GENERALIZED POLYNOMIAL IDENTITIES AND PIVOTAL MONOMIALS(1)

BY

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1. Let R be an associative ring and let $\{x\} = \{x_1, x_2, \cdots\}$ be an infinite set of noncommutative indeterminates. The now classical approach to the theory of polynomial identities of a ring R was to consider identical relations in R of the form $p[x] = 0$, where $p[x] = \sum \alpha_{(i)} x_{i_1} x_{i_2} \cdots x_{i_n}$ is a polynomial in the x_j with coefficients $\alpha_{(i)}$ which are integers or belong to a commutative field F over which R is an algebra. The main result in the theory of these identities is due to Kaplansky (e.g. [3, Theorem 1, Chapter X, p. 226]) which states that a primitive ring satisfying a polynomial identity of degree d is a finite-dimensional algebra over its center, and its dimension is $\leqq [d/2]^2$.

The purpose of the present paper is to extend this result to a more general type of polynomial relation. The generalized polynomial relations to be dealt with are of the form:

$$P[x] = \sum a_{i_1} \pi_{j_1} a_{i_2} \pi_{j_2} \cdots a_{i_k} \pi_{j_k} a_{i_{k+1}} = 0,$$

where the π_j are monomials in the indeterminates x_j and the elements $a_{i_\lambda} \in R$ appear both as coefficients and between the monomials π_j. More precisely, one considers a ring R which is an algebra over a field F, and $P[x]$ are the elements of the free product of the ring R and the free associative ring $F[x_1, x_2, \cdots]$. Thus $P[x] = 0$ is an identical relation in R if for every substitution $x_i = r_j$, $P[r] = 0$.

This type of identical relation has been first studied by A. R. Richardson [6] who determined the quadratic relation for quaternions, and later by D. E. Littlewood [5] who has considered the identical relation for matrix rings over the quaternions. In particular, they have shown that the quaternions satisfy the identity

$$(xi)^2 - (ix)^2 + (xj)^2 - (jx)^2 + (xk)^2 - (kx)^2 = 0$$

where i, j, k are the quaternion basis.

For a matrix ring F_n over a commutative field F, one easily obtains the relation: $e_{11} x_1 e_{11} x_2 e_{11} - e_{11} x_2 e_{11} x_1 e_{11} = 0$, or a relation of the form

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$$\left(\sum_{i,j} e_{ij}x_1e_{ji}\right)x_2 - x_2\sum_{i,j} e_{ij}x_1e_{ji} = 0.$$

The last relation is valid in F_n since $\sum_{j,i}e_{ij}xe_{ji}$ is the trace of x and belongs to the center; furthermore, if D is a central simple algebra of dimension n^2 over its center C, then $D\otimes_C F \cong F_n$, where F is a maximal commutative subfield of D, and by expressing the orthogonal basis e_{ij} of F_n as linear combinations of elements of D, one can obtain quadratic generalized polynomial identities which hold in D. Another generalized polynomial relation will be given later. Thus the extension of the above quoted results of Kaplansky fails to hold.

Nevertheless, the following is shown: A primitive ring R satisfies a nontrivial (generalized) polynomial identity if and only if R is a dense ring of linear transformations of a space V_D over a division ring D, and the dimension of D over its center C is bounded. The bound depends on the degree of the polynomial relation $P[x]=0$ and the number of the C-independent elements of R appearing in $P[x]$.

In particular, if R is a division ring then $R=D$ and the existence of a generalized polynomial relation is equivalent to the finiteness over the center.

A second generalization of polynomial identities was given by Drazin [2] and this is the idea of a pivotal monomial. A pivotal monomial of a ring R is a monomial $\pi(x)=x_{i_1}\cdots x_{i_k}$ such that for every substitution $x_i=r_i$, the element $\pi(r)$ belongs to the left ideal generated by all monomials $\sigma(r)$, where $\sigma(x)=x_{j_1}\cdots x_{j_q}$ is such that either $q>k$, or else, $q\leqq k$ but some $i_h\neq j_h$ for $h\leqq q$. A primitive ring R was proved to possess a pivotal monomial if and only if $R=D_n$, D a division ring.

Defining a generalized pivotal monomial with respect to a given finite set of elements $a_1,a_2,\cdots,a_r$, as a monomial $\pi(x)=a_{i_1}x_{j_1}a_{i_2}x_{j_2}\cdots x_{j_k}a_{i_{k+1}}$ such that for every substitution $x_i=r_i$ the element $\pi(r)$ belongs to the left ideal generated by all $\sigma(r)$, where the $\sigma(x)$ are generalized monomials including the a_i with evident restrictions—we show that possessing such a pivotal monomial is a necessary and sufficient condition for a primitive ring to possess a left minimal ideal.

The generalization of this result in [1] obtained by assuming that $\pi(r)$ is only left quasi-regular modulo the left ideal generated by the $\sigma(r)$ works in the present case as well.

2. **A lemma.** The main result depends heavily on the following lemma, which is interesting by itself; but, surprisingly, on first observation, it seems to be hardly related to the purpose of the present paper—yet it is of fundamental importance.

LEMMA 1. *Let V, U be two vector spaces over a field F and let $T_1,\cdots,T_r$ be F-linear independent transformations of V into U; then for any finite-dimensional* *subspace U_0 of U, either there exists $v \in V$ such that $T_1v, \cdots, T_\tau v$ are linearly independent modulo U_0, or there exists $S = \sum \alpha_i T_i \neq 0$ of finite rank. Furthermore, S can be chosen so that*

$$\dim SV < \dim U_0 + \binom{\tau+1}{2} - 1.$$

Proof. Let $\mathscr{T} = \{\sum \gamma_i T_i \mid \gamma_i \in F\} \subseteq \mathrm{Hom}_F(V, U)$ be the space of linear transformations generated by the T_i. If there is no $v \in V$ such that $T_1v, \cdots, T_\tau v$ are linearly independent modulo U_0 then $\mathscr{T}$ and U_0 have the property:

(a) For each $v \in V$ there exists $0 \neq T = \sum \gamma_i T_i \in \mathscr{T}$ such that $Tv \in U_0$. Indeed, since the set $\{T_i v\}$ are linearly dependent modulo U_0 we have $\sum \gamma_i T_i v \in U_0$ for some $\gamma_i \in F$. Assuming property (a) to be valid we proceed to prove our lemma by induction on $\tau = \dim \mathscr{T}$. If $\dim \mathscr{T} = 1$ then $\mathscr{T} = FT_1$, and by assumption it follows readily that $T_1 V \subseteq U_0$ as required.

Let $\dim \mathscr{T} > 1$. Choose $v_0 \neq 0$ arbitrarily in V, and let $0 \neq T_0 \in \mathscr{T}$ be such that $T_0 v_0 \in U_0$.

