GENERALIZED PONDEROMOTIVE FORCES AND THREE-WAVE INTERACTI N

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The subject of this article is a unified Hamiltonian approach to the theory of nonlinear interactions among waves and particles. The unifying feature of the approach is a generalization of the concept of "ponderomotive force". The formulation can be said to retain the conceptual simplicity of the familiar ponderomotive-potential method [1-3], but to remove the approximations [4]. The essence of the approach is to replace the usual method of time-averaging by the performance of a canonical transformation. The transformation is designed to eliminate the terms in the Hamiltonian of a particle which are linear in the wave potentials, replacing them with bilinear terms at combination frequencies. The new entity (the "oscillation center") thus has no first order jittering motion. The transformation formalism leads to explicit expressions for the required nonlinear currents, which can be decomposed into the current of oscillation centers and the "polarization" corrections [4]. The oscillation-center representation is thus quite analogous to the more familiar guiding-center representation in strong magnetic fields.

Such an approach has previously been applied to the theory of induced scattering of waves by resonant particles [5]. The useful extension of this point of view to other nonlinear processes is advocated here. We shall demonstrate the oscillation-center approach by deriving a compact general formula for the three-wave coupling coefficient in collisionless plasma. We wish here to emphasize ideas rather than the details of the formalism [6]. Accordingly, our subsequent discussion will be fairly schematic.

We consider a collisionless, nonuniform and possibly relativistic plasma, confined by spatially inhomogeneous electric and magnetic fields. We treat the linear normal modes of the configuration as fully electromagnetic, assuming that their eigenfrequencies are nearly real. We interpret the confinement as due to the adiabatic invariants of each particle, and shall implicitly assume the existence of the action-angle variables $(\underline{I}, \underline{\theta})$ associated with the unperturbed Hamiltonian $H_o(\underline{I})$. Accordingly, we separate all the plasma particles into two categories: the vast majority which comprise the nonresonant particles, and the small subset of "resonant" particles which satisfy [7,8]

$$\omega_a \approx \underline{\ell}_a \cdot \partial H_o/\partial \underline{I} \equiv \underline{\ell}_a \cdot d\underline{\theta}/dt \ . \qquad (1)$$

In Eq. (1), ω_a denotes the real part of the eigenfrequency for normal mode a , and the vector $\underline{\ell}_a$ represents a set of three integers. We wish to concentrate on the nonresonant particles in this article, and so we decompose (in some mathematically smooth way) the unperturbed phase-space distribution function in the form

$$f_o(\underline{r},\underline{p}) = \tilde{f}_o(\underline{r},\underline{p}) + f_o^{res}(\underline{r},\underline{p}) \ , \qquad (2)$$

where $\tilde{f}_o$ represents the nonresonant distribution.

We choose to work with conjugate variables $(\underline{r},\underline{p})$, where $\underline{r}$ denotes the Cartesian position vector in physical space. The unperturbed Hamiltonian for a plasma particle can be written

$$H_o(\underline{r},\underline{p}) = e\phi_o(\underline{r}) + \left\{[\underline{p}c - e\underline{A}_o(\underline{r})]^2 + m^2c^4\right\}^{\frac{1}{2}} \ . \qquad (3)$$

Adopting the radiation gauge $\Phi' = 0$, we consider a set of three perturbing normal modes,

$$\underline{A}'(\underline{x},t) = \sum_{a=1}^{3} \underline{A}_a(\underline{x}) e^{-i\omega_a t} + \text{c.c.} , \tag{4}$$

whose (positive) frequencies satisfy the resonant matching condition $\omega_3 = \omega_1 + \omega_2$. A small frequency mismatch $\Delta\omega << \omega_a$ can be taken into account in the usual way [9]. Representative components of the perturbed Hamiltonian are

$$H_a' = - e/c \, \underline{A}_a(\underline{r}) \cdot \underline{\partial} \, H_o(\underline{r},\underline{p}) \, e^{-i\omega_a t} , \tag{5}$$

$$H_3'' = e^2/c^2 \, \underline{A}_1(\underline{r}) \, \underline{A}_2(\underline{r}) : \underline{\partial} \, \underline{\partial} \, H_o(\underline{r},\underline{p}) \, e^{-i\omega_3 t} , \tag{6}$$

$$H_o''' = - e^3/c^3 \, \underline{A}_1(\underline{r}) \, \underline{A}_2(\underline{r}) \, \underline{A}_3^*(\underline{r}) \; \vdots \; \underline{\partial} \, \underline{\partial} \, \underline{\partial} \, H_o(\underline{r},\underline{p}) , \tag{7}$$

