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GENERALIZED PORTMANTEAU STATISTICS AND TESTS OF RANDOMNESS*

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ABSTRACT

This paper considers the problem of testing the randomness of Gaussian and non-Gaussian time series. A general class of parametric portmanteau statistics, which include the Box-Pierce and the Ljung-Box statistics, is introduced. Using the exact first and second moments of the sample autocorrelations when the observations are i.i.d. normal, the exact expected value of any portmanteau statistic is obtained for this case. Two new portmanteau statistics, which exploit the exact moments of the sample autocorrelations, are studied. For the nonparametric case, a rank portmanteau statistic is introduced. The latter has the same distribution for any series of exchangeable random variables and uses the exact moments of the rank autocorrelations. We show that its asymptotic distribution is chi-square. Simulation results indicate that the new portmanteau statistics are better approximated by the chi-square asymptotic distribution than the Ljung-Box statistic. Several analytical results presented in the paper were derived by using a symbolic manipulation program.

RÉSUMÉ

Ce texte considère le problème qui consiste à tester le caractère aléatoire de séries chronologiques gaussiennes et non-gaussiennes. Nous définissons une classe générale de statistiques "portemanteau" qui inclut la statistique de Box-Pierce et celle de Ljung-Box. Utilisant les premiers et seconds moments exacts des autocorrélations échantillonnales pour le cas d'observations i.i.d. normales, nous dérivons l'espérance mathématique exacte de toute statistique portemanteau dans ce cas. Nous étudions deux nouvelles statistiques portemanteau qui exploitent les moments exacts des autocorrélations échantillonnales. Pour le cas non-gaussien, nous introduisons une statistique portemanteau de rang. Cette dernière a la même distribution pour toutes les séries de variables aléatoires interchangeables et utilise les moments exacts des autocorrélations de rang. Nous démontrons que la distribution asymptotique de cette statistique est chi-carré. Nous présentons des résultats de simulation qui indiquent que les nouvelles statistiques portemanteau ont des distributions qui sont mieux approximées par la distribution chi-carré asymptotique que la statistique de Ljung-Box. Nous avons obtenu plusieurs de nos résultats analytiques en utilisant un logiciel de manipulation symbolique.

1. INTRODUCTION

Testing the randomness of a time series is one of the basic problems of statistical analysis. Many inference procedures apply only to independent identically distributed (i.i.d.) observations. It is the first question that gets raised when identifying a time series model. Theories in various fields can be verified by testing the randomness of certain series. For example, important economic hypotheses can be tested in this way: market efficiency (see Fama, 1970), rational expectations (Kantor, 1978), the life cycle-permanent income hypothesis (Hall, 1978), etc. In particular, one can check the efficiency of a speculative market by testing whether first differences of relevant asset prices, like stock prices or exchange rates, are independent (the random walk hypothesis).

To test the randomness of a series $X_1,\ldots,X_n$ against serial dependence alternatives, it is standard practice to look at the sample autocorrelations

$$r_k = \sum_{t=1}^{n-k} (X_t - \bar{X})(X_{t+k} - \bar{X}) \Big/ \sum_{t=1}^{n} (X_t - \bar{X})^2 \quad , \quad 1 \leq k \leq n-1, \quad (1.1)$$

where $\bar{X} = \sum_{t=1}^{n} X_t/n$. Under the assumption that $X_1,\ldots, X_n$ are i.i.d. with finite variance, the asymptotic distribution of $\sqrt{n}r_k$ is N(0,1) (see Anderson, 1971, Chap. 8). Thus, by considering normalized autocorrelation coefficients $\sqrt{n}r_k$ and asymptotic critical values from the N(0,1) distribution, one can test randomness against serial dependence at each lag k. Moreover, in many situations, a global test against serial dependence at several lags (say $k=1,\ldots,m$) is required. An especially simple statistic for this purpose is the portmanteau statistic suggested by Box and Pierce (1970):

$$Q_1 = \sum_{k=1}^{m} (\sqrt{n}r_k)^2 = n \sum_{k=1}^{m} r_k^2 \quad (1.2)$$

Under the null hypothesis of randomness, Q_1 has a $\chi^2(m)$ asymptotic distribution. However, various theoretical and simulation results suggest that the chi-square approximation is not accurate even for moderately large samples (Davies, Triggs and Newbold, 1977; Ljung and Box, 1978; Ansley and Newbold, 1979). Instead, considering autocorrelations based on uncentered data

$$\bar{r}_k = \sum_{t=1}^{n-k} a_t a_{t+k} \Big/ \sum_{t=1}^{n} a_t^2 \quad , \; 1 \leq k \leq n-1 ,$$

Ljung and Box (1978) observe that

$$E(\bar{r}_k) = 0 \quad , \quad Var(\bar{r}_k) = (n-k)/[n(n+2)]$$

when $a_1, \ldots, a_n$ are i.i.d. normal, and suggest replacing the asymptotic standard error $1/\sqrt{n}$ by $[(n-k)/n(n+2)]^{\frac{1}{2}}$. This leads to a second portmanteau statistic

$$Q_2 = n(n+2) \sum_{k=1}^{m} r_k^2/(n-k) \quad . \tag{1.3}$$

Q_2 is asymptotically equivalent to Q_1 but its finite-sample distribution appears better approximated by the $\chi^2(m)$ distribution, even though it remains far from perfect.

