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Generalized Quantifiers and Modal Logic

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Abstract. We study several modal languages in which some (sets of) generalized quantifiers can be represented; the main language we consider is suitable for defining any first order definable quantifier, but we also consider a sublanguage thereof, as well as a language for dealing with the modal counterparts of some higher order quantifiers. These languages are studied both from a modal logic perspective and from a quantifier perspective. Thus the issues addressed include normal forms, expressive power, completeness both of modal systems and of systems in the quantifier tradition, complexity as well as syntactic characterizations of special semantic constraints. Throughout the paper several techniques current in the theory of generalized quantifiers are used to obtain results in modal logic, and conversely.

Key words: Modal logic, generalized quantifiers, axiomatic completeness, complexity, definability.

1. INTRODUCTION

This paper is motivated mainly by the following question: in the modal system $S5$ the box ('$\Box$') and diamond ('$\Diamond$') may be interpreted as a universal and an existential quantifier, respectively (Goranko *et al.*, 1992); how can other quantifiers be represented within a modal language?

We will consider a number of modal languages, each designed to represent (a set of) generalized quantifiers. The prime case is the language $\mathcal{L}(QUANT)$ in which every first order definable quantifier will turn out to be definable; a more modest language between the language of $S5$ and $\mathcal{L}(QUANT)$ will also be studied. The third language we will consider contains the modal counterparts of some higher order quantifiers. Furthermore, techniques used in these modal languages will be employed to get some results about 'quantifier languages'.

This paper concentrates mainly on modal topics. Nevertheless, many issues addressed below find their origin in the theory of generalized quantifiers;

and even some of the techniques used are current in the theory of generalized quantifiers rather than in modal logic. On the other hand, we will also use our modal machinery to contribute some results to the theory of generalized quantifiers.

To be more specific, this paper is organized as follows. In Section 2 we introduce two modal languages $\mathcal{L}(QUANT)$ and $\mathcal{L}(QUANT_k)$ for dealing with (sets of) first order definable quantifiers; a quick normal form theorem for these languages is proved, after which we compare them to other languages, both modal and classical. Section 3, then, contains completeness and complexity results for systems in both languages. Next, in Section 4, we ask some questions familiar from the theory of generalized quantifiers but now in a modal setting. Also, using our modal apparatus we arrive at a complete axiomatization of the set of quantifiers $\{\, more_n : n \in \mathbb{N} \,\}$, where $more_n XY$ holds between X, Y if $|X \cap Y| > n$. Then, in Section 5, we move on to the realm of higher order quantifiers. A complete axiomatization is given for a modal operator simulating the quantifier *there are at least as many Xs as Ys*; after that some issues from earlier sections re-occur, and we have an exploratory look at modal operators simulating other higher order quantifiers. Section 6 rounds off this paper by formulating some conclusions and pointing at a number of directions for further research.

We want to thank Edith Spaan for her kind permission to include a result of hers in Section 3.3. We are also grateful to Johan van Benthem who fought several battles with text-editors in order to send us his comments on an earlier version of this paper.

2. THE SYSTEMS $QUANT$ and $QUANT_k$

2.1. *Basic Definitions and Examples*

DEFINITION 2.1. Let *Prop* be a set of proposition letters, and let *Un* and *Bin* be sets of unary and binary modal operators, respectively. The set of well-formed formulas over *Prop* and *Un*, *Form(Prop, Un, Bin)* is given by

$$
\begin{array}{ll}
\text{proposition letters:} & p \in Prop \\
\text{unary modal operators:} & U \in Un \\
\text{binary modal operators:} & B \in Bin \\
\text{formulas:} & \varphi \in Form\ (Prop,\ Un,\ Bin) \\
\end{array}
$$
$$\varphi ::= p \mid \bot \mid \varphi_0 \wedge \varphi_1 \mid \neg\varphi \mid U\varphi \mid \varphi_0 B \varphi_1.$$

Our main concern below are formulas built up using the set of unary operators $\{\, M_n, L_n : n \in \mathbb{N} \,\}$. Here, we consider L_n to be an abbreviation for '$\neg M_n \neg$'. We will sometimes also use the following abbreviations: $M!_0\varphi := \neg M_0\varphi$,

$M!_n\varphi := (M_{n-1}\varphi \wedge \neg M_n\varphi)$ $(n > 0)$. Instead of *Form(Prop,* $\{ M_n, L_n :$ $n \in \mathbb{N} \}, \emptyset)$ we write *Form*.

DEFINITION 2.2. A *model* for the elements of *Form* is a pair $\mathcal{M} = \langle W, V \rangle$ with W a non-empty set (called a *frame*), and V a function that assigns subsets of W to proposition letters. Then, $\mathcal{M}, w \models \varphi$ is defined inductively: $\mathcal{M}, w \models p$, for $p \in Prop$, if $w \in V(p)$; the Boolean cases are standard, while $\mathcal{M}, w \models M_n\varphi$ if $|\{ v : \mathcal{M}, v \models \varphi \}| > n$. Dually, $\mathcal{M}, w \models L_n\varphi$ if $|\{ v : \mathcal{M}, v \not\models \varphi \}| \le n$. (So L_0 is nothing but the usual modal box '$\Box$' with the universal relation as its interpretation.)

$\mathcal{M} \models \varphi$ is short for: for all $w \in W$, $\mathcal{M}, w \models \varphi$; and $W, w \models \varphi$ is short for: for all V, $\langle W, V \rangle, w \models \varphi$. We write $W \models \varphi$ for: for all w it holds that $W, w \models \varphi$.

Although this is not the first paper in which the operators M_n, L_n are being discussed, we believe that the above quantifier interpretation of these operators is in fact new. One of the first people to study the operators M_n, L_n was Fine (1972); he gave the following interpretation to M_n: $M_n\varphi$ is true at a world w in a Kripke model $\langle W, R, V \rangle$ if at least n R-successors of w satisfy φ. Our use of these operators is different from this interpretation in two respects: we have replaced 'at least n' in the previous sentence by 'more than n', and we only consider the special case in which R is the universal relation. In the mid 1980s Kit Fine's operators were rediscovered by several Italian logicians, and called *graded modalities* (Fattorosi-Barnaba *et al.* 1988).

Parallel to definition 2.2 we can define a translation of elements of *Form* into monadic first order formulas. To be precise, let $\mathcal{L}_0$ be the language of first order logic with identity; $\mathcal{L}_1$ is $\mathcal{L}_0$ plus unary predicate letters $P_0, P_1, P_2, \ldots$ corresponding to the elements of *Prop*.

DEFINITION 2.3. Let x be a fixed variable. The *standard translation $ST(\varphi)$* taking $\varphi \in Form$ to an $\mathcal{L}_1$-formula, is defined as follows: it maps a proposition letter p to Px, and commutes with the Boolean connectives, while

$$ST(M_n\varphi) = \exists y_0 \ldots \exists y_n \Big(\bigwedge_{i \neq j \le n} (y_i \neq y_j) \wedge \bigwedge_{i \le n} [y_i/x]ST(\varphi) \Big),$$

where the y_is are fresh variables.

Every model for $\mathcal{L}_1$ can be viewed as a model for formulas in *Form*, and conversely. A simple induction establishes that $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \models ST(\varphi)$, and $\mathcal{M} \models ST(\varphi)$ iff $\mathcal{M} \models \forall x ST(\varphi)$, for any $\varphi \in Form$.

Let's pause for a moment, and consider some examples. The binary quantifier *all A are B* can be represented as $L_0(A \to B)$, while *some A are*

B can be represented as $M_0(A \wedge B)$. Using these representations one can easily express syllogistic inferences:

all A are B	$L_0(A \rightarrow B)$
some C are not B	$M_0(C \wedge \neg B)$
some C are not A	$M_0(C \wedge \neg A)$.

Likewise, the generalized quantifier *at least k A are B* can be represented in our modal language by $M_{k-1}(A \wedge B)$; this gives us the following simulation of so-called 'numerical' syllogisms (Atzeni *et al.*, 1988):

there are 10 As	$M!_{10}A$
at least 7 Bs are As	$M_6(B \wedge A)$
at least 4 Cs are As	$M_3(C \wedge A)$
at least 1 B is C	$M_0(B \wedge C)$.

The basic principles governing the deductive behavior of the operators M_n and L_n are given in the following definition.

DEFINITION 2.4. We define the modal logic *QUANT*. As rules of inference *QUANT* has Modus Ponens ($\varphi, \varphi \rightarrow \psi / \psi$), Necessitation ($\varphi / L_0 \varphi$), and Substitution. Besides those of propositional logic, its axioms are the following:

A1 $\quad L_0 \varphi \rightarrow \varphi$
A2 $\quad M_n \varphi \rightarrow L_0 M_n \varphi$
A3 $\quad L_0(\varphi \rightarrow \psi) \rightarrow (M_n \varphi \rightarrow M_n \psi)$
A4 $\quad L_0 \neg(\varphi \wedge \psi) \rightarrow (M!_n \varphi \wedge M!_m \psi \rightarrow M!_{n+m}(\varphi \vee \psi))$
A5 $\quad M_{n+1} \varphi \rightarrow M_n \varphi$.

It may amuse the reader to show that $QUANT \vdash L_0(\varphi \rightarrow \psi) \rightarrow (L_0 \varphi \rightarrow L_0 \psi)$. Thus, the fragment of *QUANT* with only L_0, M_0 as its modal operators is precisely $S5$. For this reason *QUANT* has been called $\overline{S5}$ (van der Hoek, 1992), or also $S5n$ (Fine 1972).

It will appear below (cf. 2.16) that in the language of *QUANT* we can define all first order definable quantifiers. Following a suggestion due to Valentin Shehtman we will also consider a more modest system called $QUANT_k$ that is somewhere in between $S5$ and *QUANT*. $QUANT_k$ has modal operators M_0, L_0 and M_k, L_k, for a fixed $k > 0$. The move from $S5$ to $QUANT_k$ is motivated by a similar move in the literature on axiomatic theories of specific quantifiers (cf. also Section 4.1), where pairs of dual quantifiers are not only studied in isolation, but also on top of well understood quantifiers like *all* and *some* (Westerståhl, 1989).

For a fixed $k > 0$, let $Form_k$ abbreviate $Form(Prop, \{ M_0, L_0, M_k, L_k \}, \emptyset)$.

DEFINITION 2.5. Let $k > 0$. The system $QUANT_k$ has as inference rules Modus Ponens, Necessitation and Substitution. Besides those of propositional logic, its axioms are the following (for $i \in \{ 0, k \}$):

$B1$ $L_0\varphi \to \varphi$
$B2$ $M_i\varphi \to L_0 M_i\varphi$
$B3$ $L_0(\varphi \to \psi) \to (M_i\varphi \to M_i\psi)$
$B4$ $\bigwedge_{0 \le j \ne h \le k} L_0\neg(\psi_j \wedge \psi_h) \to \left(\bigwedge_{0 \le j \le k} M_0(\psi_j \wedge \psi) \to M_k\psi \right)$
$B5$ $M_k\varphi \to M_0\varphi$.

2.2. Normal Forms

The question whether a modal axiom system allows for a reduction of the depth of nestings of modal operators is well motivated in the literature on modal logic. In the present setting this question receives additional motivation. Quantifiers express relations between subsets of a given model; this is reflected in the standard notation QXY for 'quantifier Q holds of the sets X and Y'. In a modal setting sets are typically represented by purely propositional formulas. Hence, the proper arguments of modal operators simulating quantifiers are the purely propositional formulas, or in any case, those that are reducible to such formulas. In this section we will prove a rather general normal forms theorem saying that every formula is equivalent to one without nestings of modal operators; from this we will be able to derive normal form results for a number of modal languages.

Let $\mathcal{L}(\mathcal{O})$ be a modal language with a set of modal operators $\mathcal{O}$ such that $L_0 \in \mathcal{O}$. Elements of $\mathcal{O}$ can have arbitrary arity. Let O range over elements of $\mathcal{O}$. An element σ of $\mathcal{L}(\mathcal{O})$ is called a *prenex* modal formula if it is of the form $(\neg)O\vec{\psi}$ (note that the formulas ψ_i in $\vec{\psi}$ may still contain modalities).

DEFINITION 2.6. A logic in the language $\mathcal{L}(\mathcal{O})$ is called *neat* if it extends propositional logic, has a necessitation rule for L_0, while the following are theorems of that logic:
  1. $L_0(\varphi \to \psi) \to (L_0\varphi \to L_0\psi)$;
  2. $\sigma \leftrightarrow L_0\sigma$, if σ is a prenex formula;
  3. $L_0(\varphi \leftrightarrow \varphi') \leftrightarrow (O(\vec{\psi}, \varphi, \vec{\chi}) \leftrightarrow O(\vec{\psi}, \varphi', \vec{\chi}))$.

For the remainder of this section we will assume that all operators under consideration are in $\mathcal{O}$, and that the logics under consideration are neat in $\mathcal{L}(\mathcal{O})$.

LEMMA 2.7. *Let σ be a modal formula in prenex form. Then the following are derivable.*

1. $L_0\sigma \rightarrow (O(\vec{\psi}, \alpha \vee (\beta \wedge \sigma), \vec{\chi}) \leftrightarrow (O(\vec{\psi}, \alpha \vee \beta, \vec{\chi}) \wedge \sigma))$;
2. $L_0\neg\sigma \rightarrow (O(\vec{\psi}, \alpha \vee (\beta \wedge \sigma), \vec{\chi}) \leftrightarrow (O(\vec{\psi}, \alpha, \vec{\chi}) \wedge \neg\sigma))$.

