)]$, $(\alpha = 1, \cdots, n; \beta = 1, \cdots, r)$, which on $[a, b]$ are respectively (Lebesgue) integrable, (Lebesgue) measurable and $|M_{\alpha\beta}(t)|^p$ integrable, measurable and essentially bounded. Also, for brevity the symbols $\mathcal{L}_n[a, b]$, $\mathcal{L}^p_n[a, b]$, $\mathcal{L}^\infty_n[a, b]$ are written for the respective classes designated by indices $n, r = 1$.

2. **Formulation of the general problem.** In the following it will be supposed that on a given interval $I$ on the real line the $n \times n$ matrix functions $A(t)$, $B(t)$ and $C(t)$ are of class $\mathcal{L}^\infty_{nn}[a, b]$ on arbitrary compact subintervals $[a, b]$ of $I$, while $B(t)$ and $C(t)$ are hermitian for $t \in I$. We shall be concerned with the vector differential system

$$(2.1) \qquad \begin{aligned} L_1[u, v](t) &\equiv -v'(t) - A^*(t)v(t) + C(t)u(t) = 0, \\ L_2[u, v](t) &\equiv u'(t) - A(t)u(t) - B(t)v(t) = 0, \end{aligned} \qquad t \in I,$$

in the $n$-dimensional vector functions $u(t)$, $v(t)$, and the corresponding matrix differential system

$$(2.1_M)\quad \begin{aligned} L_1[U,V](t) &\equiv -V'(t) - A^*(t)V(t) + C(t)U(t) = 0\,, \\ L_2[U,V](t) &\equiv U'(t) - A(t)U(t) - B(t)V(t) = 0\,, \end{aligned} \qquad t \in I,$$

in general $n \times r$ dimensional matrix functions $U(t)$, $V(t)$. In the case of such differential systems, a "solution" is to be understood in the "Carathéodory sense," that is, the involved vector and matrix functions are locally a.c. on $I$, and satisfy the differential equation a.e. on this interval. In particular, in view of the assumption that the coefficient matrix functions are essentially bounded on arbitrary compact subintervals $[a, b]$, it follows that the solutions of (2.1) or $(2.1_M)$ are Lipschitzian on such subintervals.

If $y = (y_\sigma)$, $(\sigma = 1, \cdots, 2n)$, with $y_\alpha = u_\alpha$, $y_{n+\alpha} = v_\alpha$, $(\alpha = 1, \cdots, n)$, then (2.1) may be written as the $2n$-dimensional vector differential equation

$$(2.1') \qquad \mathcal{L}[y](t) \equiv \mathcal{J} y'(t) + \mathcal{A}(t)y(t) = 0, \qquad t \in I,$$

where $\mathcal{J}$ and $\mathcal{A}(t)$ are the $2n \times 2n$ matrices

$$(2.2) \qquad \mathcal{J} = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix}, \qquad \mathcal{A}(t) = \begin{bmatrix} C(t) & -A^*(t) \\ -A(t) & -B(t) \end{bmatrix}.$$

As $\mathcal{A}(t)$ is hermitian, and $\mathcal{J}$ is skew-hermitian, the vector differential operator $\mathcal{L}[y](t)$ is identical with its formal Lagrange adjoint $\mathcal{L}^*[y](t) = -\mathcal{J}^*y'(t) + \mathcal{A}^*(t)y(t)$. Correspondingly, if $Y = (Y_{\sigma j})$, $(\sigma = 1, \cdots, 2n;\ j = 1, \cdots, r)$, with $Y_{\alpha j} = U_{\alpha j}$, $Y_{n+\alpha,j} = V_{\alpha j}$, then $(2.1_M)$ may be written as

$$(2.1_M{}') \qquad \mathcal{L}[Y](t) \equiv \mathcal{J} Y'(t) + \mathcal{A}(t)Y(t) = 0, \qquad t \in I.$$

If $y_\alpha(t) = (u_\alpha(t); v_\alpha(t))$, $(\alpha = 1, 2)$, are solutions of (2.1) it follows readily that $y_2^*(t)\mathcal{J}y_1(t) = \{u_1; v_1 \mid u_2; v_2\}(t) = v_2^*(t)u_1(t) - u_2^*(t)v_1(t)$ is constant on $I$; in particular, if the constant value of $\{u_1; v_1 \mid u_2; v_2\}$ is zero then $(u_1(t); v_1(t))$ and $(u_2(t); v_2(t))$ are said to be *(mutually) conjoined solutions* of (2.1). If $Y(t) = (U(t); V(t))$ is a $2n \times r$ matrix whose column vectors are $r$ linearly independent solutions of (2.1) which are mutually conjoined, these solutions form a basis for a conjoined family of solutions of dimension $r$, consisting of the set of all solutions of (2.1) which are linear combinations of these solutions. It may be established readily, (see Reid [13, Lemma 2.3]), that the maximal dimension of a conjoined family of solutions of (2.1) is $n$; moreover, a given conjoined family of solutions of dimension

$r < n$ is contained in a conjoined family of dimension $n$. If $Y(t) = (U(t); V(t))$ is a solution of $(2.1_M)$ whose column vectors form a basis for an $n$-dimensional conjoined family of solutions, then for brevity we shall say that $Y(t)$ is a *conjoined basis for* (2.1). In particular, if $\tau \in I$ and $Y(t; \tau) = (U(t; \tau); V(t; \tau))$ is the solution of $(2.1_M)$ satisfying the initial conditions

$$(2.3) \qquad U(\tau; \tau) = 0, \qquad V(\tau; \tau) = E,$$

then $Y(t; \tau)$ is a conjoined basis for (2.1). Correspondingly, a second conjoined basis for (2.1) is given by the solution $Y_0(t; \tau) = (U_0(t; \tau); V_0(t; \tau))$ of $(2.1_M)$ specified by the initial conditions

$$(2.4) \qquad U_0(\tau; \tau) = E, \qquad V_0(\tau; \tau) = 0.$$

If $T(t)$ is a fundamental matrix solution of $T'(t) - A(t)T(t) = 0$, then one may verify readily that $y(t) = (u(t); v(t))$ is a solution of (2.1) if and only if $y^0(t) = (u^0(t); v^0(t))$ with

$$(2.5) \qquad u^0(t) = T^{-1}(t)u(t), \qquad v^0(t) = T^*(t)v(t),$$

is a solution of the differential system

$$(2.6) \qquad \begin{aligned} L_1{}^0[u^0, v^0](t) &\equiv -v^{0\prime}(t) + C^0(t)u^0(t) = 0, \\ L_2{}^0[u^0, v^0](t) &\equiv u^{0\prime}(t) - B^0(t)v^0(t) = 0, \end{aligned}$$

where $B^0 = T^{-1}BT^{*-1}$ and $C^0 = T^*CT$. Moreover, if $(u_\alpha(t); v_\alpha(t))$, $(\alpha = 1, 2)$, are solutions of (2.1) and $(u_\alpha{}^0(t); v_\alpha{}^0(t))$ are the corresponding solutions (2.5) of (2.6), $\{u_1; v_1 \mid u_2; v_2\}(t) \equiv \{u_1{}^0; v_1{}^0 \mid u_2{}^0; v_2{}^0\}(t)$. In particular, if $Y(t) = (U(t); V(t))$ is a conjoined basis for (2.1), then $Y^0(t) = (U^0(t); V^0(t)) = (T^{-1}(t)U(t); T^*(t)V(t))$ is a conjoined basis for (2.6).

3. **Matrix generalizations of the trigonometric functions.** Of particular significance is a matrix differential system of the form

$$(3.1) \qquad \begin{aligned} \Lambda_1[\Phi, \Psi](t) &\equiv -\Psi'(t) - Q(t)\Phi(t) = 0, \\ \Lambda_2[\Phi, \Psi](t) &\equiv \Phi'(t) - Q(t)\Psi(t) = 0, \end{aligned} \qquad t \in I,$$

where $Q(t)$ is an $n \times n$ hermitian matrix function of class $\mathcal{L}^\infty_{nn}[a, b]$ on arbitrary compact subintervals $[a, b]$ of a given interval $I$ on the real line. Clearly (3.1) is of the form $(2.1_M)$ with $A(t) = 0$ and $B(t) = -C(t) = Q(t)$.

If $\Phi = \Phi_0(t)$, $\Psi = \Psi_0(t)$ is a solution of (3.1), then it may be verified readily that $\Phi = -\Psi_0(t)$, $\Psi = \Phi_0(t)$ is also a solution of this system. Consequently, from the discussion of the preceding section it follows that if $(\Phi(t); \Psi(t))$ is a solution of (3.1) then the matrix functions

$\{\Phi;\ \Psi\ |\Phi;\ \Psi\}(t) = \Psi^*(t)\Phi(t) - \Phi^*(t)\Psi(t)$ and $\{\Phi;\ \Psi\ |\ -\Psi;\ \Phi\}(t) = \Phi^*(t)\Phi(t) + \Psi^*(t)\Psi(t)$ are constant on $I$. In particular, if $\Phi(t)$, $\Psi(t)$ are $n \times n$ matrix functions and $(\Phi(t); \Psi(t))$ is a solution of (3.1) for which $\Psi^*(t)\Phi(t) - \Phi^*(t)\Psi(t) \equiv 0$ and $\Phi^*(t)\Phi(t) + \Psi^*(t)\Psi(t) \equiv E$ on $I$, then the $2n \times 2n$ matrix

$$\begin{bmatrix} \Phi(t) & -\Psi(t) \\ \Psi(t) & \Phi(t) \end{bmatrix} \tag{3.2}$$

is unitary for $t \in I$, and also $\Phi(t)\Phi^*(t) + \Psi(t)\Psi^*(t) \equiv E$, $\Phi(t)\Psi^*(t) - \Psi(t)\Phi^*(t) \equiv 0$.