When the mean of the data is unknown, the portmanteau statistics Q_1 and Q_2 are based on approximate normalizations of the sample autocorrelations: both the mean and the variance of each coefficient r_k are approximate. While taken to be zero, the exact mean of r_k is $-(n-k)/[n(n-1)]$ for any series of i.i.d. variables. This result was proved several years ago by Moran (1948) but was not apparently exploited in the context of portmanteau tests. The exact variances and covariances of the sample autocorrelations for the case of a Gaussian white noise and for rank autocorrelations from an arbitrary random series were derived recently by Dufour and Roy (1984). Further, simulation results indicate that normalizing each sample autocorrelation with its exact mean and variance, instead of the usual approximate moments, can improve considerably the accuracy of the asymptotic N(0,1) distribution.

In this paper, we use the results of Moran (1948) and Dufour and Roy (1984) to obtain exact results on the distribution of portmanteau statistics and propose new parametric portmanteau tests whose distributions are better approximated by the chi-square distribution. We also introduce a nonparametric portmanteau statistic based on rank autocorrelations, whose distribution under the null hypothesis of randomness is relatively well approximated by the chi-square distribution. Besides, since several of our analytical results require long derivations, we checked them with the symbolic manipulation program MACSYMA (1983).

In section 2, we first define a generalized parametric portmanteau statistic as a positive definite quadratic form based on sample autocorrelations. Both the Box-Pierce and the Ljung-Box statistics (Q_1 and Q_2) belong to this family. We derive the exact expected value of any parametric portmanteau statistic for the case of a Gaussian white noise. We define two new portmanteau statistics: the first one (Q_3) uses the exact means and variances of the sample autocorrelations in the Gaussian case while the second one (Q_4) also takes into account the covariances between the autocorrelations. In contrast with Q_1 and Q_2, both Q_3 and Q_4 in the Gaussian case have exactly the same expected value as the $\chi^2(m)$ distribution. We observe that the distribution of any parametric portmanteau statistic is the same for all samples that follow a spherically symmetric (s.s.) distribution: since the distribution of a Gaussian white noise is s.s., any result valid in the latter case also holds for the more general class of s.s. distributions. Further, we give upper bounds on the expected values of Q_1, Q_2 and Q_3, valid for any sequence of exchangeable random variables and thus also for an arbitrary random sample. We observe that these bounds are remarkably close to the exact expected values obtained in the Gaussian case.

In section 3, we derive closed-form expressions for the first and second moments of rank autocorrelations from an arbitrary random sample. Since these derivations (as well as certain results of section 2) require tedious algebra, we use MACSYMA to check them. We exploit these results to define a nonparametric portmanteau statistic based on rank autocorrelations (Q_5). We observe that the distribution of this statistic is the same for all samples where the observations are continuous exchangeable, irrespective of the form of the distribution. We show that the asymptotic distribution of Q_5 is $\chi^2(m)$ for any series of continuous exchangeable random variables.

In section 4, we consider the problem of testing the randomness of a Gaussian time series and study by Monte-Carlo methods how well the distributions of the five portmanteau statistics discussed above are approximated by their asymptotic distribution. We find that Q_3 is better approximated by the asymptotic $\chi^2(m)$ distribution than Q_1 and Q_2: normalizing the sample autocorrelations with their exact means and variances yields more accurate critical values. On the other hand, Q_3 and Q_4 yield almost identical results; taking into account the covariances between the sample autocorrelations does not seem to improve the control of the level. Finally, we find that the critical values of the rank portmanteau statistic Q_5 are about as well approximated by the chi-square asymptotic distribution then those of Q_3.

2. PARAMETRIC GENERALIZED PORTMANTEAU STATISTICS

Both the Box-Pierce and the Ljung-Box statistics may be viewed as special cases of

$$Q = (\underline{r} - \underline{\nu})'\Sigma_0^{-1}(\underline{r} - \underline{\nu}) \quad , \qquad (2.1)$$

where $\underline{r} = (r_1,\ldots,r_m)'$, $\underline{\nu} = (\nu_1,\ldots,\nu_m)'$ is a vector of constants and Σ_0 is an $m\times m$ positive definite fixed matrix. Usually, one wishes to set $\underline{\nu}$ and Σ_0 close to the true mean and the true covariance matrix of $\underline{r}$. If we take $\underline{\nu} = \underline{0}$ and $\Sigma_0 = n^{-1}I_m$, we get the Box-Pierce statistic Q_1. If we take $\underline{\nu} = \underline{0}$ and $\Sigma_0 = \mathrm{Diag}(c_1^2,\ldots,c_m^2)$ where $c_k^2 = (n-k)/[n(n+2)]$, we get the Ljung-Box statistic Q_2.

When the underlying sample is random, whether normal or non-normal, neither Q_1 nor Q_2 uses the exact first and second moments of the sample autocorrelations. If $X_1,\ldots,X_n$ are i.i.d. (Moran, 1948) and, more generally, if they are exchangeable (Dufour and Roy, 1984), the expected value of r_k is

$$\mu_k = -(n-k)/[n(n-1)] \quad , \quad 1 \leq k \leq n-1 \quad . \qquad (2.2)$$

This holds irrespective of the form of the distribution (provided $P[X_1 = X_2 = \ldots = X_n] = 0$). We will also use the following results from Dufour and Roy (1984). The variance-covariance structure of the r_k's depends on the form of the distribution only through $E(S_4/S_2^2)$, where $S_j = \sum_{t=1}^{n} (X_t-\bar{X})^j$, $j \geq 1$. When $X_1,\ldots,X_n$ are i.i.d. $N(\mu,\sigma^2)$, the exact second moments of the sample autocorrelations are

$$\mathrm{Var}(r_k) = \frac{n^4 - (k+3)n^3 + 3kn^2 + 2k(k+1)n - 4k^2}{(n+1)n^2(n-1)^2} \; , \; 1 \leq k \leq n-1, \qquad (2.3)$$

$$\mathrm{Cov}(r_k, r_h) = \frac{2[kh(n-1) - (n-h)(n^2-k)]}{(n+1)n^2(n-1)^2} \quad , \ 1 \leq k < h \leq n-1 \ . \qquad (2.4)$$