where the primes refer to the order in the perturbation, the subscripts identify the time dependence, and $\underline{\partial} \equiv \partial/\partial\underline{p}$. Let us devise a canonical transformation to eliminate H', i.e., all first-order terms in the perturbed Hamiltonian:

$$(\underline{r},\underline{p},H) \rightarrow (\underline{R},\underline{P},K) , \tag{8}$$

$$K = H_o + K'' + \cdots . \tag{9}$$

Such a transformation corresponds to simple Hamilton-Jacobi perturbation theory. If the particle were in resonance with any of the three primary modes, then a two-time-scale refinement of the transformation [5,10] would be required. The forces derivable from the oscillation-center Hamiltonian K'' may be viewed as generalized ponderomotive forces. Note, however, that we have not ordered frequencies and averaged over time. We have simply performed a canonical transformation.

The perturbative generating function for the transformation, $S(\underline{r},\underline{P},t)$, is determined by the equations [5]

$$D_t S_a(\underline{r},\underline{p},t) = - H_a'(\underline{r},\underline{p},t) , \tag{10}$$

$$D_t = \partial/\partial t + [\;\; , H_o] . \tag{11}$$

The resultant Hamiltonian $K''(\underline{r},\underline{p},t)$ is then a sum of H'' and a known bilinear functional [5] of first-order quantities. The nonresonant phase-space distribution function can now be decomposed in the form

$$\tilde{f}(\underline{r},\underline{p},t) = F(\underline{r},\underline{p},t) + \Delta(\underline{r},\underline{p},t) \ , \tag{12}$$

$$F = \tilde{f}_o + F'' + \cdots \ , \quad \Delta = \Delta' + \Delta'' + \cdots \ , \tag{13}$$

where F denotes the solution of the Vlasov equation for oscillation centers, and Δ represents the difference between F and $\tilde{f}$ at the same phase point $(\underline{r},\underline{p})$. Again, we have explicit formulas for Δ' and Δ'' in terms of S, H and $\tilde{f}_o$ [5].

Let us apply these ideas to calculate the slow evolution of the amplitudes of the interacting normal modes. Our concern is therefore with coupled equations of the form

$$\underset{\sim}{D}(\omega_a + i\partial/\partial t)\cdot\underline{E}_a(\underline{x}) = (4\pi/i\omega_a)\underline{j}''_a(\underline{x}) \ , \tag{14}$$

where $\underset{\sim}{D}(\omega)$ denotes the linear dispersion operator (assumed nearly Hermitian), and $\underline{j}''_a$ represents the nonlinear current source at frequency ω_a due to the beating of the other two modes. Retaining only the nonresonant terms in Eq.(14), we write

$$\underset{\sim}{D}'(\omega_a + i\partial/\partial t)\cdot\underline{E}_a = (4\pi/i\omega_a)\tilde{\underline{j}}''_a \ , \tag{15}$$

where $\underset{\sim}{D}'$ denotes the Hermitian part of $\underset{\sim}{D}$. Since ω_a is a linear eigenfrequency, we have

$$\underset{\sim}{D}'(\omega_a + i\partial/\partial t)\cdot\underline{E}_a \rightarrow i\partial\underset{\sim}{D}'(\omega_a)/\partial\omega_a\cdot d\underline{E}_a/dt \ . \tag{16}$$

Now, the total energy of wave a can be written

$$W_a = (\omega_a/4\pi)\int d^3x\ \underline{E}_a^*\cdot\partial\underset{\sim}{D}'(\omega_a)/\partial\omega_a\cdot\underline{E}_a \ . \tag{17}$$

Combining relations (15) to (17), we thus obtain the action evolution equation

$$\omega_a^{-1}\ \tilde{d}W_a/dt = -2\ \mathrm{Im}\int d^3x\ c^{-1}\ \underline{A}_a^*\cdot\tilde{\underline{j}}''_a \ . \tag{18}$$

where the symbol $\hat{d}$ denotes the evolution due to nonresonant currents.