Proof. We only prove item 1. By propositional logic we have $\sigma \rightarrow ((\alpha \vee (\beta \wedge \sigma)) \leftrightarrow (\alpha \vee \beta))$. Thus, since our logic is neat, we have $L_0\sigma \rightarrow (L_0(\alpha \vee (\beta \wedge \sigma)) \leftrightarrow L_0(\alpha \vee \beta))$. By 2.6.(3) this gives $L_0\sigma \rightarrow (O(\vec{\psi}, \alpha \vee (\beta \wedge \sigma), \vec{\chi}) \leftrightarrow O(\vec{\psi}, \alpha \vee \beta, \vec{\chi}))$. Now, by 2.6.(2) we have for any φ, $L_0\sigma \rightarrow (\varphi \leftrightarrow (\varphi \wedge \sigma))$. Thus it follows that $L_0\sigma \rightarrow (O(\vec{\psi}, \alpha \vee (\beta \wedge \sigma), \vec{\chi}) \leftrightarrow O(\vec{\psi}, \alpha \vee \beta, \vec{\chi}) \wedge \sigma)$. $\dashv$

LEMMA 2.8. *Let σ be a modal formula in prenex form. Then $O(\vec{\psi}, \alpha \vee (\beta \wedge \sigma), \vec{\chi}) \leftrightarrow ((O(\vec{\psi}, \alpha \vee \beta, \vec{\chi}) \wedge \sigma) \vee (O(\vec{\psi}, \alpha, \vec{\chi}) \wedge \neg\sigma))$.*

Proof. By propositional logic and 2.6.(2) we have $L_0\sigma \vee L_0\neg\sigma$. Now apply 2.7. $\dashv$

Lemma 2.8 may be rephrased as: if σ is a prenex formula that occurs in φ, then $\varphi \leftrightarrow ((\sigma \wedge [\top/\sigma]\varphi) \vee (\neg\sigma \wedge [\bot/\sigma]\varphi))$.

DEFINITION 2.9. A formula φ in $\mathcal{L}(\mathcal{O})$ is in *normal form* (NF) if it is a disjunction of conjunctions of the general form

$$\delta \wedge (\neg)O_1\vec{\delta_1} \wedge \ldots \wedge (\neg)O_n\vec{\delta_n},$$

where δ, δ_i ($1 \leq i \leq n$) are purely propositional (possibly $\bot$, $\top$).

LEMMA 2.10. *If φ is in NF and has some maximal[1] prenex modal formula σ as a subformula, then σ must be in NF, and there exist α, β in $\mathcal{L}(\mathcal{O})$ such that α, β are in NF and φ may be assumed to have the form $\alpha \vee (\beta \wedge \sigma)$.*

THEOREM 2.11. *In any neat logic in $\mathcal{L}(\mathcal{O})$ every formula φ is equivalent to a formula in NF.*

Proof. Induction on φ. The only interesting case is $\varphi \equiv O(\vec{\psi}, \gamma, \vec{\chi})$, where γ is in NF and contains a prenex modal formula $\sigma \equiv O'(\vec{\psi}', \gamma', \vec{\chi}')$ in NF. Use 2.10 to write φ as $O(\vec{\psi}, \alpha \vee (\beta \wedge \sigma), \vec{\chi})$. Using 2.8 we see that φ is equivalent to $(O(\vec{\psi}, \alpha \vee \beta, \vec{\chi}) \wedge \sigma) \vee (O(\vec{\psi}, \alpha, \vec{\chi}) \wedge \neg\sigma)$. Repeating this argument we can remove all nested occurrences of modal operators from φ. $\dashv$

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[1] Maximal in the sense that it is not a strict subformula of a prenex formula.

COROLLARY 2.12. *Over QUANT every $\varphi \in$ Form is equivalent to a formula $\psi \in$ Form without nestings of modal operators.*

Proof. Here $\mathcal{O} = \{ L_0, M_n : n \in \mathbb{N} \}$. We leave it to the reader to check that *QUANT* is neat. ⊣

COROLLARY 2.13. *Over $QUANT_k$ every $\varphi \in Form_k$ is equivalent to a formula $\psi \in Form_k$ without nestings of modal operators.*

2.3. *Connections with Other Formalisms*

When interpreted on models *QUANT*-formulas become equivalent to a special kind of monadic first order formulas. The notion of equivalence involved here may be understood in either a *local* or *global* sense: a first order formula $\alpha(x) \in \mathcal{L}_1$ is locally equivalent to a *QUANT*-formula φ if for all $\mathcal{M}$, and all $w \in \mathcal{M}$, we have $\mathcal{M}, w \models \varphi \leftrightarrow \alpha(x)$; $\alpha \in \mathcal{L}$ and φ are called globally equivalent if for all $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \alpha$. (Clearly, if φ is locally equivalent to $\alpha(x)$, then it is globally equivalent to $\forall x\, \alpha$.)

From 2.12 we derive

PROPOSITION 2.14. *On models every $\varphi \in$ Form is (locally) equivalent to a Boolean combination of Σ_1^0-formulas, each of which has at most one free variable.*

It's the converse of this proposition that is more interesting: which monadic first order formulas are equivalent to a *QUANT*-formula on models? We can prove every $\mathcal{L}_1$-sentence to be equivalent to (the *ST*-translation of) some *QUANT*-formula by using a special case of the Ehrenfeucht-Fraïssé Theorem. For full details and a proof of this result we refer the reader to Westerståhl (1989: Section 1.7).

DEFINITION 2.15. Fix a finite set of proposition letters $Prop = \{ p_0, \ldots, p_{k-1} \}$. Let $\mathcal{L}_{1k}$ denote the monadic first order language into which the modal language with this restricted set of proposition letters translates via the *ST*-translation; so $\mathcal{L}_{1k}$ only has k unary predicate letters $P_0, \ldots, P_{k-1}$. If $X \subseteq W$, then $X^0 = X$, $X^1 = W \setminus X$; if φ is a formula, $\varphi^0 = \varphi$, $\varphi^1 = \neg\varphi$. For $s \in 2^k$, we use $\mathcal{P}_s$ to denote both the *partition set* and the *partition conjunction* associated with s:

$$P_0^{s(0)} \cap \ldots \cap P_{k-1}^{s(k-1)}, \text{ and } p_0^{s(0)} \wedge \ldots \wedge p_{k-1}^{s(k-1)}.$$

$\{ \mathcal{U}_i \}_{1 \leq i \leq 2^{2^k}}$ is used both to enumerate all possible *unions* (including the empty one) of partition sets, and to enumerate all possible *disjunctions* (including the empty one) of partition conjunctions. We use $\mathcal{P}_s^{\mathcal{M}}$ and $\mathcal{U}_i^{\mathcal{M}}$ to denote the extensions of $\mathcal{P}_s$ and $\mathcal{U}_i$ in some given model $\mathcal{M}$.

Let $\mathcal{M} = \langle W, P_0, \ldots, P_{k-1} \rangle$ and $\mathcal{M}' = \langle W', P_0', \ldots, P_{k-1}' \rangle$ be two $\mathcal{L}_{1k}$-models. We write $\mathcal{M} \equiv_n \mathcal{M}'$ if $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same $\mathcal{L}_{1k}$-sentences of quantifier rank at most n. For two sets X, Y we write $X \sim_n Y$ iff $|X| = |Y| < n$ or $|X|, |Y| \geq n$; by extension we put $\mathcal{M} \sim_n \mathcal{M}'$ iff for all $s \in 2^k$, $\mathcal{P}_s^{\mathcal{M}} \sim_n \mathcal{P}_s^{\mathcal{M}'}$.

The two notions $\equiv_n$ and $\sim_n$ are connected in the following way: for any two $\mathcal{L}_{1k}$-models $\mathcal{M}, \mathcal{M}'$, $\mathcal{M} \equiv_n \mathcal{M}'$ iff $\mathcal{M} \sim_n \mathcal{M}'$.

THEOREM 2.16. *On models every $\mathcal{L}_1$-sentence is equivalent to a formula $\varphi \in$ Form.*

Proof. To simplify our argument, assume that $\alpha = \alpha(P_0, \ldots, P_{k-1}) \in \mathcal{L}_1$ contains only the predicate letters indicated. Let n be the quantifier rank of α.

The number of $\sim_n$-equivalence classes is finite. Let $\mathcal{M}_1, \ldots, \mathcal{M}_g$ be representatives of the $\sim_n$-classes that contain models of α. Let $\mathcal{M} = \langle W, P_0, \ldots, P_k - 1 \rangle \in \{ \mathcal{M}_1, \ldots, \mathcal{M}_g \}$. For each of the 2^k partition sets $\mathcal{P}_s$ write down the corresponding partition conjunction preceded by the operator $M!_m$ in case $|\mathcal{P}_s^{\mathcal{M}}| = m < n$, or preceded by M_{n-1} in case $|\mathcal{P}_s^{\mathcal{M}}| \geq n$. Let $\psi_{\mathcal{M}}$ be the conjunction of these 2^k formulas. It follows that for any $\mathcal{M}$, $\mathcal{M} \models \alpha$ iff $\mathcal{M} \models ST(\psi_{\mathcal{M}_1} \vee \ldots \vee \psi_{\mathcal{M}_g})$. ⊣

COROLLARY 2.17. *Every first order definable quantifier is definable in $\mathcal{L}(QUANT)$.*

From the proof of 2.16 we can derive a *semantically* driven normal form for *QUANT*-formulas and first order ones: each such formula φ is equivalent to a disjunction of conjunctions of the form $O\mathcal{P}_s$, where $O \in \{ M!_k, M_{n-1} : k \leq n, n$ is the quantifier rank of $ST(\varphi) \}$.

We believe the natural setting for the system *QUANT* to be the realm of models rather than that of frames. For, one may understand (binary) quantifiers as expressing relations between subsets of some given universe – hence the natural surrounding for quantifiers are models of some monadic language, e.g., models for *QUANT* or for a monadic first order language.

Nevertheless, we do want to state some results on frame related topics. First, as far as definability of frames is concerned, the language of *QUANT* is equivalent to the language of first order logic over identity.

PROPOSITION 2.18. *On frames every QUANT-formula is equivalent to a sentence of first order logic over identity.*

Proof. All first order formulas over identity are equivalent to Boolean combinations of formulas expressing the existence of at least a certain number

of elements. These are obviously definable by means of *QUANT*-formulas. Conversely, using the ST-translation *QUANT*-formulas can be translated into equivalent closed second-order formulas containing only monadic predicate variables. These are equivalent to first order formulas over identity (Ackermann, 1954).   ⊣

Next, this result gives a connection with the modal language $\mathcal{L}(D)$ which has one unary operator D, whose semantics is based on the relation of inequality: $\mathcal{M}, w \models D\varphi$ iff for some $v \neq w$ we have $\mathcal{M}, v \models \varphi$ (see De Rijke, 1992) for more on $\mathcal{L}(D)$. In $\mathcal{L}(D)$ we can define the auxiliary operators E, A, U: $E\varphi := (\varphi \vee D\varphi)$ (there exists a point at which φ holds), $A\varphi := (\varphi \wedge \neg D\neg\varphi)$ (φ holds at all points), and $U\varphi := E(\varphi \wedge \neg D\varphi)$ (φ holds at unique point).

PROPOSITION 2.19. *Let* K *be a class of frames. Then* K *is definable by means of QUANT-formulas iff it is definable by means of* $\mathcal{L}(D)$*-formulas.*

Proof. By de Rijke (1992: Theorem 2.3) we have that, on frames, $\mathcal{L}(D)$ is equivalent to first-order logic over $=$. By 2.18 the result then follows.   ⊣

As an aside, on models, the language of *QUANT* is stronger than $\mathcal{L}(D)$: every model distinguishable in $\mathcal{L}(D)$ is distinguishable in the language of *QUANT*, as is easily verified using the above translation τ; the converse does not hold. Consider for example the following models:

$$\mathcal{M}_1 : \boxed{\quad \bullet \qquad \bullet \quad} \qquad\qquad \mathcal{M}_2 : \boxed{\quad \bullet \qquad \bullet \qquad \bullet \quad},$$

where all points have the same valuation; $\mathcal{M}_1$ and $\mathcal{M}_2$ verify the same $\mathcal{L}(D)$-formulas, but not the same *QUANT*-formulas.

Admittedly, Propositions 2.18 and 2.19 do not give any explicit information on the classes of frames definable by means of a *QUANT*-formula. The following corollary does give this; it was inspired by a remark of one of the anonymous referees. Let $\mathcal{F} \cong \mathcal{G}$ denote that the frames $\mathcal{F}, \mathcal{G}$ are isomorphic (here, this means that there is some bijection $\mathcal{F} \to \mathcal{G}$). Let K be a class of frames. Then $K/\cong$ denotes the subclass of K that contains exactly one representantive of every $\cong$-class in K. We say that a property holds of K *modulo isomorphism* if it holds of $K/\cong$.

COROLLARY 2.20. *A class of frames* K *is definable by means of a QUANT-formula iff* K *is closed under isomorphisms and modulo isomorphism either* K *or* K^c *is a finite set of finite frames.*

Proof. The direction from right to left is clear. For the converse, let K be defined by the *QUANT*-formula φ. Then K is closed under isomorphisms.

For the remainder of the proof we need the following auxiliary definition and claim. Let $\mathcal{F}_m$ denote the frame $\langle \{ 1, \ldots, m \} \rangle$ (so $M!_m\top$ defines $\mathcal{F}_m$).

Claim. If for all $k \in \mathbb{N}$ there is an $l > k$ such that $\mathcal{F}_l \in \mathsf{K}$, then for some $n \in \mathbb{N}$ we have $\mathcal{F}_m \in \mathsf{K}$ for all $m > n$.

Proof of the Claim. Assume that K satisfies the antecedent of the Claim. By 2.18 φ is equivalent to a first order sentence α. Let n be the quantifier rank of α; this will turn out to be the n we are looking for. Choose any $\mathcal{F}_l \in \mathsf{K}$ with $l > n$. Then $\mathcal{F}_l \models \varphi$. Now, consider the special case of the earlier notions $\equiv_n$ and $\sim_n$, relative, this time, to a first order language *without* unary predicate symbols $P_0, P_1, \ldots$, i.e., in a language with $=$ only. Then, the appropriate models are in fact the frames $\mathcal{F}$ for our *QUANT*-language. Moreover, $\mathcal{F} \sim_n \mathcal{F}'$ boils down to $|\mathcal{F}| = |\mathcal{F}'| \leq n$ or $|\mathcal{F}|, |\mathcal{F}'| > n$. And, of course, in this case without unary predicate letters we still have that the relations $\equiv_n$ and $\sim_n$ coincide.