The distribution of the sample autocorrelations is the same for all samples that follow a spherically symmetric (s.s.) distribution: since the distribution of a Gaussian white noise is s.s., the results in (2.3) and (2.4) hold whenever the sample has a s.s. distribution. Further, when $X_1, \ldots, X_n$ are exchangeable, the following upper bound on the variance of r_k holds:

$$\mathrm{Var}(r_k) \leq U_k^2 \equiv \frac{n^3 - (k+5)n^2 + (5k+6)n + 2k(k-4)}{n(n-1)^2(n-3)} \ , \ k \geq 1, \ n > 3. \quad (2.5)$$

Again this bound is valid irrespective of the form of the distribution. For most values of n and k, it is only slightly larger than the true variance in the normal case.

From the above results, we can suggest alternative portmanteau statistics and study the exact expected value of any portmanteau statistic. Namely, when testing the randomness of a normal sample, it seems natural to normalize the sample autocorrelations with their exact means and variances. This leads to the statistic

$$Q_3 = \sum_{k=1}^{m} (r_k - \mu_k)^2/\sigma_k^2 = (\underline{r} - \underline{\mu})' D^{-1} (\underline{r} - \underline{\mu}) \quad , \qquad (2.6)$$

where μ_k is given by (2.2), $\sigma_k^2 \equiv \mathrm{Var}(r_k)$ is given by (2.3), $D = \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_m^2)$ and $\underline{\mu} = (\mu_1, \ldots, \mu_m)'$. Q_3 however does not take into account the covariances between the autocorrelations. Since the latter are also available, we are led to consider the alternative statistic

$$Q_4 = (\underline{r} - \underline{\mu})' \Sigma^{-1} (\underline{r} - \underline{\mu}) \qquad (2.7)$$

where $\Sigma = [\sigma_{jk}]$ is the covariance matrix of $\underline{r}$ as given by (2.3) and (2.4).

For i.i.d. random variables with finite variances, $\sqrt{n}\underline{r}$ is asymptotically $N(\underline{0}, I_m)$ (see Anderson, 1971, Corollary 8.4.3.) and Q_1 is asymptotically $\chi^2(m)$ as n tends to infinity (with m fixed). Furthermore, it is easy to see that Q_2, Q_3 and Q_4 are asymptotically equivalent to Q_1 and thus have a $\chi^2(m)$ asymptotic distribution under the same conditions.

We will now study in greater detail the expected values of the portmanteau statistics described above. In general, the expected value of any portmanteau statistic Q in (2.1) can be written as

$$E(Q) = \mathrm{tr}(\Sigma_0^{-1}\Sigma) + (\underline{\mu} - \underline{\nu})'\Sigma_0^{-1}(\underline{\mu} - \underline{\nu}) \quad , \tag{2.8}$$

where Σ is the true covariance matrix of $\underline{r}$. Consider the case where $X_1, \ldots, X_n$ are i.i.d. normal. Then, we see easily that

$$E(Q_3) = E(Q_4) = m \quad , \tag{2.9}$$

so that Q_3 and Q_4 have exactly the same mean as the $\chi^2(m)$ distribution. On the other hand, from (2.3) and (2.4), we have

$$E(r_k^2) \equiv D_k^2 = \frac{n^3 - (k+1)n^2 + 3k^2}{n^2(n^2-1)} \quad , \quad 1 \leq k \leq n-1 \quad , \tag{2.10}$$

from which we get

$$E(Q_1) = n \sum_{k=1}^{m} D_k^2 \quad , \tag{2.11}$$

$$E(Q_2) = \sum_{k=1}^{m} (D_k/c_k)^2 = n(n+2) \sum_{k=1}^{m} D_k^2/(n-k) \quad . \tag{2.12}$$

Further, we can obtain a simple expression of $E(Q_1)$ in terms of m and n:

$$E(Q_1) = \frac{m\left[n^2(2n-m-3) + (m+1)(2m+1)\right]}{2n(n^2-1)} = m - \frac{m(m+3)}{2n} + O(n^{-2}). \tag{2.13}$$

It is interesting to observe here that the modified Box-Pierce statistic suggested by Li and McLeod (1981) for multivariate time series reduces in the univariate case to

$$Q_1^* = Q_1 + \frac{m(m+1)}{2n} \quad . \tag{2.14}$$

Though it does not fully correct the mean of Q_1, this modification clearly moves the mean in the right direction.

For non-normal distributions, it is interesting to note that the distribution of any generalized parametric portmanteau statistic is the same for all samples that follow a s.s. distribution. Consequently the results (2.8) - (2.12) hold exactly for this more general family of distributions, which includes probability laws that can differ markedly from that of a Gaussian white noise (e.g. the multivariate Student-t distribution, a multivariate Cauchy, etc.). Further, if we only assume that $X_1, \ldots, X_n$ are exchangeable, we can obtain from (2.5) an upper bound on $E(r_k^2)$:

$E(r_k^2) < V_k^2$, where

$$V_k^2 = U_k^2 + \{(n-k)^2/[n^2(n-1)^2]\}$$
$$= \frac{n^3 - (k+3)n^2 + 2kn + 3k^2}{n^2(n-1)(n-3)} \quad , k\geq 1,\ n>3 \quad . \tag{2.15}$$

Upper bounds for $E(Q_i)$, $i=1,2,3$, are then easily derived:

$$E(Q_1) < n \sum_{k=1}^{m} V_k^2 = \frac{m[2n^3 - (m+7)n^2 + 2(m+1)n + 2m^2 + 3m + 1]}{2n(n-1)(n-3)}, \tag{2.16}$$

$$E(Q_2) < \sum_{k=1}^{m} (V_k/c_k)^2 = n(n+2) \sum_{k=1}^{m} V_k^2/(n-k), \tag{2.17}$$

$$E(Q_3) < \sum_{k=1}^{m} (U_k/\sigma_k)^2 \quad . \tag{2.18}$$

These bounds hold for any sample of i.i.d. observations, irrespective of the form of the distribution. We did not obtain a similar bound for $E(Q_4)$.