Let $V_0 = \{v \mid v \in V,\ T_0 v \in U_0\}$. Thus $V_0 \neq 0$ since $v_0 \in V$. If $V_0 = V$ then it follows that $T_0 V \subseteq U_0$ and the lemma is proved with $S = T_0$.

Hence, assume that $V_0 \neq V$ and choose $v_1 \notin V_0$. Let

$$\mathscr{T}_0 = \{T \mid T \in \mathscr{T},\ Tv_1 \in U_0\}.$$

Thus, $T_0 \notin \mathscr{T}_0$ and note that the requirement of the lemma implies that $\mathscr{T}_0 \neq 0$.

Choose a submodule $\mathscr{S} \subseteq \mathscr{T}$ which contains $\mathscr{T}_0$ and which is a complement of the 1-dimensional module FT_0 in $\mathscr{T}$. Namely, let $\mathscr{T} = FT_0 \oplus \mathscr{S}$ and $\mathscr{T} \supset \mathscr{S} \supseteq \mathscr{T}_0$. This is always possible as $T_0 \notin \mathscr{T}_0$, hence one completes the base of $\mathscr{T}_0$ by adding T_0 and a set of independent elements of $\mathscr{T}$ to a base of $\mathscr{T}$. $\mathscr{S}$ will then be the linear space generated by this base with T_0 omitted. At this point we note first that $\dim \mathscr{S} < \dim \mathscr{T}$, and the lemma can be applied to $\mathscr{S}$.

Let $U_1 = U_0 + \mathscr{T} v_1$, then $(U_1 : F) \leqq (U_0 : F) + (\mathscr{T} : F) < \infty$. Now, if for all nonzero $v \in V$ there exists a nonzero $S \in \mathscr{S}$ such that $Sv \in U_1$, then by induction it follows that SV is finite dimensional for some $0 \neq S \in \mathscr{S}$ and the lemma is valid. If this is not the case then:

There exists $0 \neq w \in V$ which satisfies the condition: "$Sw \in U_1$, $S \in \mathscr{S}$ implies $S = 0$."

On the other hand, it follows by assumption that $Tw \in U_0 \subseteq U_1$ for some nonzero $T \in \mathscr{T}$. Let $T = \alpha T_1 + S_0$ with $\alpha \in F$ and $S_0 \in \mathscr{S}$. Thus $(\alpha T_0 + S_0) w \in U_1$ and this clearly implies, by the method by which w was chosen, that $\alpha \neq 0$. Without loss of generality we may assume that $\alpha = 1$.

Consider now the element $w + v_1$ with the above chosen element v_1.

By assumption, $T'(w+v_1) \in U_0$ for some $T' \neq 0$. Let $T' = \beta T_0 + S_1$ with $S_1 \in \mathscr{S}$. Thus:

$$(\beta T_0 + S_1)(w+v_1) - \beta(T_0+S_0)w = (S_1 - \beta S_0)w + (\beta T_0 + S_1)v_1$$

is an element of U_0. Consequently $(S_1 - \beta S_0)w \in U_0 + \mathscr{T} v_1$, and since that $S_1 - \beta S_0 \in \mathscr{S}$ it follows from the method by which w has been chosen that $S_1 - \beta S_0 = 0$. This in turn yields that $0 \neq T' = \beta T_0 + S_1 = \beta(T_0 + S_0)$ and $\beta \neq 0$. Furthermore, we also have $\beta^{-1}T'(w+v_1) = (T_0+S_0)(w+v_1)$ belongs to U_0 and, hence,

$$(T_0+S_0)v_1 = (T_0+S_0)(w+v_1) - (T_0+S_0)w \in U_0.$$

Recalling the way $\mathscr{T}_0$ was defined above, we have $T_0 + S_0 \in \mathscr{T}_0$. Now $\mathscr{T}_0$ was a submodule of $\mathscr{S}$, consequently it follows that $T_0 \in \mathscr{S}$ which is a contradiction to the fact that $\mathscr{S}$ is a complement of FT_0 in $\mathscr{T}$.

Summarizing, we observe that the last case is impossible and thus the proof that there exists an S of finite rank is completed.

The proof actually yields a bound for the dimension of the module SV which was proved to be finite-dimensional. Indeed, let $\mu = (U_0 : F)$ and $\tau = (\mathscr{T} : F)$, and let $\sigma(\mu,\tau)$ denote the minimal dimension of such a linear space SV. Then the preceding proof yields that:

$$\sigma(\mu, 1) \leqq \mu \quad \text{and} \quad \sigma(\mu,\tau) \leqq \operatorname{Max}[\mu, \sigma(\mu+\tau, \tau-1)].$$

One readily verifies that

$$\sigma(\mu,\tau) \leqq \mu + \tau + (\tau - 1) + \cdots + 2 = \mu + \binom{\tau+1}{2} - 1,$$

which shows that

$$\dim SV \leqq \dim U_0 + \binom{\tau+1}{2} - 1.$$

The preceding lemma can be extended as follows:

Lemma 2. *Let W be a cofinite submodule of V, i.e., $\dim V/W < \infty$, and let $\mathscr{T} \subseteq \operatorname{Hom}(V, U)$ and U_0 be as above. If for every $w \in W$ there exists $T \neq 0$ in $\mathscr{T}$ such that $Tw \in U_0$, then there exists $0 \neq S \in \mathscr{T}$ such that SV is finite-dimensional; actually S can be chosen such that*

$$\dim SV \leqq \dim U_0 + \tau \dim(V/W) + \binom{\tau+1}{2} - 1.$$

Proof. Indeed, let $v_1, \cdots, v_k$ be a finite set of independent elements of V such that $v_1 + W, \cdots, v_k + W$ are a linearly independent basis of V/W.

Let $\bar{U}_0 = U_0 + \mathscr{S}v_1 + \cdots + \mathscr{S}v_k$; then $\dim \bar{U}_0 \leqq \dim U_0 + \tau k$, where $k = \dim(V/W)$.

The conditions of Lemma 1 are now valid with $\bar{U}_0$. For let $v \in V$. Then $v = \alpha_1 v_1 + \cdots + \alpha_k v_k + w$ with some $w \in W$. By assumption let $Tw \in U_0$, $T \neq 0$, and hence $Tv \in \bar{U}_0$ as required and Lemma 1 yields the result.

For further application we wish to extend the fundamental lemma to the vector spaces over a noncommutative division ring.

Let Ω be an arbitrary set of operators of two abelian groups V, U. Let D be a division subring of $\mathrm{Hom}_\Omega(V, U)$. Then $\mathrm{Hom}_\Omega(V, U)$ can be considered as a D-space by setting $(dT)v = d(Tv)$ and clearly it commutes with the elements of Ω.

