Generalized polygons, SCABs and GABs

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Introduction

Recently there has been a great deal of activity in the geometric and group theoretic study of "building-like geometries". One of the directions of this activity has concerned GABs ("geometries that are almost buildings") and SCABs ("chamber systems that are almost buildings"). (Other names for these are "chamber systems of type M" and "geometries of type M" in [Ti 7], and "Tits geometries of type M" or "Tits chamber systems of type M" in [AS; Tim 1,2].) The main goal of this paper is to survey these developments with special emphasis on their relationships with finite geometries.

The study of SCABs is primarily based upon the work of Tits, and especially on the seminal paper [Ti 7]. While our Chapter B briefly describes his results, few proofs are given. (In fact, relatively few proofs will be provided throughout this survey.) On the other hand, our point of view will be somewhat elementary (especially when compared with [Ti 7] and Tits' paper in these Proceedings), in the sense that a great deal of space will be devoted to ways in which familiar geometric objects (e.g. hyperovals, difference sets or root systems) can be easily used in order to construct new or less familiar ones. Consequently, it will be clear both that the subject is still in its infancy and that there are many opportunities to enter into this area.

Generalized polygons are the building bricks from which SCABs and GABs are obtained. A great deal has been written about these: they are rank 2 buildings, they include projective planes, and they are very natural from a variety of geometric, group theoretic and combinatorial viewpoints. In Chapter A we will construct almost all of the known finite examples that are not projective planes, and indicate their group theoretic origins when appropriate. However, we will not discuss the large number of characterizations presently known -- especially of finite generalized quadrangles; that is outside the scope (and size) of this paper. Instead, Chapter A is intended to function as the source for generalized polygons arising later. Moreover, Chapter A highlights one basic question concerning the constructions in Chapter C: what nonclassical finite generalized polygons can appear in finite SCABs? (In fact, the only such SCABs that seem to be known are similar to those in (C.6.11).)

While Chapter B contains generalities concerning SCABs, Chapter C is devoted both to their construction and to properties of their more geometric counterparts, GABs. Numerous open problems are mentioned. While groups appear frequently throughout the paper, the final section contains results concerning chamber-transitive SCABs, and has an especially group theoretic flavor.

No attempt has been made to give a complete list of references or a complete account of who proved which parts of which results: the present list, and the paper, are already too long. Finally, it should be noted that, since this area is growing rapidly, this survey should be out of date even before it appears. (More precisely, this survey is based on material available to me in April, 1984.)

A. Generalized n-gons

In this chapter we will discuss generalized n-gons. These are combinatorial generalizations of classical geometries arising from forms on small-dimensional vector spaces. They are also the rank 2 objects from which geometries will be built in Chapters B and C. Consequently, our objective will be to describe some basic properties and to list the known finite models. It would be tempting to write a book on this subject (cf. [PaT]). However, we have only focused on those aspects that seem especially relevant to later chapters. In particular, we have not discussed many important topics: characterizations, projective embeddings, or general properties of automorphism groups.

A.1. Definitions.

Let Γ be a graph (undirected, with no loops, no multiple edges except in restricted circumstances discussed in B.1.4, and not necessarily finite). If V is its set of vertices, write $u \sim v$ iff u and v are joined by an edge. Also, set $\Gamma(v) = \{u \in V \mid u \sim v\}$. An r-partite graph is a graph Γ such that $V = V_1 \dot{\cup} \cdots \dot{\cup} V_r$ for r nonempty pairwise disjoint sets V_i such that no two vertices of any V_i are joined; that is, each edge joins vertices of V_i and V_j for some $i \neq j$. Connectedness, the distance $d(u, v)$ between two vertices u and v, and diameter are defined in the obvious ways. The girth of a graph is the smallest number of edges in a circuit (i.e., polygon) contained in the graph.

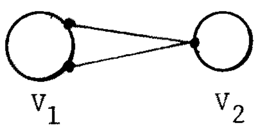
<u>Definition</u> A.1.1. A <u>rank 2 building</u> is a connected bipartite graph Γ of diameter n and girth $2n$, for some $n \geq 2$ (where $n = \infty$ is permitted), such that $|\Gamma(v)| \geq 3$ for each $v \in V$. (Here, bipartite means "2-partite.")

In a rank 2 building of diameter n, <u>any two vertices not at distance</u> n <u>are joined by a unique shortest path</u>.

Let $V = V_1 \dot{\cup} V_2$ be as above. (Of course, V_1 and V_2 are easy to recover from Γ.)

<u>Example</u>. $n = 2$. If $v_i \in V_i$ then $d(v_1, v_2)$ cannot be 3 and hence must be 1. Thus, Γ is a <u>complete bipartite graph</u>: every vertex of V_1 is joined to every vertex of V_2.

<u>Example</u>. $n = 3$. Call the vertices of V_1 "points" and those of V_2 "lines." Then two distinct points cannot be at distance ≥ 4, and hence must be at distance 2 (since a path between them must alternately pass through V_1 and V_2). Thus, two distinct points are joined to a line, which is unique because there are no 4-gons in Γ.



Similarly, any two lines are joined to a unique point. In other words, Γ is the incidence graph of a projective plane, and every projective plane produces a rank 2 building in this manner.

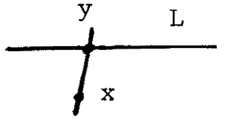
<u>Example</u>. $n = \infty$. This time Γ is just an infinite tree in which all vertices have valence ≥ 3.

Definition A.1.2. A generalized n-gon is a rank 2 building of diameter n in which the vertices in one of the sets have been called "points" and the remaining vertices have been called "lines." In other words, we have merely ordered the two sets V_1, V_2 in one of two ways. The resulting geometric object is less symmetric-looking, but slightly more satisfactory from a geometric point of view, as can be seen from the case $n = 3$: a generalized 3-gon is a single projective plane, whereas a rank 2 building of diameter 3 amounts to a pair of dual projective planes. In general, if we interchange the labels "points" and "lines," we obviously get a new generalized n-gon called the dual of the original one. A generalized n-gon is called self-dual if it is isomorphic to its dual, and self-polar if there is such an isomorphism of order 2 (a polarity).

Further examples will be given in §§ A.3, A.4. For now we will merely consider the definition from a more geometric view point.

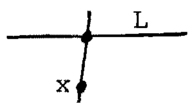
Let Γ be a generalized n-gon, where $n > 2$. Then two distinct lines are joined to at most one common point (since there are no 4-gonal circuits), and we can identify each line L with its set $\Gamma(L)$ of adjacent points. A flag of Γ is the same as an edge of the graph: an adjacent (or incident) point-line pair.

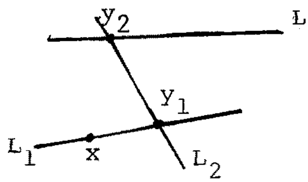
Example. $n = 4$. If a point x is not in a line L then x is collinear with a unique $y \in L$.



For, $d(x, L)$ is odd, > 1 and ≤ 4.

Example. $n = 6$. <u>If a point x is not in a line L then either x is collinear with a unique point of L or there is a unique sequence</u> $(x, L_1, y_1, L_2, y_2, L)$ <u>of points y_i and lines L_i such that successive terms are incident</u>.

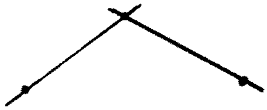




For, $d(x, L)$ is odd and ≤ 6. Uniqueness follows from the fact that the girth is 12.

Similar assertions hold whenever n is even. Here is one further example, this time with n odd.

Example. $n = 5$. <u>If two points are not collinear then there is a unique point collinear with both of them</u> (and dually).



For the points have even distance > 2 and ≤ 5; and uniqueness follows from the nonexistence of 8-gonal circuits in the graph.

A.2. Numerical restrictions

In this section we will consider *finite* generalized n-gons, where $n > 2$.

Proposition A.2.1. <u>There are integers s, $t \geq 2$ such that every line is on exactly $s + 1$ points and every point is on exactly $t + 1$ lines. If n is odd then $s = t$.</u>

Proof. If $u, v \in V$ have distance n, let $u' \in \Gamma(u)$. Then $d(u', v) < n$, and hence there is a vertex $v' \in \Gamma(v)$ at distance $n - 2$ from u'; moreover, v' is unique. Thus, $|\Gamma(u)| = |\Gamma(v)|$. Moreover, if $u'' \in \Gamma(u') - \{u\}$ then $d(u'', v) = n - 2$ or n, where the first possibility occurs only once for a given u', and $|\Gamma(u'')| = |\Gamma(v)|$ if $d(u'', v) = n$. Since $|\Gamma(u)| \geq 3$ for all u, a connectedness argument completes the proof. □

The integers s and t are the parameters of the generalized n-gon. Of course, t and s are the parameters of the dual generalized n-gon. The main results concerning n, s and t are as follows.

Theorem A.2.2 (Feit-Higman Theorem [FH]; also [Bi; Hi; KS]). $n = 3, 4, 6$ or 8.

Since projective planes $(n = 3)$ are so familiar we will only focus on the cases $n = 4, 6$ or 8.

Theorem A.2.3. Let $n = 4$. Then

(i) [FH; Hi] $st(s + 1)(t + 1)(st + 1)/\{s^2t + t^2s + s + t\} \in \mathbb{Z}$, and

(ii) [Hi; Ca] $t \leq s^2$.

Theorem A.2.4. Let $n = 6$. Then

(i) [FH; Hi] $\sqrt{st} \in \mathbb{Z}$,

(ii) [FH; Hi] $st(s + 1)(t + 1)(s^2t^2 + st + 1)/2\{s^2t + t^2s - st + s + t \pm (s - 1)(t - 1)\sqrt{st}\} \in \mathbb{Z}$

<u>for both choices of signs</u>, <u>and</u>

(iii) [Hae; HR] $t \le s^3$.

<u>Theorem A.2.5</u>. <u>Let</u> $n = 8$. <u>Then</u>

(i) [FH; Hi] $\sqrt{2st} \in \mathbb{Z}$.

(ii) [FH; Hi] $st(s+1)(t+1)(st+1)(s^2t^2+1)/2\{s^2t + st^2 + s + t\} \in \mathbb{Z}$,
$st(s+1)(t+1)(st+1)(s^2t^2+1)/4\{s^2t + st^2 - 2st + s + t \pm (s-1)(t-1)\sqrt{2st}\} \in \mathbb{Z}$

<u>for both choices of signs</u>, <u>and</u>

(iii) [Hi] $t \le s^2$.

The most remarkable and important of these results is the first one: n is highly restricted! With the exception of A.2.3(ii), all proofs of the above theorems are algebraic. Either a commutative [FH; Bi] or noncommutative [Hi] matrix algebra is associated to Γ, and multiplicities of representations are calculated. The fact that these must be integers places severe restriction on n, s and t, as can be seen in all of the theorems. A coherent exposition of all the results other than A.2.4(iii) can be found in [Hi]. (The integers in A.2.3(i), A.2.4(ii) and A.2.5(ii) are, in fact, multiplicities--as is shown in [FH; Hi].)

The fact that $t \le s^2$ in A.2.3(ii) is the only part of these theorems presently having a short, elementary proof [Ca]. In fact, that proof shows that <u>equality can hold iff the following condition holds</u>: <u>any</u> 3 <u>pairwise noncollinear points are all</u>

collinear with exactly $s + 1$ points. In [Hae] it is shown that a generalized hexagon with $t = s^3$ satisfies the following condition: for any line L and points x and y, and $0 \le i \le 2$; $0 \le j, k \le 3$;

$$|\{z \mid d(z, L) = 2i + 1,\ d(z, x) = 2j,\ d(z, y) = 2k\}|$$

depends only on i, j, k and the configuration determined by L, x and y (i.e., the isomorphism type of the subgraph whose vertices are those vertices lying on shortest paths from L to x, L to y or x to y). However, no geometric necessary and sufficient conditions seem to be known for the equalities $t = s^3$ and $t = s^2$ to hold in the cases of generalized hexagons and octagons, respectively. In the next two sections we will see that equality can indeed hold in A.2.3(ii), A.2.4(iii) and A.2.5(iii).

Roughly speaking, there seem to be large numbers of projective planes (of prime power order - e.g. in [Ka 4, 5]), reasonable numbers of generalized quadrangles, relatively few generalized hexagons and very few generalized octagons. In other words, as n increases, finite generalized n-gons appear to become scarcer - and harder to describe. We will avoid the familiar case of projective planes, and focus on generalized quadrangles (§ A.3), with a shorter discussion of the cases $n = 6$ and 8 (§ A.4).

A.3. Constructions: $n = 4$.

The classical generalized quadrangles are obtained as follows. Let V be a vector space equipped with a nondegenerate symplectic, unitary or orthogonal structure. Assume that V contains totally isotropic or totally singular 2-spaces but no totally isotropic or totally singular 3-spaces; moreover, exclude the case of an orthogonal space of dimension 4. Then let points and lines be totally isotropic or totally singular 1- and 2-spaces, respectively, with incidence (i.e., adjacence in the bipartite graph) given by inclusion. It is straightforward to check that this produces a generalized quadrangle. (The girth is at least 8 because girth 4 or 6 would imply the existence of a totally isotropic or totally singular 3-space. The diameter is 4 because if x and L are a nonincident point and line then $x, \langle x, x^{\perp} \cap L\rangle, x^{\perp} \cap L, L$ is a path of length 3.)

It is easy to construct other infinite generalized quadrangles (compare [Ti 4]). In the remainder of this section we will briefly describe the known finite ones. First of all, there are the following classical examples, whose names are those of the corresponding classical groups. We have also indicated the duals of the first two families. (Here, q is any prime power > 1.)

Name	s	t	Name of dual
$PSp(4, q)$	q	q	$P\Omega(5, q)$
$P\Omega^-(6, q)$	q	q^2	$PSU(4, q)$
$PSU(5, q)$	q^2	q^3	

The $PSp(4, q)$ quadrangles are self-dual iff q is even, and self-polar iff q has the form 2^{2e+1} for some e.

All of the above examples, and most of the known examples, can be obtained by a simple method given in [Ka2]. The following description of that method is dual to [Ka 2] in order to facilitate the use of matrices as in [CKS, § 3].

Let s and t be integers > 1. Let Q be a finite group and let $\mathcal{J}$ be a family of subgroups of Q. Assume that there is another subgroup A^* of Q associated with each $A \in \mathcal{J}$ such that the following hold for all 3-element subsets $\{A, B, C\}$ of $\mathcal{J}$:

(i) $|Q| = s^2t$, $|\mathcal{J}| = t+1$, $|A| = s$, $|A^*| = st$, $A < A^*$,

(ii) $Q = A^*B$, $A^* \cap B = 1$, and

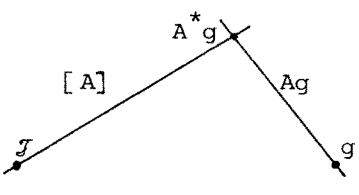
(iii) $AB \cap C = 1$.

(Here $AB = \{ab \mid a \in A, b \in B\}$; this usually will <u>not</u> be a subgroup of Q.) Given Q, $\mathcal{J}$ and $A \mapsto A^*$, we define an incidence structure $\mathcal{Q}(Q, \mathcal{J})$ as follows (where $A \in \mathcal{J}$ and $g \in Q$ are arbitrary).

<u>Point</u>. $\mathcal{J}$; coset A^*g; element g.

<u>Line</u>. Symbol $[A]$; coset Ag.

<u>Incidence</u>. $[A]$ is on $\mathcal{J}$ and A^*g; all other incidences are obtained via inclusion.



Theorem A.3.1. $\mathcal{Q}(Q,\mathcal{I})$ is a generalized quadrangle with parameters s and t.

The proof is a straightforward counting argument.

We will now present all of the known examples obtained via A.3.1. Many of these examples will appear twice, in the sense that both a quadrangle and its dual will be described in order to list all known instances of the theorem. The first two classes of examples are easy to understand. The remaining ones are fairly messy, but are included (in this form) in the hope that their availability will aid in the discovery of further examples.