The following result, which was established by Barrett [2] for the case of the equation (1.4) which he considered, is then a ready consequence of the basic existence theorem for differential systems of the form (3.1).

THEOREM 3.1. *For $\tau \in I$ let $\Phi = S(t;\ \tau)$, $\Psi = C(t;\ \tau)$ denote the solution of* (3.1) *determined by the initial conditions*

$$\Phi(\tau) = 0, \qquad \Psi(\tau) = E\,. \tag{3.3}$$

*Then for $(t, \tau) \in I \times I$ the matrix functions $S = S(t;\ \tau)$, $C = C(t;\ \tau)$ satisfy the following identities:*

$$\begin{aligned} S^*S + C^*C &\equiv E, \qquad S^*C - C^*S \equiv 0\,, \\ SS^* + CC^* &\equiv E\,, \qquad SC^* - CS^* \equiv 0\,. \end{aligned} \tag{3.4}$$

In terms of both the differential equations (3.1), and the identities (3.4), clearly the matrix functions $S(t;\ \tau)$ and $C(t;\ \tau)$ are generalizations of the trigonometric functions $\sin (t - \tau)$ and $\cos (t - \tau)$. This generalization is further emphasized by the following identities, which are ready consequences of the above stated properties and the uniqueness of solutions of (3.1) satisfying prescribed initial conditions.

COROLLARY 1. *For $(t,\ \tau,\ \sigma) \in I \times I \times I$ the matrix functions $S$ and $C$ satisfy the following identities:*

$$\begin{aligned} C(t;\sigma) &\equiv C(t;\tau)C^*(\sigma;\tau) + S(t;\tau)S^*(\sigma;\tau), \\ S(t;\sigma) &\equiv S(t;\tau)C^*(\sigma;\tau) - C(t;\tau)S^*(\sigma;\tau). \end{aligned} \tag{3.5}$$

COROLLARY 2. *For $(t,\ \tau) \in I \times I$ the matrix functions $Y(t;\ \tau) = 2S(t;\ \tau)C^*(t;\tau)$, $Z(t) = C(t;\ \tau)C^*(t;\ \tau) - S(t;\ \tau)S^*(t;\ \tau)$ satisfy the matrix differential system*

$$\begin{aligned} Y' &= Q(t)Z + ZQ(t), \qquad Y(\tau) = 0\,, \\ Z' &= -Q(t)Y - YQ(t), \qquad Z(\tau) = E\,; \end{aligned} \tag{3.6}$$

*moreover,* $Y^2(t;\tau) + Z^2(t;\tau) \equiv E$, $Y(t;\tau)Z(t;\tau) \equiv Z(t;\tau)Y(t;\tau)$.

The fact that $Y = Y(t;\tau)$, $Z = Z(t;\tau)$ is a solution of (3.6) may be verified directly, using the definitive properties of $S(t;\tau)$, $C(t;\tau)$, and certain of the identities (3.4). The final conclusion of the corollary is a consequence of the fact that the matrix functions $H(t) = Y^2(t;\tau) + Z^2(t;\tau)$, $K(t) = Y(t;\tau)Z(t;\tau) - Z(t;\tau)Y(t;\tau)$ satisfy the matrix differential system

$$(3.7) \qquad \begin{aligned} H' &= KQ(t) - Q(t)K, \qquad & H(\tau) &= E, \\ K' &= Q(t)H - HQ(t), \qquad & K(\tau) &= 0, \end{aligned}$$

and hence $H(t) \equiv E$, $K(t) \equiv 0$, in view of the uniqueness of solutions of (3.7). In particular, for systems (3.1) involving real-valued symmetric $Q(t)$ the result of Corollary 2 is that of Etgen [4, Theorem 1.1].

For given positive integers $n$ and $r$, let $M(n, r)$ denote the class of all $n \times r$ matrices with complex elements, and suppose that $Q(t;\ \Phi,\Psi)$ satisfies the following hypothesis.

(H) *If* $(t;\Phi,\Psi) \in I \times M(n,n) \times M(n,n)$ *then* $Q(t;\Phi,\Psi)$ *is an* $n \times n$ *hermitian matrix function which is (Lebesgue) integrable in* $t$ *on arbitrary compact subintervals* $[a, b]$ *of* $I$ *for fixed* $(\Phi,\Psi) \in M(n,n) \times M(n,n)$, *and continuous in* $(\Phi,\Psi)$ *on* $M(n,n) \times M(n,n)$ *for fixed* $t \in I$. *Moreover, the solutions of the matrix differential system*

$$(3.8) \qquad \begin{aligned} -\Psi' + Q(t;\Phi,\Psi)\Phi &= 0, \\ \Phi' - Q(t;\Phi,\Psi)\Psi &= 0, \end{aligned} \qquad t \in I,$$

*are locally unique.*

It is to be noted that hypothesis (H) holds for $Q(t;\ \Phi,\ \Psi) = Q_0(t) + \Psi B(t)\Psi^* + \Psi A(t)\Phi^* + \Phi A^*(t)\Psi^* - \Phi C(t)\Phi^*$, where $A(t)$, $B(t)$, $C(t)$, $Q_0(t)$ are $n \times n$ matrix functions of class $\mathcal{L}_{nn}[a, b]$ for arbitrary compact $[a, b] \subset I$, with $Q_0(t)$, $B(t)$ and $C(t)$ hermitian for $t \in I$.

In particular, one has for the nonlinear differential system (3.8) the following extension of the result of Theorem 3.1.

THEOREM 3.2. *Suppose that the matrix function* $Q(t;\ \Phi,\Psi)$ *satisfies hyposthesis* (H), *and that* $(\tau, \Phi_0, \Psi_0) \in I \times M(n, n) \times M(n, n)$. *Then there exists a unique solution* $\Phi = \Phi(t;\ \tau, \Phi_0, \Psi_0)$, $\Psi = \Psi(t;\ \tau, \Phi_0, \Psi_0)$ *of* (3.8) *such that* $\Phi = \Phi_0$, $\Psi = \Psi_0$ *for* $t = \tau$, *and the maximal interval of existence of this solution is the given interval* $I$; *moreover, the* $n \times n$ *matrix functions* $\Phi^*\Phi + \Psi^*\Psi$ *and* $\Psi^*\Phi - \Phi^*\Psi$ *are constant on* $I$. *In particular, if* $\Phi^*\Phi + \Psi^*\Psi \equiv E$ *and* $\Psi^*\Phi - \Phi^*\Psi \equiv 0$ *on* $I$, *then on this interval the corresponding* $2n \times 2n$ *matrix* (3.2) *is unitary and also* $\Phi\Phi^* + \Psi\Psi^* \equiv E$, $\Phi\Psi^* - \Psi\Phi^* \equiv 0$. *If the solution* $(\Phi;\ \Psi)$ *of* (3.8) *is such that* $\Phi(\tau)\Phi^*(\tau) + \Psi(\tau)\Psi^*(\tau) = E$ *and* $\Phi(\tau)\Psi^*(\tau) - \Psi(\tau)\Phi^*(\tau) = 0$, *then* $\Phi\Phi^* + \Psi\Psi^* \equiv E$, $\Phi\Psi^* - \Psi\Phi^* \equiv 0$, *the matrix* (3.2) *is unitary, and also* $\Phi^*\Phi + \Psi^*\Psi \equiv E, \Psi^*\Phi - \Phi^*\Psi \equiv 0$ *on* $I$.

Since $Q(t; \Phi, \Psi)$ is hermitian on the interval of existence of a solution $(\Phi(t);\Psi(t))$ of (3.8), it follows from the initial discussion of this section for system (3.1) with $Q(t) = Q(t; \Phi(t), \Psi(t))$ that the matrix functions $\Phi^*\Phi + \Psi^*\Psi$ and $\Psi^*\Phi - \Phi^*\Psi$ are constant throughout the interval of existence of the solution $(\Phi(t); \Psi(t))$. In particular, the constancy of $\Phi^*\Phi + \Psi^*\Psi$ implies that any solution $(\Phi(t); \Psi(t))$ of (3.8) is bounded throughout its maximal interval of existence, and from well-known theorems on the continuation of solutions of ordinary differential equations it follows that the maximal interval of existence must be the whole of $I$. Also, from the discussion of the second paragraph of this section it follows that if a solution $(\Phi(t); \Psi(t))$ of (3.8) is such that $\Phi^*\Phi + \Psi^*\Psi \equiv E$ and $\Psi^* \Phi - \Phi^*\Psi \equiv 0$ on $I$ then the matrix (3.2) is unitary and also $\Phi\Phi^* + \Psi\Psi^* \equiv E, \Phi\Psi^* - \Psi\Phi^* \equiv 0$ on $I$. In order to prove the result stated in the last sentence of the theorem, it is to be noted that the matrix functions $H(t) = \Phi(t)\Phi^*(t) + \Psi(t)\Psi^*(t)$, $K(t) = \Phi(t)\Psi^*(t) - \Psi(t)\Phi^*(t)$ are solutions of the linear matrix differential system (3.7) with $Q(t) = Q(t; \Phi(t), \Psi(t))$, and hence $H(t) \equiv E$, $K(t) \equiv 0$, in view of the uniqueness of solutions of (3.7).