With these notations the proof of Lemma 1, with D replacing F yields the following generalization.

Lemma 3. *Let $U_0 \subseteq U$ and $\mathscr{T} \subseteq \mathrm{Hom}(V, U)$ be two finite-dimensional D-spaces. If for every $0 \neq v \in V$ there exists $0 \neq T \in \mathscr{T}$ such that $Tv \in U_0$, then there exists $0 \neq S \in \mathscr{T}$ such that SV generates a finite-dimensional D-subspace of U, and its dimension is*

$$\leqq (U_0 : D) + \binom{\tau + 1}{2} - 1,$$

where $\tau = (\mathscr{T} : D)$.

The proof is the same as the proof of Lemma 1 with D replacing F and noting that the set $\mathscr{T}_0$ defined in the proof is actually a D-space since U_0 is such, and hence one can continue with the proof with choosing the subspace $\mathscr{S} \subseteq \mathscr{T}$, etc. Note also that for $v \in V$, $\mathscr{T}v$ is a D-subspace of U and the rest follows with no additional observations.

We shall apply Lemma 3 in the case $V = U$ in the following form:

Lemma 4. *Let D be a division subring of $\mathrm{Hom}_\Omega(V, V)$ and let $T_1, \cdots, T_\tau$ be linearly left D-independent endomorphisms of V. If $V_0 \subseteq V$ is finite-dimensional D-space then either there exist $v \in V$ such that $T_1 v, \cdots, T_\tau v$ are D-independent $\mathrm{mod}\, V_0$, or some $(\sum d_i T_i)V$ generates a D-space of dimension*

$$\leqq \dim V_0 + \binom{\tau + 1}{2} - 1.$$

3. **Finite ranked transformations in primitive rings.** Let R denote throughout this paper a primitive ring—considered as a dense ring of linear transformations of a vector space V over a division ring D, i.e., $D = \mathrm{Hom}_R(V, V)$ and $R \subseteq \mathrm{Hom}_D(V, V)$. Let C be the center of D, and let F be a maximal commutative subfield of D.

Denote by $R_F(R_D, R_C)$ the subalgebra over $F(D, C)$ of $\mathrm{Hom}_Z(V, V)$(2) generated by R. For further reference we observe that R_F and R_D are homomorphic images of $R \otimes_Z F$ and $R \otimes_Z D$ respectively. Then we have:

Lemma 5. *The canonical operation of R_F, R_D on V, turns it into an irreducible ring of linear transformations with the centralizing field F and C, respectively.*

Proof. Consider first the ring R_F whose elements are of the form $\sum r_i\alpha_i$, $r_i \in R$ and $\alpha_i \in F$. Writing the operations of R and of D on the left of the elements of V, we have by definition:

$$(\textstyle\sum r_i\alpha_i)v = \sum r_i(\alpha_i v) = \sum \alpha_i(r_i v),$$

and if $(\sum r_i\alpha_i)v = 0$ for all $v \in V$ then $\sum r_i\alpha_i = 0$.

Since $R_F \supseteq R$, one readily verifies that if $v \neq 0$, $R_F v = V$ and hence R_F is an irreducible ring of linear transformations. It remains to determine the centralizer of R_F. Let $\lambda \in \mathrm{Hom}(V, V)$ commuting with all elements of R_F; thus λ commutes with $R \subseteq R_F$ and therefore $\lambda \in D$ which is the centralizer of R. Now for $\alpha \in F$, $r \in R$, the relation $r(\lambda\alpha) = \lambda(r\alpha) = (r\alpha)\lambda$ is valid. Hence, $r(\lambda\alpha - \alpha\lambda) = 0$ which yields that $r(\lambda\alpha - \alpha\lambda)v = 0$ for all $r \in R$ and $v \in V$. This implies that $\lambda\alpha = \alpha\lambda$. But F is a maximal commutative subfield of D, hence $\lambda \in F$ as required.

A similar proof holds for R_D and both are special cases of the following whose proof is similar:

Lemma 5*. *Let $K \supset C$ be a subdivision ring of D; then R_K is a dense ring of linear transformations of V over K^*, where K^* is the centralizer of K in D.*

Lemma 6. (a) *Let $0 \neq r$, $s \in R$ be such that $sxr = rxs$ for all $x \in R$; then $s = \lambda r$ for some $\lambda \in C$.*

(b) *Let $\{\alpha_i\}$ be a C-base of F; then the elements of R_F can be expressed uniquely in the form $\sum r_i\alpha_i$, $r_i \in R$.*

Proof. To prove (a) we consider the endomorphism $\lambda \in \mathrm{Hom}(V, V)$ defined by $\lambda(\sum t_i r v_i) = \sum t_i s v_i$, for all $t_i \in R$ and $v_i \in V$. This is a well-defined homomorphism, for if $\sum t_i r v_i = 0$, then for all $x \in R$ we have:

$$0 = sx\textstyle\sum t_i r v_i = \sum r x t_i s v_i = rx(\sum t_i s v_i).$$

This being true for all $x \in R$, and since $r \neq 0$, it follows that $\sum t_i s v_i = 0$. Furthermore, for all $x \in R$, $\lambda(\sum x t_i r v_i) = \sum x t_i s v_i = x\sum t_i s v_i = x\lambda(\sum t_i r v_i)$, hence $\lambda \in \mathrm{Hom}_R(V, V) = D$. Finally, for every $d \in D$:

$$d\lambda(\textstyle\sum t_i r v_i) = \sum d t_i s v_i = \sum t_i s d v_i = \lambda(\sum t_i r d v_i) = \lambda(d\sum t_i r v_i) = \lambda d(\sum t_i r v_i).$$

(2) Z denotes the ring of integers.

which proves that λ is in the center of D, i.e., $\lambda \in C$. It is evident from the definition that $s = \lambda r$, and (a) is proved.

To prove (b) it suffices to show that if $\sum_{i=1}^{k} r_i\alpha_i = 0$ then all $r_i = 0$. If this is not the case, let k (the number of elements of the last sum) be minimal; then for all $x \in R$:

$$0 = r_k x(\sum r_i\alpha_i) - (\sum r_i\alpha_i)xr_k = \sum_{i=1}^{k-1} (r_k x r_i - r_i x r_k)\alpha_i.$$

This element is of lower length. It follows therefore that $r_k x r_i - r_i x r_k = 0$ for $i = 1, \cdots, k$. Hence, (a) yields that $r_i = \lambda_i r_k$, $\lambda_i \in C$. Thus, $\sum r_i\alpha_i = r_k \sum \lambda_i\alpha_i$. Now $r_k \neq 0$, by the minimality of k, and $\sum \lambda_i\alpha_i \in F$ which is a field from which we deduce that $\sum \lambda_i\alpha_i = 0$. But the α_i are a C-base, hence all $\lambda_i = 0$ which is impossible since in particular $\lambda_k = 1$.

