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GENERALIZED PROBABILISTIC PERTURBATION METHOD FOR STATIC ANALYSIS

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Abstract

This paper studies the static response and reliability of uncertain structures with vector-valued and matrix-valued functions. The finite element analysis method of uncertain structures is based on matrix calculus, Kronecker algebra and perturbation theory. Random variables and system derivatives are conveniently arranged into 2D matrices and generalized mathematical formulae for probabilistic perturbation are obtained.

Key words uncertain structures, response, reliability, generalized probabilistic perturbation method

I. Introduction

Up to the present, much research has been done to the finite element analysis of the deterministic parameter structures. Because of uncertain information in design stage of engineering structures, design parameters of the structures are uncertain; moreover, structural parameters are uncertain due to uncertainty of engineering material properties, errors of manufacture and installation, etc.. The investigations of effects of random structural parameters on structural response and reliability are important for engineering safe analysis. In recent twenty years many methods have been presented on this problem, for example, the probabilistic finite element method[1]-[7], the probabilistic perturbation method[7] and so on.

This paper focuses on extension of the probabilistic perturbation method to vector-valued and matrix-valued functions. Using matrix calculus, Kronecker algebra and perturbation theory generalized formulae for the probabilistic perturbation method are developed. The finite element analysis for generalized random structures becomes proposible.

II. Static Response Analysis

The linear finite element equations are

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$$Ku=P \tag{2.1}$$

where the stiffness matrix is

$$K=\int_{\Omega} B^T D B d\Omega \tag{2.2}$$

where $B(x)$, $D(x, R)$, $u(R)$, and $P(R)$ denote the generalized gradient, material property matrix, nodal displacement vector and nodal force vector, respectively; x is the spatial coordinate; Ω is the domain and a superscript T is the transpose; $R(x)$ is a random function vector of s dimension. In this formulation, $R(x)$ can be random material properties or random loads.

The random parameter vector R, stiffness matrix K, displacement vector u and force vector P are represented as

$$R=R_d+\varepsilon R_r \tag{2.3}$$

$$K=K_d+\varepsilon K_r \tag{2.4}$$

$$u=u_d+\varepsilon u_r \tag{2.5}$$

$$P=P_d+\varepsilon P_r \tag{2.6}$$

where ε is a small parameter. The determinitic portion is denoted by subscript d and the random portion with the means of zero is denoted by subscript r. Obviously, the random portion should be much less than the deterministic portion. Taking the expected value of equations (2.3), (2.4), (2.5) and (2.6) yields

$$E(R)=E(R_d)+\varepsilon E(R_r)=R_d=\bar{R} \tag{2.7}$$

$$E(K)=E(K_d)+\varepsilon E(K_r)=K_d=\bar{K}=K(\bar{R}) \tag{2.8}$$

$$E(u)=E(u_d)+\varepsilon E(u_r)=u_d=\bar{u}=u(\bar{R}) \tag{2.9}$$

$$E(P)=E(P_d)+\varepsilon E(P_r)=P_d=\bar{P}=P(\bar{R}) \tag{2.10}$$

Similarly, on the basis of Kronecker algebra[8] and corresponding stochastic analysis theory[9], taking the variance of equations (2.3), (2.4), (2.5) and (2.6) yields

$$\mathrm{Var}(R)=E[(R-E(R))^{[2]}]=\varepsilon^2 E[R_r^{[2]}] \tag{2.11}$$

$$\mathrm{Var}(K)=E[(K-E(K))^{[2]}]=\varepsilon^2 E[K_r^{[2]}] \tag{2.12}$$

$$\mathrm{Var}(u)=E[(u-E(u))^{[2]}]=\varepsilon^2 E[u_r^{[2]}] \tag{2.13}$$

$$\mathrm{Var}(P)=E[(P-E(P))^{[2]}]=\varepsilon^2 E[P_r^{[2]}] \tag{2.14}$$

where $(\cdot)^{[2]}=(\cdot)\otimes(\cdot)$ is Kronecker power, symbol $\otimes$ denotes Kronecker product. $\mathrm{Var}(\cdot)$ includes all the variances and covariances.