## 4. A generalized polar coordinate transformation for (2.1).

For $n \times n$ matrix functions $A(t)$, $B(t)$, $C(t)$ satisfying on an interval $I$ the conditions specified at the beginning of §2, let

$$(4.1)\quad \begin{aligned} Q(t; \Phi, \Psi) &= \Psi B(t)\Psi^* + \Psi A(t)\Phi^* + \Phi A^*(t)\Psi^* - \Phi C(t)\Phi^*, \\ M(t; \Phi, \Psi) &= \Phi A(t)\Phi^* + \Psi C(t)\Phi^* + \Phi B(t)\Psi^* - \Psi A^*(t)\Psi^*. \end{aligned}$$

With the aid of Theorem 3.2 one may establish the following generalized polar coordinate transformation; this theorem embodies the result of Reid [**14**, Theorem 3.1], which extended the original result of Barrett [**2**].

THEOREM 4.1. *If* $\tau \in I$ *and* $Y(t) = (U(t); V(t))$ *is a conjoined basis for* (2.1) *with* $Y(\tau) = (U_0; V_0)$, *then*

$$(4.2)\quad U_0^*U_0 + V_0^*V_0 > 0, \qquad V_0^*U_0 - U_0^*V_0 = 0.$$

*Moreover, if* $\Phi_0, \Psi_0, R_0$ *are* $n \times n$ *matrices satisfying*

$$(4.3)\quad R_0^*R_0 = U_0^*U_0 + V_0^*V_0, \quad U_0 = \Phi_0^*R_0, \qquad V_0 = \Psi_0^*R_0,$$

*then*

$$(4.4)\quad \Phi_0\Phi_0^* + \Psi_0\Psi_0^* = E, \qquad \Phi_0\Psi_0^* - \Psi_0\Phi_0^* = 0,$$

*and the solution* $(\Phi(t); \Psi(t); R(t))$ *of the differential system*

$$
(4.5)\quad
\begin{aligned}
&\text{(a) } \Lambda_1^0[\Phi,\Psi](t) \equiv -\Psi' - Q(t;\Phi,\Psi)\Phi = 0,\ \Phi(\tau) = \Phi_0,\\
&\text{(b) } \Lambda_2^0[\Phi,\Psi](t) \equiv \Phi' - Q(t;\Phi,\Psi)\Psi = 0,\quad \Psi(\tau) = \Psi_0,\\
&\text{(c) } \Lambda^0[\Phi,\Psi,R](t) \equiv R' - M(t;\Phi,\Psi)R = 0,\ R(\tau) = R_0,
\end{aligned}
$$

*where* $Q(t;\Phi,\Psi)$ *and* $M(t;\Phi,\Psi)$ *are defined by* (4.1), *is such that*

$$
(4.6)\qquad U(t) = \Phi^*(t)R(t), \qquad V(t) = \Psi^*(t)R(t) \quad \textit{for } t \in I.
$$

*Conversely, if* $(\Phi(t); \Psi(t); R(t))$ *is a solution of* (4.5), *where* $R_0$ *is nonsingular and* $(\Phi_0, \Psi_0)$ *satisfies* (4.4), *then* (4.6) *defines a conjoined basis* $Y(t) = (U(t); V(t))$ *for* (2.1) *with*

$$
(4.7)\qquad R^*(t)R(t) = U^*(t)U(t) + V^*(t)V(t) \quad \textit{for } t \in I.
$$

It is to be noted that for a conjoined basis $Y(t)$ with $Y(\tau) = (U_0; V_0)$ the conditions (4.2), (4.3) imply that $R_0$ is nonsingular and relations (4.4) hold. The result of Theorem 3.2 implies that the solution $(\Phi(t); \Psi(t))$ of the differential system (4.5 (a), (b)) satisfying the initial condition $(\Phi(\tau); \Psi(\tau)) = (\Phi_0; \Psi_0)$ has maximal interval of existence equal to $I$, and that throughout this interval one has the identities

$$
(4.8)\quad
\begin{aligned}
\Phi\Phi^* + \Psi\Psi^* &\equiv E, &\quad \Phi\Psi^* - \Psi\Phi^* &\equiv 0,\\
\Phi^*\Phi + \Psi^*\Psi &\equiv E, &\quad \Psi^*\Phi - \Phi^*\Psi &\equiv 0.
\end{aligned}
$$

Now if $U(t)$, $V(t)$, $\Phi(t)$, $\Psi(t)$, $R(t)$ are $n \times n$ matrix functions which are a.c. on arbitrary compact subintervals of $I$, and which are related by equations (4.6), one may verify directly that the following identities hold:

$$
(4.9)\quad
\begin{aligned}
L_1[U,V] &\equiv (\Lambda_1^0[\Phi,\Psi])^* R + G_1[\Phi,\Psi]R - \Psi^*\Lambda^0[\Phi,\Psi,R],\\
L_2[U,V] &\equiv (\Lambda_2^0[\Phi,\Psi])^* R + G_2[\Phi,\Psi]R + \Phi^*\Lambda^0[\Phi,\Psi,R],
\end{aligned}
$$

where

$$
(4.10)\quad
\begin{aligned}
G_1[\Phi,\Psi] &= [E - \Phi^*\Phi - \Psi^*\Psi]\,[C\Phi^* - A^*\Psi^*]\\
&\quad + [\Phi^*\Psi - \Psi^*\Phi]\,[A\Phi^* + B\Psi^*],\\
G_2[\Phi,\Psi] &= [\Phi^*\Psi - \Psi^*\Phi]\,[C\Phi^* - A^*\Psi^*]\\
&\quad - [E - \Phi^*\Phi - \Psi^*\Psi]\,[A\Phi^* + B\Psi^*].
\end{aligned}
$$

Consequently, if $Y(t) = (U(t); V(t))$ is a conjoined basis for (2.1) with $Y(\tau) = (U_0; V_0)$, and $(\Phi_0; \Psi_0; R_0)$ satisfies (4.3), then the solution $(\Phi(t); \Psi(t); R(t))$ of (4.5) is such that $G_1[\Phi,\Psi] \equiv 0$, $G_2[\Phi,\Psi] \equiv 0$ on $I$, and (4.9) implies that the matrix functions $U(t)$, $V(t)$ defined by

(4.6) are solutions of $(2.1_M)$ which agree at $t = \tau$ with the given solution of this system, and hence are indeed equal to this given solution throughout the interval $I$. Conversely, if $(\Phi(t); \Psi(t); R(t))$ is a solution of (4.5) with $R_0$ nonsingular and the matrices $\Phi_0$, $\Psi_0$ satisfying (4.4), the identities (4.8) are a consequence of Theorem 3.2 for the system (4.5 (a), (b)) so that $G_1[\Phi, \Psi] \equiv 0$, $G_2[\Phi, \Psi] \equiv 0$ on $I$, and the fact that the matrix functions $U(t)$, $V(t)$ defined by (4.6) provide a solution of $(2.1_M)$ is a direct consequence of the identities (4.9). Moreover, the nonsingularity of $R_0$ implies that the column vectors of $Y(t) = (U(t); V(t))$ are linearly independent solutions of (2.1), and the fact that $Y(t)$ is a conjoined basis for (2.1) is a direct consequence of the identity $\Phi\Psi^* - \Psi\Phi^* \equiv 0$.

It is to be pointed out that the above proof of Theorem 4.1 is entirely analogous to the usual proof of the polar coordinate transformation (1.1) for the second order scalar differential equation (1.2), in that the result is derived as a direct consequence of theorems on the existence and continuation of solutions of a nonlinear differential system. In this regard the method is at variance with those used by Barrett [2] and Reid [15], since each of these earlier proofs was by a constructive method. In particular, if one considers the system (2.1) equivalent to the second order matrix differential equation (1.4) with $A(t) \equiv 0$, $B(t) = M^{-1}(t)$, $C(t) = K(t)$, then for the solution $\Phi = S(t; \tau)$, $\Psi = C(t; \tau)$ of the corresponding system (4.5 (a), (b)) satisfying the initial condition $(\Phi(\tau); \Psi(\tau)) = (0; E)$ the method of Barrett involved the determination of the corresponding $Q(t) = Q(t; S, C)$ as the limit of a sequence of hermitian matrix functions $Q_m(t)$, $m = 0, 1, \cdots$, where $Q_0(t)$ is an arbitrary hermitian matrix function of class $\mathcal{L}_{nn}^{\infty}[a, b]$ on arbitrary compact subintervals $[a, b]$ of $I$, and

$$S_m(t) = S(t; \tau \mid Q_m), \qquad C_m(t) = C(t; \tau \mid Q_m),$$

$$Q_{m+1}(t) = C_m(t)M^{-1}(t)C_m^*(t) - S_m(t)K(t)S_m^*(t),$$

and $\Phi = S(t; \tau \mid Q_m)$, $\Psi = C(t; \tau \mid Q_m)$ is the solution of the corresponding system (3.1) with $Q(t) = Q_m(t)$ satisfying the initial conditions $(\Phi(\tau); \Psi(\tau)) = (0; E)$. In Barrett's method the associated $R(t)$ is determined as the solution of the corresponding matrix differential equation (4.5 (c)) with $R(\tau) = E$.