THEOREM 7. *Let R be a dense ring of linear transformations of V_D and let F be a maximal commutative subfield D. If R_F contains a linear transformation of finite rank over F, then R contains also a linear transformation of finite rank over D, and $(D{:}\, C) < \infty$.*

Proof. It follows by Lemma 5 that R_F is a dense ring of linear transformations. Let $T \in R_F$ such that $(TV{:}\, F) < \infty$ and let $T = \sum_{i=1}^{k} r_i\alpha_i$ with $r_i \in R$ and $\{\alpha_i\}$ a C-base of F. Among all T with this property we choose T with k minimal. We note that for $x \in R$, $(r_k x T - T x r_k) V \subseteq (r_k x) TV + TV$. TV is of finite dimension and so is $r_k x TV$ since the latter is an F-homomorphic image of TV, where the homomorphism is obtained by the mapping $Tv \rightarrow r_k x Tv$. Consequently, $r_k x T - T x r_k = \sum (r_k x r_i - r_i x r_k)\alpha_i$ is of lower length, hence $r_k x r_i = r_i x r_k$ for all $x \in R$. It follows from (a) of the preceding lemma that $r_i = r_k \lambda_i$ with $\lambda_i \in C$ and, hence, $T = r_k \sum \lambda_i\alpha_i = r_k\alpha$ with $\alpha \in F$. Since $T \neq 0$, we have also $r_k \neq 0$ and $\alpha \neq 0$.

Consequently, $TV = r_k\alpha V = r_k V$ as α^{-1} exists in F. Now $r_k V$ is as well a D-space since $r_k R$ commutes with the elements of D. Hence $\infty > (r_k V{:}\, F) = (r_k V{:}\, D)(D{:}\, F)$ which yields that both $(r_k V{:}\, D) < \infty$ and $(D{:}\, F) < \infty$. The finiteness of the first proves the first part of the theorem and the finiteness of $(D{:}\, F)$ yields (e.g. [3, Chapter VII, Theorem 9.1, p. 175]) that $(D{:}\, C) < \infty$ and since F is maximal we also have $(D{:}\, C) = (D{:}\, F)^2$.

A bound for $(D{:}\, C)$ can be obtained as follows:

Let $T = \sum_{i=1}^{k} r_i\alpha_i$ be such that $(TV{:}\, F) = m$, then the preceding proof shows that either $TV = r_k V$ or there exists $T' = \sum r_i'\alpha_i$ of lower length and $(T'V{:}\, F) \leqq 2m$. Continuing this way we get an $r \in R$ such that $(rV{:}\, F) \leqq 2^{k-1}m$. Hence from the relation $(rV{:}\, F) = (rV{:}\, D)(D{:}\, F)$, and $(D{:}\, C) = (D{:}\, F)^2$ it follows that:

COROLLARY 8. *If $T = \sum_1^k r_i\alpha_i$, and $(TV{:}\, F) = m$ then $(D{:}\, C) \leqq 2^{2k-2}m^2 = 4^{k-1}m^2$.*

A special case of Theorem 7 is of interest:

COROLLARY 8*. *Let D be a division algebra with a center C. Let F be a maximal subfield of F, then $D\otimes_C F$ is a primitive ring acting on D and it contains finite ranked transformations if and only if $(D:C)<\infty$.*

Proof. D can be considered also as a vector space over which D acts by multiplication on the left. Its centralizer is its anti-isomorphic ring D^* of all right multiplication. Now $D\otimes_C F$ can be identified with $D\otimes_C F^*=D_F$ by Lemma 6. The rest is the application of Theorem 7.

4. **Polynomial identities.** We can turn now to the main object of the paper.

Let $\{x\}=\{x_1,x_2,\cdots\}$ be an infinite set of noncommutative indeterminates, R a primitive ring which is a dense ring of linear transformations on a space V_D, and D the centralizing division ring having C as its center and let F be a maximal commutative subfield. If R is a division ring, we take $V_D=R$ and the elements of R operate by left multiplication and the centralizer is to be taken D^* the ring of all right multiplications.

Clearly, the center of R is an integral domain contained in C (and might be the zero element only), but for our purpose we assume, and as it will be seen, without loss of generality, that this center is the field C itself so that $R=R_C$ is also a C-algebra.

Let $R\langle x\rangle$ be the C-universal product of R and the free ring $C[x]$ with the x's commuting with the elements of C. Recall that in our case $R\langle x\rangle$ can be characterized uniquely up to isomorphism by the property that:

Every C-homomorphism $\phi: R\to S$ into a C-algebra S and a mapping $\psi: x_i\to s_i$ have a unique extension to a homomorphism $\bar\phi: R\langle x\rangle\to S$. An extension, in the sense that $\bar\phi\,|\,R=\phi$, $\bar\phi(x_i)=s_i$. The construction of $R\langle x\rangle$ can be obtained as follows:

Let X be the C-module generated by the x_i and let $Y_{(i)}=Y_{i_1}\otimes\cdots\otimes Y_{i_k}$ where Y_i is either R or X and the product is taken with respect to C. Let $Y=\sum Y_{(i)}$ be the direct sum taken over all possible (i) (and all k). We turn Y into an associative ring by defining multiplication: $y_{(i)}y_{(j)}=y_{(i)}\otimes y_{(j)}$ for all $y_{(i)}\in Y_{(i)}$, $y_{(j)}\in Y_{(j)}$ and extending it linearly to all Y. Let N be the two-sided ideal of Y generated by the elements

$$\{r_1\otimes r_2-r_1r_2;\,r_1,r_2\in R\}$$

and by the elements $y\otimes 1-y$, $1\otimes y-y$ for $y\in Y$, if R contains a unit 1.

$R\langle x\rangle$ is defined to be the quotient ring Y/N. Every homomorphism $\bar\phi: R\to S$ and a mapping $\psi: x_i\to s_i$ is extended to Y by setting $\Phi(y_1\otimes\cdots\otimes y_k)=\phi(y_1)\phi(y_2)\cdots\phi(y_k)$ where $\phi(y_j)$ is the ϕ-image of y_j if $y_j\in R$, and if $y_j=\sum c_ix_i\in X$ then $\phi(y_j)=\sum c_is_i$ and since $\operatorname{Ker}\Phi\supseteq N$, it follows that Φ induces the homomorphism $\bar\phi$ of $R\langle x\rangle$.

Though it is not difficult to show that $R\langle x\rangle$ is the universal product of R and $C[x]$, it is sufficient for our purpose to use only the above construction of $R\langle x\rangle$ and the property of the existence of the extension.

Henceforth, let $\{r_\lambda\}$ be a C-base of R and we shall always set $r_1 = 1$ even if R does not contain a unit.