Now, take any $\mathcal{F}_m$ with $m > n$. Then $\mathcal{F}_m \sim_n \mathcal{F}_l$, hence $\mathcal{F}_m \equiv_n \mathcal{F}_l$, and therefore $\mathcal{F}_m \models \alpha$ and $\mathcal{F}_m \models \varphi$. So $\mathcal{F}_m \in \mathsf{K}$. $\dashv_{\text{Claim}}$

Returning to the main argument, suppose that $\mathsf{K}/{\cong}$ is infinite. Then, by the Claim $\mathcal{F}_m \in \mathsf{K}$ for all $m > n$, for some n. Then $\mathsf{K}/{\cong}$ must contain all (representatives of) infinite frames as well, by arguments similar to those establishing the Claim. It follows that $\mathsf{K}^c/{\cong}$ can only contain finite frames $\mathcal{F}_l$ for $l < n$; but then $\mathsf{K}^c/{\cong}$ must be finite. If, on the other hand, $\mathsf{K}/{\cong}$ is finite, then $\mathsf{K}/{\cong}$ cannot contain infinite frames, for otherwise it would contain arbitrarily large finite ones, and thus be infinite. So, if $\mathsf{K}/{\cong}$ is finite, it must be a finite set of finite frames, as required. $\dashv$

3. COMPLETENESS AND COMPLEXITY

3.1. *Prerequisites*

A completeness proof for the system *QUANT* may be found in Fine (1972). There it is shown that *QUANT* is complete with respect to all frames of the form $\langle W, R \rangle$, where R is an equivalence relation that provides the interpretation for L_0; in this setting the operators M_n, L_n mean: 'more than n R-successors satisfy $\ldots$' and 'at most n R-successors falsify $\ldots$'. However, since *QUANT*-formulas are preserved under generated subframes of such 'non-standard' frames, we can derive from Fine's completeness result that *QUANT* is complete w.r.t. the standard frames in which the modal operators receive their quantifier interpretation.

In van der Hoek (1992) the finite model property for *QUANT* is established; there, it is shown that the size of the model needed to refute a non-theorem φ

is bounded by $g(\varphi) \cdot 2^{\#\varphi}$. Here, $g(\varphi)$, the *grade* of φ, is defined inductively as follows: $g(p) = 0$, $g(\neg\varphi) = g(\varphi)$, $g(\varphi \wedge \psi) = \max(g(\varphi), g(\psi))$, and $g(M_n\varphi) = \max(n + 1, g(\varphi))$; and $\#\varphi$ is simply the number of (occurrences of) of symbols in φ, e.g., $\#M_7(p \wedge \neg p)$ equals 5. Adopting an argument due to (Ladner, 1977), we can obtain a better upper bound:

PROPOSITION 3.1. *Let $\varphi \in Form$. Then φ is satisfiable iff φ is satisfiable in a model with at most $1 + \#\varphi \cdot g(\varphi)$ elements.*

Proof. Let φ be satisfied in a *QUANT*-model $\mathcal{M} = \langle W, V \rangle$. We will use the subformulas of φ as instructions for extracting a set of elements W' from W that will serve as the domain of the desired small model. A function Γ is defined inductively on the instances of subformulas of φ.

1. Choose some $w \in W$ with $\mathcal{M}, w \models \varphi$; put $\Gamma(\varphi) = \{ w \}$.

Now suppose that $\Gamma(\psi)$ has already been defined; then

2. $\Gamma(\chi) = \Gamma(\psi)$ if $\psi \equiv \neg\chi$;
3. $\Gamma(\chi_1) = \Gamma(\chi_2) = \Gamma(\psi)$ if $\psi \equiv \chi_1 \wedge \chi_2$;
4. $\Gamma(\chi) = \Gamma(\psi)$ if $\psi \equiv M_n\chi$ and $\mathcal{M}, w \not\models \psi$;
5. if $\psi \equiv M_n\chi$ and $\mathcal{M}, w \models \psi$, then choose $n + 1$ distinct points $w_1, \ldots, w_{n+1}$ such that $\mathcal{M}, w_i \models \chi$ $(1 \leq i \leq n + 1)$, and put $\Gamma(\chi) = \{ w_1, \ldots, w_{n+1} \}$.

Define W' to be the union of all $\Gamma(\psi)$, where ψ ranges over the subformulas of φ. Put $V' = V \upharpoonright W'$, and $\mathcal{M}' = \langle W', V' \rangle$. Then $|W'| \leq 1 + \#\varphi \cdot g(\varphi)$. Also, one may establish inductively that for all subformulas ψ of φ, and all $v \in W \cap W'$, we have $\mathcal{M}, v \models \psi$ iff $\mathcal{M}', v \models \psi$. ⊣

By 2.16 the above proposition implies that properties of first order definable quantifiers may be decided on finite models.

The method used in 3.1 may also be used to establish:

PROPOSITION 3.2. *Let $\varphi \in Form_k$, $k \in \mathbb{N}_{>0}$. Then φ is satisfiable iff it is satisfiable in a model with at most $1 + (k + 1) \cdot \#\varphi$ elements.*

3.2. *Completeness of $QUANT_k$*

We will prove the completeness of $QUANT_k$ via a Henkin-like construction. For a consistent formula φ we will build a canonical model $\mathcal{M}_c$ containing, for each maximal consistent set Δ, at most $k + 1$ copies of Δ, together with a relation R_c on $\mathcal{M}_c$ to interpret the modal operators. To obtain a model in which the modal operators receive their intended interpretations, it will then be sufficient to show that φ is true in a point in some part of the canonical model on which R_c is total.

Our completeness proof for $QUANT_k$ differs from Kit Fine's completeness proof for $QUANT$ in the following respect. If we wanted to prove the completeness of $QUANT$ using the method just sketched, we would have to construct a canonical model that may contain, for each maximal consistent set Δ, infinitely many copies of Δ. Fine, on the other hand, first introduces, for every k, an accessibility relation R_k to interpret M_k. In order to end up with a standard model he then maps these relations onto a single one.

DEFINITION 3.3. The *canonical model* $\mathcal{M}_c$ for $QUANT_k$ is a triple $\langle W_c, R_c, V_c \rangle$ such that

$W_c = \{ \langle \Gamma, j \rangle : \Gamma$ is maximal $QUANT_k$-consistent, $0 \leq j \leq k \}$;
$\langle \Gamma, j \rangle R_c \langle \Delta, h \rangle$ iff $h = 0$ and $(\delta \in \Delta \Rightarrow M_0 \delta \in \Gamma)$, or $1 \leq h \leq k$ and $(\delta \in \Delta \Rightarrow M_k \delta \in \Gamma)$;
$V_c(p) = \{ \langle \Gamma, j \rangle : p \in \Gamma \}$.

LEMMA 3.4. $QUANT_k \vdash (M_k \varphi \wedge \neg M_k \psi) \rightarrow M_0(\varphi \wedge \neg \psi)$.

Proof. We have

$$
\begin{aligned}
M_k \varphi \quad &\rightarrow \quad \big(\neg M_0(\varphi \wedge \neg \psi) \rightarrow L_0(\varphi \rightarrow \psi) \big) \\
&\rightarrow \quad \big(\neg M_0(\varphi \wedge \neg \psi) \rightarrow (M_k \varphi \rightarrow M_k \psi) \big), \quad B3 \\
&\rightarrow \quad \big(\neg M_0(\varphi \wedge \neg \psi) \rightarrow M_k \psi \big).
\end{aligned}
$$

So $M_k \varphi \wedge \neg M_k \psi \rightarrow M_0(\varphi \wedge \neg \psi)$. $\dashv$

LEMMA 3.5. *Let* $j, h, l \in \{ 0, \ldots, k \}$. *Then*
1. $\langle \Gamma, j \rangle R_c \langle \Delta, h \rangle$ *iff* $\langle \Gamma, l \rangle R_c \langle \Delta, h \rangle$;
2. $\langle \Gamma, j \rangle R_c \langle \Delta, h \rangle$ *implies* $\langle \Gamma, j \rangle R_c \langle \Delta, 0 \rangle$;
3. $\langle \Gamma, j \rangle R_c \langle \Delta, 1 \rangle$ *implies* $\langle \Gamma, j \rangle R_c \langle \Delta, h \rangle$.

Proof. By definition of R_c, R_c-successors of $\langle \Gamma, j \rangle$ don't depend on j – this proves item 1. To prove item 2, if $h \neq 0$, and $\langle \Gamma, j \rangle R_c \langle \Delta, h \rangle$, then we have that $\delta \in \Delta$ implies $M_k \delta \in \Gamma$. So by axiom $B5$, $M_0 \delta \in \Gamma$, but then $\langle \Gamma, j \rangle R_c \langle \Delta, 0 \rangle$ holds. Finally, to prove item 3, assume $\langle \Gamma, j \rangle R_c \langle \Delta, 1 \rangle$. Then $\langle \Gamma, j \rangle R_c \langle \Delta, h \rangle$ for any $h \in \{ 1, \ldots, k \}$, and by item 2 also $\langle \Gamma, j \rangle R_c \langle \Delta, 0 \rangle$. $\dashv$

Next comes our main lemma. In it we use a notion of truth $\models_n$ based on a relation R, whose important clause is: $\langle W, R, V \rangle, w \models_n M_i \varphi$ iff $|\{ v : wRv$ and $\langle W, R, V \rangle, v \models_n \varphi \}| > i$ $(i \in \{ 0, k \})$.

LEMMA 3.6 (Truth Lemma). *Let* $\varphi \in Form_k$, *let* Γ *be a maximal* $QUANT_k$-*consistent set, and assume* $j \in \{ 0, \ldots, k \}$. *Then* $\mathcal{M}_c, \langle \Gamma, j \rangle \models_n \varphi$ *iff* $\varphi \in \Gamma$.

Proof. As usual the proof is by induction on φ. The cases $\varphi \equiv p$, $\varphi \equiv \varphi_1 \wedge \varphi_2$, $\varphi \equiv \neg\varphi_1$ are straightforward.

Assume $\varphi \equiv M_0\psi$. If $\mathcal{M}_c, \langle\Gamma, j\rangle \models_n M_0\psi$, then for some $\langle\Delta, h\rangle$ we have $\langle\Gamma, j\rangle R_c\langle\Delta, h\rangle$ and, by the induction hypothesis, $\psi \in \Delta$. By 3.5.(2) it follows that $\langle\Gamma, j\rangle R_c\langle\Delta, 0\rangle$. But then $M_0\psi \in \Gamma$.

Conversely, if $M_0\psi \in \Gamma$, then the set $\{\psi\} \cup \{\gamma : L_0\gamma \in \Gamma\}$ can be extended to a maximal $QUANT_k$-consistent set Δ by standard modal arguments. Then $\langle\Gamma, j\rangle R_c\langle\Delta, 0\rangle$; hence, by the induction hypothesis we have $\mathcal{M}_c, \langle\Gamma, j\rangle \models_n M_0\psi$.

Next, assume that $\varphi \equiv M_k\psi$, and let $\mathcal{M}_c, \langle\Gamma, j\rangle \models_n M_k\psi$. We distinguish two cases. The first one is that for some Δ, $\psi \in \Delta$ and $\langle\Gamma, j\rangle R_c\langle\Delta, k\rangle$. Then, by definition of R_c, $M_k\psi \in \Gamma$. The second case is that there is no Δ such that $\psi \in \Delta$ and $\langle\Gamma, j\rangle R_c\langle\Delta, k\rangle$. By 3.5.(2) and (3) this means that there is no single Δ containing ψ that occurs more than once as the first component of an R_c-successor of $\langle\Gamma, j\rangle$. But then, there must be pairwise different sets $\Delta_0, \ldots, \Delta_k$ such that $\psi \in \Delta_i$ and $\langle\Gamma, j\rangle R_c\langle\Delta_i, 0\rangle$ $(0 \le i \le k)$. So there are formulas δ_{ih} $(0 \le i \ne h \le k)$ such that $\delta_{ih} \in \Delta_i \setminus \Delta_h$. Putting

$$\delta_i \equiv \bigwedge_{\substack{0 \le h \le k \\ h \ne i}} \delta_{ih} \wedge \bigwedge_{\substack{0 \le h \le k \\ h \ne i}} \neg\delta_{hi},$$

we have $\delta_i \in \Delta_i$, and $QUANT_k \vdash L_0\neg(\delta_i \wedge \delta_h)$, for $i \ne h$. Also, since $\langle\Gamma, j\rangle R_c\langle\Delta_i, 0\rangle$ and $\delta_i \wedge \psi \in \Delta_i$, we have that $M_0(\delta_i \wedge \psi) \in \Gamma$. Using axiom $B4$ we find that $M_k\psi \in \Gamma$.

Conversely, if $M_k\psi \in \Gamma$, then, by axiom $B1$, $M_0\psi \in \Gamma$. Reasoning as in the case of $M_0\psi \in \Gamma$ we find a Δ_0 such that $\psi \in \Delta_0$ and $\langle\Gamma, j\rangle R_c\langle\Delta_0, 0\rangle$. Now, if there is such a Δ_0 with the additional property that $\langle\Gamma, j\rangle R_c\langle\Delta_0, k\rangle$, then we are done by 3.5 and the induction hypothesis. Otherwise, there is some $\delta_0 \in \Delta_0$ with $\neg M_k\delta_0 \in \Gamma$. Hence, by 3.4, $M_0(\psi \wedge \neg\delta_0) \in \Gamma$ – but this implies the existence of a Δ_1 for which $\langle\Gamma, j\rangle R_c\langle\Delta_1, 0\rangle$, $\Delta_0 \ne \Delta_1$, and $\psi \wedge \neg\delta_0 \in \Delta_1$. By assumption we don't have $\langle\Gamma, j\rangle R_c\langle\Delta_1, k\rangle$. Repeating this argument, we find pairwise different sets $\Delta_0, \ldots, \Delta_k$ with $\langle\Gamma, j\rangle R_c\langle\Delta_i, 0\rangle$ and $\psi \in \Delta_i$ $(0 \le i \le k)$. Hence, by the induction hypothesis, $\mathcal{M}_c, \langle\Gamma, j\rangle \models_n M_k\psi$. $\dashv$

LEMMA 3.7.
1. R_c is serial (i.e., it satisfies $\forall x \exists y\, xRy$);
2. R_c is euclidean (i.e., it satisfies $\forall xyz\, (xRy \wedge xRz \to yRz)$).