The constructive method of Reid [15] involves an initial determination of the most general form of the matrix $R(t)$ belonging to a representation (4.6) of a conjoined basis $(U(t); V(t))$ for (2.1) which satisfies the conditions of Theorem 4.1. If $(U(t); V(t))$ is a conjoined basis for (2.1), and $R_0(t)$ is the unique positive definite hermitian square root of the positive definite hermitian matrix function $U^*(t)U(t) + V^*(t)V(t)$, then the most general solution of (4.7) is $R(t) = F(t)R_0(t)$, where $F(t)$ is unitary for $t \in I$. Moreover, $R_0(t)$ is a.c. on compact subintervals of $I$, and the condition that $R(t)$ satisfies the required relation

$$R^*R' = U^*AU + V^*CU + U^*BV - V^*A^*V$$

is that $F(t)$ be a unitary matrix which is a.c. on compact subintervals of $I$ and satisfies the matrix differential equation

$$F' = FP(t), \tag{4.11}$$

where $P(t)$ is the matrix function

$$P = R_0^{-1}[U^*AU + V^*CU + U^*BV - V^*A^*V - R_0R_0']R_0^{-1}, \tag{4.12}$$

which is skew hermitian in view of the fact that $R_0PR_0 = U^*U' + V^*V' - R_0R_0'$, and hence $R_0[P + P^*]R_0 = [U^*U + V^*V - R_0^2]' = 0' = 0$. If $F = F_0(t)$ is the solution of (4.11) satisfying $F(\tau) = E$, then the most general unitary solution of this equation is $F(t) = \Gamma F_0(t)$, where $\Gamma$ is an arbitrary constant unitary matrix. Having thus determined the most general form of the matrix function $R(t)$ belonging to a triple $(\Phi(t);\ \Psi(t);\ R(t))$ satisfying the conditions of Theorem 4.1, the corresponding matrix function $Q(t)$ is determined as

$$Q = R^{*-1}[V^*BV + V^*AU + U^*A^*V - U^*CU]R^{-1}. \tag{4.13}$$

As is well known, (see, for example, Reid [13, §2] and [16, §II]), if $Y(t) = (U(t);\ V(t))$ is a solution of $(2.1_M)$ with $U(t)$ nonsingular on a subinterval $I_0$ of $I$ then $W(t) = V(t)U^{-1}(t)$ is a solution of the Riccati matrix differential equation

$$K[W] \equiv W' + WA(t) + A^*(t)W + WB(t)W - C(t) = 0 \tag{4.14}$$

on $I_0$. Correspondingly, if $Y(t) = (U(t);\ V(t))$ is a solution of $(2.1_M)$ with $V(t)$ nonsingular on a subinterval $I_0$ then $W_1(t) = U(t)V^{-1}(t)$ is a solution of the Riccati matrix differential equation

$$\begin{aligned} K_1[W_1] \equiv W_1' - A(t)W_1 - W_1A^*(t) \\ + W_1C(t)W_1 - B(t) = 0 \end{aligned} \tag{4.15}$$

on this subinterval. Moreover, in each instance $Y(t)$ is a conjoined basis for (2.1) if and only if the corresponding matrix function $W(t)$, or $W_1(t)$, is hermitian on the considered subinterval.

Now if $Y(t) = (U(t);\ V(t))$ is a conjoined basis for (2.1), and $(\Phi(t);\ \Psi(t);\ R(t))$ is the corresponding solution of the differential system (4.5) such that relations (4.6) hold, then in view of the identities (4.8) we have that $V(t)U^{-1}(t) = \Psi^*(t)\Phi^{*-1}(t) = \Phi^{-1}(t)\Psi(t)$ if $U(t)$ is nonsingular and $U(t)V^{-1}(t) = \Phi^*(t)\Psi^{*-1}(t) = \Psi^{-1}(t)\Phi(t)$ if $V(t)$ is nonsingular. In particular, if $A(t) = 0$, $B(t) = E$, $C(t) = -E$ for $t \in I$, so that $(2.1_M)$ is equivalent to the second order linear homogeneous matrix differential equation $U'' + U = 0$, and $Y(t;\ \tau) = (U(t;\ \tau);\ V(t;\ \tau))$ is the solution of $(2.1_M)$ satisfying the initial conditions (2.3), then $(U_0;\ V_0;\ R_0) = (0;\ E;\ E)$ satisfies (4.3), and $(\Phi(t);\ \Psi(t);\ R(t)) = (S(t;\ \tau);\ C(t;\ \tau);\ E)$ is the solution of the corresponding differential system (4.5). In particular, on a subinterval on which $S(t;\ \tau)$ is nonsingular the matrix function $W(t;\ \tau) = S^{-1}(t;\ \tau)C(t;\ \tau)$ is a solution of the Riccati matrix differential equation

$$W' + W^2 + E = 0\,. \tag{$4.14_0$}$$

Also, on a subinterval on which $C(t;\ \tau)$ is nonsingular the matrix function $W_1(t;\tau) = C^{-1}(t;\tau)S(t;\tau)$ is a solution of the Riccati matrix differential equation

$$W_1{}' - W_1{}^2 - E = 0. \tag{$4.15_0$}$$

Consequently, in terms of the differential equations satisfied by them individually, the matrix functions $S^{-1}(t;\ \tau)C(t;\ \tau)$ and $C^{-1}(t;\ \tau)S(t;\ \tau)$ are generalizations of the scalar functions $\operatorname{ctn}(t-\tau)$ and $\tan(t-\tau)$, respectively.

5. **Disconjugacy criteria.** Two distinct points $r$ and $s$ on $I$ are said to be *(mutually) conjugate* with respect to (2.1) if there exists a solution $y(t) = (u(t);\ v(t))$ of this differential system with $u(t) \not\equiv 0$ on the subinterval with endpoints $r$ and $s$, while $u(r) = 0 = u(s)$. The system is termed *disconjugate* on a subinterval $I_0$ of $I$ provided no two distinct points of this subinterval are conjugate; moreover, (2.1) is said to be *disconjugate for large* $t$ if there exists a subinterval $(a, \infty)$ of $I$ on which this system is disconjugate.

For a nondegenerate subinterval $I_0$ of $I$, let $\Lambda(I_0)$ denote the linear space of $n$-dimensional vector functions $v(t)$ which are solutions of $v'(t) + A^*(t)v(t) = 0$, and $B(t)v(t) = 0$ for $t \in I_0$; clearly $v \in \Lambda(I_0)$ if and only if $u(t) \equiv 0$, $v(t)$ is a solution of (2.1) on $I_0$. If $\Lambda(I_0)$ is zero-dimensional then (2.1) is said to be *normal on* $I_0$, or to have *abnormality of order zero on* $I_0$, whereas if $\Lambda(I_0)$ has dimension $d = d(I_0) > 0$ the system (2.1) is said to be *abnormal*, with *order of abnormality* $d$ on $I_0$. A system (2.1) is said to be *identically normal* on $I$ if it is normal on every nondegenerate subinterval $I_0$ of $I$. If $I_0 = [r, s]$, for brevity we write $d[r, s]$ instead of the more precise $d([r, s])$, with similar contractions in case $I_0$ is of the form $[r, s)$, $(r, s]$, or $(r, s)$. For $I_0$ a subinterval of $I$, clearly $0 \leqq d(I_0) \leqq n$; indeed, if $k$ is the largest integer such that $B(t)$ has rank $k$ at a point of approximate continuity of $B(t)$ on $I_0$, then $d(I_0) \leqq n - k$. Moreover, if $s \in I$ then $d[s, t]$ is an integral-valued monotone nonincreasing function on $\{t \mid t \in I, t > s\}$ with at most $n$ points of discontinuity, at each of which $d[s, t]$ is left-hand continuous. In particular, if $[s, \infty) \subset I$ then $d[s, \infty)$ is the minimum of $d[s, t]$ for $t > s$ and $d^\infty$, the maximum of $d[s, \infty)$ for $s \in I$, is the limit of $d[r, \infty)$ as $r \to \infty$; moreover, if $r$ is such that $d[r, \infty) = d^\infty$, then $d[s, \infty) = d^\infty$ for $s \geqq r$, and there exists an $s_1 > r$ such that $d[r, t] = d^\infty$ if $t \geqq s_1$.

For a discussion of criteria for disconjugacy in the case of abnormal differential systems the reader is referred to Reid [**16**], and references contained in the bibliography of that paper. In particular, if $A(t) \equiv 0$ and $B(t) \geqq 0$ for $t$ a.e. on $I$, then it follows readily that (2.1) is identically normal on $I$ if and only if $\int_r^s B(t)dt > 0$ for arbitrary $r < s$ with $[r, s] \subset I$.

For $[a, b] \subset I$, the symbol $\mathfrak{D}[a, b]$ will denote the linear space of $n$-dimensional vector functions $\eta(t)$ which are a.c. on $[a, b]$, and for which there exists a corresponding $\zeta(t) \in \mathcal{L}_n^2[a, b]$ such that $\eta'(t) - A(t)\eta(t) = B(t)\zeta(t)$ on $[a, b]$. The subspace of $\mathfrak{D}[a, b]$ on which $\eta(a) = 0 = \eta(b)$ will be designated by $\mathfrak{D}_0[a, b]$. The fact that $\eta(t)$ belongs to $\mathfrak{D}[a, b]$ or $\mathfrak{D}_0[a, b]$ with an associated $\zeta(t)$ will be indicated by the respective symbol $\eta \in \mathfrak{D}[a, b] : \zeta$ or $\eta \in \mathfrak{D}_0[a, b] : \zeta$. For $[a, b] \subset I$ and $\eta \in \mathfrak{D}[a, b] : \zeta$, we shall denote by $J[\eta; a, b]$ the functional

$$J[\eta; a, b] = \int_a^b \{\zeta^*(t)B(t)\zeta(t) + \eta^*(t)C(t)\eta(t)\}dt. \tag{5.1}$$

It is to be noted that if $\eta \in \mathfrak{D}[a, b] : \zeta_1$ and $\eta \in \mathfrak{D}[a, b] : \zeta_2$, then $B(t)\zeta_1(t) = B(t)\zeta_2(t)$ on $[a, b]$, and the value of the integral in (5.1) is independent of the choice of the corresponding $\zeta(t)$. Moreover, if $\eta \in \mathfrak{D}[a, b]$, and $v \in \Lambda[a, b]$, then $[v^*(t)\eta(t)]' = 0$ on $[a, b]$, and hence $v^*(t)\eta(t)$ is constant on this interval.