Substituting the equations (2.4), (2.5) and (2.6) into the equation (2.1) yields

$$(K_d+\varepsilon K_r)(u_d+\varepsilon u_r)=P_d+\varepsilon P_r \tag{2.15}$$

Expanding Eq. (2.15) and comparing the coefficients of the same power of ε, and neglecting the terms of $O(\varepsilon^2)$, we obtain

$$\varepsilon^0: \quad K_d u_d=P_d \tag{2.16}$$

$$\varepsilon^1: \quad K_d u_r=P_r-K_r u_d \tag{2.17}$$

The deterministic portion (e.g. the mean value) of the response is obtained from Eq. (2.16). Obviously, the random portion (e.g. interrelated portion to the variance) of the response can not be determined from Eq. (2.17) under the available first two moments, so that it can be solved after altering the form.

According to Taylor's rule of vector-valued and matrix-valued functions[8], K_r, u_r and P_r can be expanded to $E(R)=R_d$ under the condition that the random portion of the random parameter vector is small compared with the deterministic portion of the random parameter vector, we have

$$K_r=\frac{\partial K_d}{\partial R^T}R_r \tag{2.18}$$

$$u_r=\frac{\partial u_d}{\partial R^T}R_r \tag{2.19}$$

$$P_r=\frac{\partial P_d}{\partial R^T}R_r \tag{2.20}$$

Substituting the Eqs. (2.18), (2.19) and (2.20) into the Eq. (2.17) and expanding the Eq. (2.17), we get

$$K_d\frac{\partial u_d}{\partial R_i}=\frac{\partial P_d}{\partial R_i}-\frac{\partial K_d}{\partial R_i}u_d \tag{2.21}$$

$\partial u_d/\partial R_i$ can be obtained from Eq. (2.21)

$$\frac{\partial u_d}{\partial R^T}=\left[\frac{\partial u_d}{\partial R_1}\quad\frac{\partial u_d}{\partial R_2}\quad\cdots\quad\frac{\partial u_d}{\partial R_s}\right] \tag{2.22}$$

Substituting the Eq. (2.19) into Eq. (2.13), the variance matrix of the displacement response can be obtained

$$\mathrm{Var}(u)=\varepsilon^2E[u_r^{[2]}]=\varepsilon^2E\left[\left(\frac{\partial u_d}{\partial R^T}R_r\right)^{[2]}\right]$$

$$=\left[\frac{\partial u_d}{\partial R^T}\right]^{[2]}\mathrm{Var}(R) \tag{2.23}$$

Substituting Eq. (2.22) into Eq. (2.23), the variances of the displacement response are determined. Thus, the mean values and variances of the displacement response are completely determined.

The strain and stress vectors for a typical element e are

$$\varepsilon=B(x)u^e \tag{2.24}$$

$$\sigma=D(x,R)\varepsilon \tag{2.25}$$

where u^e is the element nodal displacement vector.

Similarly, the mean value matrix and variance matrix of the strain ε and stress σ can be obtained to be

$$E(\varepsilon)=\varepsilon_d=Bu_d^e \tag{2.26}$$

$$\mathrm{Var}(\varepsilon)=\left[B\frac{\partial u_d^e}{\partial R^T}\right]^{[2]}\mathrm{Var}(R) \tag{2.27}$$

$$E(\sigma)=\sigma_d=D_d\varepsilon_d=D_dBu_d^e \tag{2.28}$$

$$\begin{aligned}\mathrm{Var}\ (\sigma)&=\left[\frac{\partial\sigma_d}{\partial R^T}\right]^{[2]}\mathrm{Var}(R)\\&=\left[\frac{\partial D_d}{\partial R^T}(I_s\otimes Bu_d^e)+D_dB\frac{\partial u_d^e}{\partial R^T}\right]^{[2]}\mathrm{Var}(R)\end{aligned} \tag{2.29}$$

where I_s is the $s\times s$ unit matrix.