(A.3.2) $s = t = q$. Q is a 3-dimensional vector space over $GF(q)$, and $\mathcal{I}$ is an oval (i.e., a set of $q+1$ 1-spaces that is an oval of the corresponding projective plane $PG(2,q)$). Also, A^* is the tangent "line" to $\mathcal{I}$ at A. These examples are all due to Tits [De, p. 304].

When $\mathcal{I}$ is a conic, $\mathcal{Q}(Q,\mathcal{I})$ is just the $P\Omega(5,q)$ quadrangle. Projectively inequivalent ovals produce nonisomorphic generalized quadrangles. For a discussion of the number of projectively inequivalent ovals, see [Hir, pp. 176-182, 416; Pa 3].

(A.3.3) $s = q$, $t = q^2$. Q is a 4-dimensional vector space over $GF(q)$, $\mathcal{I}$ is an ovoid, and A^* is the tangent "plane" to $\mathcal{I}$ at A. This example is also due to Tits [De, p. 304]. If $\mathcal{I}$ is a quadric then $\mathcal{Q}(Q,\mathcal{I})$ is the $P\Omega^-(6,q)$ quadrangle. Projectively inequivalent ovoids produce nonisomorphic generalized quadrangles. In particular, when $q = 2^{2e+1} > 2$ there is an additional generalized quadrangle obtained from the Suzuki ovoid [De, p. 52].

(A.3.4) <u>$s = q$, $t = q$</u>. The $PSp(4, q)$ quadrangle can be obtained as follows. Let

$$Q = GF(q)^3,$$

$$[\alpha, \beta, \gamma][\alpha', \beta', \gamma'] = [\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \beta\alpha' - \alpha\beta'],$$

$$A(\infty) = \{[0, \beta, 0] \mid \beta \in GF(q)\},$$

$$A(t) = \{[-\beta t, \beta, 0] \mid \beta \in GF(q)\}, \quad t \in GF(q), \text{ and}$$

$$\mathcal{T} = \{A(t) \mid t \in GF(q) \cup \{\infty\}\}.$$

Then Q has order q^3 and center $Z(Q) = \{[0, 0, \gamma] \mid \gamma \in GF(q)\}$ of order q. If $A \in \mathcal{T}$ write $A^* = AZ(Q)$. Then A.3.1 applies.

In order to see that $\mathcal{Q}(Q, \mathcal{T})$ is the $PSp(4, q)$ quadrangle, note that Q can be viewed as the set of all matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & \beta & -\alpha & 1 \end{pmatrix}$$

all of which preserve the alternating form

$$((x_i), (y_i)) = x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2$$

and induce the identity on $e_1^{\perp}/\langle e_1\rangle$, where $e_1 = (1, 0, 0, 0)$. Moreover, $A(\infty)$ is the stabilizer in Q of $\langle(0,1,0,0), (0,0,1,0)\rangle$, while $A(t)$ is the stabilizer of $\langle(0,0,0,1), (0,1,t,0)\rangle$; all of these $1 + q$ lines are totally isotropic.

(A.3.5) $\underline{s = q^2,\ t = q^3}$. The PSU(5 , q) quadrangle can be obtained as follows. Let

$$Q = \{[\alpha,\beta,\zeta,\mu] \in GF(q^2)^4 \mid \operatorname{tr}\mu + \operatorname{tr}\alpha\bar{\beta} + \zeta\bar{\zeta} = 0\}$$

$$[\alpha,\beta,\zeta,\mu][\alpha',\beta',\zeta',\mu'] = [\alpha+\alpha',\beta+\beta',\zeta+\zeta',\mu+\mu'-\bar{\alpha}\beta'-\bar{\beta}\alpha'-\bar{\zeta}\zeta']$$

where $\bar{\zeta} = \zeta^q$ and $\operatorname{tr}\zeta = \zeta + \bar{\zeta}$. Then Q is a group of order q^7 with center $Z(Q) = \{[0,0,0,\mu] \mid \operatorname{tr}\mu = 0\}$ of order q. Let $\mathcal{J}$ consist of the $1 + q^3$ groups

$$A(\infty) = \{[\alpha,0,0,0] \mid \alpha \in GF(q^2)\}$$

$$A(b,c) = \{[\beta\bar{b},\beta,\beta\bar{c},0] \mid \beta \in GF(q^2)\}$$

where $\operatorname{tr} b + c\bar{c} = 0$. Also let

$$A^*(\infty) = \{[\alpha,0,\zeta,\mu] \mid \operatorname{tr}\mu + \zeta\bar{\zeta} = 0\}$$

$$A^*(b,c) = \{[-b\beta-c\zeta,\beta,\zeta,\mu] \mid \operatorname{tr}\mu + (\bar{c}\beta - \zeta)(c\bar{\beta} - \bar{\zeta}) = 0\}.$$

Then A.3.1 applies.

In order to see that $\mathcal{Q}(Q , \mathcal{J})$ is as asserted, note that Q can be viewed as the group of all matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 0 \\ \zeta & 0 & 1 & 0 & 0 \\ \beta & 0 & 0 & 1 & 0 \\ \mu & -\bar{\beta} & -\bar{\zeta} & -\bar{\alpha} & 1 \end{pmatrix} \quad \text{with } \operatorname{tr}\mu + \operatorname{tr}\alpha\bar{\beta} + \zeta\bar{\zeta} = 0 .$$

This group preserves the hermitian form

$$((x_i),(y_i)) = \sum_{i=1}^{5} x_i\bar{y}_{6-i}$$

while inducing the identity on $e_1^{\perp}/\langle e_1\rangle$, where $e_1 = (1,0,0,0,0) \in GF(q^2)^5$. Moreover, $A^*(\infty)$ is the stabilizer in Q of

$(0,1,0,0,0)$, $A(\infty)$ is the stabilizer of $\langle(0,1,0,0,0), (0,0,0,0,1)\rangle$, $A^*(b, c)$ is the stabilizer of $\langle(0,1,c,b,0)\rangle$, and $A(b, c)$ is the stabilizer of $\langle(0,1,c,b,0), (0,0,0,0,1)\rangle$.

(A.3.6) $\underline{s = q^2, t = q}$. The $PSU(4, q)$ generalized quadrangle is obtained as follows, in the notation of (A.3.5):

$$Q_1 = \{[\alpha,\beta,0,\mu] \mid tr(\mu + \alpha\bar{\beta}) = 0\}$$
$$A_1(\infty) = A(\infty), A_1^*(\infty) = A^*(\infty) \cap Q_1$$
$$A_1(b) = A(b, 0), A_1^*(b) = A^*(b,0) \cap Q_1$$

whenever $tr\, b = 0$. This produces the required set $\mathcal{T}_1$ of $q + 1$ subgroups of order q^2 of the group Q_1 of order q^5.

(A.3.7) $\underline{s = q^3, t = q^2}$. The dual of the $PSU(5, q)$ quadrangle can be obtained as follows: use

$$Q = \{[\alpha,\beta,\sigma,\mu,\nu] \in GF(q^2)^5 \mid tr\,\alpha + \mu\bar{\mu} = 0 = tr\,\beta + \nu\bar{\nu}\},$$

$$[\alpha,\beta,\sigma,\mu,\nu][\alpha',\beta',\sigma',\mu',\nu']$$
$$= [\alpha+\alpha' - \bar{\mu}\mu', \beta+\beta' - \bar{\nu}\nu', \sigma+\sigma' - \bar{\nu}\mu', \mu+\mu', \nu+\nu'],$$

$$A(\infty) = \{[\alpha,0,0,\mu,0] \mid tr\,\alpha + \mu\bar{\mu} = 0\}$$

$$A^*(\infty) = \{[\alpha,0,\sigma,\mu,0] \mid tr\,\alpha + \mu\bar{\mu} = 0\},$$

$$A(t) = \{[t\bar{t}\beta, \beta, -t\beta, -t\nu, \nu] \mid tr\,\beta + \nu\bar{\nu} = 0\},$$

$$A^*(t) = \{t\bar{\sigma} - \bar{t}\sigma + t\bar{t}\beta,\beta,\sigma,-t\nu,\nu] \mid tr\,\beta + \nu\bar{\nu} = 0\}, \text{ and}$$

$$\mathcal{T} = \{A(t) \mid t \in GF(q^2) \cup \{\infty\}\}$$

The proof is similar to the preceding ones. Restricting to the subgroup with $\mu = \nu = 0$ brings us back to the situation in

(A.3.3) with a quadric as ovoid. (Note that $\mu = \nu = 0$ defines $Z(Q)$. The quadric is $\{[\alpha,\beta,\sigma,0,0] \mid \mathrm{tr}\,\alpha = 0 = \mathrm{tr}\,\beta,\ \alpha\beta + \sigma\bar{\sigma} = 0\}$.)

<u>Remark</u>. Now that we have indicated that every classical example arises via A.3.1, it seems appropriate to provide an explanation for this fact. In each classical case, the stabilizer of a point (or line) is a parabolic subgroup, with Levi decomposition $Q \rtimes L$ [Car, p. 119]. Here, Q is regular on the points (or lines) at distance 4 from the original one. As in (A.3.2) - (A.3.6), this produces the desired description.

<u>There is presently only one other situation in which</u> A.3.1 <u>has been applied</u> (in order to obtain a non-classical generalized quadrangle):

(A.3.8) <u>$s = q^2$, $t = q$</u>, with q a prime power, $q > 2$ and $q \equiv 2 \pmod 3$. Let

$$Q = \{(\alpha,\beta,\gamma,\delta,\epsilon) \mid \alpha,\beta,\gamma,\delta,\epsilon \in GF(q)\}$$

$$(\alpha,\beta,\gamma,\delta,\epsilon)(\alpha',\beta',\gamma',\delta',\epsilon')$$
$$= (\alpha + \alpha',\ \beta + \beta',\ \gamma + \gamma' + \alpha'\epsilon - 3\beta'\delta,\ \delta + \delta',\ \epsilon + \epsilon').$$

Then Q has order q^5 and $Z(Q) = \{(0,0,\gamma,0,0) \mid \gamma \in GF(q)\}$ has order q. Let

$$A(\infty) = \{(0,0,0,\delta,\epsilon) \mid \delta,\epsilon \in GF(q)\}$$

$$A(t) = \{(\alpha,\ \alpha t,\ -\alpha^2t^3,\ \alpha t^2,\ \alpha t^3)(0,\ \beta,\ -3\beta^2 t,\ 2\beta t,\ 3\beta t^2) \mid \alpha,\beta \in GF(q)\}$$
for $t \in GF(q)$,

$$\mathcal{T} = \{A(t) \mid t \in GF(q) \cup \{\infty\}\}$$

$$A^* = AZ(Q) \quad \text{for } A \in \mathcal{T}.$$

In [Ka 2] it is shown that A.3.1 applies and produces <u>a generalized quadrangle not isomorphic to any of those in</u> (A.3.6) <u>nor to the dual of any of those in</u> (A.3.3).

For further discussion of all of the above examples see [Ka 2].

It seems as if A.3.1 should produce still more examples. However, conditions (i) - (iii) are not very easy to work with. In particular, (iii) is very awkward. This condition does <u>not</u> say that the set ABC has size s^3; after all, s^3 may be larger than $s^2t = |Q|$. Of course, ABC <u>is</u> a group if Q is abelian, in which case (iii) states that the members of $\mathcal{J}$ are "triple-wis independent," just as (ii) states that A^* and B are "independent" whenever $A \neq B$. This independence is especially visible in (A.3.2) and (A.3.3), and is tantalizingly similar to that in the spreads involved in the construction of translation planes [De, p. 133]. However, in our situation nothing structural is known about Q. For example, it is not even known that Q must be solvable. However, much more should be true:

<u>Conjecture</u>. The conditions in A.3.1 force Q to be a p-group for some prime p.

In each classical case, and many others in (A.3.2), (A.3.3) and (A.3.8), the hypotheses of one of the following conjectures holds.

<u>Conjecture</u>. If Aut Q has a subgroup of order t fixing one member of $\mathcal{J}$ and transitive on the remaining ones, then Q is a p-group.

Conjecture. If Aut Q has a subgroup inducing a rank 1 group of Lie type on $\mathcal{T}$, in its usual 2-transitive representation, then $\mathcal{Q}(Q, \mathcal{T})$ is either classical or as in (A.3.3) (with Sz(q) induced) or (A.3.8).

Finally, we turn to the only other type of finite generalized quadrangle known at present. These have parameters $s = q - 1$, $t = q + 1$ or $s = q + 1$, $t = q - 1$, where q is a prime power, and were found in [ASz], [Hal] and [Pa 1].

(A.3.9) $\underline{s = q - 1,\ t = q + 1}$. Start with a $PSp(4, q)$ quadrangle, and one of its points p. Consider those points not in $p^{\perp}$, and those lines not containing p as well as those lines of $PG(3, q)$ containing p but not inside $p^{\perp}$. It is very easy to check that these points and lines (and ordinary inclusion) produce a generalized quadrangle with $s = q - 1$ and $t = q + 1$.

(A.3.10) $\underline{s = q - 1,\ t = q + 1}$, $q = 2^e > 2$. Let Ω be a hyperoval of $PG(2, q)$, viewed as a set of $q + 2$ 1-spaces of $V = GF(q)^3$ any 3 of which span V. Call vectors "points," and translates of members of Ω "lines." Once again, inclusion produces a generalized quadrangle. If Ω arises from a conic then we obtain the example in (A.3.9). Projectively inequivalent hyperovals produce nonisomorphic generalized quadrangles [Pa 1].

All of these generalized quadrangles arise from ones in (A.3.2) exactly as in (A.3.9), using the distinguished point $p = \mathcal{T}$.

When $q = 4$ a subgroup of $PSL(3, q)$ induces A_6 on Ω. In this case the automorphism group of the generalized quadrangle is flag-transitive. There is exactly one other example (A.3.10) having a flag-transitive automorphism group, arising when $q = 16$ [Hir, p. 177].

(A.3.11) <u>$s = q + 1$, $t = q - 1$, $q = 2^e$</u>. Let V and Ω be as in (A.3.10). Choose $A, B \in \Omega$. Define points and lines as follows:

points: vectors; translates of those 2-spaces $\neq \langle A, B\rangle$ containing A or B;

lines: translates of members of $\Omega - \{A, B\}$.

As usual, incidence is just inclusion. The verification is straightforward.

Projectively inequivalent pairs $(\Omega, \{A, B\})$ produce non-isomorphic generalized quadrangles with $s = q + 1$, $t = q - 1$. The dual of one of these is isomorphic to one in (A.3.10) iff there are q elations of $PG(2, q)$ fixing A and B and preserving Ω. This fact, and the construction, can be found in [Pa 1, 2].

<u>Summary</u>. We have now listed all of the known types of finite generalized quadrangles. The only known finite ones admitting flag-transitive automorphism groups are the classical ones and two in (A.3.10) with parameters 3, 5 and 15, 17.

A.4. Constructions: $n = 6, 8$

Up to duality, only three classes of finite generalized n-gons are known when $n = 6$ or 8. These arise from rank 2 groups of Lie type via their BN-pairs [Ti 2, p. 40; Car, p. 107], and these groups act flag-transitively.

Name	n	s	t	prime power q > 1
$G_2(q)$	6	q	q	arbitrary
${}^3D_4(q)$	6	q	q^3	arbitrary
${}^2F_4(q)$	8	q	q^2	$q = 2^{2e+1}$

The construction of these groups and their BN-pairs is rather complicated [Car], especially in the case of octagons. We will outline a construction of the dual $G_2(q)$ and the ${}^3D_4(q)$ hexagons analogous to A.3.1 but significantly more complicated (see [Ti 3]). (N.B.--The $G_2(q)$ hexagon is self-dual iff $q = 3^e$, and self-polar iff e is odd. In general, the $G_2(q)$ hexagon is distinguished from its dual by the following property: it can be realized by the set of all totally singular points and certain totally singular lines of an $\Omega(7, q)$-space; see [Ti 1] or [CaK, pp. 409, 420 - 421].)