For the extension of the Sturmian theory to selfadjoint differential systems as initiated by the fundamental work of Marston Morse ([**9**], [**10**]), the basic result concerning disconjugacy on a compact subinterval $[a, b]$ of $I$ is presented in the following theorem, (see Reid [**16**, Theorem 5.1]).

THEOREM 5.1. *For a selfadjoint vector differential system* (2.1), *and* $[a, b] \subset I$, *the functional* $J[\eta; a, b]$ *is positive definite on* $\mathfrak{D}_0[a, b]$ *if and only if* $B(t) \geqq 0$ *for* $t$ *a.e. on* $[a, b]$, *and one of the following conditions holds:*

(a) (2.1) *is disconjugate on* $[a, b]$;

(b) *there exists no point* $s \in (a, b]$ *which is conjugate to* $t = a$;

(c) *there exists a conjoined basis* $Y(t) = (U(t); V(t))$ *for* (2.1) *with* $U(t)$ *nonsingular on* $[a, b]$.

If (2.1) is identically normal, $\tau \in I$, and $\Phi = S(t; \tau)$, $\Psi = C(t; \tau)$ is the solution of (4.5 (a), (b)) satisfying the initial conditions $\Phi(\tau) = 0$, $\Psi(\tau) = E$, then a value $s$ on $I$ distinct from $\tau$ is conjugate to $\tau$ if and only if $S(s; \tau)$ is singular. If $S(s; \tau)$ is of rank $n - k$, or equivalently $U(s; \tau) = S^*(s; \tau)R(s; \tau)$ is of rank $n - k$, then $s$ is said to be a conjugate point to $\tau$ of order $k$. Moreover, in view of the above theorem, we have that if $B(t) \geqq 0$ for $t$ a.e. on $I$, and (2.1) is identically normal, then this system is disconjugate on a compact subinterval $[a, b]$ of $I$ if and only if $U(t; a)$ is nonsingular on $(a, b]$, where $Y(t; a) = (U(t; a); V(t; a))$ denotes the conjoined basis for (2.1) determined by the initial conditions (2.3) with $\tau = a$.

Let $c \in (a, \infty)$ be such that (2.1) is disconjugate on $[c, \infty)$, and $d[c, \infty) = d^\infty$. In the following it will be supposed that $d^\infty > 0$, since in the contrary case the argument simplifies in that certain matrix functions do not exist, and there is an obvious corresponding deletion of details in the argument. For $\tau \in [c, \infty)$, let $\Delta(\tau)$ denote an $n \times d^\infty$ matrix such that $\Delta^*(\tau)\Delta(\tau) = E_{d^\infty}$ and the solution $V = V_{d\tau}(t)$ of the differential system $V'(t) + A^*(t)V(t) = 0$, $V(\tau) = \Delta(\tau)$ is such that $B(t)V(t) = 0$ for $t \in [c, \infty)$, so that the column vectors of $V_{d\tau}(t)$ form a basis for $\Lambda[c, \infty)$. For simplicity of notation, if $\tau \in [c, \infty)$ let $\mu(\tau)$ be a value on $(\tau, \infty)$ such that $d[\tau, \mu(\tau)] = d^\infty$. Moreover, for $\tau \in [c, \infty)$, let $Y_{2\tau}(t) = (U_{2\tau}(t); V_{2\tau}(t))$ be the solution of ($2.1_M$) satisfying the initial conditions $U_{2\tau}(\tau) = \Delta(\tau)$, $V_{2\tau}(\tau) = 0$. As in the proof of Theorem 5.3 of Reid [**16**], one may establish the following auxiliary result.

LEMMA 5.1. *If* $s \in [\mu(c), \infty)$ *and* $Q$ *is an* $n \times (n - d^\infty)$ *matrix such that* $\Delta^*(s)Q = 0$ *and* $Q^*Q = E_{n-d^\infty}$, *then there exists an* $n \times (n - d^\infty)$ *matrix* $V^0_{s\infty}$ *such that* $\Delta^*(s)V^0_{s\infty} = 0$, $Q^*V^0_{s\infty}$ *is hermitian, and if* $Y^0_{s\infty}(t) = (U^0_{s\infty}(t); V^0_{s\infty}(t))$ *is the solution of* ($2.1_M$) *satisfying the initial condition*

$$Y^0_{s\infty}(s) = (Q; V_{s\infty}) \text{ and } Y_{s\infty}(t) = (U_{s\infty}(t); V_{s\infty}(t))$$

*with* $U_{s\infty}(t) = [U^0_{s\infty}(t)\ U_{2s}(t)]$, $V_{s\infty}(t) = [V^0_{s\infty}(t)\ V_{2s}(t)]$ *then*:

(i) $Y_{s\infty}(t)$ *is a conjoined basis for* (2.1) *with* $U_{s\infty}(t)$ *nonsingular on* $[s, \infty)$, *and* $V^*_{ds}(t)U^0_{s\infty}(t) = 0$ *for* $t \in [s, \infty)$.

(ii) $Y_{s\infty}(t)$ *is a principal solution of* ($2.1_M$) *at* $\infty$ *in the sense of Reid* [**16**]; *that is, if*

$$S(t, s; U_{s\infty}) = \int_s^t U_{s\infty}^{-1}(r)B(r)\, U_{s\infty}^{*-1}(r)dr\,,$$

*and*

$$\theta(t, s; U_{s\infty}) = U_{s\infty}(s)S^*(t, s; U_{s\infty})U_{s\infty}^*(s)\,,$$

*then the E. H. Moore generalized inverse* $\theta^{\#}(t, s; U_{s\infty})$ *of* $\theta(t, s; U_{s\infty})$ *tends to* 0 *as* $t \to \infty$.

Now since $V_{ds}^*(t)\, U_{s\infty}^0(t) = 0$ and $B(t)V_{ds}(t) = 0$ for $t \in [c, \infty)$, we have that there exists an $n \times (n - d^\infty)$ continuous matrix function $\Phi_s(t)$ such that

$$U_{s\infty}^{-1}(t) = \begin{bmatrix} \Phi_s^*(t) \\ V_{ds}^*(t) \end{bmatrix}\,,$$

and consequently,

$$S(t, s; U_{s\infty}) = \int_s^t \operatorname{diag}\,\{\Phi_s^*(r)B(r)\Phi_s(r); 0\}dr\,.$$

Also, since $U_{s\infty}(t)$ is nonsingular on $[s, \infty)$, as in Reid [14, §3] it follows that if $Y(t) = (U(t); V(t))$ is a solution of $(2.1_M)$ then

(5.2) $U(t) = U_{s\infty}(t)[U_{s\infty}^{-1}(s)U(s) - S(t, s; U_{s\infty})\{U; V \mid U_{s\infty}; V_{s\infty}\}]\,.$

Moreover, if $\hat{U}(t)$, $\hat{V}(t)$ are $n \times (n - d^\infty)$ matrix functions such that $Y(t) = (U(t); V(t))$ is of the form

(5.3) $\qquad U(t) = [\hat{U}(t) \;\; U_{2s}(t)], \qquad V(t) = [\hat{V}(t) \;\; V_{2s}(t)]$

with also

(5.4) $\qquad V_{ds}^*(t)\hat{U}(t) = 0 \quad \text{for } t \in [s, \infty)\,,$

then

(5.5) $\qquad U_{s\infty}^{-1}(t)U(t) = \operatorname{diag}\,\{\Phi_s^*(t)\hat{U}(t), E_{d\infty}\}, \quad \text{for } t \in [s, \infty);$

in addition,

(5.6) $\qquad \{U; V \mid U_{s\infty}; V_{s\infty}\} = \operatorname{diag}\,\{\{\hat{U}; \hat{V} \mid U_{s\infty}^0; V_{s\infty}^0\}; 0\}, \quad \text{if } \Delta^*(s)\hat{V}(s) = 0\,.$

Let $Y_{3s}^0(t) = (U_{3s}^0(t);\; V_{3s}^0(t))$ be the solution of $(2.1_M)$ such that $U_{3s}^0(s) = 0$, $V_{3s}^0(s) = Q$, and $Y_{3s}(t) = (U_{3s}(t);\; V_{3s}(t))$ with $U_{3s}(t) = [U_{3s}^0(t) \;\; U_{2s}(t)]$, $V_{3s}(t) = [V_{3s}^0(t) \;\; V_{2s}(t)]$. Then $Y_{3s}(t)$ is a conjoined basis for (2.1), and equation (5.2) implies the relation

$$U_{3s}(t) = U_{s\infty}(t)[\operatorname{diag}\{0; E_{d^\infty}\} - S(t, s; U_{s\infty})\{U_{3s}; V_{3s} \mid U_{s\infty}; V_{s\infty}\}] ;$$

moreover, $\Delta^*(s)V_{3s}^0(s) = 0$ and $\{U_{3s}; V_{3s} \mid U_{s\infty}; V_{s\infty}\} = \operatorname{diag}\{-Q^*Q; 0\}$. Now, if $t_0 \in (s, \infty)$ and $U_{3s}(t_0)$ is singular, it follows that there exist vectors $\xi_1$ and $\xi_2$ of respective dimensions $n - d^\infty$ *and* $d^\infty$ such that not both are null vectors and $0 = U_{3s}^0(t_0)\xi_1 + U_{2s}(t_0)\xi_2$. Since $V_{ds}^*(t)U_3{}^0(t) = 0$ and $V_{ds}^*(t)U_{2s}(t) = E_{d^\infty}$ for $t \in [s, \infty)$, it then follows that $\xi_2$ is the null vector and $\xi_1$ is nonnull, so that $U_{3s}^0(t_0)$ has rank less than $n - d^\infty$. Moreover, if $t_0 \in [\mu(s), \infty)$ then $u(t) = U_{3s}^0(t)\xi_1$ is not identically zero throughout $[s, t_0]$, for if such were the case then there would exist a vector $\xi_1'$ satisfying $V_{3\,s}^0(t)\xi_1 = V_{ds}(t)\xi_1'$ for $t \in [s, t_0]$ and, in particular, $0 = V_{3s}^0(s)\xi_1 - V_{ds}(s)\xi_1' = Q\xi_1 - \Delta(s)\xi_1'$ and $\xi_1 = 0$, $\xi_1' = 0$ since the matrix $[Q\ \ \Delta(s)]$ is nonsingular. Consequently, on $[\mu(s), \infty)$ the $n \times n$ matrix function $U_{3s}(t)$ is nonsingular, the $n \times (n - d^\infty)$ matrix function $U_{3s}^0(t)$ is of rank $n - d^\infty$, and the $(n - d^\infty) \times (n - d^\infty)$ matrix function