III. Reliability Analysis

A fundamental problem in reliability analysis is the computation of the multi-fold integral of the reliability R

$$R=\int_{g(X)>0}f_X(X)dX \tag{3.1}$$

in which $f_X(X)$ denotes the probability density function of the vector of random variables X (response and threshold etc.), $g(X)$ defines the state function, representing the safe state and failure state

$$\left.\begin{array}{ll}g(X)\leqslant 0 & \text{failure state}\\ g(X)>0 & \text{safe state}\end{array}\right\} \tag{3.2}$$

where $g(X)=0$ is the limit-state equation, representing limit-state surface or failure surface.

The first passage problem for uncertain structures on the basis of the interference theory of the reliability is defined as

$$g_i(X)=|A_i|-|Z_i| \tag{3.3}$$

where A_i is the threshold of Z_i of random response (displacement, strain or stress) $Z=(Z_1,Z_2,\cdots,Z_n)^T$. The response Z_i and the threshold A_i are mutual independent random variables.

The mean value and variance of the state function $g_i\ (X)$ are determined

$$E(g_i)=E|A_i|-E|Z_i| \tag{3.4}$$

$$\mathrm{Var}(g_i)=\mathrm{Var}(A_i)+\mathrm{Var}(Z_i) \tag{3.5}$$

The reliability index is defined

$$\beta_i=\frac{E(g_i)}{\sqrt{\mathrm{Var}\ (g_i)}} \tag{3.6}$$

Thus, on the one hand the structural reliability can be directly measured using the reliability index; on the other hand, when the random variable vector X is normal, the limit-state surface is replaced with the tangent plane at the failure point. The first-order estimate of the reliability is

$$R_i=\Phi(\beta_i) \tag{3.7}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

If the random variable vector X is nonnormal, a Rosenblatt transformation[10] must be made from the correlated and nonnormal random variables to uncorrelated and normal random variables.

$$Y_i=\Phi^{-1}[F_{X_i}|X_1,\cdots,X_{i-1}(X_i|X_1,\cdots,X_{i-1})] \tag{3.8}$$

where $F_X(X)$ is the joint cumulative distribution function for X, Y_i is uncorrelated, standard normal random variable with zero mean and unit standard deviation for reliability analysis, it is convenient to transform the limit-state function $g(X)$ into the limit-state function $G(Y)$ in a standard, uncorrelated normal space $Y=(Y_1, Y_2, \ldots, Y_n)^T$. This transformation is as follows

$$g(X)=G(Y) \tag{3.9}$$

IV. Numerical Examples

Example 1 The planar truss is shown in Fig. 1. When a vertically downward load of P is applied at node 4, and the mean value is 9.8(kN). The mean value of material strength r is 230.3(MPa). The mean value of random material properties is given in Table 1. The random parameter vector $R=(P, E, A, r)^T$ is normally distributed with a coefficient of variation equal to 0.05.

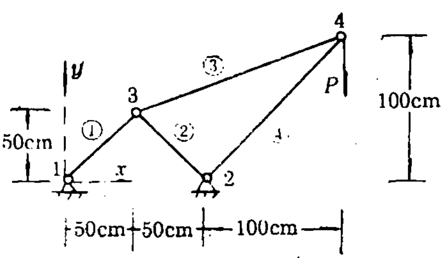

Fig. 1 The planar truss

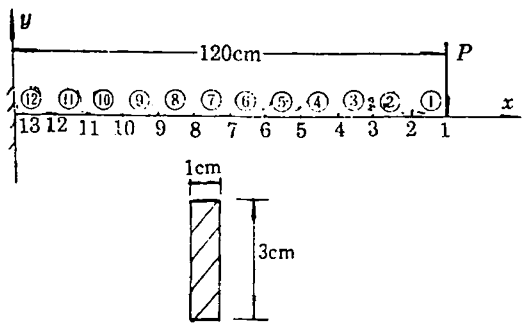

Fig. 2 Cantilever beam

Table 1

Member number c	Cross-sectional area $A^{(c)}$ (cm²)	Young's modulus $E^{(c)}$ (GPa)
1	2.0	196.0
2	2.0	196.0
3	1.0	196.0
4	1.0	196.0