(A.4.1) The ${}^3D_4(q)$ hexagon. Let

$$Q = \{(a,\beta,c,\delta,e) \mid a,c,e \in GF(q);\ \beta,\delta \in GF(q^3)\}$$

$$(a,\beta,c,\delta,e)(a',\beta',c',\delta',e')$$
$$= (a+a', \beta+\beta', c+c'+a'e-\mathrm{tr}\beta'\delta, \delta+\delta', e+e'),$$

where $\operatorname{tr}\gamma = \gamma + \gamma^q + \gamma^{q^2}$. Then Q is a group of order q^9. If $1 \le i \le 5$ let x_i be the element of Q whose i^{th} coordinate is x and all others 0, and let X_i be the set of all such x_i. Then $Q' = Z(Q) = X_3$ has order q. If $x \in GF(q^3)$ then $x_6 \in \operatorname{Aut} Q$, where x_6 is defined by

$$\begin{aligned}x_6\colon (a,\beta,c,\delta,e) \mapsto (&a, \beta + ax,\\ &c - a^2x^{1+q+q^2} - \operatorname{tr}\beta^{q+q^2}x - \operatorname{tr} a\beta^{q+q^2},\\ &\delta + ax^{q+q^2} + \beta^2x^{q^2} + \beta^{q^2}x^q,\\ &e + ax^{1+q+q^2}\operatorname{tr}\beta x^{q+q^2} + \operatorname{tr}\delta x).\end{aligned}$$

Now identify (for typographical reasons) $t \in GF(q^3)$ with t_6, and define

$$A_1(\infty) = X_5,\ A_2(\infty) = X_4X_5,\ A_3(\infty) = X_3X_4X_5,\ A_4(\infty) = X_2X_3X_4X_5,$$

$$A_1(t) = X_1^t,\ A_2(t) = (X_1X_2)^t,\ A_3(t) = (X_1X_2X_3)^t,\ A_4(t) = (X_1X_2X_3X_4)^t.$$

As in A.3.1, define points and lines using cosets $A_j(t)g$, where $j \in \{1,2,3,4\}$, $t \in GF(q^3) \cup \{\infty\}$ and $g \in Q$:

<u>point</u>: Symbol $\mathcal{J}$; $A_4(t)g$; $A_2(t)g$; g;

<u>line</u>: t; $A_3(t)g$; $A_1(t)g$;

<u>incidence</u>: t is on $\mathcal{J}$ and $A_4(t)g$; all other incidences are obtained via inclusion.

For more details, see [Ti 3].

There is a similar description of the dual generalized hexagon using $Q^* = X_2X_3X_4X_5X_6$ and conjugates of $\prod_{i=j}^{6} X_i$, $3 \le j \le 6$, by the elements of X_1.

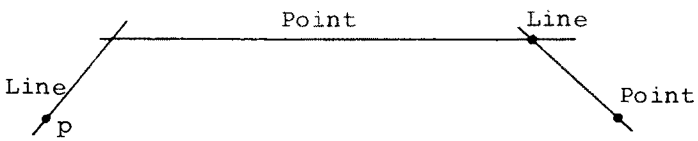
(A.4.2) Dual $G_2(q)$ hexagon. In (A.4.1), restrict β, δ and t to $GF(q)$. This produces four families of $q + 1$ groups of orders q, q^2, q^3 and q^4. The desired generalized hexagon is constructed exactly as in (A.4.1).

Remark. The group in (A.4.2), and the subgroups of order q^2 and q^3, are the same as those in (A.3.8). This produces the following strange construction of the generalized quadrangle (A.3.8) from the generalized hexagon (A.4.2). Fix a point p of the generalized hexagon, and define Points, Lines and Incidence as follows.

Point: p; points at distance 6 from p; lines at distance 3 from p.

Line: lines on p; points at distance 4 from p.

Incidence: a line on p is incident with p and all Points at distance 2 from it; a Line not incident with p is incident with all Points at distance 1 or 2 from it.



This produces a generalized quadrangle with parameters q^2, q iff $q \equiv 2 \pmod 3$.

Summary. Relatively few finite generalized hexagons or octagons are known, all requiring fairly intricate constructions. The ${}^3D_4(q)$ and ${}^2F_4(q)$ examples are especially complicated and group-related.

An analogue of A.3.1 can be proved, but involves 4 or 6 families of subgroups of a group, satisfying complicated generalizations of the conditions in A.3.1. I have made many attempts at finding new examples, using either these analogues or other, very different methods. My lack of success has led to the following

Conjecture. The only finite generalized hexagons or octagons are those naturally associated with $G_2(q)$, ${}^3D_4(q)$ or ${}^2F_4(q)$.

However, it is not at all clear how such a characterization could ever be proved. No automorphism group is hypothesized. Consequently, there is presently only one relevant characterization known of all of the above generalized hexagons: the beautiful result in [Ro 1], involving an additional but natural geometric assumption. No non-group theoretic characterization is known for the ${}^2F_4(q)$ generalized octagons.

B. Buildings and coverings

This chapter contains a summary of results of Tits [Ti 7]. These fundamental results form the theoretical superstructure upon which the remaining chapters are built. We will define building-like "geometries" (called "SCABs") and their universal covering SCABs, which usually are buildings.

B.1. Chamber systems; SCABs.

Definitions B.1.1. A chamber system $(\mathcal{C}, \{\pi_i \mid i \in I\})$ over the finite set I consists of a set $\mathcal{C}$ of chambers, together with a family $\{\pi_i \mid i \in I\}$ of partitions of $\mathcal{C}$. We will also call this "the chamber system $\mathcal{C}$". Two chambers in the same member of π_i are called i-adjacent.

If $J \subseteq I$ let π_J be the join of the partitions π_j, $j \in J$ (i.e., the set of connected components of the graph $(\mathcal{C}, \underset{J}{\sim})$, where $c \underset{J}{\sim} d \Leftrightarrow$ c and d are j-adjacent for some $j \in J$).

Call $\mathcal{C}$ connected if $\pi_I = \{\mathcal{C}\}$.

The rank of $\mathcal{C}$ is $|I|$.

If $J \subseteq I$ and $X \in \pi_J$ then the residue $\mathrm{res}_J X = \mathrm{res}\, X$ is the chamber system $(X, \{\pi_j' \mid j \in J\})$, where π_j' consists of those members of π_j lying in X. A rank r residue of $\mathcal{C}$ is any residue $\mathrm{res}_J X$ with $|J| = r$.

Construction B.1.2. Consider a group G, a subgroup B, and a finite family $\{P_i \mid i \in I\}$ of subgroups containing B. If $J \subseteq I$ write $P_J = \langle P_j \mid j \in J\rangle$. The corresponding chamber system $\mathcal{C}(G/B, \{P_i \mid i \in I\})$ has

$$\mathcal{C} = \text{the set } G/B \text{ of cosets of } B, \text{ and}$$

$$Bg \text{ and } Bh \text{ are } i\text{-adjacent} \Leftrightarrow gh^{-1} \in P_i .$$

If $J \subseteq I$ then Bg and Bh lie in the same member of π_J iff $gh^{-1} \in P_J$. In particular, $\mathcal{C}$ is connected iff $G = \langle P_i \mid i \in I \rangle$.

<u>Definition</u> B.1.3. The chamber system of a generalized n-gon (§ A.1) consists of the set of edges of the graph, together with the equivalence relations "have a common point" and "have a common line." This chamber system is said to have <u>diagram</u> $\overset{n}{\bullet\!\!-\!\!-\!\!-\!\!\bullet}$.

Note that the equivalence classes here correspond bijectively to the points and lines of the generalized n-gon.

<u>Definition</u> B.1.4. A diagram D over I, where $|I| \geq 1$, is the "graph" with "vertex set" I such that distinct vertices i and j are joined by an "edge" labeled $D(i, j)$, where $D(i, j) = D(j, i)$ is either ∞ or an integer ≥ 2.

Diagrams are abbreviated by replacing

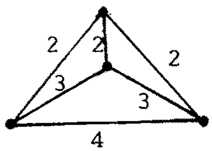
$\overset{2}{\bullet\!\!-\!\!-\!\!-\!\!\bullet}$ by $\bullet \quad \bullet$

$\overset{3}{\bullet\!\!-\!\!-\!\!-\!\!\bullet}$ by $\bullet\!\!-\!\!-\!\!-\!\!\bullet$

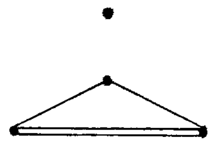
$\overset{4}{\bullet\!\!-\!\!-\!\!-\!\!\bullet}$ by $\bullet\!\!=\!\!=\!\!=\!\!\bullet$

and $\overset{6}{\bullet\!\!-\!\!-\!\!-\!\!\bullet}$ by $\bullet\!\!\equiv\!\!\equiv\!\!\equiv\!\!\bullet$.

(n = 2 , 3 , 4 , and 6 are the most prevalent instances. By A.2.2 and B.1.3, these values of n and n = 8 are the only ones relevant to finite situations.) For example,



becomes



Moreover, we now have the notion of a connected diagram.

Definition B.1.5. A SCAB with diagram D over I (or just a D-SCAB) is a connected chamber system $(\mathcal{C}, \{\pi_i \mid i \in I\})$ such that $\mathrm{res}_{\{i\,,\,j\}}X$ has diagram $\overset{D(i,j)}{\bullet\!\!-\!\!-\!\!-\!\!\bullet}$ (cf. B.1.4) whenever $X \in \pi_{\{i\,,\,j\}}$, $i \neq j$.

Note that, whenever $J \subset I$, $|J| \geq 2$, and $X \in \pi_J$, the chamber system $\mathrm{res}_J X$ is a SCAB whose diagram D_J consists of the vertices in J, with the corresponding edges labeled as before. Also note that two different chambers are i-adjacent for at most one $i \in I$.

Whenever possible--and especially in Chapter C--we will draw the diagram instead of specifying I and the $D(i\,,\,j)$.

Remark. Tits [Ti 7] considers a more general situation: I need not be finite, and only "weak" generalized polygons are required. However, (B.1.5) seems to be adequate for our purposes.

Example. A projective geometry $PG(d\,,\,K)$ produces a SCAB $\bullet\!-\!\!-\!\bullet \;\cdots\; \bullet\!-\!\!-\!\bullet$ of rank d whose chambers are the maximal flags, two flags being i-adjacent if they agree at all but (perhaps) their $(i-1)$-spaces. For example, if $1 \leq i < d-1$ and

$X \in \pi_{\{i, i+1\}}$ then res X corresponds to the projective plane $PG(2, K)$ of all $i-1$- and i-spaces lying between the $i-2$- and $i+1$-spaces in X.

Definition B.1.6. A type-preserving automorphism of $\mathcal{C}$ is a permutation of $\mathcal{C}$ leaving each π_i invariant. An automorphism of $\mathcal{C}$ is a permutation φ of $\mathcal{C}$ preserving $\{\pi_i \mid i \in I\}$; if φ is not type-preserving then it induces an automorphism of D, called a graph-automorphism (or diagram-automorphism) of $\mathcal{C}$. Sometimes a non-type-preserving automorphism is called a graph automorphism.

The group of all type-preserving automorphisms of $\mathcal{C}$ will be denoted by Aut $\mathcal{C}$.

B.2. Coxeter groups

In order to state the results in Tits [Ti 7] some additional notation is needed. Consider a diagram D over I.

Definitions B.2.1. (i) The Coxeter group with diagram D is the group $W = W(D)$ having the following presentation:

generators r_i, $i \in I$;

relations $r_i^2 = 1$, $(r_i r_j)^{D(i,j)} = 1$ whenever $i \neq j$ and $D(i,j) < \infty$.

(ii) If $J \subseteq I$ write $W_J = \langle r_j \mid j \in J \rangle$. Call J spherical if W_J is a finite subgroup of W. (All such W_J are known.)

Definition B.2.2. Let $M = M(D)$ be the free monoid on the alphabet I. The map $i \mapsto r_i$ extends to a homomorphism $m \mapsto \overline{m}$ from M onto W.

Definition B.2.3. The length $\ell(m)$ of a word $m \in M$ is the number of $i \in I$ used to "spell" m. The length $\ell(w)$ of $w \in W$ is $\min\{\ell(m) \mid \overline{m} = w\}$, and $m \in M$ is reduced if $\ell(m) = \ell(\overline{m})$. (Compare [Car, p. 109].)

B.3. Buildings; coverings

Let D, $W = W(D)$ and $M = M(D)$ be as in § B.2.

Let $(\mathcal{C}, \{\pi_i \mid i \in I\})$ be a SCAB with diagram D.

If $m = i_1 \circ \cdots i_\ell \in M$, a gallery of type m in $\mathcal{C}$ is a sequence $c_0 c_1 \cdots c_\ell$ of chambers such that c_{j-1} and c_j are i_j-adjacent for each j. If $c_{j-1} \neq c_j$ for each j then the gallery will be called nonrepetitive.

Definition B.3.1. $\mathcal{C}$ is a building with diagram D if, for any two nonrepetitive galleries starting at the same chamber, ending at the same chamber, and having reduced types m_1 and m_2, we have $\overline{m}_1 = \overline{m}_2$.

This is not the standard definition of buildings! Tits [Ti 7, Theorem 2] showed that this one is indeed equivalent to the usual one [Ti 2]. We have used the present definition in order to avoid discussing complexes and apartments (see Tits' paper in these Proceedings).

Definition B.3.2. Consider two D-SCABs $(\mathcal{C}, \{\pi_i \mid i \in I\})$ and $(\mathcal{C}', \{\pi_i' \mid i \in I\})$. A covering SCAB $\varphi: (\mathcal{C}', \{\pi_i' \mid i \in I\}) \to (\mathcal{C}, \{\pi_i \mid i \in I\})$ is a map $\varphi: \mathcal{C}' \to \mathcal{C}$ such that

(i) φ is surjective, and

(ii) If $J \subseteq I$ and $|J| \leq 2$, then each member of π_J' is mapped bijectively onto a member of π_J.

In [Ti 7] and [Ro 2] this is called a 2-covering or 2-cover (since $|J| \leq 2$ in (ii)). Of course, we will say that $\mathcal{C}'$ is a covering SCAB of $\mathcal{C}$. Note that this is not the same as topological covering spaces (but see [Ro 2]).

Definition B.3.3. $\varphi : \mathcal{C}' \to \mathcal{C}$ is a *universal covering SCAB* if it has the usual universal property: if $\psi : \mathcal{C}'' \to \mathcal{C}$ is a covering SCAB and $c'^{\varphi} = c''^{\psi}$ for some $c' \in \mathcal{C}'$, $c'' \in \mathcal{C}''$, then there is a covering SCAB $\lambda : \mathcal{C}' \to \mathcal{C}''$ such that

$$\begin{tikzcd} \mathcal{C}' \arrow[rr, "\varphi"] \arrow[dr, "\lambda"'] & & \mathcal{C} \\ & \mathcal{C}'' \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes. Then $\mathcal{C}'$ is unique up to isomorphism, and hence will be called *"the" universal covering SCAB* of $\mathcal{C}$.

Proposition B.3.4. *Every* D-SCAB $\mathcal{C}$ *has a universal covering SCAB* $\tilde{\mathcal{C}}$.

Proof. Fix a chamber c_0. Consider all galleries Λ of $\mathcal{C}$ starting at c_0. Call two such galleries Λ, Λ' equivalent if Λ' can be obtained from Λ by a sequence of replacements of the following sort: if $\Lambda = \Lambda_1\Lambda_2\Lambda_3$ and $\Lambda' = \Lambda_1\Lambda_2'\Lambda_3$ for subgalleries Λ_i and Λ_2', if Λ_2 and Λ_2' have the same first chamber and the same last chamber, and if Λ_2 and Λ_2' both lie in a member of π_J for some 2-set $J \subseteq I$, then replace Λ by Λ'.