LEMMA 9. *The polynomials $p[x] \in R\langle x\rangle$ can be written in the form*

$$(*)\qquad p[x] = \sum \alpha_i r_{i_k} x_{j_k} x_{j_{k-1}} r_{i_{k-2}} \cdots x_{j_1} r_{i_0},$$

where $\alpha_{(i)} \in C$ and r_{i_j} is one of the C-base $\{r_\lambda\}$ (or $r_{i_j} = 1$)(3).

The proof is evident if it is shown that every $y_1 \otimes \cdots \otimes y_k \in Y_{(i)}$ has its representation mod N, and this is trivial since every y_i is a linear combination of the x_λ's if it belongs to X and a linear combination of the r_λ's if it belongs to R(3).

At this point we do not raise the question of uniqueness, but clearly we may always assume that in the representation (*) of $p[x]$ for any two terms with two nonzero coefficients $\alpha_{(i)}, \alpha'_{(i)}$, we have $(i_k, j_k, i_{k-1}, j_{k-1}, \cdots, i_1, j_1, i_0) \neq (i'_h, j'_h, i'_{h-1}, j'_{h-1}, \cdots, i'_1, j'_1, i'_0)$ since one can sum all similar terms into a single term. When the representation (*) of $p[x]$ satisfies this condition, we shall say that it is *a standard form* of $p[x]$.

Each term $r_{i_k} x_{j_k} \cdots x_{j_1} r_{i_0}$ is referred to as a monomial and k is called its degree. The degree of a standard form is the maximal degree of its monomials (which appear with a nonzero coefficient).

DEFINITION. A polynomial $p[x] \in R\langle x\rangle$ is said to be a *polynomial relation*(4) of R, if $p[x]$ is not trivially zero and if $p[x] = 0$ hold identically in R; in other words, for every homomorphism $\phi: R\langle x\rangle \to R$, $\phi(p) = 0$.

We shall also say that $p = 0$ is a polynomial identity in R.

Our main theorem is:

THEOREM 10. *A primitive ring R satisfies a polynomial identity if and only if it is isomorphic with a dense ring of linear transformations over a division ring D which is finite over its center, and R contains a linear transformation of finite rank.*

Proof. If R is as above, let $e \in R$ be a primitive idempotent; then $eRe \cong D$ [3, p. 77]. If $(D:C) < \infty$, then D satisfies a standard identity $[y_1, y_2, \cdots, y_h] = \sum \pm y_{i_1} y_{i_2} \cdots y_{i_k}$ (e.g., for $h > (D:C)$ [3, p. 227]). Hence, R satisfies the polynomial identity

$$\sum \pm e x_{i_1} e x_{i_2} e x_{i_3} \cdots e x_{i_h} e = 0.$$

(3) The case that R does not contain a unit should not cause any misunderstanding as 1 always appears in one of the form $x1, 1x, r1$ which have an obvious meaning.

(4) We shall refer to these generalized polynomial relations as the polynomial relations throughout this paper.

To prove the converse, assume that R satisfies a polynomial relation $p[x]=0$ and that R acting on V_D either does not contain a linear transformation of finite rank or $(D:C)=\infty$.

First note that we may assume that $p[x]$ is linear in each of its indeterminates x_i. Indeed, suppose $p[x]$ given in a standard form which is of degree $\geqq 2$ in x_1; then, as in the usual linearization process, one chooses x_j which does not appear in $p[x]$ and then $p[x_1+x_j, x_2, \cdots]-p[x_1, x_2, \cdots]-p[x_j, x_2, \cdots]$ $=\bar{p}[x_1, x_j, x_2, \cdots]$ is again a polynomial which is not trivially zero, and which holds identically in R. Furthermore, it is of lower degree both in x_1 and x_j and of the same degree in the other indeterminates. Continuing this way one obtains a multilinear identity. So henceforth we assume that $p[x]$ is multilinear.

We turn to the ring R_F, whose definition was given in the beginning of the section, and consider it as acting on V_D. The elements of R_F are of the form $\sum d_i\alpha_i$, $d_i \in R$, $\alpha_i \in F$ (Lemma 6), and the elements of F commute with all the elements of R; hence one readily verifies that any multilinear identity which holds in R, holds also in R_F.

Let the standard form of the multilinear polynomial p be:

$$(*) \qquad p[x]=\sum \alpha_{(i)} r_{i_1} x_{j_1} r_{j_2} \cdots r_{i_1} x_{j_1} r_{i_0}, \qquad \alpha_{(i)} \in C.$$

Consider the finite set of the r_i's which appear in (*) in monomials with a nonzero coefficient. Without loss of generality we may assume that these are $r_1=1, r_2, \cdots, r_\tau$. As linear transformations they are also independent over F. Indeed, if $\sum r_i\lambda_i=0$, $\lambda_i \in F$, then $\lambda_i=\sum c_{i_j}\alpha_j$, $c_{i_j} \in C$ for a C-base $\{\alpha_\lambda\}$ of F; hence $\sum(\sum r_i c_{i_j})\alpha_j=0$. Consequently, it follows by Lemma 6 that $\sum r_i c_{i_j}=0$, but the r_i's are C-independent, which implies that all $c_{i_j}=0$. Thus also all $\lambda_i=0$.

We proceed with the proof by showing first:

Lemma 11. *Let $r_1, \cdots, r_\tau$ be C-independent elements in the primitive ring R. If R does not contain a finite ranked transformation, then for every integer h, there exists $v_1, \cdots, v_h \in V$ such that the set $\{r_i v_j\}$ are τh D-independent vectors in V.*

Proof. We apply the fundamental Lemma 4 to the following situation: Let $\mathscr{T}=\{\sum r_i\beta_i\}$ be the τ-dimensional linear subspace of R_F generated by the r_i's. Let $V_0=0$; then it follows by Lemma 4 that there exist $v_0 \in V$ such that $r_1v_0, r_2v_0, \cdots, r_\tau v_0$ are F-linearly independent. Put $V_0=\mathscr{T}v_0$.

Applying again Lemma 4 with $W_0=V_0$ we obtain $v_1 \in V$ such that $\{r_i v_1\}$ are linearly independent modulo V_0. Thus all r_iv_1, r_jv_0 are linearly independent. Continuing this way with $V_1=\mathscr{T}v_0+\mathscr{T}v_1$, etc., $\cdots$, we obtain v_1, $v_2, \cdots, v_h$ such that all r_iv_j are F-linearly independent.

We return to the proof of the theorem. Assume that R does not contain a finite ranked transformation; then it follows by Theorem 7 that R_F also does not have a finite ranked transformation and we proceed to obtain a contradiction by applying the preceding lemma to R_F (and F replacing D) and choosing h too large.