(5.7) $$\hat{S}(t, s; U_{s\infty}) = \int_s^t \Phi_s{}^*(r)B(r)\phi_s(r)dr$$

is nonsingular on this interval. In view of these results, we have that

(5.8) $$S^\#(t, s; U_{s\infty}) = \operatorname{diag}\{\hat{S}^{-1}(t, s; U_{s\infty}); 0\} \quad \text{for } t \in [\mu(s), \infty),$$

and, since $S^\#(t, s; U_{s\infty}) \to 0$ as $t \to \infty$, it follows that

(5.9) $$\lambda[\hat{S}(t, s; U_{s\infty})] \to \infty \quad \text{as } t \to \infty ,$$

where, in general, for an hermitian matrix $M$ the smallest proper value of $M$ is denoted by $\lambda[M]$. Also, since the matrix $U_{s\infty}(s) = [Q\ \ \Delta(s)]$ is unitary, it follows readily that

$$\theta^\#(t, s; U_{s\infty}) = U_{s\infty}^{*-1}(s)[S^\#(t, s; U_{s\infty})]^* U_{s\infty}^{-1}(s), \quad \text{for } t \in [\mu(s), \infty).$$

Moreover, in view of the above relations (5.2), (5.5), (5.6) we have the following result.

LEMMA 5.2. *If* $Y(t) = (U(t); V(t))$ *is a solution of* $(2.1_M)$ *of the form* (5.3) *satisfying* (5.4), *with* $\Delta^*(s)\hat{V}(s) = 0$, *and such that the constant* $(n - d^\infty) \times (n - d^\infty)$ *matrix*

(5.10) $$K = \{\hat{U}; \hat{V} \mid U_{s\infty}^0; V_{s\infty}^0\}$$

*is nonsingular, then there exists a* $b \in (s, \infty)$ *such that* $U(t)$ *is nonsingular on* $[b, \infty)$ *and*

(5.11) $$U^{-1}(t)U_{s\infty}(t) \to \operatorname{diag}\{0; E_{d^\infty}\} \quad as\ t \to \infty\ ;$$

*moreover, if* $Y(t)$ *is also a conjoined basis for* (2.1) *then the hermitian matrix function*

(5.12) $$S(t, b; U) = \int_b^t U^{-1}(r)B(r)U^{*-1}(r)dr$$

*converges to a finite value* $S(\infty, b; U)$ *as* $t \to \infty$.

If (5.10) defines a nonsingular matrix $K$ then in view of relations (5.2), (5.5) and (5.6) it follows that $U(t)$ is nonsingular on $[b, \infty)$ if $b$ is chosen so large that $K^{-1}\hat{S}^{-1}(t, s; U_{s\infty})\Phi_s^*(s)\hat{U}(s)$ has norm less than 1 for $t \in [b, \infty)$, and conclusion (5.11) is then a direct consequence of the relation (5.2).

If in addition $Y(t)$ is a conjoined basis for (2.1), then corresponding to (5.2) one has the formula

(5.13) $$U_{s\infty}(t) = U(t)[U^{-1}(b)U_{s\infty}(b) + S(t, s; U)\operatorname{diag}\{K^*, 0\}]\ , \quad \text{for } t \in [b, \infty)\ ,$$

since $\{U^0_{s\infty};\ V^0_{s\infty} \mid \hat{U};\ \hat{V}\} = -\{\hat{U};\ \hat{V} \mid U^0_{s\infty};\ V^0_{s\infty}\}^* = -K^*$. Also, there exists an $n \times (n - d^\infty)$ matrix function $\hat{\Phi}(t)$ such that

$$U^{-1}(t) = \begin{bmatrix} \hat{\Phi}^*(t) \\ V^*_{ds}(t) \end{bmatrix}\ , \quad \text{for } t \in [b, \infty)\ ,$$

and

(5.14) $$S(t,\ b;\ U) = \int_b^t \operatorname{diag}\{\hat{\Phi}^*(r)B(r)\hat{\Phi}(r); 0\}dr\ ;$$

consequently, in view of the relations (5.11) and (5.13) it follows that

$$S(t, b; U) \to \operatorname{diag}\{-\hat{\Phi}^*(b)\, U^0_{s\infty}(b)K^{*-1}; 0\} \quad \text{as } t \to \infty\ .$$

If $Y(t) = (U(t);\ V(t))$ is the solution of $(2.1_M)$ of the form (5.3) with $\hat{U}(s) = V_{s\infty}$, $\hat{V}(s) = -Q$, then it follows readily that $Y(t)$ is a conjoined basis for (2.1) and $\{U;\ V \mid U_{s\infty};\ V_{s\infty}\} = \operatorname{diag}\{Q^*Q + V^*_{s\infty}V_{s\infty};\ 0\}$, so that the matrix $K$ of (5.10) is the positive definite hermitian matrix $Q^*Q + V^*_{s\infty}V_{s\infty}$.

In particular, the above results imply the first conclusion of the following theorem, and the further stated results are ready algebraic consequences of this first conclusion.

THEOREM 5.2. *Suppose that* $I = [a, \infty)$, $B(t) \geqq 0$ *for t a.e. on I, and* (2.1) *is disconjugate for large t. Then there exists a conjoined basis* $Y(t) = (U(t);\ V(t))$ *of* (2.1) *such that* $U(t)$ *is nonsingular for large t, and*

$$(5.15) \qquad \int^{\infty} U^{-1}(t)B(t)U^{*-1}(t)dt < \infty .$$

*Moreover, for any such conjoined basis,*

$$(5.16) \qquad \int^{\infty} \{\nu[B(t)]/\nu^2[U(t)]\}dt < \infty ;$$

*in particular, if for arbitrary solutions* $y(t) = (u(t); v(t))$ *of* (2.1) *we have that* $|u(t)|$ *is bounded on* $[a, \infty)$ *then*

$$(5.17) \qquad \int^{\infty} \nu[B(t)]\, dt < \infty .$$

Since for a nonnegative definite $n \times n$ hermitian matrix $B$ the trace of $B$ satisfies the inequality $(1/n)\,\mathrm{Tr}\{B\} \leqq \nu[B] \leqq \mathrm{Tr}\{B\}$, inequalities (5.16), (5.17) may be stated equally well in terms of $\mathrm{Tr}\{B(t)\}$. In this connection, it is to be noted that $\nu[B(t)]$ is integrable on arbitrary compact subintervals of $I$, in view of the similar condition satisfied by $B(t)$, (see Reid [17, Theorem 3.1]).

Corresponding to Theorem 5.3 of Reid [15], we now have the following result.

THEOREM 5.3. *If* $I = [a, \infty)$ *and* $Q(t) \geqq 0$ *for* $t$ *a.e. on* $I$, *then a differential system* (3.1) *is disconjugate for large* $t$ *if and only if*

$$(5.18) \qquad \int^{\infty} \nu[Q(t)]\, dt < \infty .$$

In view of the remarks preceding the statement of the theorem, condition (5.18) may equally well be phrased as

$$(5.18') \qquad \int^{\infty} \mathrm{Tr}\{Q(t)\}dt < \infty ,$$

or as the matrix condition

$$(5.18'') \qquad \int^{\infty} Q(t)dt < \infty .$$

As all solutions of (3.1) are bounded in view of the unitary nature of the matrix (3.2), from Theorem 5.2 it follows that disconjugacy of (3.1) for large $t$ implies relation (5.18). Conversely, if $r(t) = \nu[Q(t)]$ is such that (5.18) holds, let $\hat{r}(t)$ be such that $\hat{r}(t) - r(t)$ is continuous, $\hat{r}(t) > r(t)$, and $\int^{\infty} \hat{r}(t)dt < \infty$; for example, one might choose $\hat{r}(t) = r(t) + (1 + t^2)^{-1}$. Now for $[a, b] \subset I$ the functional $J[\eta; a, b] = \int_a^b \{\zeta^*(t)Q(t)\zeta(t) - \eta^*(t)Q(t)\eta(t)\}dt$ corresponding to (5.1) is such that if $\eta(t)$ belongs to the corresponding linear space $\mathfrak{D}[a, b]$ with $\zeta(t)$ then $\eta'(t) = Q(t)\zeta(t)$ so that $|\eta'|^4 = |\eta^{*\prime}Q\zeta|^2 \leqq (\eta^{*\prime}Q\eta')(\zeta^*Q\zeta) \leqq \hat{r}|\eta'|^2(\zeta^*Q\zeta)$ and $J[\eta; a, b] \geqq \int_a^b \{[\hat{r}(t)]^{-1}|\eta'(t)|^2 - \hat{r}(t)|\eta(t)|^2\}dt.$

As in the proof of Theorem 5.1 of Reid [**15**], from the fact that the related scalar differential equation $[u'/\hat{r}(t)]' + \hat{r}(t)u = 0$ admits the solution $\sin(-\int_t^\infty \hat{r}(s)ds)$ it follows that (3.1) is disconjugate for large $t$.