Let $\tilde{\mathcal{C}}$ be the set of all such equivalence classes $[\Lambda]$.

Define the partition $\tilde{\pi}_i$ of $\tilde{\mathcal{C}}$ as follows: $[\Lambda]$ and $[\Lambda']$ are in the same member of $\tilde{\pi}_i$ iff $[\Lambda'] = [\Lambda c]$ for some chamber c i-adjacent to the last term of Λ.

Define $\varphi : \tilde{\mathcal{C}} \to \mathcal{C}$ by $[\Lambda]^{\varphi} =$ last term of Λ.

We must verify that B.3.2 (i) - (ii) hold and that $\tilde{\mathcal{C}}$ is a D-SCAB. Statement B.3.2 (i) is obvious. Consider $J \subseteq I$ with $|J| \le 2$, and let $X \in \tilde{\pi}_J$. If $[\Lambda] \in X$, then every member of X has the form $[\Lambda c_1 \cdots c_\ell]$, where all adjacencies between c_{k-1} and c_k, and c_1 and the last term c of Λ, are j-adjacencies for $j \in J$. Thus, $c, c_1, \cdots,$ and c_ℓ lie in a member of π_J, and every chamber in that member of π_J arises in this manner: $X^\varphi \in \pi_J$. If $[\Lambda c_1 \ldots c_\ell]^\varphi = [\Lambda c_1' \ldots c_\ell']^\varphi$ then use $\Lambda_2 = c_1 \cdots c_\ell$ and $\Lambda_2' = c_1' \cdots c_\ell'$ in the definition of equivalent galleries, together with A.1.1, in order to see that $[\Lambda c_1 \ldots c_\ell] = [\Lambda c_1' \ldots c_\ell']$. This proves B.3.2(ii). Moreover, if $|J|=2$ we see that $\mathrm{res}_J X \cong \mathrm{res}_J X^\varphi$. Finally, $\tilde{\mathcal{C}}$ is connected since every gallery can be shrunk to the 1-element gallery c_0 by a sequence of adjacencies. Now use B.1.5 in order to see that $\tilde{\mathcal{C}}$ is a D-SCAB.

If $\varphi: \mathcal{C}' \to \mathcal{C}$ is a covering SCAB let $(c_0')^\varphi = c_0$. Then every gallery beginning at c_0 lifts to a unique gallery starting at c_0', with equivalent galleries lifting to equivalent galleries. Thus, the mapping $\lambda: \tilde{\mathcal{C}} \to \mathcal{C}'$, defined by $[\Lambda]^\lambda$ = last term of the lift of Λ, behaves as required in B.3.3. □

Of course, the above proof shows that any connected chamber system has a connected universal covering [Ti 7; Ro 2].

<u>Definition</u> B.3.5. If $\varphi: \mathcal{C}' \to \mathcal{C}$ is a covering SCAB then a <u>deck transformation</u> is an automorphism α of $\mathcal{C}'$ such that $\alpha\varphi = \varphi$. These clearly form a group.

<u>Proposition</u> B.3.6. <u>Let</u> $\varphi: \tilde{\mathcal{C}} \to \mathcal{C}$ be a <u>universal covering</u> SCAB. <u>Let</u> $c_0 \in \mathcal{C}$. <u>Then the group of deck transformations is regular on</u> $\{c \in \tilde{\mathcal{C}} \mid c^\varphi = c_0\}$.

<u>Proof</u>. This is a straightforward consequence of the lifting property of galleries already noted as the end of B.3.4. □

It should be noted that the uniqueness of $\tilde{\mathcal{C}}$ (up to isomorphism) is also a consequence of the aforementioned lifting property.

Corollary B.3.7. *Every automorphism group* G *of* $\mathcal{C}$ *lifts to an automorphism group* $\tilde{G}$ *of* $\tilde{\mathcal{C}}$ *containing the group* T *of deck transformations as a normal subgroup. Moreover,* $G \cong \tilde{G}/T$, *and* $\tilde{G}$ *is chamber-transitive if* G *is*.

Proof. If $g \in G$ then the universal property of $\tilde{\mathcal{C}}$ implies that there are mappings ψ and $\tilde{g}$ making the diagrams

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & \mathcal{C} \\ & \mathcal{C} \arrow[ur, "g^{-1}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \tilde{\mathcal{C}} \arrow[rr, "\varphi"] & & \mathcal{C} \\ & \tilde{\mathcal{C}} \arrow[ul, "\tilde{g}"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commute. Then $c^{\tilde{g}\varphi} = c^{\psi} = c^{\varphi g}$ and $\tilde{g}$ is an automorphism. Different choices of $\tilde{g}$ differ by a deck transformation. This proves the first assertion, and the remaining ones follow immediately. □

Finally, we come to one version of the main result in [Ti 7].

Theorem B.3.8 [Ti 7]. $\mathcal{C}$ *is covered by a building if and only if that is true for* $\mathrm{res}_J X$ *whenever* J *is a spherical set of size* 3 *and* $X \in \pi_J$.

Proposition B.3.9 [Ti 7]. *Let* $\tilde{\mathcal{C}}$ *be a building*.

(i) $\tilde{\mathcal{C}} \xrightarrow{1} \tilde{\mathcal{C}}$ *is a universal covering* SCAB.

(ii) *If* $\varphi : \tilde{\mathcal{C}} \to \mathcal{C}$ *is a universal covering* SCAB *then so is the restriction of* φ *to each residue of* $\tilde{\mathcal{C}}$.

The property B.3.9(i) is called *simple connectedness* (or 2-connectedness [Ti 7]) of a SCAB. There are SCABs having spherical diagrams that are not buildings but that are simply connected ([Ti 7] and C.2.4). We will partly eliminate these from consideration in § C.2.

B.4. Examples.

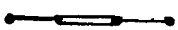
Examples of buildings will be discussed by Tits in these Proceedings. For spherical examples (i.e., having finite W -- compare B.2.1) see [Ti 2] and [Car]. We will give some "affine" examples later, when they will be needed (C.3.9, § C.5). For now, we will just note that there is at least one rank $r \geq 3$ building for each prime power q and each of the following diagrams.

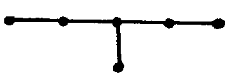
(B.4.1) Connected spherical diagrams of rank $r \geq 3$.

A_r 

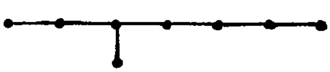
C_r

D_r 

F_4 

E_6 

E_7 

E_8 

The corresponding automorphism group contains a Chevalley group as a normal subgroup acting chamber-transitively. For a precise description of this situation, see [Car].

For the case of affine buildings we defer to Tits' paper in these Proceedings.

C. SCABs and GABs.

This chapter is the heart of this survey. It contains examples of SCABs (B.1.5) and GABs (geometries that are almost buildings), many of which are closely related to familiar geometric objects. The emphasis is on finite examples. (However, infinite examples arise in the study of the finite ones, either in the construction or implicitly as universal covering SCABs (B.3.4).) The chapter concludes with a section of problems and another containing more information of a group theoretic nature.

C.1. Definitions; criteria for GABs

Definition C.1.1. Let $(\mathcal{C}, \{\pi_i \mid i \in I\})$ be a rank r SCAB. Recall that π_J was defined in B.1.1. A vertex of type i is a member of $\pi_{I-\{i\}}$. Let V_i be the set of vertices of type i, and turn $V = V_1 \dot\cup \cdots \dot\cup V_r$ into a graph $(V, \sim)$ by requiring that (for $u, v \in V$)

$$u \sim v \Leftrightarrow u \neq v \ \underline{\text{and}}\ u \cap v \neq \emptyset.$$

(Recall that v is a set of chambers: a member of some $\pi_{I-\{i\}}$.) If $u \sim v$ we will say that u and v are incident. A flag (or clique) is a set of pairwise incident vertices.

It is important to note that V is the disjoint union of the sets V_i. In § C.3 we will see many situations in which the following situation occurs:

Definition C.1.2. $\mathcal{C}$ is tight if $V_i = \{\mathcal{C}\}$ (i.e., $\pi_{I-\{i\}} = \{\mathcal{C}\}$) for each $i \in I$. For example, in B.1.2 this occurs iff any $r - 1$ of the groups P_i generate G.

However, we will be primarily interested in a situation diametrically opposite to that in C.1.2:

Definition C.1.3. A GAB with diagram D (over I) (or D-GAB) is a triple $\Gamma = (V, \sim, \tau)$ satisfying the following conditions:

(i) $(V, \sim)$ is a graph (with $\sim$ called incidence, and flags or cliques defined as in C.1.1);

(ii) $\tau: V \to I$ is the type function: $u, v \in V$, $u \sim v \Rightarrow u^\tau \neq v^\tau$ (r-partite graph, where $r = |I|$ is the rank of Γ);

(iii) If X is any flag of size $\leq |I| - 2$ and $\Gamma(Y) = \{v \in V \mid v \sim y$ for all $y \in Y\}$ is its set of neighbors (or residue), then $\Gamma(Y)$ is connected (using the restriction of $\sim$ to Y); and

(iv) If Y is any flag of size $|I| - 2$, and if $Y^\tau = I - \{i,j\}$ (cf. (ii)), then $\Gamma(Y)$ is the graph of a generalized $D(i,j)$ - gon $\overset{D(i,j)}{\bullet\!\!-\!\!-\!\!-\!\!\bullet}$ (cf. § A.1).

Lemma C.1.4. If $\Gamma = (V, \sim, \tau)$ is a D-GAB, then a D-SCAB $\mathcal{C}(\Gamma)$ is obtained as follows: chambers are maximal flags of V; and two chambers c and d are i-adjacent $\Leftrightarrow (c \cap d)^\tau \supseteq I - \{i\}$.

Proof. Let $i, j \in I$, $i \neq j$, and let $X \in \pi_{\{i,j\}}$. If $c, d \in X$ then $d \supset Y = \{v \in c \mid v^\tau \neq i,j\}$. Thus, $X = \{Y \cup \{u,v\} \mid u,v \in \Gamma(Y)$ and $u \sim v\}$, so that $\mathrm{res}_{\{i,j\}} X$ is the generalized $D(i,j)$-gon $\Gamma(Y)$.

In order to prove that $\mathcal{C}(\Gamma)$ is connected it suffices (by the connectedness of Γ) to prove that any two intersecting chambers lie in the same member of π_I, and this is immediate by induction. □

In view of C.1.4, we can view each GAB as a SCAB. Conversely, by C.1.1 GABs "are" precisely those SCABs in which the vertices determine all of the chambers (i.e., in which chambers are precisely the maximal flags). If X is any flag of the GAB Γ, then $\Gamma(X)$ is a GAB arising from a residue of $\mathcal{C}(\Gamma)$.

With this correspondence in mind, we can state one of the motivations for the study of GABs (whose proof is, however, elementary):

<u>Proposition</u> C.1.5 [Ti 2]. <u>All buildings "are" GABs</u> (i.e., arise as $\mathcal{C}(\Delta)$ for a building Δ).

In general, it is not easy to construct other "nice" GABs explicitly, and still harder to show that a SCAB arises from a GAB as in C.1.4. The remainder of this section is concerned with two criteria concerning the latter problem.

A diagram is <u>linear</u> (or a "string") if it has the form $\bullet\!\!-\!\!\bullet\cdots\bullet\!\!-\!\!\bullet$ once all the numbers $D(i,j) > 2$ are erased. Thus, the underlying graph is just a path.

<u>Definition</u> C.1.6. Given a group G and a finite collection $\{G_i \mid i \in I\}$ of subgroups generating G, let $\Gamma = \Gamma(G; \{G_i \mid i \in I\})$ be the graph whose vertex set is the disjoint union of the sets of cosets of the G_i in G, with $G_i g \sim G_j h$ iff $i \neq j$ and $G_i g \cap G_j h \neq \emptyset$. There is also a natural <u>type</u> function $\tau \colon G_i g \mapsto i$.

If $J \subseteq I$ let $G_J = \bigcap \{G_j \mid j \in J\}$.

<u>Theorem</u> C.1.7 [A 1;MT]. <u>Assume that G and G_i are as before, and satisfy the conditions</u>

(i) _If_ $J \subseteq I$ _and_ $|I - J| \le 2$ _then_ $G_J = \langle G_{J \cup \{i\}} \mid i \in I - J \rangle$;

(ii) _If_ $J \subset I$ _and_ $I - J = \{i,k\}$ _with_ $i \neq k$, _then_ $\Gamma(G_J; \{G_{J \cup \{i\}}, G_{J \cup \{k\}}\})$ _is the graph of a generalized_ $D(i,k)$-_gon for some_ $D(i,k) \ge 2$; _and_

(iii) _The diagram_ D _determined by the_ $D(i,k)$ (as in B.1.4) _is linear_.

Then the following hold:

(1) $\Gamma(G; \{G_i \mid i \in I\})$ _is a_ D-GAB;

(2) G _is chamber-transitive on_ Γ; _and_

(3) _The generalized polygon in_ (ii) _is isomorphic to the residue_ $\Gamma(G_J)$ _of the vertex_ G_J.

The preceding result fails if D is not linear (e.g., by C.3.4). Another GAB criterion can frequently be applied, when sufficient amounts of transitivity are available:

Theorem C.1.8 [A 1] _Assume that_ G _and_ $\{G_i \mid i \in I\}$ _satisfy the following conditions_:

(i) _If_ $i \in I$ _then_ $\Gamma(G_i; \{G_i \cap G_j \mid j \in I - \{i\}\})$ _is a GAB_;

(ii) _If_ $i \neq j$ _then_ $G \neq G_iG_j$; _and_

(iii) _Any_ 3-_element subset of_ I _can be ordered_ i,j,k _so that_ $G_i \cap G_j$ _has at most two orbits on_ $G_i/G_i \cap G_k$ and on $G_j/G_j \cap G_k$.

Then the following hold:

(1) $\Gamma(G; \{G_i | i \in I\}$ is a GAB on which G acts chamber-transitively; and

(2) The GAB in (i) is isomorphic to the GAB determined by the subgraph $\Gamma(G_i)$.

Proof. We first show that every flag of size 3 has the form $\{G_i, G_j, G_k\}h$ for some $h \in G$ and some 3-set $\{i,j,k\} \subseteq I$. Since G is transitive on $\{i,j\}$-edges, we may assume that the flag is $\{G_i, G_j, G_k g\}$, where $g \in G_i$ and i and j are as in (iii). Suppose that this is not in the same G-orbit as $\{G_i, G_j, G_k\}$. Then $G_i \cap G_j$ has at least two orbits on $\Gamma(G_j) \cap (G/G_k)$, with representatives G_k and $G_k g$, each of which is also in $\Gamma(G_i)$. By (iii), $\Gamma(G_j) \cap (G/G_k) \subseteq \Gamma(G_i)$. By symmetry, $\Gamma(G_i) \cap (G/G_k) = \Gamma(G_j) \cap (G/G_k)$. Then $G_k G_j = G_k G_i$, so that $G_k G_j = G_k \langle G_i, G_j \rangle$. It is easy to see that $G = \langle G_i, G_j \rangle$, and this contradicts (ii).

We can now prove (2) For, if Γ_i denotes the GAB in (i), there is a natural map $\varphi : \Gamma_i \to \Gamma(G_i)$ defined by $(G_i \cap G_j)g \mapsto G_i g$ for $g \in G_i$. Moreover, φ is bijective and maps edges to edges. We claim that every edge of $\Gamma(G_i)$ arises in this manner. In fact, this is clear, since we have seen that every clique of size 3 containing G_i can be sent to $\{G_i, G_j, G_k\}$ by some element of G and hence of G_i, for some $j, k \in I$.