Indeed, without loss of generality we may assume that the standard form $p[x]$ given in (*) is such that

$$(**)\qquad p[x] = \sum \beta_\nu r_\nu x_k r_{i_{k-1}} x_{k-1} r_{i_{k-2}} \cdots r_{i_1} x_1 r_{i_0} + \cdots$$

and at least one $\beta_\nu \neq 0$, and the terms appearing after the plus sign either contain a different r_{i_h} or contain a permutation of the indeterminates $x_1, \cdots, x_k$.

Since R_F is a dense ring of linear transformations of V_F, and the $r_i v_j$ are linearly independent, there are well-defined k elements d_j of R_F satisfying for each $j = 1, \cdots, k$:

$$d_j(r_{i_{j-1}} v_{j-1}) = v_j \quad \text{and} \quad d_j(r_\lambda v_\mu) = 0 \ \text{elsewhere.}$$

If $h \geqq k$ this leads to a contradiction. Indeed consider the element $p[d_1, \cdots, d_k]v_0$. Clearly, all monomials $\alpha_{(i)} r_{l_k} x_{j_k} \cdots r_{l_1} x_{j_1} r_{l_0}$ when substituted in $x_i = d_i$ and are acted on by v_0 will yield the zero unless it is one of the first terms appearing in (**); that is, $r_{l_t} = r_{i_t}$, $1 \leqq t \leqq k-1$ and $x_{j_t} = x_t$, and if it is one of these terms it yields $\beta_\nu r_\nu v_k$. Consequently $p[d_j]v_0 = \sum \beta_\nu r_\nu v_k$ and since one of the $\beta_\nu \neq 0$ and $r_\nu v_k$ are all linearly independent we obtain $p[d_j]v_0 \neq 0$ which implies that $p[d_j] \neq 0$ in contradiction with the fact that $p[x] = 0$ holds identically in R_F as well as in R. This concludes the proof of the theorem.

The preceding proof, modified a little, together with Lemma 4 and Corollary 8, yields actually a bound for $(D\colon C)$.

Indeed, let $p[x] = 0$ given in the standard form (**). Note that the linearization process does not increase the total degree and the number of C-independent elements among the elements of R which appear in the original polynomial before linearization.

Consider again the F-module $\mathscr{T} = \{\sum r_i \alpha_i\}$ generated by the r_i, $i = 1, 2, \cdots, \tau$. As above we have, either: (I) a vector $v_0 \in V$ such that $\{r_i v_0\}$ are linearly independent, or (II) some $T = \sum_{i=1}^{\tau} \alpha_i r_i$ has finite rank and then its rank is, by Lemma 4,

$$\leqq \binom{\tau+1}{2} - 1.$$

Consequently, in this case Corollary 8 implies that

$$(D\colon C) \leqq 4^{\tau-1}\left[\binom{\tau+1}{2} - 1\right]^2.$$

Continuing as in the preceding proof we obtain: (I) $v_1 \in V$ such that $\mathscr{S}v_0 + \mathscr{S}v_1$ is of dimension 2τ, or else (II) some $T \in \mathscr{S}$ of finite rank, and again by Lemma 4 its rank is

$$\leqq \tau + \binom{\tau+1}{2} - 1$$

(here $\mathscr{S}v_0$ is taken for W_0 of Lemma 4) and by Corollary 8,

$$(D\colon C) \leqq 4^{\tau-1}\left[\tau + \binom{\tau+1}{2} - 1\right]^2.$$

Generally, we obtain that either

$$(D\colon C) \leqq 4^{\tau-1}\left[h\tau + \binom{\tau+1}{2} - 1\right]^2$$

or V contains vectors $v_0, v_1, \cdots, v_h$ such that $r_i v_j$ are all F independent. But, modifying the above proof following the proof of Theorem 1 of [1] we shall show that this is possible only if $h \leqq [k/2]$ where k is the degree of $p[x]$ and $[k/2]$ is the greatest integer in $k/2$.

Indeed, let $h > [k/2]$. One observes that the space $\mathscr{S}v_0 \oplus \mathscr{S}v_1 \oplus \cdots \oplus \mathscr{S}v_h$ is isomorphic with the tensor product $\mathscr{S} \otimes U$ where U is a C-vector space of dimension $h+1$ and can be taken to be $U = \sum Cv_i$. Furthermore, U can be identified with $r_1 U$. For our purpose we may identify $\mathscr{S} \otimes U$ with $\sum \mathscr{S}v_i$. Let E be the linear transformation given by $Ev_i = v_{i+1}$ for $i+1 \leqq h$ and $Ev_i = 0$ if $i+1 \geqq h$.

Since R_F is dense in $\mathrm{Hom}(V_F, V_F)$ we can determine k elements $d_1, \cdots, d_k$ in R_F such that for all $v \in U$:

If $j = 2l$, then $d_j(r_{i_{j-1}}v) = E^l S E^{h-l} v$ and for $j = 2l+1$; then $d_j(r_{i_{j-1}}v) = E^{l+1} S E^{h-l} v$, and in both cases $d_j(r_\mu v) = 0$ for all $r_\mu \neq r_{i_{j-1}}$; and S is given by $Sv_h = v_0$ and zero otherwise.

Consider now $p[d_1, d_2, \cdots, d_k]v_0$ and as in the preceding proof we show that, since $h > [k/2] = m$:

$$p[d_j]v_0 = \sum \beta_\nu r_\nu E^m S E^{h-m} \cdot \cdots \cdot ESE^{h-1} \cdot ESE^h \cdot SE^h v_0.$$

The other terms yield the zero, since either they contain a term of the form ESE^tS with $t > h$ which is zero since $E^{h+1} = 0$, or else the d_j will act on an $r_\mu v$ with $\mu \neq i_{j-1}$. Now the sum

$$\sum \beta_\nu r_\nu E^m (SE^h SE^h \cdot \cdots \cdot SE^h) v_0 = \sum \beta_\nu r_\nu E^m v_0 = \sum \beta_\nu r_\nu v_m \neq 0$$

since one $\beta_\nu \neq 0$ and all $r_\nu v_m$ are linearly independent. Consequently, $p[d_j]v_0 \neq 0$ which contradicts the assumption that $p = 0$ holds in R.

Thus either we get as far as $h = [k/2]$ or we obtain a linear transformation of finite rank. We have:

Corollary 12. *A primitive ring R satisfying a nontrivial identity $p = 0$ of degree k which includes τ C-independent elements* (*including* 1), *is isomorphic with a dense ring of linear transformations over a division ring D containing a finite ranked transformation and* ([5])

$$(D: C) \leqq 4^{\tau-1}\left(\binom{\tau+1}{2} - 1 + \tau\left[\frac{k}{2}\right]\right)^2.$$

Note that for $\tau = 1$, which includes the case of a polynomial identity in the old sense, i.e., with coefficients in C, we get Levitzky's bound of $[k/2]^2$.