Corresponding to Theorem 5.4 of Reid [**15**], one has for the differential system (2.1) the following result.

THEOREM 5.4. *Suppose that $Y(t) = (U(t); V(t))$ is a conjoined basis for* (2.1) *on an interval $I = [a, \infty)$, and that there is a subinterval $I_1 = [a_1, \infty)$ of $I$ on which the hermitian matrix function*

$$G(t; Y) = V^*(t)B(t)V(t) + V^*(t)A(t)U(t) + U^*(t)A^*(t)V(t) - U^*(t)C(t)U(t) \tag{5.19}$$

*is nonnegative definite for $t$ a.e. on $I_1$. If $H(t)$ is a matrix function such that $H^*(t)H(t) = U^*(t)U(t) + V^*(t)V(t)$, then there exists a subinterval $I_0 = [a_0, \infty)$ of $I_1$ on which* (2.1) *is disconjugate if and only if the matrix function $Q(t) = H^{*-1}(t)G(t; Y)H^{-1}(t)$ satisfies condition* (5.18).

In particular, it is to be remarked that the hermitian matrix function (5.19) is nonnegative definite on $I$ if $A(t) = 0$, and $B(t) \geqq 0$, $C(t) \leqq 0$ for $t \in I$. This fact, together with the comments of the last paragraph of §2, provide for general systems (2.1) with $B(t) \geqq 0$, $C(t) \leqq 0$ on an interval $[a, \infty)$ a necessary and sufficient condition for this system to be disconjugate for large $t$.

6. **Additional oscillation theorems.** If $Y(t) = (U(t); V(t))$ is a conjoined basis for (2.1), a value $t = c$ is called a *focal point* of this basis of order $k$ if $U(c)$ has rank $n - k$. If (2.1) is identically normal on $I$ then the focal points of any conjoined basis are isolated; indeed, with the aid of Theorem 5.1 it may be established readily that if $I_0$ is a subinterval of $I$ on which (2.1) is disconjugate then on $I_0$ there are at most $n$ focal points of a given conjoined basis, each focal point being counted a number of times equal to its order. This separation theorem is one of the basic results of the Morse generalization of the Sturmian theory for selfadjoint differential systems (2.1); for a proof of this result for systems (2.1) equivalent to equations of the form (1.4), see Morse [9, Theorem 8]. Application of this result to the differential system (3.1) yields the following result.

THEOREM 6.1. *Suppose that $Q(t) \geqq 0$ for $t$ a.e. on $I$, and $\int_r^s Q(t)dt > 0$ for $r < s$, $[r, s] \subset I$, while $M_\alpha$, $N_\alpha$, $(\alpha = 1, 2)$, are $n \times n$ matrices such that each $n \times 2n$ matrix $[M_\alpha^* \; N_\alpha^*]$ is of rank $n$ and $M_\alpha^* N_\alpha - N_\alpha^* M_\alpha = 0$. If $(\Phi(t); \Psi(t))$ is a solution of* (3.1) *such that $\Phi^*\Phi + \Psi^*\Psi \equiv E$, $\Psi^*\Phi - \Phi^*\Psi \equiv 0$, then the matrix functions $\Phi_\alpha(t) = \Phi(t)M_\alpha - \Psi(t)N_\alpha$, $\Psi_\alpha(t)$*

$= \Psi(t)M_\alpha + \Phi(t)N_\alpha$ *are such that each* $(\Phi_\alpha(t); \Psi_\alpha(t))$ *is a conjoined basis for* (3.1), *and if* $I_0$ *is a subinterval of* $I$ *then the number of focal points of* $(\Phi_1(t); \Psi_1(t))$ *on* $I_0$ *differs from the number of focal points of* $(\Phi_2(t); \Psi_2(t))$ *on* $I_0$ *by at most* $n$.

As a consequence of the generalized polar coordinate transformation of Theorem 4.1 for selfadjoint differential systems (2.1), the above theorem yields the following result, which goes beyond the results obtained previously by other methods.

THEOREM 6.2. *Suppose that* $B(t) \geqq 0$ *for* $t$ *a.e. on* $I$, *the system* (2.1) *is identically normal on* $I$, *and* $Y(t) = (U(t); V(t))$ *is a conjoined basis for this system for which there is a subinterval* $I_1$ *of* $I$ *on which the corresponding matrix function* $Q(t) = Q(t; \Phi, \Psi)$ *of Theorem* 4.1 *satisfies* $Q(t) \geqq 0$ *for* $t$ *a.e. on* $I_1$ *and* $\int_r^s Q(t)dt > 0$ *for* $r < s$, $[r, s] \subset I_1$. *If* $M_\alpha$, $N_\alpha$ $(\alpha = 1, 2)$ *are* $n \times n$ *matrices such that each* $n \times 2n$ *matrix* $[M_\alpha^* \; N_\alpha^*]$ *is of rank* $n$ *and* $M_\alpha^* N_\alpha - N_\alpha^* M_\alpha = 0$, *then the matrix functions* $F_\alpha(t) = M_\alpha^* U(t) - N_\alpha^* V(t)$ *are such that if* $I_0$ *is a subinterval of* $I$ *then the number of zeros of* det $F_1(t)$ *on* $I_0$ *differs from the number of zeros of* det $F_2(t)$ *on* $I_0$ *by at most* $n$, *where a zero* $t_0$ *of* det $F_\alpha(t)$ *is counted* $k$ *times if* $F_\alpha(t_0)$ *is of rank* $n - k$.

7. **Results for a special system (2.1).** It is to be noted that the discussion of §§2-5 involves no assumption of normality for the differential system (2.1). In particular, the results of these sections are applicable to a system (2.1) with $B(t) = 0$ for $t \in I$, in which case on arbitrary nondegenerate subintervals $I_0$ of $I$ this system has order of abnormality equal to $n$. Moreover, in the particular case for which

$$B(t) = 0, \qquad C(t) = 0, \quad \text{for } t \in I, \tag{7.1}$$

the system (2.1) reduces to

$$\begin{aligned} L^*[v](t) &\equiv -v'(t) - A^*(t)v(t) = 0, \\ L[u](t) &\equiv u'(t) - A(t)u(t) = 0, \end{aligned} \qquad t \in I, \tag{7.2}$$

consisting of the direct sum of the first order linear homogeneous $n$th order vector differential equation $L[u](t) = 0$ and its adjoint $L^*[v](t) = 0$. Although (7.2) may seem to be a trivial special case of (2.1), we shall proceed to show that the results of earlier sections applied to this system yield nontrivial results.

If $\tau \in I$ and $U = T(t; \tau)$ is the fundamental matrix solution of $L[U](t) = 0$ satisfying $U(\tau) = E$, then $V = T^{*-1}(t; \tau)$ is the fundamental matrix solution of $L^*[V](t) = 0$ satisfying $V(\tau) = E$, and the most general solution $Y(t) = (U(t); V(t))$ of (7.2) is of the form

$(T(t)K_1;\ T^{*-1}(t)K_2)$. In particular, $Y(t)$ is a conjoined basis for (7.2) if and only if $K_1$ and $K_2$ are $n \times n$ matrices such that the $n \times 2n$ matrix $[K_1^* \quad K_2^*]$ is of rank $n$ and the $n \times n$ matrix $K_1^* K_2$ is hermitian.

For (7.2) the corresponding matrices $Q(t; \Phi, \Psi)$, $M(t; \Phi, \Psi)$ of (4.1) are

$$\begin{aligned} Q(t; \Phi, \Psi) &= \Psi A(t)\Phi^* + \Phi A^*(t)\Psi^*, \\ M(t; \Phi, \Psi) &= \Phi A(t)\Phi^* - \Psi A^*(t)\Psi^*. \end{aligned} \tag{7.3}$$

In particular, for the solution $(U(t);\ V(t)) = (T(t;\ \tau);\ 0)$ of (7.2) the relations (4.2), (4.3) hold with $U_0 = \Phi_0 = R_0 = E$, $V_0 = \Psi_0 = 0$, and the corresponding solution of the differential system (4.5) is $\Phi(t) \equiv E$, $\Psi(t) \equiv 0$, $R(t) = T(t;\ \tau)$. Similarly, for the solution $(U(t);\ V(t)) = (0;\ T^{*-1}(t;\ \tau))$ of (7.2) the relations (4.2), (4.3) hold with $U_0 = \Phi_0 = 0$, $V_0 = \Psi_0 = R_0 = E$, and the corresponding solution of (4.5) is $\Phi(t) \equiv 0, \Psi(t) \equiv E$, $R(t) = T^{*-1}(t; \tau)$.