Consequently, G_i is transitive on the maximal flags of $\Gamma(G_i)$, and hence G is transitive on the maximal flags of Γ. Moreover, condition C.1.3 (iii) is now clear. Finally, since G is transitive on the edges of type $\{i,j\}$, C.1.3 (iv) follows from (i). This proves (1). □

Remarks. The hypotheses in C.1.7 and C.1.8 are designed for especially group theoretic situations. For the most part, they concern structural properties of the groups G_J, $J \subseteq I$, along with various relations among these groups. Note that C.1.8 (ii) amounts to the fact that the diagram for Γ is connected.

Isomorphisms such as those in C.1.7 (3) and C.1.8 (2) are crucial ingredients in much more general results proved in [A 1; MT] for graphs that are not necessarily GABs. It is an important open problem to find further criteria for a graph $(V, \sim, \tau)$ to be a GAB.

C.2. Covering GABs.

Tits' theorem B.3.8 on universal covering SCABs has an almost immediate consequence for GABs. In order to describe this we will first define "quotient GAB by an automorphism group."

Let $\Delta = (V, \sim, \tau)$ be a D-GAB. Let H be a group of type-preserving automorphisms of Γ, and let $\Delta/H = (V/H, \sim, \tau)$ be the graph with

$$V/H = \{v^H \mid v \in V\},$$

edges $\{u,v\}^H = \{u^H, v^H\}$ for edges $\{u,v\}$,

and

$$(v^H)^\tau = v^\tau.$$

In general, Δ/H will not be a GAB. However, it is a D-GAB if the following hold:

Conditions C.2.1. (i) If X is a flag of rank $|I| - 2$ then $H_X = 1$;

(ii) If X is a flag and $v \in V$ then $\{u \in v^H \mid \{u\} \cup X$ is a flag$\}$ is either $\emptyset$ or an orbit of H_X; and

(iii) If $u, v \in V$, $u \sim v$, and X is a flag such that X forms a flag together with some element of u^H and also with some element of v^H, then X forms a flag with some element of $\{u,v\}^H$. (Here, u and X "form a flag" if $\{u\} \cup X$ is a flag.)

Each building "is" a GAB (C.1.5). One version of the main theorem of [Ti 7] is a partial converse of this fact.

Theorem C.2.2 [Ti 7]. *Let* Γ *be a* GAB *of* rank ≥ 2. *Assume that each residue with diagram* *is covered by a building, and that there is no residue with diagram*  . *Then* $\Gamma \cong \Delta/H$ *for a building* Δ *and a group* H *satisfying* C.2.1.

Here, H arises as a group of deck transformations B.3.6.

One important consequence of C.2.2 is the following result, which says that $H = 1$ when Δ is finite.

Theorem C.2.3 [BCT]. *Let* Γ *be a* D-GAB.

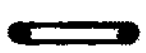
(i) Γ *is a building if* D *is* A_n, D_n *or* E_6 (cf. B.4.1).

(ii) Γ *is a building if it is finite*, D *is spherical* (cf. B.2.1) *and all residues with diagram* C_3 *are covered by buildings*.

There are several proofs of parts of C.2.3 (ii). In [BCT] adjacency matrices of graphs are used to show that C.2.1 can only hold when $H = 1$ if Δ is a finite building. A different proof that $H = 1$ for buildings Δ arising from classical groups, noticed independently by Surowski and myself, uses the fact that [Ka 1, 2.1] immediately applies. Finally, there is an elegant geometric proof (again in the classical group case) in [CS], using correlations of finite projective spaces; the same proof was also found by Brouwer and Cohen.

On the other hand, the proof of C.2.3(i) given in [Ti 7] and [BCT] for infinite Γ is primarily geometric.

There is an unfortunate hypothesis in C.2.3(ii) concerning the diagram  . That this hypothesis is essential is seen in the following remarkable GAB.

Example C.2.4 [Ne]. The exceptional GAB . Let $G = A_7$, acting on 7 points. Call triples of points "lines". Fix a conjugacy class of subgroups PSL(3,2), and call the corresponding family of $|A_7 : PSL(3,2)| = 15$ structures of PG(2,2)s "planes". Incidence is just inclusion. Each  is a PG(2,2), and each  is a PSp(4,2) quadrangle.

Conjecture. The "A_7-GAB" C.2.4 is the only finite GAB with a spherical diagram that is not covered by a building.

Some results concerning this conjecture in the case of the diagram C_3 are found in [Ot 2], [Re] and [A 2].

C.3. Difference sets.

Some of the simplest examples of SCABs arise from difference sets. These are easy to construct and exist in great profusion. Many of them are tight (C.1.2), and hence give the impression of being uninteresting. However, they are all covered by buildings (by B.3.8), and probably will turn out to be covered by many finite GABs (cf. C.6.2). We will give examples of the latter situation in the second part of this section.

Let A be an additive group of order $n^2 + n + 1$. Recall that a difference set in A is a set D of $n + 1$ elements of A such that every nonzero element of A can be uniquely written $d - d'$ for $d, d' \in D$. This uniquely determines a projective plane $\pi(A,D)$ whose points are the elements of A and whose lines are the translates $D + a$, $a \in A$.

If $E = (e_i)$ and $F = (f_i)$ are ordered $n + 1$-tuples of elements of A write $E - F = (e_i - f_i)$.

Theorem C.3.1. Let $E^\alpha = (e_i^\alpha)$, $1 \le \alpha \le r$, be ordered $n + 1$-tuples from A such that the elements in $E^\alpha - E^\beta$ form a difference set whenever $1 \le \alpha < \beta \le r$. Write

$$\mathcal{C} = \{(e_i^1 + a, \ldots, e_i^r + a) \in A^r \mid a \in A, \ 1 \le i \le n + 1\},$$

and let π_j arise from the equivalence relation "have the same j^{th} component". Then

(i) $(\mathcal{C}, \{\pi_j \mid 1 \le j \le r\})$ is a SCAB whose diagram is a complete graph on r vertices, and

(ii) Each rank 2 residue of $\mathcal{C}$ is a projective plane of the form $\pi(A, E^\alpha - E^\beta)$.

Proof. By restricting to the first two components of A^r we obtain the chamber system for $\pi(A, E^2 - E^1)$. This proves that $\pi_{\{i,j\}} = \{\mathcal{C}\}$ for $i \neq j$, and that $\mathrm{res}_{\{i,j\}}\mathcal{C} = \pi(A, E^i - E^j)$.

The preceding theorem is due to Ronan [Ro 3] when $r = 3$ and A is abelian. (There is also another use of difference sets in [Ro 3], due to Ott.)

Corollary C.3.2. Let $q = p^e$ for a prime p, let $r \geq 3$, and let $n_{r,q}$ be the number of unordered r-tuples $\{0,1,m_3,\ldots,m_r\}$ of integers (mod $q^2 + q + 1$) such that the difference of any two of them is a unit. Then C.3.1 produces at least $n_{r,q}/3e$ pairwise nonisomorphic rank r SCABs all of whose rank 2 residues are $PG(2,q)$.

Proof. First consider $\mathrm{Aut}\,\mathcal{C}$ in C.3.1 when $\pi(A, E^1 - E^2)$ is $PG(2,q)$ and A is cyclic. By restricting to $\mathrm{res}_{\{1,2\}}\mathcal{C}$ we find that $\mathrm{Aut}\,\mathcal{C} \leq P\Gamma L(3,q)$. We claim that $A \trianglelefteq \mathrm{Aut}\,\mathcal{C}$. For otherwise $\mathrm{Aut}\,\mathcal{C} \geq PSL(3,q)$ in view of the subgroup structure of $PSL(3,q)$. By considering the action of a group of q^2 elations of $\mathrm{res}_{\{1,2\}}\mathcal{C}$ on $\mathrm{res}_{\{2,3\}}\mathcal{C}$ and on $\mathrm{res}_{\{1,3\}}\mathcal{C}$, it is easy to obtain a contradiction.

Now let $E^0 = 0$ and $E^1 = D$ in C.3.1, and consider two rank r SCABs constructed there using this E^0, this E^1 and the same $A < PGL(3,q)$. Then any isomorphism $\varphi: \mathcal{C} \to \mathcal{C}'$ must send $\mathrm{Aut}\,\mathcal{C}$ to $\mathrm{Aut}\,\mathcal{C}'$, and hence must lie in $N_{P\Gamma L(3,q)}(A)$. Since $A \leq \mathrm{Aut}\,\mathcal{C} \cap \mathrm{Aut}\,\mathcal{C}'$, we may assume that φ fixes D and hence is induced by a multiplier of D; there are $3e$ such multipliers (forming $\mathrm{Aut}\,GF(q^3)$). Finally, let $E^\alpha = m_\alpha D$ in order to construct $\mathcal{C}$ in C.3.2. □

Remarks. It is easy to see that r can be made arbitrarily large in C.3.2 by using sufficiently large prime powers q.

All of the SCABs in C.3.1 and C.3.2 are tight (C.1.2). By B.3.8, each of them produces a building. When $r = 3$ it would be very interesting to know which, if any, of these buildings are "classical" ones obtained from $PSL(3,K)$ for complete local (skew) fields K [BT] (cf. (C.3.9)). Some of them certainly do: those with $q = 2$, $r = 3$ appearing in C.3.2 and its proof. For, in those cases $\mathrm{Aut}\,\mathcal{C}$ is transitive on $\mathcal{C}$, and other constructions can be used (cf. C.3.11).

Definition C.3.3. For $q = 2$ or 8, a $PG(2,q)$-family consists of a group G and a family $\{X_i \mid 1 \le i \le r\}$ of subgroups of G such that

(i) $G = \langle X_1, \ldots, X_r \rangle$,

(ii) $|X_i| = q + 1$ for each i, and

(iii) $\langle X_i, X_j \rangle$ is a Frobenius group of order $(q^2 + q + 1)(q + 1)$ whenever $i \ne j$.

Proposition C.3.4. If $\{X_i \mid 1 \le i \le r\}$ is a $PG(2,q)$-family in G then $\mathcal{C}(G/1, \{X_i \mid 1 \le i \le r\})$ is a SCAB whose diagram is a complete graph on r vertices such that each rank 2 residue is $PG(2,q)$. Moreover, G is transitive on chambers.

Proof. $\mathcal{C}(\langle X_i, X_j \rangle / 1, \{X_i, X_j\})$ is $PG(2,q)$ for $i \ne j$. □

The preceding result needed $q = 2$ or 8 since only then can $PG(2,q)$ admit a sharply flag-transitive group.

Problem. Generalize C.3.4 to all q.

Problem. Construct finite SCABs $\triangle$ with nondesarguesian residues.

The remainder of this section is concerned with examples of C.3.4. These are easy to find:

Example C.3.5. Let d be any positive integer. Let $G = GF(q^2 + q + 1)^d \rtimes Z_{q+1}$ where the Z_{q+1} corresponds to scalar action (e.g., G could even be a Frobenius group of order $(q^2 + q + 1)(q + 1)$). Then the family of all subgroups of G of order $q + 1$ behaves as required in C.3.4. This and many equally trivial constructions can be found in [Ka 7].

Only one example is presently known of a $PG(2,2)$-family with $r \geq 4$ (in fact, $r = 4$) that produces a finite GAB [KMW 1].

Problem. Find $PG(2,q)$-families of arbitrary size that produce finite GABs.

Problem. Study those buildings arising (via B.3.8) from C.3.4 when $r \geq 4$. (In some cases all rank 3 residues produce extremely well-behaved buildings - see C.3.11. These are probably the most interesting cases of this problem.)

The case of $\triangle$ GABs in C.3.4 is much better behaved than the case $r \geq 4$. When $q = 2$ there are at most 4 buildings that arise (and two of them have been determined; cf. C.3.10, C.3.11):

Theorem C.3.6 [Ro 3]. Every $PG(2,2)$-family of three groups produces a SCAB covered by exactly one of four buildings $\mathcal{C}(G/1,\{X_1,X_2,X_3\})$, where $X_1 = \langle a \rangle$, $X_2 = \langle b \rangle$, $X_3 = \langle c \rangle$, and G has one of the following presentations: generators a,b,c, and relations $a^3 = b^3 = c^3 = 1$, $(ab)^7 = (bc^{\pm 1})^7 = (ca)^7 = 1$, and one of

$$\text{(Type I)}\quad (ab)^2 = ba,\quad (bc)^2 = cb,\quad (ca)^2 = ac,$$

$$\text{(Type II)}\quad (ab)^2 = ba,\quad (bc)^2 = cb,\quad (ac)^2 = ca,$$

$$\text{(Type III)}\quad (ab)^2 = ba,\quad (c^{-1}b)^2 = bc^{-1},\quad (ac)^2 = ca,\quad \text{or}$$

$$\text{(Type IV)}\quad (ab)^2 = ba,\quad (bc^{-1})^2 = c^{-1}b,\quad (ac)^2 = ca.$$

Remark C.3.7. There are obvious graph automorphisms in Type I ($a \mapsto b \mapsto c \mapsto a$ and $a \leftrightarrow b^{-1}$, $c \mapsto c$), Type II ($a \leftrightarrow c^{-1}$, $b \mapsto b^{-1}$), Type III ($a \mapsto b^{-1} \mapsto c^{-1} \mapsto a$ and $a \leftrightarrow b^{-1}$, $c \mapsto c$), and Type IV ($a \mapsto a$, $b \leftrightarrow c$).

Theorem C.3.8 [KMW 1]. If $\{X_1,X_2,X_3\}$ is a $GF(2)$-family in G and if $|G| > 168$ then the corresponding SCAB "is" a GAB.

This is proved by a careful examination of the possible relations among the X_i, with some help from a computer.

We now turn to explicit constructions. Since we will be able to identify the universal covers in Types I and III of C.3.6, we first give a brief description of some affine buildings.

<u>Definition</u> C.3.9. Let K be the p-adic field, the field of formal Laurent series over $GF(p)$, or more generally any complete local field. Let $\mathcal{O}$ be its valuation ring with uniformizer π (e.g., $\pi = p$ in the p-adic case). Let e_1, e_2, e_3 be the standard basis for K^3. Let R_i be the stabilizer in $R = PSL(3,K)$ of the $\mathcal{O}$-module L_i, where

$$L_0 = \langle e_1, e_2, e_3 \rangle_{\mathcal{O}},\ L_1 = \langle \tfrac{1}{\pi} e_1, e_2, e_3, \rangle_{\mathcal{O}},\ L_2 = \langle \tfrac{1}{\pi} e_1, \tfrac{1}{\pi} e_2, e_3 \rangle_{\mathcal{O}}.$$

(Here the subscript $\mathcal{O}$ refers to the $\mathcal{O}$-submodule generated by the indicated vectors.) Let $B = R_1 \cap R_2 \cap R_3$. Then the <u>affine building for</u> $PSL(3,K)$ is $\mathcal{C}(R/B, \{R_1, R_2, R_3\})$ (cf. B.1.2). Note that each $R_i \cong SL(3, \mathcal{O})$, and that $\mathcal{C}$ has diagram $\triangle$ and rank 2 residues $PG(2, \mathcal{O}/\pi\mathcal{O})$ - i.e., $PG(2,p)$ in the cases mentioned above. (See [BT].)

<u>Notation</u>. Let $K = \mathbb{Q}_2$, and let λ be a unit in $\mathcal{O} = \mathbb{Z}_2$ such that $\lambda^2 + \lambda + 2 = 0$. Define $a, \tau, b, c \in GL(3,K)$ by

$$a = \begin{pmatrix} 1 & 0 & -\lambda-1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (-\lambda-2)/2 & 0 & 0 \end{pmatrix}, b = a^{\tau},\ c = b^{\tau}.$$

Let $G = \langle a,b,c \rangle$. Then $a^3 = b^3 = c^3 = 1$ and τ^3 is a scalar matrix.

<u>Theorem</u> C.3.10 [KMW2]. (i) $\mathcal{C}(G/1, \{\langle a \rangle, \langle b \rangle, \langle c \rangle\})$ <u>is the affine building for</u> $PSL(3, \mathbb{Q}_2)$.