Proof. Item 1 is immediate: $\varphi \in \Gamma$ implies $M_0\varphi \in \Gamma$ by axiom $B1$; hence, we have $\langle\Gamma, j\rangle R_c\langle\Gamma, 0\rangle$. To prove item 2, suppose $\langle\Gamma, j\rangle R_c\langle\Delta, l\rangle$ and $\langle\Gamma, j\rangle R_c\langle\Sigma, m\rangle$. If $m = 0$ then $\sigma \in \Sigma$ implies $M_0\sigma \in \Gamma$ which implies $L_0M_0\sigma \in \Gamma$ (axiom $B2$), hence $M_0\sigma \in \Delta$. But then $\langle\Delta, l\rangle R_c\langle\Sigma, m\rangle$. If

$m \neq 0$ then $\sigma \in \Sigma$ implies $M_k\sigma \in \Gamma$, hence $L_0 M_k\sigma \in \Gamma$. Thus $M_k\sigma \in \Delta$, which means that $\langle \Delta, l \rangle R_c \langle \Sigma, m \rangle$. ⊣

To prove that a consistent formula φ has a model, it suffices to find a model $\mathcal{M} = \langle W, R, V \rangle$ such that for some $w \in W$, $\mathcal{M}, w \models_n \varphi$, and such that R is total on $\mathcal{M}$. Now, a relation R that is euclidean and serial need not be total. However, for our purposes it suffices that such an R is 'almost total' in the following sense: $\forall xyz\, (xR^n y \wedge xR^m z \rightarrow yRz)$. The proof that any serial, euclidean relation is almost total is left to the reader.

THEOREM 3.8. *Let* $\varphi \in Form_k$. *Then* $QUANT_k \vdash \varphi$ *iff* $QUANT_k \models \varphi$.

Proof. Proving soundness is left to the reader. To show completeness, assume φ is $QUANT_k$-consistent. Then, by axiom $B1$, so is $M_0\varphi$. Thus for some maximal $QUANT_k$-consistent set Γ we have $M_0\varphi \in \Gamma$. Lemma 3.6 gives $\mathcal{M}_c, \langle \Gamma, 0 \rangle \models_n M_0\varphi$. We may of course assume that $\mathcal{M}_c$ is R_c-generated by $\langle \Gamma, 0 \rangle$; by 3.7 $\mathcal{M}_c$ is serial and euclidean.

$\mathcal{M}_c, \langle \Gamma, 0 \rangle \models_n M_0\varphi$ implies that for some $\langle \Delta, i \rangle$ we have $\langle \Gamma, 0 \rangle R_c \langle \Delta, i \rangle$ and $\mathcal{M}_c, \langle \Delta, i \rangle \models_n \varphi$. Let $\mathcal{M}$ be the submodel R_c-generated by $\langle \Delta, i \rangle$. Then, we have $\mathcal{M}, \langle \Delta, i \rangle \models_n \varphi$, and on $\mathcal{M}$, R_c is the universal relation, so the modal operators receive their intended interpretations in $\mathcal{M}$, i.e., $\mathcal{M}, \langle \Delta, i \rangle \models \varphi$. ⊣

Note that, by Lemma 3.6, Theorem 3.8 generalizes to *strong* completeness, i.e., to the case of deductions from arbitrary sets of sentences.

COROLLARY 3.9. *Let* $k > 0$, *and* $\varphi \in Form_k$. *Then* $QUANT \vdash \varphi$ *iff* $QUANT_k \vdash \varphi$.

3.3. *Complexity*

Recall that in §3.1 we gave upper bounds for the size of the model needed to satisfy a consistent formula φ in terms of #φ (the number of symbols in φ) and g(φ): $1+$ the highest n occurring as subscript in an operator M_n in φ. From a computational point of view it is more natural to have a bound in terms of the length $|\varphi|$ of the representation of φ. (So $|M_n|$ is the length of the representation of n.)

PROPOSITION 3.10. *The problem of determining whether a formula* $\varphi \in Form_k$, $k \in \mathbb{N}_{>0}$, *is satisfiable is NP-complete.*

Proof. It suffices to show that the problem is in NP. But this follows from 3.1. First guess a model with at most $1 + (k + 1) \cdot |\varphi|$ elements. Then determine

the validity of each subformula in each element, starting with the proposition letters occurring in φ. This can be done in polynomial time. ⊣

What about $QUANT$-satisfiability? By an argument due to Edith Spaan $QUANT$-satisfiability is in PSPACE. Below, an algorithm is given that tests for $QUANT$-satisfiability, and is in PSPACE. The main idea behind this algorithm is that the truth-value of a $QUANT$-formula (possibly containing modal operators) in a model, is completely determined by Boolean combinations of proposition letters, and the number of occurrences of such combinations in the model. This idea will be implemented in our test for $QUANT$-satisfiability as follows.

Given a formula φ we first consider certain propositional counterparts of φ and its subformulas; we then *guess* a number (intuitively, the size of the model) and valuations, in the meantime determining how often these propositional combinations will occur in the resulting model. Finally, we reconsider the original formula φ, and show how its truth-value is determined by its propositional counterpart.

Fix some formula φ. We need some preliminary notions. For $\psi \in$ *Form*, $Cl(\psi)$ is the smallest set containing ψ and closed under subformulas. To determine the propositional counterpart of φ, let λ be a new symbol.

- For $\psi \in$ *Form* define $strip(\psi)$ as follows:

  - $strip(p) := p$, for $p \in Prop$,
  - $strip(\neg\psi) := $ **if** $strip(\psi) = \lambda$ **then** λ

    else $\neg strip(\psi)$,
  - $strip(\psi_1 \wedge \psi_2) := $ **if** $strip(\psi_1) = \lambda$ **and** $strip(\psi_2) = \lambda$

    then λ

    else if $strip(\psi_1) = \lambda$

      then $strip(\psi_2)$

      else if $strip(\psi_2) = \lambda$ **then** $strip(\psi_1)$

        else $strip(\psi_1) \wedge strip(\psi_2)$,
  - $strip(M_n\psi) := \lambda$.

- Put $STRIP(\varphi) = \{ strip(\psi) : \psi \in Cl(\varphi) \} \setminus \{ \lambda \} \cup \{ \top, \bot \}$. To give a simple example, $STRIP(M_2p \wedge q) = \{ p, q, \top, \bot \}$. Note that $STRIP(\varphi)$ contains propositional formulas only. These will be used to guess a kind of table that contains all information relevant to building a model for φ.

- Here's the **Algorithm**:
  - Guess $worlds \le 1 + |\varphi| \cdot 2^{|\varphi|}$ (intuitively, $\{ w_1, \ldots, w_{worlds} \}$ will be the domain of the model satisfying φ, if φ is satisfiable).
  - For $\psi \in \text{STRIP}(\varphi)$ put $count(\psi) := 0$;
    for $i := 1$ **to** $worlds$ **do**

    guess a propositional valuation V_i for w_i,
    for all $\psi \in \text{STRIP}(\varphi)$ **if** $w_i \models \psi$ **then** $count(\psi) := count(\psi)+1$.

    Since we 'forget' about the valuations, we only need polynomial space to store this 'table' containing the *counts* for $\psi \in \text{STRIP}(\varphi)$.

- The next step is to check that this works. To this end, let $\mathcal{M}$ be a model that is derived from this table, in the sense that for $\psi \in \text{STRIP}(\varphi)$, the number of worlds in $\mathcal{M}$ that satisfy ψ equals $count(\psi)$. (Of course, $\mathcal{M}$ need not be determined uniquely by the table, but for our purposes it suffices to have it satisfy the "*counts*".) Next, we have to check that this model is a model for φ – to this end we connect up the propositional formulas in $\text{STRIP}(\varphi)$ with the *modal* formulas from $\text{Cl}(\varphi)$. This is done using a function f. To be precise, we define a function $f : \text{Cl}(\varphi) \rightarrow \text{STRIP}(\varphi)$ for which it is easily seen that the number of worlds satisfying ψ in the model $\mathcal{M}$ equals $count(f(\psi))$ for every $\psi \in \text{Cl}(\varphi)$:

  - for $p \in Prop$, $f(p) := p$
  - $f(\neg\psi) := \neg f(\psi)$ (to have $range(f) \subseteq \text{STRIP}(\varphi)$ we take $\neg\top \equiv \bot$, and $\neg\bot \equiv \top$),
  - to define $f(\psi_1 \wedge \psi_2)$ we have to distinguish a number of cases:

    if $f(\psi_1) = \top$ then $f(\psi_1 \wedge \psi_2) = f(\psi_2)$,
    if $f(\psi_1) = \bot$ then $f(\psi_1 \wedge \psi_2) = \bot$,
    if $f(\psi_2) = \top$ then $f(\psi_1 \wedge \psi_2) = f(\psi_1)$,
    if $f(\psi_2) = \bot$ then $f(\psi_1 \wedge \psi_2) = \bot$,
    otherwise $f(\psi_1 \wedge \psi_2) = f(\psi_1) \wedge f(\psi_2)$.

    (The above definition may seem somewhat laborious, but it really is necessary to distinguish the various cases, since $f(\psi_1 \wedge \psi_2)$ can be in $\text{STRIP}(\varphi)$ while $f(\psi_1) \wedge f(\psi_2)$ need not be in it, cf. the example below.)
  - if $count(f(\psi)) > n$ we define $f(M_n\psi)$ to be $\top$, otherwise $f(M_n(\psi)) = \bot$.

  By simply following the inductive definition of f it is clear that the number of worlds satisfying ψ in $\mathcal{M}$ equals $count(f(\psi))$ for any $\psi \in \text{Cl}(\varphi)$.
  To continue our earlier example, let $\varphi' \equiv M_2p \wedge q$, and assume that $count(\psi)$ is known for all $\psi \in \text{STRIP}(\varphi')$. Then $f(\varphi') = q$ if $count(p) > 2$, otherwise $f(\varphi') = \bot$.

- Finally, $count(f(\varphi)) > 0$ iff φ is satisfiable in $\mathcal{M}$.

   To conclude our example, since $count(\bot) = 0$, we have that $count(f(M_2 p \wedge q)) > 0$ iff $count(p) > 2$ and $count(q) > 0$.

   To sum up, given a formula φ as input, run the Algorithm on φ, and verify whether $count(f(\varphi)) > 0$. Since the Algorithm is in PSPACE and the function f is obviously in P, the entire procedure must be in PSPACE.

THEOREM 3.11. *The problem of determining whether a formula $\varphi \in Form$ is satisfiable is in PSPACE.*

It is still open whether or not $QUANT$-satisfiability is also PSPACE-hard.

4. SEMANTIC CONSTRAINTS AND INFERENTIAL PATTERNS

In this section some topics familiar from generalized quantifier theory are addressed in a modal setting; also, some applications are given of the systems $QUANT$ and $QUANT_k$ to these topics.

4.1. *Semantic Constraints*

In this subsection we consider some well-known semantic constraints on quantifiers, and try to match them up with syntactic restrictions on modal formulas. On the way we will give some examples of how our modal apparatus allows us to translate our semantic (Boolean) intuitions into syntactic ones. Most results will be stated for $QUANT$-formulas only, but they have an immediate analogue for $QUANT_k$-formulas.

   Let us fix some terminology first. Following Westerståhl (1989) we define a (binary) *generalized quantifier* to be a function assigning to every set $\mathcal{M}$ a binary relation $Q_\mathcal{M}$ between subsets of $\mathcal{M}$, and we recall that the conditions imposed to obtain so-called *logical* quantifiers are

1. CONSERV $Q_\mathcal{M} P_0 P_1$ iff $Q_\mathcal{M} P_0 (P_1 \cap P_0)$;
2. ISOM $Q_\mathcal{M} P_0 P_1$ iff $Q_{\mathcal{M}'} f[P_0] f[P_1]$ for all bijections $f : \mathcal{M} \to \mathcal{M}'$;
3. EXT if $P_0, P_1 \subseteq \mathcal{M} \subset \mathcal{M}'$ then $Q_\mathcal{M} P_0 P_1$ iff $Q_{\mathcal{M}'} P_0 P_1$.

   A first order sentence $\alpha(P_0, P_1)$ satisfies the combined conditions CONSERV (for P_0) and EXT iff it is logically equivalent to some sentence with all quantifiers P_0-restricted (Westerståhl 1989: Theorem 3.2.3). An obvious question here is whether a similar characterization exists for $QUANT$-formulas.

   We say that a $QUANT$-formula φ satisfies CONSERV if $\langle W, P_0, P_1 \rangle \models \varphi$ iff $\langle W, P_0, P_1 \cap P_0 \rangle \models \varphi$; it satisfies EXT if $P_0, P_1 \subseteq W \subseteq W'$ implies $\langle W, P_0, P_1 \rangle \models \varphi$ iff $\langle W', P_0, P_1 \rangle \models \varphi$. Note that we only consider *global*

truth of *QUANT*-formulas in this context; this corresponds to the fact that quantifiers are usually defined using sentences rather then with formulas that may contain free variables.

Define an $\mathcal{L}(QUANT)$-formula $\varphi(p_0, p_1)$ to be p_0-*restricted* if it is a Boolean combination of formulas of the form $M_n(p_0 \wedge \psi)$, where ψ is a purely propositional formula.

PROPOSITION 4.1. *A formula* $\varphi(p_0, p_1) \in \mathcal{L}(QUANT)$ *satisfies CONSERV and EXT iff it is logically equivalent to a formula that is p_0-restricted.*

Proof. The 'easy' direction may be proved as follows. If $\varphi(p_0, p_1)$ is p_0-restricted, then $ST(\varphi)$ may be written as an $\mathcal{L}_1$-sentence in which all quantifiers are P_0-restricted. Thus $ST(\varphi)$ satisfies CONSERV and EXT by the result quoted above. But then the same holds for φ itself.