Table 1 provides numerical evaluations of the exact means $E(Q_i)$, $i=1,2,3$, and the corresponding upper bounds, for a number of values of m and n. We see that $E(Q_1)$ is appreciably smaller than m, in particular for n small and m large with respect to n. Even the upper bound is smaller than m. $E(Q_2)$ is always larger than m, though the distortion is clearly less pronounced than for Q_1. Of course, $E(Q_3)$ is exactly equal to m. For all cases examined, the upper bounds are remarkably close to the exact values in the normal case.

Some derivations in this section require lengthy algebraic manipulations, especially (2.3), (2.4), (2.5), (2.13) and (2.16). The latter were all checked with the symbolic manipulation program MACSYMA (1983).

3. RANK PORTMANTEAU STATISTIC

Rank autocorrelations were studied by several authors as an attractive nonparametric alternative to standard autocorrelation coefficients; see Wald and Wolfowitz (1943), Stuart (1956), Knoke (1977), Gupta and Govindarajulu (1980), Aiyar (1981), Dufour (1981), Bartels (1982), Dufour, Lepage and Zeidan (1982),

TABLE I

EXACT VALUES AND UPPER BOUNDS FOR MEANS OF DIFFERENT PORTMANTEAU STATISTICS

n	m	$E(Q_1)$	$\bar{E}(Q_1)$	$E(Q_2)$	$\bar{E}(Q_2)$	$E(Q_3)$	$\bar{E}(Q_3)$
10	1	0.8	1.0	1.1	1.3	1.0	1.3
	3	2.2	2.6	3.2	4.0	3.0	3.7
	5	3.2	3.9	5.5	6.7	5.0	6.2
20	1	0.9	1.0	1.0	1.2	1.0	1.1
	3	2.6	2.8	3.1	3.5	3.0	3.3
	5	4.0	4.5	5.2	5.8	5.0	5.6
	10	6.9	7.6	10.5	11.6	10.0	11.1
30	1	0.9	1.0	1.0	1.1	1.0	1.1
	3	2.7	2.9	3.1	3.3	3.0	3.2
	5	4.3	4.6	5.1	5.5	5.0	5.4
	10	7.9	8.4	10.3	11.0	10.0	10.7
	15	10.6	11.4	15.5	16.6	15.0	16.1
50	1	1.0	1.0	1.0	1.1	1.0	1.0
	3	2.8	2.9	3.1	3.2	3.0	3.1
	5	4.6	4.8	5.1	5.3	5.0	5.2
	10	8.7	9.1	10.2	10.6	10.0	10.4
	15	12.3	12.8	15.3	15.9	15.0	15.6
	25	18.1	18.9	25.5	26.5	25.0	26.0
100	1	1.0	1.0	1.0	1.0	1.0	1.0
	3	2.9	3.0	3.0	3.1	3.0	3.1
	5	4.8	4.9	5.0	5.2	5.0	5.1
	10	9.4	9.5	10.1	10.3	10.0	10.2
	15	13.7	13.9	15.1	15.4	15.0	15.3
	25	21.5	22.0	25.2	25.7	25.0	25.5
	50	36.9	37.6	50.5	51.5	50.0	51.1

Q_1 and Q_2 are respectively the Box-Pierce and the Ljung-Box statistics defined in (1.2) and (1.3). Q_3 is defined by (2.6). $\bar{E}(Q_i)$, $i=1,2,3$, refer to the corresponding upper bounds from (2.16) - (2.18).

Govindarajulu (1983), Bhattacharyya (1984), Dufour and Roy (1984), Hallin, Ingenbleek and Puri (1985a, b). The rank autocorrelation at lag k is obtained by replacing each observation X_t by its rank R_t:

$$\tilde{r}_k = \sum_{t=1}^{n-k} (R_t - \bar{R})(R_{t+k} - \bar{R}) \Big/ \sum_{t=1}^{n} (R_t - \bar{R})^2 \quad , \; 1 \le k \le n-1 \quad , \qquad (3.1)$$

where $\bar{R} = \sum_{t=1}^{n} R_t/n = (n+1)/2$ and $\sum_{t=1}^{n} (R_t - \bar{R})^2 = n(n^2-1)/12$. Though a number of alternative definitions have been considered in the literature, the latter is the simplest and most natural one. The distribution of the rank autocorrelations is the same whenever $X_1,\ldots,X_n$ are continuous i.i.d. random variables, irrespective of the form of the distribution. Actually, this holds under the weaker assumption that $X_1,\ldots,X_n$ are continuous exchangeable, because all rank permutations in this case are equally probable. When testing randomness (or exchangeability) against serial dependence at several lags, it is natural to combine rank autocorrelations in a portmanteau statistic. To do that however, we need the first and second moments of $\tilde{r}_k$ under the null hypothesis.