We obtain an interesting consequence for division rings.

Theorem 13. *A division ring D satisfies a polynomial identity if and only if it is finite-dimensional over its center C, and then the bound is as given in Corollary* 12.

This is an immediate consequence of applying Theorem 10 and Corollary 12 to $R = D$, $V = D$ and R acts on V by left multiplication. Here the centralizer of R is D^*, the ring of right multiplications, which is anti-isomorphic with D; and thus $(D^*: C) = (D: C) < \infty$.

In all preceding results we have assumed that R is a C-algebra and then one verifies by Lemma 6 that $R_F = R \otimes_C F$. In the general case, the center of R is only an integral domain contained in C. Nevertheless, if $p = 0$ holds in R then the linearization process of p yields a multilinear relation which will hold also in R_C and R_F. This shows that there was no loss of generality by assuming that R was a C-algebra, since we can start from R_C.

5. **Applications.** In this section we apply the preceding result to obtain some information on the structure of the ring $R\langle x\rangle$. We assume, henceforth, that R is a primitive ring that either does not contain a minimal left ideal or else its commuting ring is not finite over the center. These conditions mean by the Structure Theorem [3, p. 75] that the result of the preceding section holds for R.

A simple application is:

Corollary 14. *The representation of $p[x] \in R\langle x\rangle$ in the standard form* (*) *of Lemma* 9 *is unique.*

Indeed, it suffices to show that the monomials $r_{i_k}x_{j_k}r_{i_{k-1}} \cdots x_{j_1}r_{i_0}$ are C-independent. If this is not the case then some linear combination of them will yield a polynomial $p[x]$ of standard form (*) which is not trivially zero by definition but for which $p = 0$ in $R\langle x\rangle$. Hence, it is evident that

([5]) As one considers only the r_μ appearing in the same place, in the monomials of (**) one can replace τ by the number of independent elements in the same place of the monomials, and a bound for this is the number of nonzero monomials in (**).

under all homomorphisms $\phi: R\langle x\rangle \to R$, $\phi(p) = 0$, i.e., $p = 0$ holds in R. But the proof of the main theorem shows that with our assumption about R this is impossible.

In the next theorem we consider only division rings D and our next object is to show that:

THEOREM 15. *$D\langle x\rangle$ is a ring without zero divisors imbeddable in a division ring; furthermore, if D is ordered then $D\langle x\rangle$ can be imbedded in an ordered division ring.*

Proof. We consider first the case that D is infinite over its center. There are different methods to prove that $D\langle x\rangle$ has no zero divisors. We present here a method which is a simple application of the main theorem.

Suppose $p[x]$, $q[x]$ are nonzero elements in $D\langle x\rangle$ and $pq = 0$ in $D\langle x\rangle$. Then for every homomorphism $\phi: D\langle x\rangle \to D$, $\phi(pq) = \phi(p)\phi(q) = 0$. Since $\phi(p), \phi(q)$ belong to a division ring it follows that either $\phi(p) = 0$ or $\phi(q) = 0$. Choose now x_j which does not appear in $p[x]$ and $q[x]$ then we have also $\phi(px_jq) = 0$ for all ϕ. Consequently, the polynomial relation $p[x]x_jq[x] = 0$ holds in D, this leads to a contradiction if we can prove that $p[x]x_jq[x]$ is not trivially zero in $D\langle x\rangle$. Indeed let $p[x] = \alpha r_{i_k}x_{j_k}r_{i_{k-1}} \cdots x_{j_1}r_{i_0} + \cdots$ and $q[x] = \beta r_{i_h}x_{j_h} \cdots x_{j_1}r_{i_0} + \cdots$ and let the monomials written be of maximal degree; then one readily verifies that px_jq will contain the monomial $\alpha\beta r_{i_k}x_{j_k} \cdots r_{i_0}x_{j_h} \times r_{i_h}x_{i_{h-1}} \cdots r_{i_0}$ once and only once, and hence $px_jq \neq 0$.

To prove the imbeddability, we follow a method of imbedding rings without zero divisors due to M. Rabin who used it to prove the following general result (unpublished) ([6]):

"*A ring without zero divisors which is a subring of a complete product of division rings is imbeddable in a division ring; furthermore, if the division rings of the product are ordered then the ring can be imbedded in an ordered division ring.*"

His proof goes as follows:

Let $S = \prod D_\alpha$ be the complete product of division rings D_α, where α ranges over a set I. That is: $S = \{f\}$, the ring of all functions f, such that $f(\alpha) \in D_\alpha$. Let $R \subseteq S$ be a subring without zero divisors.

To each $0 \neq f \in R$ let $I_f = \{\alpha \mid \alpha \in I, f(\alpha) \neq 0\}$. The sets I_f form a base-filter([7]) in I. Indeed, $I_{f_1} \cap I_{f_2} \cap \cdots \cap I_{f_k} \supseteq I_{f_1 \cdots f_k}$ and since R is without zero divisors, $f_1 f_2 \cdots f_k \neq 0$. Finally, $I_f \neq \emptyset$, as otherwise for all $\alpha \in I$, $f(\alpha) = 0$ which means that $f = 0$, and this case was excluded. Let F be

([6]) Quoted and proved by a different method in A. Robinson, *A note on embedding problems*, Fund. Math. **50** (1962), 461.

([7]) For basic facts on filters, see e.g., N. Bourbaki, *Topologie générale*, Vol. III, Hermann, Paris, 1961; Ch. I, §6, p. 63.

any ultrafilter containing all the sets I_f; then the set of all functions, $h \in S$ such that $\{\alpha \mid h(\alpha) = 0\} \in F$, form a maximal two-sided ideal in S. The quotient ring of S modulo this ideal (known as an *ultraproduct* of the D_α) which will be denoted by $\prod D_\alpha/F$, is a division ring. Indeed, if $f \not\equiv 0 \pmod F$ then define g by $g(\alpha) = f(\alpha)^{-1}$ when $f(\alpha) \neq 0$ and zero otherwise. The set $\{\alpha \mid f(\alpha) = 0\} \notin F$; hence, since F is an ultrafilter, it follows that its complement belongs to the filter F from which it is readily seen that $gf \equiv 1 \pmod F$, since $\{\alpha \mid (gf)(\alpha) = 1\} = \{\alpha \mid f(\alpha) \neq 0\} \in F$. Thus, $g = f^{-1}$ in $\prod D_\alpha/F$.

This ultraproduct has the required properties, and the imbedding $R \to \prod D_\alpha \to \prod D_\alpha/F$ is a monomorphism. Furthermore, if all D_α are ordered, then so is the ultraproduct $\prod D_\alpha/F$. The order is obtained by setting $f < g$ if $\{\alpha \mid f(\alpha) < g(\alpha)\} \in F$. The proof is immediate and we shall not produce it here.