To prove the 'hard' direction, assume that $\varphi(p_0, p_1)$ satisfies CONSERV and EXT. By our remarks following 2.16 φ has a 'semantic' normal form $\Psi \equiv \psi_0 \vee \ldots \vee \psi_g$. For a disjunct ψ in Ψ, define ψ' to be ψ with the conjuncts in which p_0 occurs negated left out. Put $\Psi' := \psi_0' \vee \ldots \vee \psi_g'$. Then, for any model $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \Psi'$. Obviously, $\mathcal{M} \models \varphi$ implies $\mathcal{M} \models \Psi'$. To prove the converse, assume $\mathcal{M} = \langle W, P_0, P_1 \rangle \models \psi_i'$, for some i. Now $\psi_i \equiv \psi_i' \wedge O_m(\neg p_0 \wedge p_1) \wedge O_n'(\neg p_0 \wedge \neg p_1)$, where $O, O' \in \{ M, M! \}$. Let $\mathcal{M}_1$ be $\mathcal{M}$ with $|P_0^c \cap P_1| = m$ if $O \equiv M!$, and $|P_0^c \cap P_1| = m + 1$ otherwise. Let $\mathcal{M}_2$ be $\mathcal{M}_1$ with $|P_0^c \cap P_1^c| = n$ if $O' \equiv M!$, and $|P_0^c \cap P_1^c| = n + 1$ otherwise. Then $\mathcal{M}_2 \models \psi_i$. But then $\mathcal{M}_2 \models \varphi$. By EXT this implies $\mathcal{M}_1 \models \varphi$, which yields $\mathcal{M} \models \varphi$ by CONSERV. $\dashv$

An important condition on quantifiers that has figured prominently in the literature is *monotonicity*. A binary quantifier Q is *upward monotone in its left argument* (or $\uparrow$MON) if $Q_{\mathcal{M}}P_0P_1$ and $P_0 \subseteq P_0'$ imply $Q_{\mathcal{M}}P_0'P_1$; the modal version is: a modal formula is $\uparrow$MON in p_0 if $\langle W, P_0, \ldots \rangle \models \varphi$ and $P_0 \subseteq P_0'$ imply $\langle W, P_0', \ldots \rangle \models \varphi$. As an application of the Lyndon Theorem for first order logic we have that a first order sentence $\alpha(P)$ is $\uparrow$MON (in P) iff it is equivalent to a sentence in which P occurs only positively (in the usual syntactic sense). A similar result holds in $\mathcal{L}(QUANT)$, and can be read off from the earlier semantically driven normal forms:

THEOREM 4.2. *A formula* $\varphi(p) \in \mathcal{L}(QUANT)$ *satisfies $\uparrow$MON in p iff it is equivalent to a formula in which p occurs only positively.*

Proof. To prove the direction from right to left we first introduce a *local* version of monotonicity. Define a formula φ to be $\uparrow$LMON ($\downarrow$LMON) if $\langle W, P_0, \ldots \rangle, x \models \varphi$, $P_0 \subseteq P_0'$ ($P_0' \subseteq P_0$) implies $\langle W, P_0', \ldots \rangle, x \models \varphi$, for any model $\langle W, P_0, \ldots \rangle$, and $x \in W$. One can prove by induction on φ that if all

occurrences of p_0 in φ are positive (negative) then φ is $\uparrow$LMON ($\downarrow$LMON). This implies one half of the theorem.

Conversely, let $\varphi(p)$ satisfy $\uparrow$MON. Rewrite the disjuncts in the *semantic normal form* Φ of φ according to the following recipe. Let N be the maximal number occurring as the index of some modal operator in Φ. Replace

$$M_N(p \wedge D) \wedge M_N(\neg p \wedge D)$$

by

$$M_N(p \wedge D) \wedge M_{2N+1}D,$$

where D is the remaining part of the partition conjunction. Then, rewrite conjuncts of the form $M!_q((\neg)p \wedge D)$ according to the definition of $M!$. The resulting conjuncts

$$M_{k-1}(p \wedge D) \wedge \neg M_k(p \wedge D) \wedge M_{l-1}(\neg p \wedge D) \wedge \neg M_l(\neg p \wedge D)$$

should be rewritten as

$$M_{k-1}(p \wedge D) \wedge M_{k+l-1}D \wedge \neg M_{k+l}D \wedge \neg M_l(\neg p \wedge D).$$

Other combinations may be rewritten in a similar way. Let Φ' be the formula that arises from Φ by applying the above rewriting recipe. Then all occurrences of p in Φ' are positive. By elementary logic we have $\models \Phi \rightarrow \Phi'$. To prove the converse, assume $\langle W, V \rangle \models \psi'$ where ψ' is a disjunct in Φ'. Choose $V'(p) \subseteq V(p)$ minimal so as to still have $\langle W, V' \rangle \models \psi'$. Then, in W, there are enough elements left to have $\langle W, V' \rangle \models \psi$, where ψ is the disjunct in Φ that was rewritten to ψ'. But then, by $\uparrow$MON, $\langle W, V \rangle \models \varphi$ – hence $\langle W, V \rangle \models \Phi$. $\dashv$

A related topic in the theory of generalized quantifiers is the *relational behavior* of quantifiers. A typical result in this area is the following: on the finite sets the quantifier *all* is the only logical quantifier that is both transitive ($\forall XYZ$, ($QXY \wedge QYZ \rightarrow QXZ$)) and reflexive ($\forall X$ (QXX)) (van Benthem 1984: Theorem 3.1.4). Here, we put our modal apparatus to work to characterize the logical (first order) quantifiers that are symmetric, i.e., that satisfy $\forall XY$ ($QXY \rightarrow QYX$).

Let $\alpha(P_0, P_1)$ be a first order sentence with quantifier rank q. From our remarks following 2.16 we know that $\alpha(P_0, P_1)$ has a semantic normal form (in $\mathcal{L}(QUANT)$). Using this normal form one can construct a set R_α of 4-tuples describing the models of α. Let

$$O_k(p_0 \wedge p_1) \wedge O'_l(p_0 \wedge \neg p_1) \wedge O''_m(\neg p_0 \wedge p_1) \wedge O'''_n(\neg p_0 \wedge \neg p_1).$$

be a disjunct in the semantic normal form of α. This disjunct gives rise to adding a 4-tuple $\langle a, b, c, d \rangle$ to R_α as follows

- **if** $O = M!$ **then** $a := k$ **else** $O = M$ and k must equal $q - 1$, and we put $a := q$;
- similarly for O', O'', O''' and b, c, and d respectively.

(Note that the highest number occurring in any 4-tuple in R_α is q, the quantifier rank of α.) A look at the semantic normal form of α may lead one to conjecture that α is symmetric just in case we may swap the arguments of the second and third conjunct in any disjunct in the semantic normal form of α, and still retain an equivalent of α. To see that this is indeed the case, define for a given set R_α, the set R_α^* to be $\{ \langle a, c, b, d \rangle : \langle a, b, c, d \rangle \in R_\alpha \}$.

PROPOSITION 4.3. *Let* $\alpha(P_0, P_1)$ *be an* $\mathcal{L}_1$-*sentence. Then* α *is symmetric iff* $R_\alpha = R_\alpha^*$.

Proof. We only prove the direction from right to left. Suppose $R_\alpha = R_\alpha^*$. Assume $\mathcal{M} \models \alpha(P_0, P_1)$; we want to show that $\mathcal{M} \models \alpha(P_1, P_0)$. $\mathcal{M}$ is accounted for in R_α by some tuple $\langle k, l, m, n \rangle$; by assumption $\langle k, m, l, n \rangle \in R_\alpha$. Let $\mathcal{M}'$ be a model for $\alpha(P_0, P_1)$ witnessing this:
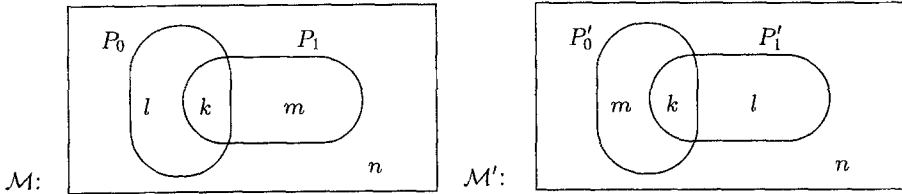
We may assume that $\mathcal{M}$ and $\mathcal{M}'$ have the same universe W. Choose a bijection $f : \mathcal{M}' \to \mathcal{M}$ that maps $P_0' \cap P_1'$ to $P_0 \cap P_1$, and $P_0'^c \cap P_1'^c$ to $P_0^c \cap P_1^c$, but $P_0' \cap P_1'^c$ to $P_0^c \cap P_1$, and $P_0'^c \cap P_1'$ to $P_0 \cap P_1^c$. Then $f[P_1'] = P_0$ and $f[P_0'] = P_1$. From this and $\mathcal{M}' \models \alpha(P_0, P_1)$ it follows that $\mathcal{M} \models \alpha(P_1, P_0)$. $\dashv$

The kind of reasoning our modal language has lead us to in the previous proof is pretty much the same as the type of argument that is usually employed in connection with the the *Tree of Numbers* (see van Benthem 1986 for details).

THEOREM 4.4. *Let* $\alpha(P_0, P_1)$ *define a logical first order quantifier. Then* α *is symmetric iff* α *is equivalent to a disjunction of formulas of the form* **atleast** k As are Bs, *and* **exactly** k As are Bs.

Proof. It is obvious that the listed forms are symmetric. So assume that α is symmetric, and consider R_α. Then $\langle a, b, c, d \rangle \in R_\alpha$ iff $\langle a, b, c, 0 \rangle \in R_\alpha$

(by EXT) iff $\langle a, b, 0, 0 \rangle \in R_\alpha$ (by CONSERV) iff $\langle a, 0, b, 0 \rangle \in R_\alpha$ (by 4.3) iff $\langle a, 0, 0, 0 \rangle \in R_\alpha$ (by CONSERV). So we may assume that R_α consists entirely of 4-tuples of the form $\langle a, 0, 0, 0 \rangle$ – but then α must have the desired form. ⊣

4.2. *Inferential Patterns*

The inferential patterns satisfied by some fixed quantifier Q have been studied on at least three levels of analysis. A purely relational (or syllogistic) level is the minimal one, where the admissible formulas are Boolean combinations of formulas of the form QXY with X, Y without any structure. A typical result here says that symmetry and quasi-reflexivity (QXY/QXX) completely axiomatize the syllogistic theory of *some* (van Benthem 1984: Theorem 3.3.5). On a second level of analysis one adds Boolean structure to the arguments X, Y of Q; to give an example: the property CONSERV ($QAB/QA(B \cap A)$ and $QA(B \cap A)/QAB$) resides at this level, as well as irreflexivity ($QAA/\bot$) (Westerståhl 1989: Section 4). To express even stronger properties of quantifiers one can move up to richer languages. For example, one might add constants for *all* and *some* to the Boolean level, and analyze one's favorite quantifier on top of this enriched Boolean language. But, the modal approach of the present paper also resides on this third level. We obviously allow for more 'types' of formulas than those allowed for in the Boolean approach. However, since in principle we can do without nestings of modal operators according to 2.16, the modal approach is rather close to the Boolean one.

This close connection between the two approaches suggests at least two lines of investigations as far as the inferential theory of specific quantifiers is concerned. For a start, we can ask questions familiar from the Boolean approach, but now lifted to the modal level. An example of such a question concerns the extent to which the syntactic behavior of a quantifier (or a set of quantifiers) *determines* its (their) semantic behavior. The completeness results for $QUANT$ and $QUANT_k$ given in Section 3 fall under this heading; what they amount to is that the respective sets of axioms say all one can say about the sets of operators $\{ M_n : n \geq 0 \}$ and $\{ M_n : n = 0, k \}$ in $\mathcal{L}(QUANT)$ and $\mathcal{L}(QUANT_k)$. Note that these sets of operators are not *determined* by their respective axiomatizations in the sense of Westerståhl (1989: Section 4.5). For these axioms are also satisfied by the modal operators $\Diamond_n$, where $\langle W, R, \ldots \rangle, v \models \Diamond_n \varphi$ if there are more than n R-successors of v that satisfy φ, where R is an equivalence relation. Even if we restrict our attention to models for monadic first order logic there is no determination of $\{ M_n : n \geq 0 \}$ or $\{ M_0, M_k \}$ ($k > 0$) by their respective axiomatizations; to see this one can

adapt the arguments of Westerståhl (1989: Corollary 4.5.10).

Another option suggested by the close connection between the Boolean and modal approach to quantifiers, is to try and solve questions from the Boolean level of analysis using our modal intuitions and results. Along this line we will present a complete axiomatization of the Boolean counterparts $more_n$ of our modal operators M_n; so $more_n XY$ denotes the quantifier $|X \cap Y| > n$.

The language $\mathcal{L}_B$ is built up as follows. It has *primitives* $(X, Y, \ldots)$ built up from unary predicate letters $P_0, P_1, \ldots$ using $(\cdot)^c, \cap$; below we will often pretend that primitives are propositional formulas built up from the 'proposition letters' $P_0, P_1, \ldots$. The *atomic formulas* of $\mathcal{L}_B$ have the form $more_n XY$, where $n \in \mathbb{N}$, and X, Y are primitives. From these, formulas are built up in the usual way. Some useful abbreviations are $allbut_n XY := \neg more_n XY^c$, and $precisely_n XY$, which is defined as $\neg more_0 XY$ if $n = 0$, and as $more_{n-1} XY \wedge \neg more_n XY$ otherwise.

Loosely speaking, $\mathcal{L}_B$ corresponds to a fragment of $\mathcal{L}(QUANT)$ in which every formula is a Boolean combination of formulas of the form $M_n \varphi$, where φ is purely propositional. So given the fact that the axioms $A1$–$A5$ axiomatize the complete theory of the operators M_n, an obvious conjecture for a complete set of axioms in $\mathcal{L}_B$ is arrived at by deleting from the list of $QUANT$-axioms those by which the number of nestings of operators may be altered, i.e., leave out $A1$ and $A2$. Apart from one additional axiom governing the way in which the operators $more_n$ combine with Boolean operators inside their arguments, this is in fact all we will need!