Since $R_1,\ldots,R_n$ are exchangeable when $X_1,\ldots,X_n$ are exchangeable and continuous, the mean of $\tilde{r}_k$ is given by (2.2). Further, using (2.3), (2.5) and (3.3) of Dufour and Roy (1984), we get after some tedious algebra:

$$\mathrm{Var}(\tilde{r}_k) = \frac{5n^4 - (5k+9)n^3 + 9(k-2)n^2 + 2k(5k+8)n + 16k^2}{5(n-1)^2 n^2 (n+1)} \; , \; 1 \le k \le n-1 , \qquad (3.2)$$

$$\mathrm{Cov}(\tilde{r}_k, \tilde{r}_h) = - \frac{2\left[5n^3 - (5h-6)n^2 - (5hk-k+6h)n - 8hk\right]}{5(n-1)^2 n^2 (n+1)} \; , \; 1 \le k < h \le n-1 , \qquad (3.3)$$

We checked the two last expressions with the program MACSYMA. If we expand these formulas up to order n^{-2}, we see that

$$\mathrm{Var}(\tilde{r}_k) = \frac{5(n-k) - 4}{5n^2} + O(n^{-3}) \quad , \qquad (3.4)$$

$$\mathrm{Cov}(\tilde{r}_k, \tilde{r}_h) = - \frac{2}{n^2} + O(n^{-3}) \quad . \qquad (3.5)$$

Given the above results, we define the following portmanteau statistic

$$Q_5 = \sum_{k=1}^{m} (\tilde{r}_k - \mu_k)^2 / \tilde{\sigma}_k^2 \qquad (3.6)$$

where $\tilde{\sigma}_k^2 = \text{Var}(\tilde{r}_k)$ is given by (3.2). It is easy to see that $E(Q_5) = m$. Further, we can show that the asymptotic distribution of Q_5 is $\chi^2(m)$, when $X_1,\ldots,X_n$ are continuous exchangeable. We state this result in the following theorem.

THEOREM: Let $X_1,\ldots,X_n$ be exchangeable and continuous random variables, $\tilde{r}_k$ a rank autocorrelation obtained from these by (3.1) and $\underline{\tilde{r}} = (\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_m)'$ where $m < n$. Then, for m fixed, the asymptotic distribution of $\sqrt{n}[\underline{\tilde{r}} - E(\underline{\tilde{r}})]$ as $n\to\infty$ is $N[\underline{0}, I_m]$ and the asymptotic distribution of Q_5 is $\chi^2(m)$.

PROOF: To prove this theorem, we will find convenient to use the results of Hallin et al (1985b). Consider the alternative definition

$$\tilde{R}_k = \left(\frac{n}{n-k}\right) \frac{\sum_{t=1}^{n-k} R_t R_{t+k} - (n-k)\bar{R}^2}{\sum_{t=1}^{n} (R_t - \bar{R})^2}, \quad 1 \leq k \leq n-1 , \tag{3.7}$$

and let $\underline{\tilde{R}} = (\tilde{R}_1,\ldots,\tilde{R}_m)'$ where m is fixed. Under the assumption that $X_1,\ldots,X_n$ are i.i.d. with a density satisfying certain regularity conditions, Hallin et al (1985b) show that $\sqrt{n}[\underline{\tilde{R}} - E(\underline{\tilde{R}})]$ is asymptotically $N[\underline{0},I_m]$. However, in this case, the distribution of $\underline{\tilde{R}}$ is determined by the distribution of the rank vector which is uniform over the set of all permutations of $1,2,\ldots,n$. Since the latter property also holds in the more general case where $X_1,\ldots,X_n$ are continuous exchangeable, we can infer that the asymptotic distribution of $\sqrt{n}[\underline{\tilde{R}} - E(\underline{\tilde{R}})]$ is $N[\underline{0},I_m]$ whenever $X_1,\ldots,X_n$ are continuous exchangeable.

The latter result also holds for $\underline{\tilde{r}} = (\tilde{r}_1,\ldots,\tilde{r}_m)'$. To see this, let $r_k^* = [(n-k)/n]\tilde{R}_k$ and observe that

$$\sum_{t=1}^{n-k} (R_t - \bar{R})(R_{t+k} - \bar{R}) = \sum_{t=1}^{n-k} R_t R_{t+k} - (n-k)\bar{R}^2 + D,$$

where

$$D = -2k\bar{R}^2 + \bar{R}\left[\sum_{t=1}^{k} R_t + \sum_{t=n-k+1}^{n} R_t\right].$$

Since $1 \leq R_t \leq n$ and $\bar{R} = (n+1)/2$, we have $|D| \leq (k/2)(n+1)(3n+1)$, hence

$$|\tilde{r}_k - r_k^*| = O(n^{-1}) \ . \tag{3.8}$$

Further, we can write

$$\tilde{r}_k - \tilde{R}_k = (\frac{n}{n-k})(\tilde{r}_k - r_k^*) - \frac{k\tilde{r}_k}{n-k} \ ,$$

hence, using (3.8) and $|\tilde{r}_k| < 1$,

$$|\tilde{r}_k - \tilde{R}_k| = O(n^{-1}) \quad , \quad |E(\tilde{r}_k) - E(\tilde{R}_k)| = O(n^{-1})$$

and

$$\sqrt{n}[\tilde{r}_k - E(\tilde{r}_k)] = \sqrt{n}[\tilde{R}_k - E(\tilde{R}_k)] + O(1/\sqrt{n}) \ . \tag{3.9}$$

Consequently, $\sqrt{n}[\underline{\tilde{r}} - E(\underline{\tilde{r}})]$ and $\sqrt{n}[\underline{\tilde{R}} - E(\underline{\tilde{R}})]$ have the same asymptotic distribution, and $\sqrt{n}[\underline{\tilde{r}} - E(\underline{\tilde{r}})]$ is asymptotically $N[\underline{0}, I_m]$. Further, the asymptotic covariance matrix of $\underline{\tilde{r}}$ may also be obtained by directly taking the limits of (3.2) and (3.3). In particular, $\lim_{n\to\infty} (n\tilde{\sigma}_k^2) = 1$. Since $E(\tilde{r}_k) = \mu_k$ and $T_k \equiv [\tilde{r}_k - \mu_k]/\tilde{\sigma}_k = \sqrt{n}[\tilde{r}_k - E(\tilde{r}_k)]/(n\tilde{\sigma}_k^2)^{\frac{1}{2}}$, $k=1,\ldots,m$, we can conclude that the vector $\underline{T} = (T_1, \ldots, T_m)'$ is also asymptotically $N[\underline{0}, I_m]$ and Q_5 is asymptotically $\chi^2(m)$. Q.E.D.