(ii) <u>If m is any odd integer and φ is the map "passage mod m", then</u> $\mathcal{C}(G^{\varphi}/1, \{\langle a \rangle^{\varphi}, \langle b \rangle^{\varphi}, \langle c \rangle^{\varphi}\})$ <u>is a</u> $\triangle$ SCAB <u>with rank 2 residues $PG(2,2)$ and Type I</u> (cf. C.3.6).

(iii) <u>The above SCAB is a GAB.</u>

(iv) <u>If m is an odd prime then $G^{\wp}$ is $SL(3,m)$ for $m \equiv 1, 2, 4 \pmod 7$, $SU(3,m)$ for $m \equiv 3, 5, 6 \pmod 7$, and $7^2SL(2,7)$ for $m = 7$.</u>

<u>Proof</u>. Easy calculations show that $a \in R_0 \cap R_1$, $b \in R_1 \cap R_2$ and $c \in R_3 \cap R_1$. Moreover, $\langle a,b\rangle$ is a Frobenius group of order 21 that is flag-transitive on the "residue" $L_1/2L_1$. It follows from the connectedness of $\mathcal{C} = \mathcal{C}(R/B,\{R_1,R_2,R_3\})$ that G is chamber-transitive on $\mathcal{C}$.

This proves (i). Moreover, since $G \leq SL(3, \mathbb{Z}[\frac{1}{2},\sqrt{-7}])$, passage mod m is possible and preserves the relations among a, b and c. Now an easy calculation proves (ii), and (iii) follows from C.3.8. For (iv) see [KMW 2]. $\square$

For another view of C.3.10, see C.5.9. Note that τ "is" just the graph automorphism of order 3 in C.3.7, and that it greatly simplified the construction. The same is true in the next situation.

<u>Notation</u>. Let K be the field of formal Laurent series over $GF(2)$ in the indeterminate x. Write

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 & x \\ 0 & x+1 & 1 \\ x+1+x^{-1} & x & x+1 \end{pmatrix}, \quad a = (b^{-1})^{\tau^{-1}}, \quad c = b^{\tau} \text{ and}$$

$G = \langle a,b,c\rangle$. Let R be the ring $GF(2)[x^{-1},(x^2 + x + 1)^{-1}]$. Then $G \leq SL(3,R)$.

<u>Theorem</u> C.3.11 [KMW 3]. (i) <u>$\mathcal{C}(G/1, \{\langle a\rangle,\langle b\rangle,\langle c\rangle\})$ is the affine building for</u> $PSL(3,K)$.

(ii) *Passage modulo suitable ideals of* R *produces finite* $\triangle$ *GABs in arbitrarily large solvable groups*.

(iii) *Passage modulo* Rf *for the irreducible polynomials* $f \in GF(2)[x]$ *of degree* $n \geq 10$ *produces more than* $2^{n/4}$ *pairwise nonisomorphic* $\triangle$ *GABs with chamber-transitive group* $SL(3,2^n)$.

The proof of (i) is similar to that of C.3.10(i), and passage modulo suitable ideals produces finite GABs. Parts (ii) and (iii) are discussed in [KMW 3], where a better but messier estimate is given; in particular, examples exist for all $n \geq 3$ (but $PSL(3,4)$ cannot occur [KMW 1]). It seems unexpected and remarkable that exponentially many examples can arise from the same group in the same manner.

Construction C.3.12. The elements $(2\ 3\ 5)(4\ 7\ 6)$, $(1\ 2\ 6)(4\ 7\ 5)$, and $(1\ 3\ 7)(2\ 6\ 4)$ produce a $GF(2)$-family in A_7 of Type II. The 2-, 3-, and 6-fold perfect central extensions of A_7 produce covers of this A_7 one [Ro 3]. All of these examples arise from GABs [Ro 3].

Problem. Are there infinitely many finite examples of C.3.6 Type II?

Problem. Are there *any* examples of Type IV?

In [KMW 1] all simple groups of order less than a million were examined to see which produce $PG(2,2)$-families of size 3.

We have just discussed $PG(2,2)$-families in some detail. No analogue of C.2.6 is known for $PG(2,8)$-families. However, an analogue of C.3.11 is known for $GF(8)$ [KMW 4].

C.4. A table of SCABs.

In this section we will list all the known finite SCABs (of rank ≥ 3) admitting a chamber-transitive automorphism group, other than spherical buildings and those in §C.3.

The first column gives a location in this paper where the example is mentioned (if it is). The second column gives chamber-transitive automorphism groups. If there is an integer "m" here then it is a prime in the table, but more general values of m also occur - see the reference in the fifth column. (For example, m is merely odd in (1)-(3).) The fourth column names all of the residual generalized polygons (other than generalized 2-gons). Except in (26), this is a classical polygon and the group induced on the residue contains the commutator subgroup of the corresponding Chevalley group.

Column 6 indicates whether or not the SCAB is a GAB - and occasionally points out that the SCAB is tight (C.1.2). Finally, the last column concerns the universal cover, giving the associated 2-adic group (in (1)-(6)) or, more frequently, as follows:

? = unknown, but is a building

?? = unknown, but is not a building

(cf. B.3.8).

In (1)-(5) the quadratic forms are as follows:

$$f_k = \sum_1^k x_i^2$$

$$f'_k = 2\sum_1^k x_i^2 - \sum_{i=1}^{k-1} x_i x_{i+1}.$$

The order in which the examples have been listed is as follows: first those with known 2-adic universal covers; then lexicographically in terms of the rank and the size of the fields of the residual polygons.

Numbers (25) and (26) are exceptional, in that the rank is any number at least 3.

The table does not indicate rank 3 (or higher) residues, due to considerations of space. Some of those residues merit further comments:

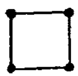
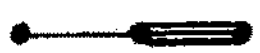
Nos. (14), (15), (18) - (21), (23), (24). One of the types of rank 3 residues is No. (7).

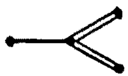
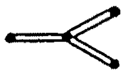
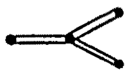
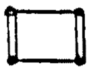
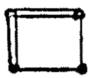
No. (15). One type of rank 3 residue is No. (3). The universal covering SCAB $\varphi: \tilde{\mathcal{C}} \to \mathcal{C}$ does <u>not</u> have the property that rank 3 residues are always mapped isomorphically by φ (compare B.3.2(ii)).

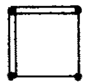
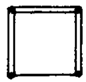
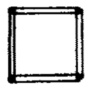
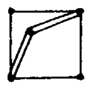
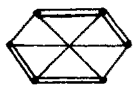
No. (19). One of the types of rank 3 residues is No. (3) with $m = 3$, and hence has a known, 2-adic universal covering building. However, there is no reason to expect that building to arise as a residue of the universal covering SCAB of (19); in fact, (19) may even by simply connected! (N.B. - This is a situation in which B.3.8 does not apply. Whenever B.3.8 does, in fact, apply, B.3.9 states that residues of universal coverings are universal coverings of residues.)

<u>Problem</u>. Given a SCAB $\mathcal{C}$, find a simple way to detect whether or not it is simply connected. In particular, if $\tilde{\mathcal{C}}$ is a building (by B.3.8) , find a simple way to decide whether or not $\mathcal{C} = \tilde{\mathcal{C}}$. (In other words, criteria are needed, involving relatively few assumptions and only simple tests, that will guarantee that $\mathcal{C}$ is a building.)

Chamber-transitive SCABs

No.	Diagram	Group	Polygons	References	GAB?	Universal cover
(1)C.5.6		$P\Omega^+(8,m)$, $m>2$	PG(2,2)	[Ka 6]; [AS] for m=3	Y	$P\Omega^+(8,\mathbb{Q}_2)$ $=P\Omega(\mathbb{Q}_2,f_8)$
(2)C.5.7		$P\Omega(7,m)$, $m>2$	PG(2,2), PSp(4,2)	[Ka 6]; [AS] for m=3	Y	$B_3(\mathbb{Q}_2)$ $=P\Omega(\mathbb{Q}_2,f_7)$
(3)C.5.7		$P\Omega^\pm(6,m)$, $m>2$ $\Omega^\pm(6,m)$, $m>2$	PSp(4,2)	[Ka 6]; [RS; Ka 2; AS] for m=3	Y	$P\Omega(\mathbb{Q}_2,f_6)$
(4)C.5.7		$P\Omega(5,m)$, $m>3$ 3^4A_6, $m=3$	PSp(4,2)	[KMW 4]	Y	$P\Omega(\mathbb{Q}_2,f_5')$
(5)C.5.7		$P\Omega^\pm(6,m)$, $m>2, m\neq 7$ $\Omega^\pm(6,m)$, $m>2, m\neq 7$ $7^5\Omega(5,7)$ $\Omega(5,7)$	PG(2,2)	[Ka 7]; [AS] for m=3	Y	$P\Omega(\mathbb{Q}_2,f_6')$
(6)		$G_2(m)$, $m>2$	PG(2,2), $G_2(2)$	[Ka 6]; [Co ; AS] for m=3	Y	$G_2(\mathbb{Q}_2)$
(7)C.2.4		A_7	PG(2,2), PSp(4,2)	[Ne; AS; Ka 6]	Y	Self [Ro 4]
(8)		Suz	PSp(4,2), PSp(4,2)	[RSm]	Y	?
(9)		$P\Omega^+(8,2)$	PSp(4,3), $\Omega(5,3)$	[Ka 3]	Y	?

No.	Diagram	Group	Polygons	References	GAB?	Universal cover
(12)		$G_2(3)$	$G_2(2)$ and its dual	[AS]	Y	?
(13)		LyS	PG(2,5), $G_2(5)$	[Ka 3]	Y	?
(14)		A_7	PSp(4,2), PG(2,2)	[KMW 4; Ro 4]	Tight	??
(15)		$P\Omega(7,m)$	PG(2,2), PSp(4,2)	[Ka 6]; [AS] for m=3	Y	Known [Li]
(16)		PSU(6,m), $m\neq 3$	PSp(4,3)	Compare [Ka 7] and No. (10)	Tight	?
(17)		$P\Omega^+(8,2)$	PSp(4,3)	Compare [Ka 3] and No. (9)	Tight	?
(18)		PSU(3,5)	PG(2,2), PSp(4,2)	[Ne; Ka 6]	Y	??
(19)		M^c	PG(2,2), PSp(4,2)	[RSt]	Y	??

No.	Diagram	Group	Polygons	References	GAB?	Universal cover
(20)		$P\Omega^{\pm}(6,m)$ $m>2$	PG(2,2) PSp(4,2)	[KMW 4]	Y	Known [Li]
(21)		$P\Omega^{-}(6,3)$	PG(2,2) PSp(4,2)	[KMW 4]		??
(22)			PSp(4,2)	[KMW 4]		?
(23)		$P\Omega^{-}(6,3)$	PG(2,2) PSp(4,2)	[KMW 4]	Tight	??
(24)		$P\Omega^{-}(6,3)$	PG(2,2) PSp(4,2)	[KMW 4]	Tight	??
(25)			PSp(4,2)	[KMW 4]		?
(26)C.6.10			(s,t) = (3,5) or (15,17)		Tight	?

C.5. E_8 root lattices.

In [Ka 6] several families of chamber-transitive GABs were constructed using E_8 root lattices and 2-adic buildings. In this section we will provide a slightly different view of that paper and of later work [Ka 7; KMW 4].

Following C.3.9 we introduced the 2-adic integers λ and $\bar{\lambda}$, where $\lambda - \bar{\lambda} = \sqrt{-7}$. Consider the vector space $\mathbb{Q}_2^8$, equipped with the usual dot product (u,v) for $u,v \in \mathbb{Q}_2^8$. Let

$$\begin{aligned} e_1 &= (\lambda,\lambda,\bar{\lambda},\lambda,\bar{\lambda},\bar{\lambda},3,0) \\ e_2 &= (3,\lambda,\lambda,\bar{\lambda},\lambda,\bar{\lambda},\bar{\lambda},0) \\ e_3 &= (\bar{\lambda},3,\lambda,\lambda,\bar{\lambda},\lambda,\bar{\lambda},0) \\ e_0 &= (1,1,1,1,1,1,1,\sqrt{-7}) \end{aligned}$$

and let $\bar{e}_i$ be obtained from e_i by interchanging λ and $\bar{\lambda}$. Write $f_0 = \bar{e}_0/14$ and

$$f_i = \sum_j a_{ij} e_j \ , \quad (a_{ij}) = \begin{pmatrix} 21 & 7\bar{\lambda} & 7\bar{\lambda} \\ 7\lambda & 21 & 7\bar{\lambda} \\ 7\lambda & 7\lambda & 21 \end{pmatrix}^{-1}$$

where $1 \le i,\ j \le 3$. Finally, let

$$\begin{aligned} &\Lambda = \{(a_i) \in \mathbb{Q}^8 \mid a_i + a_j \in \mathbb{Z} \text{ for all } i,j, \text{ and } \sum_1^8 a_i \in 2\mathbb{Z}\} \\ \text{(C.5.1)}\quad &\tilde{\Lambda} = \{(a_i) \in \mathbb{Q}_2^8 \mid a_i + a_j \in \mathbb{Z}_2 \text{ for all } i,j, \text{ and } \sum_1^8 a_i \in 2\mathbb{Z}_2\}, \end{aligned}$$

so that Λ is the usual E_8 root lattice (compare [Car, §3.6]) and $\tilde{\Lambda} \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_2$.

As in C.3.9, if $S \subseteq \mathbb{Q}_2^8$ let $\langle S \rangle_{\mathbb{Z}_2}$ be the $\mathbb{Z}_2$-submodule of $\mathbb{Q}_2^8$ generated by S. (For example, $\tilde{\Lambda} = \langle \Lambda \rangle_{\mathbb{Z}_2}$.) If S is a basis $s_1, \ldots, s_8$, write $\det\langle S \rangle_{\mathbb{Z}_2} = \det((s_i, s_j))$. This is determined only up to multiplication by an element of the group $\mathbb{Z}_2^*$ of 2-adic units.

<u>Lemma</u> C.5.2. (i) $(e_i, e_j) = 0 = (f_i, f_j)$ <u>and</u> $(e_i, f_i) = \delta_{ij}$ <u>for</u> $0 \leq i,\ j \leq 3$.

(ii) $\tilde{\Lambda} = \bigoplus_{i=0}^{3} \langle e_i, f_i \rangle_{\mathbb{Z}_2}$.

<u>Proof</u>. (i) This is just a straightforward calculation.

(ii) Let

$$L_0 = \langle e_0, e_1, e_2, e_3, \overline{e}_0/14, \overline{e}_1, \overline{e}_2, \overline{e}_3 \rangle_{\mathbb{Z}_2}.$$

Since $\det(a_{ij}) = 7^{-4} \in \mathbb{Z}_2^*$, by (i) we have $\det L_0 \in \mathbb{Z}_2^*$. By another simple calculation, each generator of L_0 is in $\tilde{\Lambda}$, so that $L_0 \subseteq \tilde{\Lambda}$. Here, $\det \tilde{\Lambda} \in \mathbb{Z}_2^*$ since Λ is a unimodular integral lattice. Thus, $L_0 = \tilde{\Lambda}$. □

Let $\mathcal{B} = \{u_i \mid 1 \leq i \leq 8\}$ be the standard basis of $\mathbb{Q}_2^8$. Let $r = r_8$ be the reflection $v \to v - 2(v, u_8)u_8$ in $u_8^{\perp}$. Then r interchanges e_0 and $\overline{e}_0$, and hence sends L_0 to

$$L_1 = L_0^r = \langle \tfrac{1}{2}e_0, e_1, e_2, e_3, 2f_0, f_1, f_2, f_3 \rangle_{\mathbb{Z}_2}.$$

The group $\Omega^+(8, \mathbb{Q}_2)$ is the commutator subgroup of the group of all isometries of $\mathbb{Q}_2^8$. The corresponding (affine) building Δ_8 has as its chambers all images of $\{L_0, L_1, L_2, L_3, L_4\}$ under this group, where

$$L_2 = \langle \tfrac{1}{2}e_0, \tfrac{1}{2}e_1, e_2, e_3, f_0, f_1, f_2, f_3 \rangle_{\mathbb{Z}_2}$$

$$L_3 = \langle \tfrac{1}{2}e_0, \tfrac{1}{2}e_1, \tfrac{1}{2}e_2, \tfrac{1}{2}e_3, f_0, f_1, f_2, f_3 \rangle_{\mathbb{Z}_2}$$

$$L_4 = \langle \tfrac{1}{2}e_0, \tfrac{1}{2}e_1, \tfrac{1}{2}e_2, \tfrac{1}{2}f_3, f_0, f_1, f_2, e_3 \rangle_{\mathbb{Z}_2}.$$

The vertices of Δ_8 are just the images of these L_i.