DEFINITION 4.5. The logic B–$QUANT$ (for the Boolean counterpart of $QUANT$) is defined as follows. Its rules of inference are Modus Ponens, Substitution, and a restricted version of Necessitation: if the primitive X (considered as a propositional formula) is derivable in propositional logic, then $allbut_0 \top X$ is a theorem of $B - QUANT$. Besides those of propositional logic its axioms are:

$A3'$ $allbut_0 XY \rightarrow (more_n \top X \rightarrow more_n \top Y)$;

$A4'$ $allbut_0 XY^c \rightarrow$
$$\left(precisely_n \top X \wedge precisely_m \top Y \rightarrow precisely_{n+m} \top (X \cup Y) \right);$$

$A5'$ $more_{n+1} XY \rightarrow more_n XY$;

$A6$ $more_n XY \leftrightarrow more_n \top (X \cap Y)$.

Here's a result we will need later on:

PROPOSITION 4.6. *Let $n \in \mathbb{N}$. The following are derivable in $B-QUANT$:*

1. $\neg more_n XY \rightarrow precisely_0 XY \vee \ldots \vee precisely_n XY$;

2. $allbut_n \top (X \cap Y)^c \leftrightarrow allbut_n XY^c \leftrightarrow allbut_n YX^c$;

3. $allbut_0 XY^c \rightarrow \left(more_n ZX \wedge more_m ZY \rightarrow more_{n+m+1} Z(X \cup Y) \right)$.

Proof. We only prove item 1; item 2 is straightforward, and item 3 follows from item 1. By definition we have $\neg precisely_0 XY \rightarrow more_0 XY$ and

$$\neg precisely_1 XY \rightarrow \neg(more_0 XY \wedge \neg more_1 XY).$$

Putting this together gives $\neg precisely_0 XY \rightarrow (\neg precisely_1 XY \rightarrow more_1 XY)$. Continuing in this fashion, we end up with

$$\neg precisely_0 XY \wedge \ldots \wedge \neg precisely_n XY \rightarrow more_n XY,$$

the contrapositive of which is item 1. (By applying axiom $A5'$ one can in fact show that the disjunctions in the consequence of the formula in item 1 are *exclusive*). $\dashv$

DEFINITION 4.7. The models for $\mathcal{L}_B$ are pairs $\mathcal{M} = \langle W, V \rangle$ where W is as usual, and V is a function assigning subsets of W to unary predicate letters, and thus, by extension, to all primitives. The only interesting case in the truth definition is the atomic one:

$$\mathcal{M} \models more_n XY \text{ iff } |V(X) \cap V(Y)| > n.$$

We say that φ is *valid* iff for all $\mathcal{M}$, $\mathcal{M} \models \varphi$.

As with *QUANT*-formulas we can define a notion of *grade* for $\mathcal{L}_B$-formulas: $\mathrm{gr}(\varphi) = 1 + \max\{ n : more_n XY \text{ occurs in } \varphi \}$. A formula $\varphi \in \mathcal{L}_B$ is said to be in *disjunctive normal form* (DNF) if it is a disjunction of literals (i.e., of (negated) atomic formulas). Using the fact that every propositional formula has a DNF, we have that every $\varphi \in \mathcal{L}_B$ has a DNF.

To prove the completeness of $B-QUANT$ we assume that $\varphi \in \mathcal{L}_B$ is consistent, and try to find a model for φ. To this end it suffices to find a model for a disjunct ψ in the DNF of φ. For the time being we fix ψ to be such a conjunction of literals in $\mathcal{L}_B$.

Let $P_0, \ldots, P_{k-1}$ be the proposition letters occurring in ψ. Recall from Section 2.3 that $\mathcal{P}_s$ ($s \in 2^k$) denotes a partition set, and $\mathcal{U}_i$ ($1 \leq i \leq 2^{2^k}$) a (possibly empty) union of partition sets. For the remainder of this section we write $\bot$ for the empty union of partition sets, and $\top$ for the union of all partition sets. Define $\mathrm{MORE}(\psi) = \{ (\neg) more_n \mathcal{U}_i \mathcal{U}_j : 1 \leq i, j \leq 2^{2^k}, n \leq \mathrm{gr}(\psi) \}$.

So $|\text{MORE}(\varphi)| = 2 \cdot \text{gr}(\varphi) \cdot 2^{2^{k+1}}$. Define a subset Ψ of $\text{MORE}(\psi)$ as follows. First of all, it contains all conjuncts occurring in ψ, and secondly, it is maximal consistent in $\text{MORE}(\psi)$.

DEFINITION 4.8. The *canonical model* $\mathcal{M}_c = \langle W_c, V_c \rangle$ is defined as follows. To each partition set $\mathcal{P}_s$ ($s \in 2^k$) associate a set of primitives Π_s in such a way that $\mathcal{P}_s \in \Pi_s$, and Π_s is maximal consistent (in propositional logic, and in the fragment containing only the 'proposition letters' $P_0, \ldots, P_{k-1}$).

W_c is a set pairs $\langle \Pi_s, n \rangle$ such that $\langle \Pi_s, n \rangle \in W_c$ iff $more_n \top \mathcal{P}_s \in \Psi$; V_c is defined by putting $\langle \Pi_s, n \rangle \in V_c(P)$ iff $P \in \Pi_s$ ($0 \leq n \leq \text{gr}(\psi)$, $s \in 2^k$).

LEMMA 4.9 (Truth Lemma). *Let* $\chi \in \text{MORE}(\psi)$. *Then* $\chi \in \Psi$ *iff* $\mathcal{M}_c \models \chi$.

Proof. Assume $\chi \equiv more_n \mathcal{U}_i \mathcal{U}_j$. Then for some $\mathcal{P}_1, \ldots, \mathcal{P}_s$ we have $\vdash (\mathcal{U}_i \cap \mathcal{U}_j) \leftrightarrow (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$ in propositional logic.

Assume $\mathcal{M}_c \models more_n \mathcal{U}_i \mathcal{U}_j$, i.e., $\mathcal{M}_c \models more_n \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$, by the soundness of axiom $A6$. Then there are $n_1, \ldots, n_s$ such that $\langle \Pi_t, n_t - 1 \rangle \in W_c$ ($1 \leq t \leq s$), and $n_1 + \cdots + n_s = m > n$. By the definition of W_c we have $more_{n_t - 1} \top \mathcal{P}_t \in \Psi$ ($1 \leq t \leq s$). Now obviously, if $u \neq v$ ($1 \leq u < v \leq s - 1$) then $\vdash ((\mathcal{P}_1 \vee \ldots \vee \mathcal{P}_u) \wedge \mathcal{P}_v) \leftrightarrow \bot$ in propositional logic; hence $\vdash allbut_0 (\mathcal{P}_1 \vee \ldots \vee \mathcal{P}_u)(\mathcal{P}_v)^c$ in $B-QUANT$. By repeated applications of 4.6.(3) this yields $more_{m-1} \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s) \in \Psi$. But $m - 1 \geq n$, hence axiom $A5'$ gives $more_n \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s) \in \Psi$; but then, $more_n \mathcal{U}_i \mathcal{U}_j \in \Psi$, by the maximal consistency of Ψ.

For the converse we have to do a little more work. Suppose $\chi \in \Psi$. By $A6$ and Substitution we have $more_n \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s) \in \Psi$. We distinguish two possibilities.

1. For some t, $1 \leq t \leq s$, $more_n \top \mathcal{P}_t \in \Psi$. Then, by axiom $A5'$, the fact that Ψ is deductively closed, and the definition of W_c, we have $\langle \Pi_t, 0 \rangle, \ldots \langle \Pi_t, n \rangle \in \mathcal{M}_c$. Hence, $\mathcal{M}_c \models more_n \top \mathcal{P}_t$; thus $\mathcal{M}_c \models more_n \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$, and so $\mathcal{M}_c \models more_n \mathcal{U}_i \mathcal{U}_j$.

2. For no t ($1 \leq t \leq s$), $more_n \top \mathcal{P}_t \in \Psi$. Then, by 4.6.(1), we can conclude that there are $n_1, \ldots, n_{s-1}$ such that $precisely_{n_t} \top \mathcal{P}_t \in \Psi$ ($1 \leq t \leq s - 1$). Put $m = n_1 + \cdots + n_{s-1}$. If $m > n$, we are done. For then we have $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s$ occurring in n_t copies of Π_t for each $t \in \{1, \ldots, s - 1\}$; this implies $\mathcal{M}_c \models more_n \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$ and $\mathcal{M}_c \models more_n \mathcal{U}_i \mathcal{U}_j$. If, on the other hand, $m \leq n$, then we argue as follows. We first show that over $B-QUANT$ we have that $more_n \mathcal{U}_i \mathcal{U}_j$ implies

$$precisely_{n_1} \top \mathcal{P}_1 \wedge \ldots \wedge precisely_{n_{s-1}} \top \mathcal{P}_{s-1} \rightarrow more_{n-m} \top \mathcal{P}_s.$$

Reason 'inside' $B-QUANT$. Assume $more_n \mathcal{U}_i \mathcal{U}_j$, and $precisely_{n_1} \top \mathcal{P}_1$, $\ldots, precisely_{n_{s-1}} \top \mathcal{P}_{s-1}$. Note that by 4.6.(1) we have

$$\neg more_{n-m} \top \mathcal{P}_s \rightarrow precisely_0 \top \mathcal{P}_s \vee \ldots \vee precisely_{n-m} \top \mathcal{P}_s. \quad (1)$$

Since $u \neq v$ implies $\vdash \ allbut_0 \mathcal{P}_u(\mathcal{P}_v)^c$, axiom $A4'$ gives that for $r \in \{\, 0, \ldots, n - m \,\}$, the conjunction of the formulas $precisely_{n_1} \top \mathcal{P}_1 \wedge \ldots \wedge precisely_{n_{s-1}} \top \mathcal{P}_{s-1}$ and $precisely_r \top \mathcal{P}_s$ implies the formula $precisely_{m+r} \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$. Together with (1) and our assumptions this yields

$$\neg more_{n-m} \top \mathcal{P}_s \rightarrow$$
$$precisely_m \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s) \vee \ldots \vee precisely_n \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s).$$

But the latter disjunction implies $\neg more_n \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$, i.e. $\neg more_n \mathcal{U}_i \mathcal{U}_j$, which is a contradiction. Hence, we have $more_{n-m} \top \mathcal{P}_s$ as required. It follows that $more_{n-m} \top \mathcal{P}_s \in \Psi$. All in all we have $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s$ occurring in n_t copies of Π_t ($1 \leq t \leq s - 1$); this gives m elements of W_c 'verifying' $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s$. The fact that $more_{n-m} \top \mathcal{P}_s \in \Psi$ adds more than $n - m$ copies of Π_s to W_c, in each of which $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s$ occurs. This implies $\mathcal{M}_c \models more_n \top (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$, and hence, $\mathcal{M}_c \models more_n \mathcal{U}_i \mathcal{U}_j$. ⊣

THEOREM 4.10. *Let $\varphi \in \mathcal{L}_B$. Then $B-QUANT \vdash \varphi$ iff $B-QUANT \models \varphi$.*

Proof. As before, proving soundness is left to the reader. To prove completeness, assume that $B-QUANT \nvdash \varphi$. So $\neg\varphi$ is $B-QUANT$-consistent. But then, some disjunct ψ in the DNF of $\neg\varphi$ has a model by 4.9. Hence, $B-QUANT \nvDash \varphi$. ⊣

COROLLARY 4.11. *Let $\varphi \in \mathcal{L}_B$. Then $B-QUANT \vdash \varphi$ iff $QUANT \vdash \varphi$.*

COROLLARY 4.12. *$B-QUANT$ is strongly complete.*

Proof. Let Δ be an arbitrary set of sentences. Then $\Delta \vdash \varphi$ in $B-QUANT$ iff $\Delta \vdash \varphi$ in $QUANT$, by 4.12, iff $\Delta \models \varphi$ in $QUANT$, by the strong completeness of $QUANT$ (cf. Fine (1972)), iff $\Delta \models \varphi$ in $B-QUANT$. ⊣

The method used to prove $B-QUANT$ complete in 4.10 may also be used to give an alternative completeness proof for $QUANT$ or $QUANT_k$. We preferred to prove the completeness of $QUANT_k$ the way we did it in Section 3.2, simply because the method used there is somewhat closer to the modal tradition.

The question settled by Theorem 4.10 is related to a question due to Johan van Benthem, who asked for a complete axiomatization of the schemes common to all quantifiers $some_n$ ($n \in \mathbb{N}_{>0}$). $A6$ and 4.6.(2) are two such schemes; and, in the richer language where one has all and $some$ available, $A3'$ is a further example.

5. BEYOND THE FIRST ORDER BOUNDARY

In this section we will consider some higher order quantifiers as modal opera-
tors. The leading character in this section will be the quantifier *there are at
least as many Xs as Ys*. The choice to consider this particular quantifier
is motivated by the fact that we can use an existing calculus to axiomatize the
valid inference patterns that hold of this quantifier. Also, using the quantifier
there are at least as many Xs as Ys, a number of other higher order
quantifiers can be defined and studied.

The plan for this section is as follows. We first introduce some notation and
an axiom system QM for a modal operator $atleast(\cdot, \cdot)$. After that we prove
a completeness theorem for this system. Then, some themes from sections 2.2
and 4.1 re-emerge when we prove a normal form theorem, characterize the
QM-formulas satisfying CONSERV and EXT, and prove a (partial) Lyndon
Theorem for the modal language with *atleast*. We complete this section by
taking an exploratory look at some modal operators representing other higher
order quantifiers.