Of course, one rejects the null hypothesis of randomness (or exchangeability) against serial dependence alternatives (e.g. an ARMA model) when Q_5 is greater than the appropriate critical value determined by using a table of the chi-square distribution.

The statistic Q_5 is derived in a way analogous to the parametric statistic Q_3. We could also consider statistics analogous to Q_1, Q_2 or Q_4. However, in view of the simulation results obtained for the parametric statistics (see section 4), we did not study these alternatives in detail.

4. SIMULATION RESULTS

Even if the asymptotic distribution of all the portmanteau statistics described above is the same, the reliability of the chi-square approximation to set critical values may differ markedly in finite samples. In particular, we wish to know whether the new parametric portmanteau statistics are better approximated by the chi-square distribution than the Ljung-Box statistic and whether the rank portmanteau statistic (whose distribution is actually discrete) is reasonably well approximated by the chi-square distribution.

To investigate these issues, we conducted the following Monte-Carlo experiment. For each of five different series lengths (n=10, 20, 30, 50, 100), 10000 independent realizations of a Gaussian white noise were generated using the subroutine GGUBS of IMSL (1984). For each realization, the parametric statistics Q_i, $i=1,\ldots,4$, were computed with a number of values of m. To appreciate the accuracy of the $\chi^2(m)$ approximation, we examined the empirical frequencies of rejection of the null hypothesis of randomness by each test with five different nominal levels (1, 5, 10, 20 and 30 percent). We also computed the empirical variances of the test statistics.

To minimize the computation costs, we considered the rank portmanteau statistic (Q_5) separately. Under the hypothesis of randomness, the vector of ranks $(R_1,\ldots,R_n)$ is a random permutation of the integers 1,...,n. For each series length, 10000 independent random permutations of the integers 1,...,n were generated directly using the subroutine GGPER of IMSL. For each permutation and each value of m, Q_5 was computed. To evaluate the accuracy of the chi-square approximation, we considered the same indicators as for the parametric statistics.

The resulta of the experiment are presented in Table 2. We make the following observations. First, for the Box-Pierce statistic (Q_1), the experiment confirms that the $\chi^2(m)$ distribution is a poor approximation, even for series of 100 observations: in most cases, the empirical levels and variances are much lower than the theoretical ones. Second, the distribution of the Ljung-Box statistic (Q_2) is much closer to the $\chi^2(m)$ distribution. However, for n small (n=10), for m>5 with n=20, and for m>10 with n=30, 50, 100, the empirical significance levels can be appreciably larger than the theoretical ones. For all values of n and for m>1, the empirical variance of Q_2 is larger than the theoretical value 2m. Third, Q_3 yields the best set of results. The agreement between the empirical and the theoretical levels is in general better than for Q_2. However, for levels less than or equal to 10 percent and for m large with respect to n, the empirical frequencies of

TABLE 2: EMPIRICAL SIGNIFICANCE LEVELS (IN PERCENTAGE) AND VARIANCES OF DIFFERENT PORTMANTEAU STATISTICS FOR A NORMAL WHITE NOISE

n=10

% level	Test	m=1	m=3	m=5
1	Q_1	0.0	0.2	0.1
	Q_2	0.6	1.2	1.5
	Q_3	0.5	1.1	1.1
	Q_4	0.5	1.2	1.1
	Q_5	0.4	1.2	1.5
5	Q_1	2.1	1.1	0.5
	Q_2	5.3	5.3	6.7
	Q_3	4.2	4.7	5.3
	Q_4	4.2	4.5	4.8
	Q_5	4.7	4.7	5.6
10	Q_1	6.2	2.6	1.3
	Q_2	11.8	11.1	12.1
	Q_3	9.8	9.3	10.2
	Q_4	9.8	9.1	9.3
	Q_5	10.2	9.6	10.5
20	Q_1	16.9	7.7	4.3
	Q_2	23.7	22.2	23.5
	Q_3	21.7	19.9	19.1
	Q_4	21.7	19.4	18.6
	Q_5	22.1	19.8	19.8
30	Q_1	27.0	15.5	8.5
	Q_2	34.3	33.5	35.2
	Q_3	32.7	30.4	29.5
	Q_4	32.7	29.5	28.7
	Q_5	31.6	30.1	29.2
Variance	Q_1	1.0	2.8	4.0
	Q_2	1.8	6.4	11.2
	Q_3	1.6	6.0	10.7
	Q_4	1.6	5.9	9.9
	Q_5	1.6	6.2	11.8