The calculation results are as follows:

The stress, reliability index and reliability of element 3 are

$$E[\sigma^{(3)}]=154.95(\text{MPa}),\ \sqrt{\text{Var}(\sigma^{(3)})}=39.50(\text{MPa}),\ \beta_3=1.831,\ R_3=0.966$$

The stress, reliability index and reliability of element 4 are

$$E[\sigma^{(4)}]=-207.89(\text{MPa}),\ \sqrt{\text{Var}(\sigma^{(4)})}=53.00(\text{MPa})$$

$$\beta_4=0.413,\ R_4=0.660$$

Because the structure is static state determination structure, the structure is not safe as seen from the element 4.

Example 2 A cantilever beam is shown in Fig. 2. When a vertically downward load of $P=78.4$(N) is applied at node 1. The mean value of cross-sectional area A, Young's modulus

E, shearing modulus G, the threshold H of displacement y at node 1 are $E(A)=3.0$ (cm^2), $E(E)=196.0$(GPa), $E(G)=78.4$(GPa), $E(H)=13$ (cm). The random parameter vector $R=(A, E, G, H)^T$ is normally distributed with a coefficient of variation equal to 0.05.

The calculation results are as follows:

The displacement of node 1, reliability index and reliability are

$$E(y_1)=9.2164\ (\text{cm}),\quad \sqrt{\text{Var}(y_1)}=0.7982(\text{cm}),$$

$$\beta=3.6757,\quad R=0.99988$$

Because the transverse deflection at node 1 is largest, the structure is safe by using the transverse deflection to measure.

V. Conclusion

The structural material and geometry properties are random due to the errors for manufacture, measure, statistics, model and other uncertain factors. The loads on structures are probably random. Thus, it is inevitable to form uncertain structure systems with many random parameters. This paper employes matrix calculus, Kronecker algebra the perturbation technique to systematically develop the probabilistic perturbation method of vector-valued and matrix-valued functions and obtain the generalized mathematical formulae. The perfect numerical results are obtained. As seen from theory and numerical analysis in this paper, the probabilistic perturbation theory of 2D matrices is more generalized and complete than the probabilistic perturbation theory that has ever been researched.

References

[1] K. Handa and K. Anderson, Application of finite element methods in the statistical analysis of structures, *Proc. 3rd International Conference on Structural Safety and Reliability*, Trondheim, Norway (1981), 409.

[2] H. Contreras, The stochastic finite-element method, *Comput. & Sturct.*, **12** (1980), 341—348.

[3] W. K. Liu, T. Belytschko and A. Mani, Random field finite elements, *Int. J. Numer. Methods Eng.*, **23** (1986), 1831—1845.

[4] Zhu Weiqiu and Ren Yongjian, The stochastic finite element method based on local average of random field, *Acat Mechanics Solida Sinia*, **4** (1988), 285—294. (in Chinese)

[5] A. Der Kiureghian and J. -B. Ke, The stochastic finite element method in structural reliability, *Prob. Engrg. Mech.*, **3**, 2 (1988), 83—91.

[6] Zhang Yimin and Liu Qiaoling, The stochastic finite element method for reliability analysis of structures with many random parameters, *Journal of Northeast University of Technology*, **13**, S1 (1992), 97—99. (in Chinese)

[7] Chen Suhuan, *Vibration Theory of Structures with Random Parameters*, Jilin Science and Technology Press (1992). (in Chinese)

[8] W. J. Vetter, Matrix calculus operations and Taylor expansions. *SIAM Review*, **15**, 2 (1973), 352—369.

[9] F. Ma, Extension of second moment analysis to vector-valued and matrix-valued functions, *Int. J. Non-Linear Mech.*, **22** (1987), 251—260.

[10] M. Rosenblatt, Remarks on a multivariate transformation, *Annals of Math. Statistics*, **23** 3 (1952), 470—472.