5.1. *Axioms and Notation*

First, let us set up our language. Let *Form* $_{\geq}$ abbreviate *Form* (*Prop*,
$\emptyset, \{ atleast \}$). Here are some useful abbreviations we will use:

$$
\begin{aligned}
L_0\varphi &\equiv atleast(\varphi, \top) \\
more(\varphi, \psi) &\equiv atleast(\varphi, \psi) \wedge \neg atleast(\psi, \varphi) \\
most(\varphi, \psi) &\equiv more(\varphi \wedge \psi, \varphi \wedge \neg\psi) \\
equal(\varphi, \psi) &\equiv atleast(\varphi, \psi) \wedge atleast(\psi, \varphi).
\end{aligned}
$$

Given that the intended reading of $atleast(\varphi, \psi)$ is: there are at least as many
φs as ψs, the intended interpretations of the above abbreviations should be
obvious from the notation.

Before plunging into axiomatics, let us briefly answer two questions that
may arise at this point. First, is *there are at least as many Xs as Ys*
indeed higher order? Suppose it is not; then it has a first order definition
$\alpha(X, Y)$, say of quantifier rank n. Let $\mathcal{M} = \langle W, P, \ldots \rangle$ be a model for
monadic first order logic with $|W| = 2n + 3$, $|P| = n + 1$. Then $\mathcal{M} \not\models$
there are at least as many Ps as $\neg Ps$, hence $\mathcal{M} \not\models \alpha(P, \neg P)$. Let
$\mathcal{M}' = \langle W', P', \ldots \rangle$ with $|W'| = 2n + 4$, $|P'| = n + 2$. Then $\mathcal{M} \sim_n \mathcal{M}'$
(for the restricted fragment containing only the predicate letter P). But then
$\mathcal{M}' \not\models \alpha(P, \neg P)$, by our remarks preceding Theorem 2.16, and so $\mathcal{M}' \not\models$
there are at least as many Ps as $\neg Ps$ – a contradiction.

Second, one might well wonder why we don't use a *unary* modal operator to simulate *there are at least as many* Xs *as* Ys, – just like we used the unary operator L_0 to simulate the quantifier *all* XY. An obvious candidate would be the operator O_a with $O_a\varphi$ true at a world in a model iff there are at least as many worlds that verify φ as there are worlds verifying $\neg\varphi$. But, although O_a is certainly definable in terms of *atleast*, the latter can not be defined in terms of the former; to see this one can adapt a result due to Barwise and Cooper saying that the binary quantifier *most* is not definable using the Rescher quantifier Q_R (Westerståhl 1989: Section 1.7).

DEFINITION 5.1. We define the logic QM (for *Qualitative Modalities*). Like *QUANT*, QM has Modus Ponens, Necessitation $(\varphi/L_0\varphi)$ and Substitution as rules of inference. Besides those of propositional logic, its axioms are

$C1$ $L_0(\varphi \leftrightarrow \varphi') \wedge L_0(\psi \leftrightarrow \psi') \to (\textbf{\textit{atleast}}(\varphi,\psi) \leftrightarrow \textbf{\textit{atleast}}(\varphi',\psi'))$
$C2$ $\textbf{\textit{atleast}}(\varphi,\psi) \vee \textbf{\textit{atleast}}(\psi,\varphi)$
$C3$ $\textbf{\textit{atleast}}(\varphi,\perp)$
$C4$ $\textbf{\textit{more}}(\top,\perp)$
$C5$ $L_0\varphi \to \varphi$
$C6$ $(\textbf{\textit{atleast}}(\varphi,\psi) \leftrightarrow L_0\textbf{\textit{atleast}}(\varphi,\psi))$
        $\wedge\ (\neg\textbf{\textit{atleast}}(\varphi,\psi) \leftrightarrow L_0\neg\textbf{\textit{atleast}}(\varphi,\psi))$

$D(m)$ for sequences of formulas $\vec{\varphi}, \vec{\psi}$ both of length $m + 1$,

$$\vec{\varphi}\mathcal{E}\vec{\psi} \to (\textbf{\textit{atleast}}(\varphi_0,\psi_0) \wedge \ldots \wedge \textbf{\textit{atleast}}(\varphi_{m-1},\psi_{m-1}) \to$$
$$\textbf{\textit{atleast}}(\psi_m,\varphi_m)).$$

Here, for $m \in \mathbb{N}$, $\vec{\varphi}\mathcal{E}\vec{\psi}$ expresses a kind of generalized equivalence. It is defined as follows. For a sequence $\vec{\gamma} = \langle\gamma_0, \ldots, \gamma_m\rangle$ of $m + 1$ formulas, let $T_i(\vec{\gamma})$ be a statement that is true iff exactly i elements in $\vec{\gamma}$ are true. E.g. if $\vec{\gamma} = \langle p_0, p_1\rangle$ then $T_1(\vec{\gamma}) = (p_0 \wedge \neg p_1) \vee (\neg p_0 \wedge p_1)$, and $T_2(\vec{\gamma}) = (p_0 \wedge p_1)$. Then

$$\vec{\varphi}\mathcal{E}\vec{\psi} := L_0 \bigvee_{0 \le i \le m+1} (T_i(\vec{\varphi}) \wedge T_i(\vec{\psi})). \tag{2}$$

Loosely speaking, when interpreted on a model, the right-hand side of (2) says that every point of the model is *balanced* in the sense that i formulas from the sequence $\vec{\varphi}$ are true in a point iff i formulas from the sequence $\vec{\psi}$ are true in that point ($0 \le i \le m + 1$). Hence, what $D(m)$ expresses is that if each point is balanced, and if, in addition, for each of the first m components of $\vec{\varphi}$ we have that their extension is at least as big as the extension of the corresponding components of $\vec{\psi}$, – then the extension of the last component

of $\vec{\psi}$ should not be smaller than the extension of the last component of $\vec{\varphi}$. At least for finite models $D(m)$ is a perfectly sound principle; that it is not sound on infinite models is shown in our remarks preceding 5.3.

Let's see this system in action. We will derive a formula expressing additivity of *there are at least as many* As as Bs: $L_0\neg(\varphi \wedge \chi) \wedge L_0\neg(\psi \wedge \chi) \rightarrow (atleast(\varphi, \psi) \rightarrow atleast(\varphi \vee \chi, \psi \vee \chi))$. (We use $\vec{\varphi}\mathcal{E}^-\vec{\psi}$ to denote $\vec{\varphi}\mathcal{E}\vec{\psi}$ with the operator L_0 left out; PL is short for propositional logic.)

Obviously we have,

$$(\varphi_0 \leftrightarrow \varphi_0') \wedge \ldots \wedge (\varphi_m \leftrightarrow \varphi_m') \rightarrow \langle\varphi_0,\ldots,\varphi_m\rangle\mathcal{E}^-\langle\varphi_0',\ldots,\varphi_m'\rangle, \quad (3)$$

$$\neg(\varphi \wedge \chi) \wedge \neg(\psi \wedge \chi) \rightarrow \neg\chi \vee (\neg\varphi \wedge \neg\psi \wedge \chi), \quad (4)$$

Now, by PL, $\neg\chi$ implies $(\varphi \leftrightarrow (\varphi \vee \chi)) \wedge ((\psi \vee \chi) \leftrightarrow \psi)$, so by (3) we have

$$\neg\chi \rightarrow \langle\varphi, \psi \vee \chi\rangle\mathcal{E}^-\langle\varphi \vee \chi, \psi\rangle,$$
$$\rightarrow \langle\varphi, \psi \vee \chi\rangle\mathcal{E}^-\langle\psi, \varphi \vee \chi\rangle, \quad (5)$$

Then, again by PL, we have

$$\neg\varphi \wedge \neg\psi \wedge \chi \rightarrow (\neg\varphi \wedge (\psi \vee \chi)) \wedge (\neg\psi \wedge (\varphi \vee \chi)),$$
$$\rightarrow T_1(\varphi, (\psi \vee \chi)) \wedge T_1(\psi, (\varphi \vee \chi)),$$
$$\rightarrow \langle\varphi, \psi \vee \chi\rangle\mathcal{E}^-\langle\psi, \varphi \vee \chi\rangle.$$

So by (4) and (5) this implies

$$\neg(\varphi \wedge \chi) \wedge \neg(\psi \wedge \chi) \rightarrow \langle\varphi, \psi \vee \chi\rangle\mathcal{E}^-\langle\psi, \varphi \vee \chi\rangle. \quad (6)$$

Applying Necessitation and $D(2)$ to (6), we find

$$L_0\neg(\varphi \wedge \chi) \wedge L_0\neg(\psi \wedge \chi) \rightarrow \Big(atleast(\varphi, \psi) \rightarrow atleast(\varphi \vee \chi, \psi \vee \chi)\Big).$$

To complete this introductory section on QM, let us briefly mention an alternative proposal to analyze the quantifier *there are at least as many* Xs *as* Ys, that is due to Johan van Benthem (private communication). He suggested to consider some mixture of modal logic and additive arithmetic, with atomic statements of the form $atleast(A, B)$ and $\Sigma_i x_i \geq \Sigma_j y_j$, where the x_i and y_j are numerical variables ranging over cardinalities of subsets of the universe.

5.2. *Completeness*

In Gärdenfors (1975) a completeness result for QM is given with respect to a special class of so-called probability models. Combining this result with a

result from (van der Hoek, 1991), we can derive a completeness result for QM with respect to models in which $atleast$ and L_0 receive their intended interpretations.

To state these results, we need some definitions. Recall that a *probability measure* on a set W is a function $P : 2^W \to [0, 1]$ that satisfies (i) $P(W) = 1$, (ii) $P(\emptyset) = 0$, and (iii) if for countable I, $X_i \in 2^W$, $X_i \cap X_j = \emptyset$ whenever $i \neq j$, then $P(\bigcup_{i \in I} X_i) = \Sigma_{i \in I} P(X_i)$. A *probability model* $\mathcal{M}$ is a tuple $\langle W, F, V \rangle$ where W and V are as usual, and where F is a collection of probability measures $\{ P_w : w \in W \}$ on W. The interesting case in the truth definition is

$$\mathcal{M}, w \models atleast(\varphi, \psi) \text{ iff } P_w(V(\varphi)) \geq P_w(V(\psi)).$$

So, on probability models $atleast$ is interpreted as 'at least as likely as'. The following result may be found in Gärdenfors (1975: Section V):

THEOREM 5.2. *QM is complete w.r.t. the class of finite probability models in which all probability measures satisfy $\forall x \, (P_x(\{\, x \,\}) > 0)$, and for all $S \subseteq W$, $\forall xy \, (P_x(S) = P_y(S))$.*

A *qualitative model* is a tuple $\mathcal{M} = \langle W, R, V \rangle$ with W a finite set, V as usual, $R \subseteq W^2$, and in which $atleast$ is interpreted as follows:

$$\mathcal{M}, w \models atleast(\varphi, \psi) \text{ iff}$$
$$|\{\, v : Rwv \text{ and } \mathcal{M}, v \models \varphi \,\}| \geq |\{\, v : Rwv \text{ and } \mathcal{M}, v \models \psi \,\}|.$$

In qualitative models W has to be finite to ensure the soundness of $D(m)$. For, let W be infinite, and pick $w \in W$; put $V(p_0) = W \setminus \{\, w \,\}$, $V(p_1) = \{\, w \,\}$, $V(q_0) = W$, and $V(q_1) = \emptyset$. Then $\langle W, W \times W, V \rangle \models \langle p_0, p_1 \rangle \mathcal{E} \langle q_0, q_1 \rangle \wedge atleast(p_0, q_0)$, but $\langle W, W \times W, V \rangle \not\models atleast(q_1, p_1)$, which refutes axiom $D(1)$.

Our next aim is to prove the completeness of QM with respect to models in which the modal operators receive their intended quantifier interpretations. To do this it suffices to show that QM is complete w.r.t. qualitative models in which R is an equivalence relation. For then, $QM \not\vdash \varphi$ implies that for some qualitative model $\mathcal{M}$ in which R is an equivalence relation, $\mathcal{M}, w \not\models \varphi$. Taking the submodel generated by w gives a model $\mathcal{M}'$ in which φ is refuted, and in which R is the universal relation. Hence, $atleast$ and L_0 receive their intended interpretations in $\mathcal{M}'$.

THEOREM 5.3. *QM is complete w.r.t. finite qualitative models in which R is an equivalence relation.*

Proof. If $QM \not\vdash \varphi$ then by 5.2 there is a finite probabilistic model $\mathcal{M}_p$ $(= \langle W, F, V \rangle)$ satisfying the conditions stated in 5.2, such that for some $w \in \mathcal{M}_p$ we have $\mathcal{M}_p, w \not\models \varphi$. By van der Hoek (1991: Lemma 3.7) there is a finite qualitative model $\mathcal{M}_q$ $(= \langle W', R, V' \rangle)$, where W' contains a number of copies w' of certain $w \in W$, such that $\forall x'y' \in W'$ $(Rx'y' \leftrightarrow P_x(\{y\}) > 0)$ and for each subformula ψ of φ, $\mathcal{M}_p, w \models \psi$ iff $\mathcal{M}_q, w' \models \psi$. Moreover, using the above condition on R it can be seen that if $\mathcal{M}_p$ satisfies $\forall x (P_x(\{x\}) > 0)$ and for all $S \subseteq W$, $\forall xy (P_x(S) = P_y(S))$, then in $\mathcal{M}_q$ we have that R is an equivalence relation.   ⊣

COROLLARY 5.4. *QM is complete w.r.t. finite models $\langle W, V \rangle$ (or $\langle W, P_0, P_1, \ldots \rangle$) in which $atleast(\varphi, \psi)$ is interpreted as "there are ate least as many worlds satisfying φ as there are worlds verifying ψ."*

A few remarks are in order here. First, Corollary 5.4 does not generalize to a *strong* completeness result. For QM is not compact: define Δ to be the set

$$\{atleast(p_{i+1}, p_i), \neg atleast(p_i, p_{i+1}) : i \in \mathbb{N}\}.$$

Then, over QM, $\Delta \models \bot$, since QM is not sound on infinite models, while Δ only has infinite ones. But obviously, $\Delta_0 \not\models \bot$, for all finite $\Delta_0 \subseteq \Delta$. Hence, $\Delta \not\vdash \bot$, and strong completeness fails for QM.