n=20

% level	Test	m=1	m=3	m=5	m=10
1	Q_1	0.4	0.4	0.4	0.2
	Q_2	0.7	1.1	1.9	2.6
	Q_3	0.7	1.2	1.7	2.3
	Q_4	0.7	1.3	1.9	2.3
	Q_5	0.6	1.1	1.7	3.0
5	Q_1	3.5	2.6	2.4	1.1
	Q_2	5.1	5.2	6.2	8.1
	Q_3	4.8	5.0	5.7	7.1
	Q_4	4.8	5.2	5.7	6.8
	Q_5	4.8	4.7	5.5	7.5
10	Q_1	8.2	5.6	4.6	2.4
	Q_2	10.7	10.1	10.8	13.0
	Q_3	10.1	9.6	10.0	11.4
	Q_4	10.1	9.5	9.9	11.3
	Q_5	9.6	9.6	10.1	12.0
20	Q_1	18.2	13.4	10.1	5.3
	Q_2	21.4	21.0	20.7	22.6
	Q_3	20.4	19.4	19.4	19.8
	Q_4	20.4	19.4	19.0	19.5
	Q_5	20.4	18.7	18.9	19.9
30	Q_1	28.1	22.5	17.1	9.0
	Q_2	31.7	31.6	30.8	32.2
	Q_3	30.6	29.4	28.6	28.9
	Q_4	30.6	29.2	28.3	28.3
	Q_5	31.6	29.2	28.0	27.8
Variance	Q_1	1.4	4.1	6.9	12.5
	Q_2	1.9	6.1	11.6	27.5
	Q_3	1.8	6.0	11.3	26.8
	Q_4	1.8	6.2	11.8	27.1
	Q_5	1.7	6.2	12.2	30.5

n=30

% level	Test	m=1	m=3	m=5	m=10	m=15
1	Q_1	0.7	0.6	0.7	0.8	0.5
	Q_2	1.0	1.3	1.6	2.8	3.5
	Q_3	1.0	1.2	1.5	2.7	3.1
	Q_4	1.0	1.1	1.7	2.7	3.1
	Q_5	0.8	1.2	1.6	2.5	3.4
5	Q_1	4.5	3.1	3.0	2.5	1.7
	Q_2	5.4	5.2	6.0	7.6	8.7
	Q_3	5.2	4.9	5.5	7.1	7.9
	Q_4	5.2	4.8	5.4	7.0	7.9
	Q_5	4.9	4.8	5.6	6.9	8.3
10	Q_1	9.4	7.2	6.3	4.5	2.8
	Q_2	11.1	10.5	11.0	12.2	13.9
	Q_3	10.2	9.8	10.3	11.3	12.5
	Q_4	10.2	9.8	10.1	11.6	12.4
	Q_5	10.1	9.6	10.2	11.6	12.9
20	Q_1	19.4	15.6	13.3	8.8	5.5
	Q_2	21.5	21.3	20.6	21.8	23.0
	Q_3	20.7	19.7	19.6	20.0	20.9
	Q_4	20.7	19.7	19.3	19.7	20.7
	Q_5	20.4	19.3	19.3	20.1	20.9
30	Q_1	29.5	25.6	21.4	14.1	8.6
	Q_2	32.0	31.2	30.4	31.0	31.4
	Q_3	30.9	29.9	28.6	28.9	28.7
	Q_4	30.9	29.9	28.6	28.2	28.6
	Q_5	30.4	29.2	28.5	28.8	28.9
Variance	Q_1	1.7	4.9	8.2	16.9	23.5
	Q_2	2.1	6.3	11.5	28.8	47.9
	Q_3	2.0	6.3	11.3	28.4	46.8
	Q_4	2.0	6.4	11.8	29.8	48.4
	Q_5	1.8	6.4	11.6	28.5	48.6

n=50

% level	Test	m=1	m=3	m=5	m=10	m=15	m=25
1	Q_1	0.8	0.8	0.8	0.8	0.8	0.5
	Q_2	0.9	1.2	1.4	2.2	2.9	4.0
	Q_3	0.8	1.1	1.3	2.0	2.7	3.7
	Q_4	0.8	1.1	1.3	2.0	2.7	3.5
	Q_5	0.9	1.3	1.5	2.3	3.1	4.4
5	Q_1	4.6	3.9	3.6	3.2	2.6	1.5
	Q_2	5.3	5.1	5.4	6.6	7.7	9.4
	Q_3	5.1	4.9	5.1	6.0	7.3	8.6
	Q_4	5.1	4.8	5.1	6.2	7.3	8.6
	Q_5	4.6	5.2	5.4	6.9	7.8	9.1
10	Q_1	9.2	7.8	7.1	6.0	5.0	2.5
	Q_2	10.3	9.9	10.1	11.3	12.5	14.5
	Q_3	10.2	9.7	9.8	10.7	11.8	13.4
	Q_4	10.2	9.5	9.9	10.9	11.8	13.3
	Q_5	9.9	10.1	10.1	11.5	12.2	13.9
20	Q_1	19.3	17.3	15.5	12.0	9.5	4.8
	Q_2	20.7	20.4	20.0	20.6	21.5	23.3
	Q_3	20.3	19.6	19.3	19.8	20.4	21.9
	Q_4	20.3	19.4	19.3	19.7	20.4	21.6
	Q_5	20.8	19.8	19.4	19.7	20.5	21.8
30	Q_1	29.8	27.1	24.4	19.1	14.6	7.4
	Q_2	31.5	30.4	30.3	29.8	30.0	31.6
	Q_3	30.9	29.7	29.1	28.6	28.7	29.6
	Q_4	30.9	29.8	28.8	28.3	28.8	29.7
	Q_5	31.2	29.7	29.1	28.4	28.4	29.4
Variance	Q_1	1.7	5.3	9.0	18.8	28.6	45.2
	Q_2	2.0	6.2	11.0	25.5	43.7	86.3
	Q_3	1.9	6.2	10.9	25.2	43.1	84.9
	Q_4	1.9	6.2	11.0	25.5	43.4	83.8
	Q_5	1.9	6.2	11.0	26.6	46.2	93.0