Let W_0 consist of those 8×8 orthogonal matrices of determinant 1 which when viewed as linear transformations (using $\mathcal{B}$) send Λ to itself. Then W_0 is the commutator subgroup of the Weyl group of type E_8, and $W_0 \cong 2\Omega^+(8,2)$. (In fact, $W_0/\langle -1 \rangle \cong \Omega^+(8,2)$, acting on the 8-dimensional GF(2)-space $\Lambda/2\Lambda$ in the natural manner, preserving the quadratic form $Q(\lambda + 2\Lambda) = \tfrac{1}{2}(\lambda,\lambda) \pmod 2$.)

Set $W_1 = W_0^r$ and $G = \langle W_0, W_1 \rangle$.

<u>Lemma</u> C.5.3. (i) G <u>consists of</u> 8×8 <u>orthogonal matrices with entries in the ring</u> $\mathbb{Z}[\tfrac{1}{2}]$.

(ii) G <u>is chamber-transitive on</u> Δ_8.

(iii) W_0 <u>is the stabilizer</u> G_{L_0}.

<u>Proof</u>. (i) For, $r = \mathrm{diag}(1,1,1,1,1,1,1,-1)$, and W_0 consists of matrices of the indicated type.

(ii) By C.5.2, W_0 acts on L_0 and induces $\Omega^+(8,2)$ on $L_0/2L_0 \cong \Lambda/2\Lambda$. Thus, W_0 is transitive on the chambers containing L_0. Also, W_1 is transitive on the chambers containing L_1. Thus, the connectedness of Δ (as a chamber system) implies (ii).

(iii) The Weyl group of type E_8 is precisely the stabilizer of Λ in the group of all 8×8 orthogonal matrices with entries in $\mathbb{Q}$. □

Now define $W_i = G_{L_i}$ for $0 \le i \le 4$.

Lemma C.5.4. $W_0 \cong W_1 \cong W_3 \cong W_4 \cong 2\Omega^+(8,2)$.

Proof. If $i = 3$ or 4 then $W_0 \cap W_i$ is the stabilizer in W_0 of a totally singular 1-space of $L_0/2L_0$. Also, $W_0 \cap W_i$ induces on $L_i/2L_i$ the stabilizer in $\Omega^+(8,2)$ of a totally singular subspace; so does $W_1 \cap W_i$. If follows that $\langle W_0 \cap W_i, W_1 \cap W_i \rangle$ induces $\Omega^+(8,2)$ on $L_i/2L_i$. It is not difficult to check that only ± 1 in W_i can induce 1 on $L_i/2L_i$, and this readily implies the lemma. □

In [Ka 6] a linear transformation $\mathcal{E}$ is defined that lies in $GL(8,\mathbb{Q})$ and interchanges W_0 and W_3 as well as W_1 and W_4, while normalizing W_2. That transformation provides another proof of C.5.4, while showing that G has an automorphism group transitive on $\{W_0, W_1, W_3, W_4\}$.

Definition C.5.5. Let m be any odd integer > 1, and let $G \to G^{(m)}$ be the homomorphism induced by passage mod m (cf. C.5.3(i)). Let

$$\Delta_8^{(m)} = \Gamma(G^{(m)}; \{W_i^{(m)} \mid 0 \le i \le 4\})$$

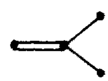
(cf. C.1.6), and let $\mathcal{C}_8^{(m)} = \mathcal{C}(\Delta_8^{(m)})$ be the corresponding chamber system obtained as in C.1.4.

Theorem C.5.6. [Ka 6]. $\Delta_8^{(m)}$ is a finite GAB with diagram , PG(2,2) residues, chamber-transitive group $G^{(m)}$, and universal covering SCAB Δ_8.

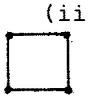
<u>Proof</u>. The map $G \to G^{(m)}$ induces a covering $\mathcal{C}_8 \to \mathcal{C}_8^{(m)}$, where $\mathcal{C}_8 = \mathcal{C}(\Delta_8)$ (cf. C.1.4). Thus, $\mathcal{C}_8^{(m)}$ is a SCAB. It is a GAB by C.1.8. By B.3.9, $\mathcal{C}_8$ is its universal covering SCAB. □

In [Ka 6] it is noted that $G^{(p)} = \Omega^+(8,p)$ for each odd prime p, and that $G^{(p)}$ has an automorphism group transitive on $\{W_i^{(p)} \mid i = 0,1,3,4\}$ and normalizing $W_2^{(p)}$. (This group of graph automorphisms can consist of inner automorphisms, depending upon the prime p.) The above GABs are No. (1) in §C.4. Of course, $G^{(m)}$ is not in $\mathrm{Aut}\,\Delta_8^{(m)}$: only $G^{(m)}/\langle -1\rangle$ is. Keeping in mind the fact that faithfulness is not an essential part of the construction of GABs via C.1.6, we can now define further "subGABs" of the above ones. (Notation: $G^{(1)} = G$.)

<u>Theorem</u> C.5.7 [Ka 6; Ka 7; KMW 4]. <u>Each of the following is a GAB with the indicated diagram and universal covering</u> SCAB:

(i) $\Gamma((G_{u_8})^{(m)}; \{(W_i)_{u_8}^{(m)} \mid i = 0,2,3,4\})$, <u>diagram</u> , <u>universal covering via</u> $m = 1$ (cf. §C.4, No. (2));

(ii) $\Gamma((G_{u_7,u_8})^{(m)}; \{((W_i)_{u_7,u_8})^{(m)} \mid i = 0,2,3\})$, <u>diagram</u> , <u>universal covering via</u> $m = 1$ (cf. §C.4, No. (3));

(iii) $\Gamma((G_{e_0,f_0})^{(m)}; \{((W_i)_{e_0,f_0})^{(m)} \mid i = 0,1,3,4\})$, <u>diagram</u> , <u>universal covering via</u> $m = 1$ (cf. §C.4, No. (5));

<u>and</u>

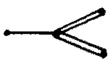
(iv) $\Gamma((G_{u_7,u_8,e_0})^{(m)}; \{((W_i)_{u_7,u_8,e_0})^{(m)} \mid i = 0,2,3\})$, <u>diagram</u> , <u>universal covering via</u> $m = 1$ (cf. §C.4, No. (4)).

In [Ka 6] there is also a construction of a family of  GABs covered by the $G_2(\mathbb{Q}_2)$ building (§C.4, No. (6)). These are obtained from $\Delta_8^{(m)}$ and $G^{(m)}$ by using a triality automorphism of $P\Omega^+(8,\mathbb{Q}_2)$.

<u>All of the preceding GABs are covered by 2-adic buildings</u>. There is a closely related family, whose universal covering SCAB is <u>not</u> a building:

<u>Theorem</u> C.5.8 [Ka 6; Li].

$$\Gamma(G_{u_8}^{(m)}; \{(G_{\langle u_7\rangle, u_8})^{(m)}, (W_i)_{u_8}^{(m)} \mid i = 0,2,3\})$$

<u>is</u> a  GAB <u>whose universal covering SCAB is obtained via $m = 1$ but is not a building</u>.

The proof in [Li] concerning the universal covering SCAB is very intriguing. It is noted that the GABs in C.5.7(i) and C.5.8 share three of their four types of vertices, and that any covering of a C.5.8 GAB induces a covering of the corresponding C.5.7(i) GAB. The same idea is used in [Li] to obtain the universal covering SCAB of §C.4, No. (20) by means of C.5.7(iii). Moreover, a general lemma in [Li] may have applications to other situations of this sort.

<u>Remark</u> C.5.9. The SCABs in C.3.10 arise in the following manner. Let H be the stabilizer in G of each of the following: $e_0, f_0, \langle e_1,e_2,e_3\rangle$, and $\langle f_1,f_2,f_3\rangle$. Then H acts transitively on the chambers of the affine $SL(3,\mathbb{Q}_2)$ building defined on $\langle e_1,e_2,e_3\rangle$. The element τ used in C.3.10 can be viewed as

$e_1 \mapsto e_2 \mapsto e_3 \mapsto e_1$, and extends in an obvious manner to an element of H. All of the SCABs and GABs in C.3.10 are visible inside $\Delta_8^{(m)}$. Moreover, H centralizes the orthogonal transformation $e_i \leftrightarrow \bar{e}_i$, and this accounts for the unitary groups appearing in C.3.10(iv).

Many further GABs have been found "inside" $\Delta_8^{(m)}$ [KMW 4]: §C.4, Nos. (20) - (24).

There are 3-adic versions of the GABs in C.5.6, admitting automorhism groups having 3 orbits on chambers. These are obtained using the same W_0, a new r, and the resulting new group $\langle W_0, W_0^r \rangle$. The small number of chamber-orbits is due to the transitivity properties of the subgroup $\Omega^+(8,2)$ of $P\Omega^+(8,3)$.

C.6. Miscellaneous problems and examples.

There are many open problems concerning either the SCABs or GABs we have constructed or the existence of other SCABs and GABs. In this section we will discuss some of these problems.

Problem C.6.1. Explicitly determine the universal covering SCABs of more of the SCABs in §C.4.

All that is known about C.6.1 is listed in §C.4. There are several reasons for studying C.6.1. In the cases of the 3 sporadic simple groups (Suz, M^C, LyS) appearing in §C.4, the universal covering SCABs may give new insight into these groups. In the majority of the cases it would be desirable to have a better understanding of the relationships between the SCABs on the list and the "classical" affine buildings defined over local fields [BT]. Of course, a few of the universal covering SCABs are already known to be such buildings (§§C.3, C.5). Others are related in more indirect manners, as indicated after C.5.8. (As another instance of examples somehow related to classical affine buildings, we remark that a relationship was noted in [Ka 7] between Nos. (10) and (11) and certain 5- and 6-dimensional 3-adic unitary groups.)

Finally, knowledge of the universal covering SCABs should produce many more finite SCABs having chamber-transitive groups (somewhat as in §C.5 or C.6.3 below). There is also the following related question (compare C.6.3):

Problem C.6.2. Under what circumstances will a finite SCAB covered by a building also be covered by infinitely many pairwise nonisomorphic finite SCABs (or even GABs)?

This is really a question concerning the fundamental group of a SCAB $\mathcal{C}$ (which is isomorphic to the group of deck transformations of the covering SCAB $\tilde{\mathcal{C}} \to \mathcal{C}$ (B.3.6); cf. [Ro 2] for the standard type of correspondence between coverings and subgroups). Namely, C.6.2 asks whether the fundamental group has infinitely many finite homomorphic images.

"Classical" affine buildings cover large numbers of GABs:

Construction C.6.3 [Ti 6]. Let G be a simple algebraic group defined over a locally compact field of characteristic 0, having rank ≥ 3, affine diagram D and affine building Δ. Then there are many discrete subgroups H of G such that the quotient Δ/H (§C.2) is a finite D-GAB. If H is such a group, so is any subgroup of finite index in H, so that Δ covers many GABs. Moreover, if H is sufficiently "small" then Δ/H even satisfies the Intersection Property (C.6.4 below). However, $\mathrm{Aut}\,(\Delta/H)$ is generally very small; in particular, only rarely will it be transitive on vertices of any type.

In particular, every classical finite generalized n-gon, $n = 3$, 4 or 6, arises as residues in infinitely many GABs.

Examples of C.6.3 can be seen in C.3.10 and C.5.6. The restriction to characteristic 0 fields is not essential: it was only assumed in order to guarantee that a group H exists. For an example in characteristic 2, see C.3.11.

We next consider structural properties of the GABs in §C.4. Buildings have various "intersection properties." We will only state one especially geometric property of this sort of a D-GAB $\Gamma = (V, \sim, \tau)$:

<u>Intersection Property</u> C.6.4. Let $i \in I$ and $V_i = \{v \in V \mid v^{\tau} = i\}$; if $v \in V - V_i$ let $\Gamma_i(v) = \Gamma(v) \cap V_i$, and let $\Gamma_i(v) = \{v\}$ if $v \in V_i$. Then, for any $u, v \in V$, $\Gamma_i(u) \cap \Gamma_i(v)$ is either $\emptyset$ or $\{x \in V \mid \{x\} \cup F$ is a flag$\}$ for some flag F of Γ for which both $\{u\} \cup F$ and $\{v\} \cup F$ are flags. Moreover, $\Gamma_i(u) \neq \Gamma_i(v)$ for $u \neq v$.

As we just indicated, <u>buildings satisfy</u> C.6.4 [Ti 2, 7].

<u>Problem</u> C.6.5. Which GABs in §C.4 have the Intersection Property? Obtain characterizations in terms of special cases of C.6.4 (or any other geometric conditions!).

Only in a few instances of §C.4 is it known whether or not some special case of the Intersection Property holds (Nos. (3) for $m = 3$, (6) for $m = 3$, (7), (8), (9), (10) for $m = 2$, (13)). Moreover, in most known situations in which any case of C.6.4 holds there is an especially nice vector space realization of the GAB involved in the proof of this property, with vertices being certain subspaces.

For further discussion of the Intersection Property see [Bu; Ti 7].

Buildings are usually defined in terms of apartments, each of which is isomorphic to $\mathcal{C}(D) = \mathcal{C}(W(D)/1, \{\langle r_i \rangle \mid i \in I\})$ -- see [Ti 2, 7]. Various attempts have been made to define apartments in GABs as "nicely embedded" morphic images of $\mathcal{C}(D)$ (e.g., [Ka 3; Ro 3]). It is not clear how "apartments" should be defined -- and undoubtedly there is no definition that can cover all of §C.4. However, it may be that usable definitions can be found only for rare classes of GABs, and that characterizations can be obtained for especially interesting GABs (other than buildings) in terms of suitable "apartments."

Another natural property of a SCAB $\mathcal{C}$ is the diameter -- either of $\mathcal{C}$, or of a suitably defined graph if $\mathcal{C}$ happens to be a GAB. Both of these notions of diameter are very appropriate in the case of buildings (where the chamber system has diameter $|W|$). The second notion has been considered for a few additional examples (e.g., in [Ka 3]). However, very little is known. The examples in C.6.3 suggest the following

<u>Problem</u> C.6.6. Characterize those SCABs of very small diameter.

Finally we return to existence questions. Probably the most basic one is suggested by the fact that only classical generalized polygons arose in C.6.3:

<u>Problem</u> C.6.7. It seems likely that <u>every</u> finite generalized polygon occurs as a residue in some finite GAB. Prove this -- or at least prove special cases of this involving either nondesarguesian projective planes or some of the generalized quadrangles in §A.3.

The case of nondesarguesian planes seems especially interesting. For example, can each translation plane occur as a residue in a $\triangle$ GAB?

Another intriguing special case of C.6.7 involves difference sets. By C.3.1, each difference set produces $\triangle$ SCABs. Probably these lift to finite GABs (cf. C.6.2). At this point, representation theoretic methods [Lie; Ot 1] may apply:

Problem C.6.8. Study finite $\triangle$ GABs Γ whose residual planes are difference set planes. Show that the residual planes of Γ are desarguesian (of order 2 or 8) if Γ admits a sharply chamber-transitive automorphism group.