This basic result can be applied to our case as follows:

Let $I = \{\phi\}$ be the set of all homomorphisms $\phi: D\langle x\rangle \to D$, then consider the product $D^I = \{f \mid f: I \to D\}$. This is a product of division rings and $D\langle x\rangle$ can be imbedded in D^I in the natural way by setting $p[x](\phi) = \phi(p[x])$ for all homomorphisms $\phi \in I$. This is a monomorphism of $D\langle x\rangle$ into D^I since $p[x](\phi) = 0$ for all ϕ means that $p = 0$ holds in D, but no such nonzero relation exists in D unless $p = 0$ in $D\langle x\rangle$. The rest follows now immediately by the above-quoted result.

To conclude the proof of Theorem 15 for arbitrary division rings we note that our result will follow from the simple fact that any division ring can be embedded in a division ring which is infinite over its center, and an ordered finite-dimensional division ring (which is necessarily commutative) can be embedded in an ordered noncommutative division ring (which is necessarily infinite over its center). We shall prove here only the first fact:

Let $\xi_1, \xi_2, \cdots$ be an infinite sequence of commutative indeterminates and consider the ring $E = D(\xi)$ of all rational functions in the ξ_i's: namely, the quotient ring of the polynomial ring $D[\xi_1, \xi_2, \cdots]$. Let δ be the derivation of D and let it be extended to $D(\xi) = E$ which can be written symbolically as $\sum_{i=1}^{\infty} \xi_{i-1}(\partial/\partial\xi_i)$, and where we denote $\xi_0 = 1$. Consider now the ring $E[\delta]$ of all differential polynomials in δ with multiplication defined by the relation: $\delta a = a\delta + a'$. The ring $E[\delta]$ is a Euclidean ring [7] and thus a principal right and left ideal ring; hence it satisfies the Öre condition and can be imbedded in a quotient ring $E(\delta)$. It remains to show that $E(\delta)$ is infinite over its center.

Indeed, note first that the ξ_i, $i > 0$, do not belong to the center as $\delta\xi_i = \xi_i\delta + \xi_{i-1}$. Now, $\xi_0, \xi_1, \cdots$ are linearly independent over the center. If it is not so, let $\sum_{i=0}^{n} \xi_i q_i = 0$ be a linear dependence relation with $q_i \in$ center of E and $q_n \neq 0$ of minimal n. Then since $\delta q_i = q_i\delta$ as elements of the center, we get:

$$0 = \delta(\sum \xi_i q_i) = (\sum \xi_i q_i)\delta + \sum_{i=1}^{n} \xi_{i-1} q_i = \sum_{i=0}^{n-1} \xi_i q_{i+1}$$

which is a linear dependence of lower length. Contradiction.

6. **Pivotal monomials.** (We turn now to the extension of the notion of a pivotal monomial [2].)

Let R be a primitive ring with a unit 1 and a center C which is a field. Let $r_1 = 1, r_2, \cdots, r_\tau$ be a finite set of C-independent elements in R.

A monomial $\pi(x)$ will stand for a monomial of the form

$$\pi(x) = r_{i_k} x_{j_k} r_{i_{k-1}} x_{j_{k-1}} \cdots r_{i_1} x_{i_1} r_{i_0}.$$

The complement P_π of π will by definition include all monomials $\sigma(x) = r_{n_l} x_{m_l} \cdots r_{n_1} x_{m_1} r_{n_0}$ for which $l > k$, or $h \leqq k$ but then some $j_t \neq m_t$ $(t \leqq l)$, or, $i_t \neq n_t$ for $t < l$ (!).

Definition. $\pi(x)$ is a pivotal monomial of R if for every substitution $x_i = d_i$, $\pi(d)$ belongs to the left ideal generated by all $\sigma(d)$, $\sigma \in P_\pi$.

Following the proof of [1, Theorem 4] in collaboration with the proof of Theorem 11, we show that:

Theorem 16. *A necessary and sufficient condition that a primitive ring possesses a minimal left ideal is that it possesses a (left) pivotal monomial.*

Proof. In one direction the proof is trivial. Indeed, if Re is a minimal left ideal then eRe is a division ring and, therefore, $exeye \in Reyexe$ for all x, y in R, i.e., $exeye$ is a pivotal monomial.

To prove the converse, we shall show that the existence of a pivotal monomial yields the fact that R is a dense ring of linear transformations of a space V_D, D a division ring and R contains a finite ranked transformation. This yields by the Structure Theorem [3, p. 75] that R possesses a minimal left ideal.

To achieve this we follow the proof of Theorem 10 with the use of Lemma 11. The situation we are dealing with in this proof is as follows: Let R be a dense ring of linear transformations of a space V_D, D the centralizer ring of R. As in the proof of Theorem 10 we consider here the ring R_D as a subring of $\mathrm{Hom}(V, V)$. The set $r_1, \cdots, r_\tau$ are also D-independent and one verifies this by the same method as in the proof of Theorem 10 (where they were shown to be F-independent). From Lemma 11, we obtain that either: (1) R_D contains a finite ranked transformation of the form $\sum r_i d_i$, $d_i \in D$, or else: (2) for every integral h, V_D contains a set of vectors $v_0, v_1, \cdots, v_h$ such that $\{r_i v_j\}$ are linearly D-independent in V.

In the first case, we repeat the proof of Theorem 7 to show that R contains a finite ranked transformation. Indeed, if for arbitrary $x \in R$ consider $(r_k x T - T x r_k) V = \sum_{i=1}^{k-1} (r_k x r_i - r_i x r_k) d_i V \subseteq r_k x T V + T V$ which generate a finite-dimensional D-subspace of V, and note that d_i commute with the elements of R (as do the α_i's in the proof of Theorem 7). The rest follows in this case as in the above-mentioned proof.

In the second case, choose $h > k$ and use a substitution similar to the one used by Drazin (e.g., the proof of [1, Theorem 4]). Namely, we choose $d_j \in R$ determined by the relation:

$$d_j(r_{i_{\nu-1}}v_{\nu-1}) = v_\nu \text{ and zero otherwise,}$$

where i_ν and j_ν are those which appear in the monomial π only. These d_j are well defined since $h > k$, and a contradiction is obtained by comparing $\pi(d)v_0 = r_{i_k}v_k \neq 0$ and $\sigma(d)v_0 = 0$ for all $\sigma \in P_\pi$ which is impossible since $\pi(d) \in \sum R\sigma(d)$. Consequently, the second case cannot happen and thus the proof that R contains a finite ranked transformation and, hence, a minimal left ideal, is completed.

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