n=100

% level	Test	m=1	m=3	m=5	m=10	m=15	m=25	m=50
1	Q_1	0.8	0.8	1.0	1.2	1.2	0.9	0.4
	Q_2	1.0	1.0	1.3	1.8	2.2	2.9	4.0
	Q_3	0.9	1.0	1.3	1.7	2.1	2.8	3.9
	Q_4	0.9	1.0	1.2	1.7	2.2	2.8	3.9
	Q_5	1.0	1.2	1.5	1.8	1.9	2.8	4.0
5	Q_1	4.7	4.2	4.2	4.1	3.8	3.0	0.9
	Q_2	5.0	4.9	5.2	6.0	6.8	7.9	9.8
	Q_3	4.8	4.8	5.0	5.9	6.5	7.5	9.3
	Q_4	4.8	4.9	5.0	5.9	6.5	7.4	9.1
	Q_5	5.2	5.3	5.4	5.9	6.5	7.6	9.7
10	Q_1	9.9	8.9	8.4	7.7	7.1	5.3	1.6
	Q_2	10.4	9.9	10.1	11.0	11.8	13.1	15.0
	Q_3	10.1	9.6	9.8	10.8	11.4	12.4	14.1
	Q_4	10.1	9.6	9.9	10.7	11.3	12.2	13.9
	Q_5	10.2	10.2	10.1	10.7	11.1	12.0	14.6
20	Q_1	19.6	18.2	17.5	15.8	14.0	10.0	3.2
	Q_2	20.3	19.7	19.8	20.3	20.4	21.3	23.9
	Q_3	20.3	19.5	19.4	19.7	19.7	20.6	22.8
	Q_4	20.3	19.5	19.4	19.5	19.8	20.5	22.5
	Q_5	20.3	20.1	20.2	19.7	20.4	21.0	22.5
30	Q_1	29.4	28.0	27.2	24.2	20.9	15.7	5.0
	Q_2	30.2	29.8	30.2	29.6	29.6	29.6	31.5
	Q_3	30.3	29.2	29.4	29.0	28.9	28.9	30.1
	Q_4	30.3	29.1	29.5	29.0	29.0	28.9	30.0
	Q_5	31.0	30.0	30.0	29.1	29.4	29.9	29.9
Variance	Q_1	1.8	5.6	9.4	19.9	30.4	52.0	97.7
	Q_2	1.9	6.0	10.4	23.2	37.4	71.3	176.8
	Q_3	1.9	6.0	10.3	23.2	37.2	70.6	175.2
	Q_4	1.9	6.0	10.3	23.4	37.7	71.1	173.8
	Q_5	2.0	6.3	11.0	23.7	38.4	73.7	184.8

Q_1 and Q_2 are the Box-Pierce and the Ljung-Box statistics defined in (1.2) and (1.3). Q_3, Q_4 and Q_5 are defined by (2.6), (2.7) and (3.6) respectively. The standard error of the empirical levels is 0.1 % for the nominal level 1 %, 0.2 for 5, 0.3 for 10, 0.4 for 20 and 0.5 for 30.

rejection remain larger than the theoretical values. Fourth, the sampling distribution of Q_4 is almost identical to the distribution of Q_3: taking into account the covariances between the sample autocorrelations does not seem to improve the control of the level. This may be explained by the fact that the covariances are of order $O(n^{-2})$. Finally, the empirical significance levels of the rank portmanteau statistic Q_5 are very close to those of Q_3; indeed, the $\chi^2(m)$ distribution provides a better approximation of the distribution of Q_5 than of Q_2.

5. CONCLUDING REMARKS

For cases where the true underlying distribution is normal, the above results suggest that the new portmanteau statistic Q_3, which uses the exact means and variances of the sample autocorrelations, is better approximated by the $\chi^2(m)$ distribution than the Ljung-Box statistic and, **a fortiori**, the Box-Pierce statistic. Q_4 which takes into account the covariances between the sample autocorrelations is equally well approximated by the same distribution but is computationally less convenient. Besides, the statistic Q_5 based on rank autocorrelations behaves almost as well as Q_3 and has the advantage of having the same distribution for all random series. Consequently, when testing randomness with portmanteau statistics, we suggest to use Q_3 when the distribution is reasonably close to the normal and Q_5 when more robustness is required.

Our results also underscore a number of topics worthwhile further investigation. First, even if the distributions of the new portmanteau statistics are better approximated by the $\chi^2(m)$ distribution, the approximation is far from perfect. Obtaining the exact distributions of pormanteau statistics remains an important though probably difficult problem. We may note here that deriving the distribution of the parametric portmanteau statistics for a Gaussian white noise and the distribution of the rank portmanteau statistic constitute quite separate problems. Second, it is possible that alternative portmanteau statistics have different power characteristics against various serial dependence alternatives, especially in finite samples. This topic may require extensive analytical and/or simulation studies. However, it is certainly worthwhile further study.

Third, though this paper focused on the problem of testing randomness, the modified portmanteau tests suggested above could also be applied to the residuals from a fitted ARMA model (with an appropriate degrees-of-freedom correction). Indeed, portmanteau tests were originally suggested as a way of checking the specification of time-series models; see Box and Pierce (1970), Ljung and Box (1978), McLeod (1978), Ansley and Newbold (1979), Davies and Newbold (1979), Clarke and Godolphin (1982), Ljung (1982), Newbold (1983). It is easy to see that the modified parametric portmanteau statistics Q_3 and Q_4 are asymptotically equivalent to Q_1 and Q_2 (at least under the null hypothesis) and thus the asymptotic null distribution is $\chi^2(m-p-q)$ when an ARMA(p,q) model has been fitted. Since a large proportion of empirical ARMA models assume a non-zero mean, we can conjecture that using the exact first and second moments for the special case where only the mean is estimated may lead to statistics whose distributions are better approximated by the chi-square distribution. This question is the topic of on-going research.

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