The last part of C.6.8 refers to the fact [Ka 8] that each finite flag-transitive projective plane is either desarguesian or admits a sharply flag-transitive Frobenius group of prime degree p. It is well-known that, in the latter case, if $p = n^2 + n + 1$ then the subgroup of $GF(p)^*$ of order $n + 1$ is a difference set in the additive group of $GF(p)$.

Of course, C.6.8 suggests the more general

Problem C.6.9. Prove theorems whose conclusions are: "all rank 2 residues must be classical." In other words, prove results reminiscent of the one that asserts that planes of a projective 3-space are desarguesian.

We conclude this section by constructing the only known finite SCABs with nonclassical residues.

Construction C.6.10. Let $q > 2$ be a power of 2, let $m \geq 2$, let V_α be a 3-dimensional vector space over $GF(q)$ for $1 \leq \alpha \leq m$, and let $\mathcal{O}_\alpha$ be a hyperoval in V_α (viewed as a set of $q + 2$ subspaces of dimension 1; cf. (A.3.10)). Write

$$\mathcal{O}_\alpha = \{ T_j^\alpha \mid 1 \leq j \leq q + 2 \}$$

$$\mathcal{C} = \{ (v_1, \ldots, v_m, j) \mid v_\alpha \in V_\alpha,\ 1 \leq j \leq q + 2 \}$$

Call $c = (v_1, \ldots, v_m, j)$ and $c' = (v_1', \ldots, v_m', j)$ "α-adjacent" iff either

$1 \le \alpha \le m$ and c and c' disagree at most in the α^{th} component, while $v_\alpha - v_\alpha' \in T_\alpha^j$, or

$\alpha = m + 1$ and c and c' disagree at most in the $(m + 1)^{st}$ component.

<u>Theorem</u> C.6.11. $\mathcal{C}$ <u>is a rank</u> $m + 1$ SCAB <u>with diagram</u>

: <u>whose quadrangle residues have parameters</u> $q - 1$, $q + 1$.

<u>Proof</u>. If $1 \le \alpha < \beta \le m$ then each member of $\Pi_{\{\alpha,\beta\}}$ is isomorphic to the generalized 2-gon $\mathcal{C}(T_j^\alpha T_k^\beta/1, \{T_j^\alpha, T_k^\beta\})$. Each member of $\Pi_{\{1,m+1\}}$ can be identified with the generalized quadrangle A.3.10 whose chambers are the pairs $(v_1, T_j^1 + v_1)$, $v_1 \in V_1$, $1 \le j \le q + 2$, with i-adjacence corresponding to having at most the i^{th} components differ. Finally, it is easy to see that $\mathcal{C}$ is connected. ☐

<u>Remarks</u>. When $m = 2$, $\mathcal{C}$ is a GAB by C.1.7 . Note that, for $1 \le \alpha \le m$, the quadrangles in $\Pi_{\{\alpha,m+1\}}$ are all isomorphic, but that the quadrangles occurring for different subscripts α can be nonisomorphic. Moreover, since we have arbitrarily ordered each of the hyperovals $\mathcal{O}_\alpha$, the construction will produce enormous numbers of nonisomorphic SCABs having the same residues and the same numbers of chambers.

<u>Corollary</u> C.6.12. <u>If</u> $q = 4$ <u>or</u> 16 <u>and</u> $m \geq 2$ <u>then there is a rank</u> $m + 1$ <u>chamber-transitive</u> SCAB <u>having diagram</u>

: <u>and quadrangle residues with parameters</u> $q - 1$, $q + 1$.

<u>Proof</u>. C.6.11 and (A.3.10). ☐

There are other variations on the preceding construction. For example, let V be a 3-dimensional vector space over $GF(q)$, q even and $q > 2$, let $\mathcal{O} = \{T_j \mid 1 \le j \le q + 2\}$ be a hyperoval in V, and let $\sigma_1 = 1, \sigma_2, \ldots, \sigma_m$ be $m \geq 2$ permutations of $\{1, \ldots, q + 2\}$ such that $\sigma_i \sigma_j^{-1}$ is fixed-point-free for $i \neq j$. Write $\mathcal{C} = \{(T_{i\sigma_1} + v, \ldots, T_{i\sigma_m} + v, v) \mid 1 \le i \le q + 2,\ v \in V\}$, and call two elements of $\mathcal{C}$ "j-adjacent" iff they disagree in at most the j^{th} component. Then $\mathcal{C}$ is again a rank $m + 1$ SCAB with diagram  and quadrangle residues having parameters $q - 1$, $q + 1$. Of course, $m \le q + 2$ this time. There are many choices for the σ_i; the simplest has σ_2 a $q + 2$-cycle and $\sigma_i = \sigma_2^{i-1}$ for each i.

As another variation, let q be a power of an odd prime p, let V_α be isomorphic to $O_p(G_x)$ with $G \cong PSp(4,q)$ and x a point of the 4-dimensional symplectic space. Let $\mathcal{T}$ be the family of $q + 1$ subgroups of order q appearing in A.3.4, write $\mathcal{O}_\alpha = \mathcal{T} \cup \{Z(V_\alpha)\} = \{T_j^\alpha \mid 1 \le j \le q + 2\}$, and let $\mathcal{C}$ be as in C.6.10. This time define $c = (v_1, \ldots, v_m, j)$ and $c' = (v_1', \ldots, v_m', j')$ to be $m + 1$-adjacent iff $j = j'$ and $v_\alpha \in (T_j^\alpha) v_\alpha'$. This produces a SCAB as before.

C.7. Connections with finite group theory.

There has been an increasing amount of research directed towards studying and characterizing finite groups acting chamber-transitively (and faithfully) on SCABs or GABs. This has, in effect, been a new view of the theory of BN-pairs [Car] or, more precisely, of the Curtis-Steinberg-Tits relations [Cu; Ti 2, § 13]. The goal has generally been to obtain results of potential value to "revisionism": the process of reworking and improving the approaches used in the classification of finite simple groups. Unfortunately, some of these results involve additional hypotheses of a group theoretic nature, and most of the proofs are highly group theoretic. Therefore in this section we will only give a brief summary of this research.

In order to deal with chamber-transitive groups, it is first necessary to have a firm hold on the case of buildings. Here, the fundamental result is due to Seitz. (The original version of [Se] was incomplete; the following statement of his result is quoted in part from [KL], where numerous variations can also be found.)

Theorem C.7.1 [Se]. *Let G be a finite group of Lie type of rank ≥ 2 having a connected diagram and corresponding building $\mathcal{C}$. Let K be a group of type-preserving automorphisms acting chamber-transitively on $\mathcal{C}$. Then one of the following holds:*

(i) $K \geq G$;

(ii) $|K| = 7\cdot 3$, $G = PSL(3,2)$;

(iii) $|K| = 73\cdot 9$, $G = PSL(3,8)$;

(iv) K is A_7, *inside* $PSL(4,2) \cong P\Omega^+(6,2)$;

(v) K is A_6, *inside* $PSp(4,2)$;

(vi) K *is a semidirect product of an elementary abelian group of order* 16 *or* 32 *with* A_5, S_5, *or a Frobenius group of order* 20, *inside* $\mathrm{Aut}PSp(4,3) \cong \mathrm{Aut}P\Omega(5,3)$;

(vii) $K \trianglerighteq PSL(3,4)\cdot 2$, *inside* $\mathrm{Aut}PSU(4,3) \cong \mathrm{Aut}P\Omega^-(6,3)$;

(viii) $K = G_2(2)'$, *inside* $G_2(2)$; *or*

(ix) $K = {}^2F_4(2)'$, *inside* ${}^2F_4(2)$.

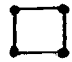
Of course, C.7.1(i) is the case of greatest interest. However, note that (ii) - (v) occurred in §§C.3, C.5.

Throughout the remainder of this section we will be concerned with the following

Hypothesis C.7.2. Let $\mathcal{C}$ be a finite SCAB of rank $r \geq 3$ having a connected diagram D, all of whose rank 2 residues (other than generalized 2-gons) arise from groups of Lie type. Let G be a chamber-transitive group of automorphisms of $\mathcal{C}$.

Recall that $\mathcal{C}$ may "be" a GAB, by C.1.4.

Timmesfeld [Tim 2] showed that, if $r = 3$, then one of the following occurs: D is linear, D is  and groups C.7.1 (ii or iii) are induced on the residual planes, or some member of some π_i has size 4. (The latter possibility probably cannot occur for nonlinear diagrams. It was not eliminated in [Tim 2] for reasons that will be explained at the end of this section.)

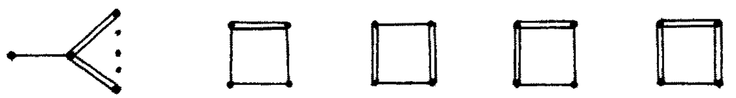
Timmesfeld also classified all diagrams D having no multiple edges [Tim 1]. He showed that $\mathcal{C}$ must be a building, except in the case of diagrams of the form  ,  or  . (In the latter cases, we saw examples in §§C.3, C.5.) Moreover, he determined all of the groups P_J for $J \neq I$. In [Ka 6, 7] it was noted that G and $\mathcal{C}$ could also be classified in the cases  and  , using the following approach. Form $\tilde{\mathcal{C}}$. This is a building by B.3.8, and hence has been determined by Tits (see his paper in these Proceedings). Some manipulation shows that $\tilde{\mathcal{C}}$ and the lifted group $\tilde{G} \leq \mathrm{Aut}\,\tilde{\mathcal{C}}$ (cf. B.3.7) are uniquely determined (up to conjugacy). Moreover, $\tilde{G}$ coincides with the appropriate group in C.5.3(i) or C.5.7(iii). Now G is a finite homomorphic image of $\tilde{G} = P\Omega(\mathbb{Z}[\frac{1}{2}],f)$ for a suitable quadratic form f (compare this with the last column in the table in §C.4). All such homomorphisms are now known, in view of the recent work of Prasad on the Congruence Subgroup Problem [Pr]. (N.B. - It is also possible to use the above approach to prove the aforementioned result in [Tim 1]. One first reduces to the case of affine diagrams of rank > 3. Various groups P_J lift to finite subgroups of $\mathrm{Aut}\,\tilde{\mathcal{C}}$, and [LaS] can then be applied to greatly restrict the possible groups P_J and buildings $\tilde{\mathcal{C}}$.)

Example C.2.4 (which we will call "$\mathcal{A}_7$" in the next few paragraphs) is clearly both beautiful and a nuisance in the present context. Several listings in §C.4 show that $\mathcal{A}_7$ can reappear in rank 4 SCABs and even rank 4 GABs (although no rank 5 occurrence in a GAB is presently known).

Recall that we are assuming C.7.2. Aschbacher [A 2] showed that, if $\mathcal{C}$ is a SCAB having a <u>linear</u> diagram and an

$\mathcal{A}_7$ residue, then $\mathcal{C}$ is $\mathcal{A}_7$. In view of C.2.3, it follows that only buildings and $\mathcal{A}_7$ can occur if D is spherical.

Along the same lines, Stroth [St 1] showed that, if $\mathcal{C}$ is a GAB having an $\mathcal{A}_7$ residue, and all rank 2 residues are generalized 2-gons, PG(2,2)'s or Sp(4,2) quadrangles, then D is one of the following:



Examples can be found in §C.4 for some of these diagrams: the first when $r = 3$ or 4, and the last three. Stroth [St 2] has also obtained further results when there is no $\mathcal{A}_7$ residue, there is an Sp(6,2) residue, and all rank 2 residues are as before.

The principal focus in [Tim 1,2] and [St 1,2] is on situations in which all rank 2 residues have characteristic 2. One reason for this is that a variety of representation theoretic techniques can be applied in that case. Another reason is that the intended group theoretic applications of these results is to groups of characteristic 2 type. An additional reason is provided by a beautiful result of Niles [Ni] which we will now describe.

Definition C.7.3. A parabolic system of characteristic p in a finite group G is a family $\{P_1,\ldots,P_r\}$ of $r \geq 2$ subgroups of G such that the following conditions hold for all distinct i and j:

(i) $G = \langle P_1,\ldots,P_r\rangle$;

(ii) $\bigcap_1^r P_i$ contains a Sylow p-subgroup of each group $P_{\{i,j\}} = \langle P_i, P_j \rangle$;

(iii) $O^{p'}(P_i/O_p(P_i)) = L_i$ is a central extension of a rank 1 group of Lie type of characteristic p; and

(iv) $O^{p'}(P_{\{i,j\}}/O_p(P_{\{i,j\}})) = L_{ij}$ is a central extension of a rank 2 group of Lie type or the product of the projections of L_i and L_j into L_{ij}.

(N.B. - If T is a group then $O^{p'}(T)$ is the smallest normal subgroup modulo which T is a p'-group.)

The product in (iv) is <u>not</u> assumed to be direct. It should be clear that (i) - (iv) are very natural in the context of C.7.2, where P_i is the stabilizer in G of that member of π_i containing a given chamber $c \in \mathcal{C}$.

In fact, there is a SCAB implicit in C.7.3. Namely, let S be the Sylow group in C.7.3(ii), and write $B = \langle N_{P_i}(S) \mid 1 \le i \le r \rangle$ and $P_i^* = P_i B$. <u>Then</u> $\mathcal{C} = \mathcal{C}(G/B, \{P_i^* \mid 1 \le i \le r\})$ <u>is a</u> SCAB. In particular, we see that each parabolic system has a diagram D (which is, of course, obvious from the definition) and produces an instance of C.7.2.

The main result of Niles is the following

<u>Theorem</u> C.7.4 [Ni]. <u>Assume that</u> $\{P_1, \ldots, P_r\}$ <u>is a parabolic system in</u> G <u>such that</u>

(a) <u>Each product appearing in</u> C.7.2(iv) <u>is direct modulo</u> $Z(L_{ij})$,

(b) *No* $L_i/Z(L_i)$ *is isomorphic to* $PSL(2,2)$, $PSL(2,3)$, $PSU(3,2)$, $Sz(2)$ *or* ${}^2G_2(3)$, *and*

(c) *No* $L_{ij}/Z(L_{ij})$ *is isomorphic to* $PSL(3,4)$.

Then $\mathcal{C}(G,\{P_1,\ldots,P_r\})$ *is a building and the group induced on* $\mathcal{C}$ *by* G *contains the corresponding Chevalley group*.

The proof in [Ni] is elegant and short: it is shown that G has a BN-pair. However, assumptions (a) - (c) are both unfortunate and at least somewhat necessary:

Examples where C.7.4 *fails without one of* (a) - (c). We refer to the table in §C.4.

$L_i/Z(L_i)$ is $PSL(2,2)$ in much of the table.

$L_i/Z(L_i)$ is $PSL(2,3)$ in Nos. (9) - (11), (16), (17).

In No. (13), with diagram , (b) and (c) hold. However, $L_{13} = SL(2,9) = SL(2,5)\cdot SL(2,5)$ occurs as a product that is not direct. Thus, the theorem "barely" fails in this case (L_{12} and L_{23} behave properly). There are many other examples with $p = 2$ in §C.4 in which *both* (a) and (b) fail.

Problems. Eliminate the restriction C.7.4(c). Characterize No. (13) in terms of the failure of condition C.7.4(a) with $p \geq 5$. Eliminate the restrictions $L_i/Z(L_i) \not\cong PSU(3,2)$, $Sz(2)$, in C.7.4(b).

Note that, when the diagram of C.7.3 is connected, ${}^2G_2(q)$ cannot occur in C.7.4 since it never appears in any L_{ij}.

The restrictions in C.7.4(b) explain why GF(3) was avoided in the result of Timmesfeld [Tim 2] mentioned earlier. They also provide further motivation for focusing on the case of characteristic 2.

It is not clear that C.7.3 automatically holds in the situation of C.7.2, although this seems likely (compare [Tim 1,2; St 2]). Finally, we note that there is also a possibility that more of the exceptional situations in C.7.1 can be induced on residues in chamber-transitive SCABs. This situation is clearly of interest, at least from a geometric point of view.

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