Now consider the particular instance of a system (7.2) with $n = 2m$, and

$$A(t) = \begin{bmatrix} \hat{A}(t) & \hat{B}(t) \\ \hat{C}(t) & -\hat{D}(t) \end{bmatrix}, \tag{7.4}$$

where $\hat{A}(t)$, $\hat{B}(t)$, $\hat{C}(t)$, $\hat{D}(t)$ are $m \times m$ matrix functions of class $\mathcal{L}^{\infty}_{mm}[a, b]$ on arbitrary compact subintervals $[a, b]$ of $I$. If we set $u_\alpha = \eta_\alpha$, $u_{m+\alpha} = \zeta_\alpha$, $(\alpha = 1, \cdots, m)$, then the differential equation $L[u](t) = 0$ may be written in terms of the $m$-dimensional vector functions $\eta(t) = (\eta_\alpha(t))$, $\zeta(t) = (\zeta_\alpha(t))$ as

$$\begin{aligned} \eta'(t) - \hat{A}(t)\eta(t) - \hat{B}(t)\zeta(t) &= 0, \\ \zeta'(t) - \hat{C}(t)\eta(t) + \hat{D}(t)\zeta(t) &= 0, \end{aligned} \qquad t \in I. \tag{7.5'}$$

Correspondingly, if $v_\alpha = \rho_\alpha$, $v_{m+\alpha} = \sigma_\alpha$, $(\alpha = 1, \cdots, m)$, then $L^*[v](t) = 0$ may be written in terms of the $m$-dimensional vector functions $\rho(t) = (\rho_\alpha(t)), \sigma(t) = (\sigma_\alpha(t))$ as

$$\begin{aligned} -\rho'(t) - \hat{A}^*(t)\rho(t) - \hat{C}^*(t)\sigma(t) &= 0, \\ -\sigma'(t) - \hat{B}^*(t)\rho(t) + \hat{D}^*(t)\sigma(t) &= 0. \end{aligned} \qquad t \in I, \tag{7.5''}$$

If $(\eta(t);\ \zeta(t))$ is a solution of (7.5′) and $(\rho(t);\ \sigma(t))$ is a solution of (7.5″), then it follows readily that $\rho^*(t)\eta(t) + \sigma^*(t)\zeta(t)$ is constant on $I$. In particular, if

$$\hat{B}(t) = \hat{B}^*(t), \quad \hat{C}(t) = \hat{C}^*(t), \quad \hat{D}(t) = \hat{A}^*(t), \quad \text{for } t \in I, \tag{7.6}$$

then whenever $(\eta(t);\ \zeta(t))$ is a solution of (7.5′) the vector function $(\rho(t);\ \sigma(t)) = (-\zeta(t);\ \eta(t))$ is a solution of (7.5″), and the above cited property reduces to the condition that if $(\eta^\beta(t);\ \zeta^\beta(t))$, $(\beta = 1, 2)$, are solutions of (7.5′) then the function $\{\eta^1;\ \zeta^1 \mid \eta^2;\ \zeta^2\}(t) = \zeta^{2*}(t)\eta^1(t) - \eta^{2*}(t)\zeta^1(t)$, as introduced in §2, is constant on $I$.

THEOREM 7.1. *Suppose that the column vectors of the $2m \times m$ matrices $(H(t);\ Z(t))$ and $(P(t);\ \Sigma(t))$ are linearly independent solutions of (7.5′) and (7.5″), respectively, and*

$$P^*(t)H(t) + \Sigma^*(t)Z(t) = 0, \quad \text{for } t \in I\,. \tag{7.7}$$

*Moreover, suppose that $(H(\tau);\ Z(\tau)) = (H_0;\ Z_0)$, $(P(\tau);\ \Sigma(\tau)) = (P_0;\ \Sigma_0)$ and that $\Phi_{01}, \Psi_{01}, R_{01}, \Phi_{02}, \Psi_{02}, R_{02}$ are $m \times$ matrices such that*

$$\begin{aligned} R_{01}^* R_{01} &= H_0{}^*H_0 + Z_0{}^*Z_0, \qquad H_0 = \Phi_{01}^* R_{01}, \qquad Z_0 = \Psi_{01}^* R_{01}\,, \\ R_{02}^* R_{02} &= P_0{}^*P_0 + \Sigma_0{}^*\Sigma_0, \qquad P_0 = \Phi_{02}^* R_{02}, \qquad \Sigma_0 = \Psi_{02}^* R_{02}\,; \end{aligned} \tag{7.8}$$

*in particular,*

$$\begin{aligned} \Phi_{01}\Phi_{01} + \Psi_{01}\Psi_{01} = E, \qquad \Phi_{02}\Phi_{02}^* + \Psi_{02}\Psi_{02}^* &= E, \\ \Phi_{02}\Phi_{01}^* + \Psi_{02}\Psi_{01}^* &= 0\,. \end{aligned} \tag{7.9}$$

*If the matrix functions $\hat{Q} = \hat{Q}(t;\ \Phi_1, \Psi_1, \Phi_2, \Psi_2)$, $M_1 = M_1(t;\ \Phi_1, \Psi_1)$, $M_2(t; \Phi_2, \Psi_2)$ are defined as*

$$\begin{aligned} \hat{Q} &= \Phi_1[\hat{A}^*(t)\Phi_2{}^* + \hat{C}^*(t)\Psi_2{}^*] + \Psi_1[\hat{B}^*(t)\Phi_2 - \hat{D}^*(t)\Psi_2{}^*]\,, \\ M_1 &= \Phi_1[\hat{A}(t)\Phi_1{}^* + \hat{B}(t)\Psi_1{}^*] + \Psi_1[\hat{C}(t)\Phi_1{}^* - \hat{D}(t)\Psi_1{}^*]\,, \\ M_2 &= -\Phi_2[\hat{A}^*(t)\Phi_2{}^* + \hat{C}^*(t)\Psi_2{}^*] - \Psi_2[\hat{B}^*(t)\Phi_2{}^* - \hat{D}^*(t)\Psi_2{}^*], \end{aligned} \tag{7.10}$$

*then the solution $(\Phi_1(t);\ \Psi_1(t);\ \Phi_2(t);\ \Psi_2(t);\ R_1(t);\ R_2(t))$ of the differential system*

$$\begin{aligned} -\Psi_2{}' - \hat{Q}^*(t; \Phi_1, \Psi_1, \Phi_2, \Psi_2)\Psi_1 &= 0, \qquad \Psi_2(\tau) = \Psi_{02}, \\ -\Phi_2{}' - \hat{Q}^*(t; \Phi_1, \Psi_1, \Phi_2, \Psi_2)\Phi_1 &= 0, \qquad \Phi_2(\tau) = \Phi_{02}, \\ \Phi_1{}' - \hat{Q}(t; \Phi_1, \Psi_1, \Phi_2, \Psi_2)\Phi_2 &= 0, \qquad \Phi_1(\tau) = \Phi_{01}, \\ \Psi_1{}' - \hat{Q}(t; \Phi_1, \Psi_1, \Phi_2, \Psi_2)\Psi_2 &= 0, \qquad \Psi_1(\tau) = \Psi_{01}, \\ R_1{}' - M_1(t; \Phi_1, \Psi_1)R_1 &= 0, \qquad R_1(\tau) = R_{01}, \\ R_2{}' - M_2(t; \Phi_2, \Psi_2)\, R_2 &= 0, \qquad R_2(\tau) = R_{02}, \end{aligned} \tag{7.11}$$

*is such that*

$$\begin{aligned} H(t) &= \Phi_1{}^*(t)R_1(t), \qquad Z(t) = \Psi_1{}^*(t)R_1(t), \\ P(t) &= \Phi_2{}^*(t)R_2(t), \qquad \Sigma(t) = \Psi_2{}^*(t)R_2(t)\,. \end{aligned} \tag{7.12}$$

*Conversely, if* $(\Phi_1(t); \Psi_1(t); \Phi_2(t); \Psi_2(t); R_1(t); R_2(t))$ *is a solution of* (7.11), *where* $R_{01}$, $R_{02}$ *are nonsingular* $m \times m$ *matrices, and the* $m \times m$ *matrices* $\Phi_{01}$, $\Psi_{01}$, $\Phi_{02}$, $\Psi_{02}$ *satisfy* (7.9), *then* (7.12) *defines solutions* $(H(t); Z(t))$ *and* $(P(t); \Sigma(t))$ *of the matrix differential systems corresponding to* (7.5′) *and* (7.5″), *respectively, which satisfy* (7.7) *and*

$$\begin{aligned} R_1^*(t)R_1(t) &= H^*(t)H(t) + Z^*(t)Z(t), \\ R_2^*(t)R_2(t) &= P^*(t)P(t) + \Sigma^*(t)\Sigma(t). \end{aligned} \tag{7.13}$$

The results of the above theorem are direct consequences of those of Theorem 4.1, applied to a solution $Y(t) = (U(t); V(t))$ of the matrix system ($7.2_M$) corresponding to (7.2), with

$$U(t) = \begin{bmatrix} H(t) & 0 \\ Z(t) & 0 \end{bmatrix}, \qquad V(t) = \begin{bmatrix} 0 & P(t) \\ 0 & \Sigma(t) \end{bmatrix}.$$

In particular, (7.7) is the condition that the column vectors of $(U(t); V(t))$ are mutually conjoined solutions of (7.2). Theorem 7.1 is indeed a true extension of Theorem 4.1 to nonselfadjoint differential systems of the form (7.5′). Moreover, whenever the selfadjointness conditions (7.6) hold, and $(H(t); Z(t))$ is a conjoined basis for (7.5′), the identification $(P(t); \Sigma(t)) = (-Z(t); H(t))$ leads to the corresponding identification $(\Phi_2(t); \Psi_2(t)) = (-\Psi_1(t); \Phi_1(t))$, and the result of Theorem 7.1 reduces to that of Theorem 4.1.

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