We shall proceed to evaluate the right-hand side of Eq.(18) using the oscillation-center transformation. The physical current density $\tilde{\underline{j}}(\underline{x},t)$ of nonresonant particles can be written

$$\tilde{\underline{j}}(\underline{x},t) = e \int d^3r \int d^3p \; \delta(\underline{x} - \underline{r}) \; \tilde{f}(\underline{r},\underline{p},t) \; \underline{\partial}H(\underline{r},\underline{p},t) \; . \qquad (19)$$

Invoking the decomposition (12) of $\tilde{f}$, we can break up the second-order contributions to Eq.(19) in the form

$$\tilde{\underline{j}}_3'' = \underline{J}_3'' + \Delta\underline{j}_3'' , \qquad (20)$$

where

$$\underline{J}_3'' \equiv e \int d^3r \int d^3p \; \delta(\underline{x} - \underline{r}) \; F_3'' \; \underline{\partial}H_o , \qquad (21)$$

and

$$\Delta\underline{j}_3'' \equiv e \int d^3r \int d^3p \; \delta(\underline{x} - \underline{r})$$

$$\times \, (\tilde{f}_o \underline{\partial}H_3'' + \Delta_1' \underline{\partial}H_2' + \Delta_2' \underline{\partial}H_1' + \Delta_3' \underline{\partial}H_o) \; . \qquad (22)$$

We rewrite the currents (21) and (22) using our generating-function formulas, and substitute the results into Eq.(18). A crucial sequence in the subsequent manipulations is the following:

$$\begin{aligned} - \int d^3x \; c^{-1} \underline{A}_3^* \cdot \underline{J}_3'' &= + \int d^3r \int d^3p \; H_3'^* \; F_3'' \\ &= - \int d^3r \int d^3p (D_t \; S_3^*) \; D_t^{-1} \; [K_3'', \tilde{f}_o] \\ &= + \int d^3r \int d^3p \; S_3^* \; [K_3'', \tilde{f}_o] \\ &= + \int d^3r \int d^3p \; \tilde{f}_o \; [S_3^*, K_3''] \; . \qquad (23) \end{aligned}$$

Judicious manipulation and partial integration lead finally to the following compact and general formula:

$$
\begin{aligned}
- \int d^3x\, c^{-1}\, \underline{A}_3^{*}\cdot\tilde{\underline{j}}_3'' = & \int d^3r \int d^3p\, \tilde{f}_c(\underline{r},\underline{p}) \\
& \times \Big\{ H_0''' + [S_1, H_1''^{*}] + [S_2, H_2''^{*}] + [S_3^{*}, H_3''] \\
& + \tfrac{1}{3}\big([S_2, [S_3^{*}, H_1']] + [S_3^{*}, [S_2, H_1']] \\
& \quad + [S_3^{*}, [S_1, H_2']] + [S_1, [S_3^{*}, H_2']] \\
& \quad + [S_1, [S_2, H_3'^{*}]] + [S_2, [S_1, H_3'^{*}]]\big) \Big\} \; .
\end{aligned} \tag{24}
$$

This expression is to be inserted into Eq.(18).

Formula (24) is manifestly symmetric under interchange of the subscripts $(1,\ 2,\ 3^{*})$. This symmetry implies conservation of action in the three-wave process (the Manley-Rowe relations [11]). We have presented here the essence of a purely classical and quite general proof of that conservation law. It is a consequence of the static nature of the equilibrium, and of separating out the dissipation to resonant particles. For purely electrostatic modes, a formula essentially equivalent to Eq. (24) has been derived by Laval and Pellat [9]. Only the triple Poisson bracket terms involving H_a' survive in the electrostatic limit.

In the limiting case of a uniform, nonrelativistic, magnetized plasma, the expression (24) can be shown to reduce to the following:

$$
\begin{aligned}
c^{-1}\, \underline{A}_3^{*}\cdot\tilde{\underline{j}}''(\underline{k}_3,\omega_3) = & \; m^{-2} \int d^3v\, \tilde{f}_0(v_\perp, v_z) \\
& \times \Big\{ (i\underline{k}_1\cdot\underline{\partial}S_1)\,(D_t\underline{\partial}S_2)\cdot(D_t\underline{\partial}S_3^{*}) \\
& + (i\underline{k}_2\cdot\underline{\partial}S_2)\,(D_t\underline{\partial}S_3^{*})\cdot(D_t\underline{\partial}S_1) \\
& + (-i\underline{k}_3\cdot\underline{\partial}S_3^{*})\,(D_t\underline{\partial}S_1)\cdot(D_t\underline{\partial}S_2) \\
& + \Omega_B\underline{\partial}S_3^{*}\cdot[ik_{1z}(D_t\underline{\partial}S_2)\times\underline{\partial}S_1 + (1\leftrightarrow 2)] \Big\} ,
\end{aligned} \tag{25}
$$

where, in this equation only, we have defined $\underline{\partial} \equiv \partial/\partial\underline{v}$. The generating function S_a can now be written explicitly as an infinite sum over Bessel functions [5]. Formula (25) was recently derived by Larsson [12] using different notation and a different method.

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