The proof of 5.3 is a special version of a rather complex argument used to prove the completeness of QM minus the axioms $C5$ and $C6$. It is still open whether 5.3 may be proved in a simpler, more direct way, for example using some version of the method used in 4.10. More specifically, is the *infinite* schema $D(m)$ really necessary, or is there some *finite* axiomatization after all?

Although we do not want to discuss the complexity of QM-satisfiability in this paper, we feel that it may be shown to be in one or other complexity class in pretty much the same way as $QUANT$-satisfiability was shown to be in PSPACE in Section 3.3.

5.3. *Normal Forms and Semantic Constraints*

Using our general result on normal forms from Section 2.2 we give a quick proof for the existence of syntactic normal forms for formulas in *Form $_\geq$*. After that we determine 'semantic' normal forms for such formulas, and use these to obtain syntactic characterizations of various semantic constraints.

DEFINITION 5.5. A formula $\varphi \in Form_\geq$ is in *normal form* (NF) if it is a disjunction of conjunctions of the general form

$$\alpha \wedge atleast(\alpha_1, \beta_1) \wedge \ldots \wedge atleast(\alpha_n, \beta_n) \wedge$$

$$\neg atleast(\gamma_1, \delta_1) \wedge \ldots \wedge \neg atleast(\gamma_m, \delta_m),$$

where $\alpha, \alpha_i, \beta_i, \gamma_j, \delta_j$ $(1 \leq i \leq n, 1 \leq j \leq m)$ are purely propositional formulas.

THEOREM 5.6. *Over QM every $\chi \in Form_\geq$ is equivalent to a formula in normal form.*

Proof. Let $\mathcal{O} = \{ L_0, atleast \}$. Prove that QM is neat, and apply 2.11. ⊣

Our next aim is to find an Ehrenfeucht-Fraïssé-like characterization for QM-formulas, and use this to find syntactic counterparts for a number of semantic constraints, as we did in Section 4.1.

First, we have to give some definitions. To simplify matters we assume that we are working in a restricted language with proposition letters $p_0, \ldots, p_{k-1}$; the appropriate models then have the form $\langle W, P_0, \ldots, P_{k-1} \rangle$, with W finite. Recall from Section 2.3 that we use $\mathcal{P}_i$ to denote partition sets (or partition conjunctions), and $\mathcal{U}_j$ to denote unions of partition sets (or disjunctions of partition conjunctions). Define

$$\mathcal{M} \sim_{\text{atleast}} \mathcal{M}' \text{ iff for all unions of partition sets } \mathcal{U}_i, \mathcal{U}_j \ (1 \leq i, j \leq 2^{2^k})$$
$$\text{we have } |\mathcal{U}_i^{\mathcal{M}}| \geq |\mathcal{U}_j^{\mathcal{M}}| \text{ iff } |\mathcal{U}_i^{\mathcal{M}'}| \geq |\mathcal{U}_j^{\mathcal{M}'}|;$$

also,

$$\mathcal{M} \equiv_{\text{atleast}} \mathcal{M}' \text{ iff } \mathcal{M} \text{ and } \mathcal{M}' \text{ verify the same } QM\text{-formulas in}$$
$$p_0, \ldots, p_{k-1}.$$

LEMMA 5.7. *For any two finite models $\mathcal{M}, \mathcal{M}'$ we have*

$$\mathcal{M} \sim_{\text{atleast}} \mathcal{M}' \text{ iff } \mathcal{M} \equiv_{\text{atleast}} \mathcal{M}'.$$

Proof. Let $\perp$ denote the empty disjunction of partition conjunctions, and $\top$ the disjunction of all partition conjunctions. Assume $\mathcal{M} \equiv_{\text{atleast}} \mathcal{M}'$. Then, if $|\mathcal{U}_i^{\mathcal{M}}| \geq |\mathcal{U}_j^{\mathcal{M}}|$, we have $\mathcal{M} \models atleast(\mathcal{U}_i, \mathcal{U}_j)$. Thus $\mathcal{M}' \models atleast(\mathcal{U}_i, \mathcal{U}_j)$, i.e., $|\mathcal{U}_i^{\mathcal{M}'}| \geq |\mathcal{U}_j^{\mathcal{M}'}|$. Since the converse may be proved similarly we have $\mathcal{M} \sim_{\text{atleast}} \mathcal{M}'$.

Conversely, assume $\mathcal{M} \sim_{\text{atleast}} \mathcal{M}'$. For φ a formula, let $[\varphi]_{\mathcal{M}}$ abbreviate $\{ x : \mathcal{M}, x \models \varphi \}$, and similarly for $[\varphi]_{\mathcal{M}'}$. To each formula φ (in $p_0, \ldots, p_{k-1}$) we will associate a union of partition sets $\mathcal{U}_i$ such that $[\varphi]_{\mathcal{M}} = \mathcal{U}_i^{\mathcal{M}}$, and $[\varphi]_{\mathcal{M}'} = \mathcal{U}_i^{\mathcal{M}'}$. Then, given the assumption that $\mathcal{M} \sim_{\text{atleast}} \mathcal{M}'$, it follows that $\mathcal{M} \equiv_{\text{atleast}} \mathcal{M}'$. For $\mathcal{M} \models \varphi$ implies $|[\varphi]_{\mathcal{M}}| = |\mathcal{U}_i^{\mathcal{M}}| \geq |[\top]_{\mathcal{M}}|,$

so, since $\mathcal{M} \sim_{\text{atleast}} \mathcal{M}'$, $|[\varphi]_{\mathcal{M}'}| = |\mathcal{U}_i^{\mathcal{M}'}| \geq [\top]_{\mathcal{M}'}$, which means that $\mathcal{M}' \models \varphi$.

With proposition letters we associate unions of partition sets as follows: let $\mathcal{U}_i = \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_{2^k - 1}$, where $\mathcal{P}_1, \ldots, \mathcal{P}_{2^k - 1}$ are all the proposition conjunctions in which p occurs positively. Then $[p]_{\mathcal{M}} = \mathcal{U}_i^{\mathcal{M}}$, and $[p]_{\mathcal{M}'} = \mathcal{U}_i^{\mathcal{M}'}$. Next, assume $[\varphi]_{\mathcal{M}} = \mathcal{U}_i^{\mathcal{M}}$, and $[\varphi]_{\mathcal{M}'} = \mathcal{U}_i^{\mathcal{M}'}$; then $[\neg\varphi]_{\mathcal{M}} = (\mathcal{U}_i^{\mathcal{M}})^c$, and $[\neg\varphi]_{\mathcal{M}'} = (\mathcal{U}_i^{\mathcal{M}'})^c$. Using some standard procedure one can bring $(\mathcal{U}_i)^c$ into a 'disjunctive' normal form, consisting of disjunctions of conjunctions of $(\neg)p_0, \ldots, (\neg)p_{k-1}$ – thus $(\mathcal{U}_i)^c = \mathcal{U}_j$, for some j, and we are done. Next assume that $[\varphi]_{\mathcal{M}} = \mathcal{U}_i^{\mathcal{M}}$, $[\psi]_{\mathcal{M}} = \mathcal{U}_j^{\mathcal{M}}$, and $[\varphi]_{\mathcal{M}'} = \mathcal{U}_i^{\mathcal{M}'}$, $[\psi]_{\mathcal{M}'} = \mathcal{U}_j^{\mathcal{M}'}$. As in the previous case one can use a standard procedure to show that for some $\mathcal{U}_k, \mathcal{U}_i \cap \mathcal{U}_j = \mathcal{U}_k$; but then $[\varphi \wedge \psi]_{\mathcal{M}} = \mathcal{U}_k^{\mathcal{M}}$ and $[\varphi \wedge \psi]_{\mathcal{M}'} = \mathcal{U}_k^{\mathcal{M}'}$. Finally, under the assumptions of the previous case we have to associate a union of partition sets to $atleast(\varphi, \psi)$. We distinguish two cases, the first one being $\mathcal{M} \models atleast(\varphi, \psi)$. Then $|\mathcal{U}_i^{\mathcal{M}}| \geq |\mathcal{U}_j^{\mathcal{M}}|$, so $|\mathcal{U}_i^{\mathcal{M}'}| \geq |\mathcal{U}_j^{\mathcal{M}'}|$; hence, $[atleast(\varphi, \psi)]_{\mathcal{M}} = [\top]_{\mathcal{M}}$, and $[atleast(\varphi, \psi)]_{\mathcal{M}'} = [\top]_{\mathcal{M}'}$. The second case is $\mathcal{M} \not\models atleast(\varphi, \psi)$. Then $|\mathcal{U}_i^{\mathcal{M}}| \not\geq |\mathcal{U}_j^{\mathcal{M}}|$, so $|\mathcal{U}_i^{\mathcal{M}'}| \not\geq |\mathcal{U}_j^{\mathcal{M}'}|$. But then we have $[atleast(\varphi, \psi)]_{\mathcal{M}} = [\bot]_{\mathcal{M}}$, and $[atleast(\varphi, \psi)]_{\mathcal{M}'} = [\bot]_{\mathcal{M}'}$. $\dashv$

COROLLARY 5.8. *Let $l > 0$. Then the modal operator M_l is not definable by means of QM-formulas.*

Proof. Let $\mathcal{M} = \langle W, V \rangle$ be a model with $|V(p)| = l+1$, and $|W| = 2 \cdot (l+1)$. Obviously, $\mathcal{M} \models M_l p$. Let $\mathcal{M}' = \langle W', V' \rangle$ be a model with $|W'| = 2$, $|V'(p)| = 1$. Then $\mathcal{M}' \not\models M_l p$. To show that there is no definition of M_l by means of QM-formulas, it is sufficient to prove $\mathcal{M} \equiv_{\text{atleast}} \mathcal{M}'$ (w.r.t. the fragment over the single proposition letter p). To see this, it suffices to show that $\mathcal{M} \sim_{\text{atleast}} \mathcal{M}'$ (w.r.t. the same fragment), by 5.7. But this is simple, since for all relevant unions of partition sets $\mathcal{U}_i$, we have $|\mathcal{U}_i^{\mathcal{M}}| = n$ iff $|\mathcal{U}_i^{\mathcal{M}'}| = n/(l+1)$. $\dashv$

Let φ be a formula in $p_0, \ldots, p_{k-1}$. The number of $\sim_{\text{atleast}}$-equivalence classes is finite; let $\mathcal{M}_1, \ldots, \mathcal{M}_g$ be representatives of the $\sim_{\text{atleast}}$-classes that contain models of φ. For $\mathcal{M} \in \{ \mathcal{M}_1, \ldots, \mathcal{M}_g \}$ write down a conjunction $\psi_{\mathcal{M}}$ of formulas of the form $(\neg)atleast(\mathcal{U}_i, \mathcal{U}_j)$, depending on whether or not $|\mathcal{U}_i^{\mathcal{M}}| \geq |\mathcal{U}_j^{\mathcal{M}}|$ in $\mathcal{M}$. (Note: for any $\mathcal{M}'$, $\mathcal{M}' \models \psi_{\mathcal{M}}$ iff $\mathcal{M}' \sim_{\text{atleast}} \mathcal{M}$.) This results in a semantic normal form for φ as follows: for any $\mathcal{M}$: $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi_{\mathcal{M}_1} \vee \ldots \vee \psi_{\mathcal{M}_g}$.

Using these semantic normal forms one can try and find syntactic counterparts (in *Form* $\geq$) of semantic constraints, just like we did in Section 4.1.

However, the semantic normal forms for QM-formulas are much more complex than those found for $QUANT$-formulas in Section 2.3. (Indeed, the proof of the Ehrenfeucht-Fraïssé Theorem for QM was already more complex than the corresponding result for $QUANT$, or first order logic (Westerståhl 1989: Section 1.7). Consequently, manipulations on semantic normal forms for QM have to be more abstract and involved than they were in the proofs of e.g. 4.1 and 4.2, as is witnessed below.

Call a formula in $Form \geq p_0$-restricted if it is a Boolean combination of formulas of the form $atleast(p_0 \wedge \varphi, p_0 \wedge \psi)$, where φ, ψ are purely propositional.

PROPOSITION 5.9. *On finite models, a formula $\varphi(p_0, p_1) \in Form_\geq$ satisfies CONSERV and EXT iff it is equivalent to a p_0-restricted formula.*

Proof. The simple proof that all p_0-restricted QM-formulas satisfy CONSERV and EXT is left to the reader. Assume φ satisfies CONSERV and EXT. Let $\Psi \equiv \psi_0 \vee \ldots \vee \psi_g$ be a semantic normal form for φ. Since we are restricting ourselves to the fragment containing only p_0, p_1, the disjunctions $\mathcal{U}_i$ occurring in the disjuncts $\psi_0, \ldots, \psi_g$ are disjunctions of formulas of the form $(\neg)p_0 \wedge (\neg)p_1$. Now, let ψ be a disjunct in Ψ. Let ψ' be obtained from ψ by deleting all conjuncts of the form $\neg p_0 \wedge (\neg)p_1$. Let Ψ' be the result of substituting ψ' for ψ in Ψ. Hence, Ψ' is p_0-restricted. Our claim is that for any $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \Psi'$. To prove this we use the Figure displayed in the proof of 4.3. Assume first that $\mathcal{M} \models \varphi$, say $\mathcal{M} \models \psi$, where ψ is a disjunct in Ψ. $\mathcal{M} \models \psi$ means that a number of inequalities involving k, l, m, n must be satisfied in $\mathcal{M}$. By CONSERV and EXT these inequalities are still satisfied if we leave out m and n. On the level of formulas this means that $\mathcal{M} \models [\bot / \neg p_0 \wedge p_1][\bot / \neg p_0 \wedge \neg p_1]\psi$. That is, $\mathcal{M} \models \psi'$.

For the converse, assume $\mathcal{M} \models \psi'$, where ψ' is some disjunct in Ψ'. This means that certain inequalities involving *only* k and l must be satisfied in $\mathcal{M}$. Define $\mathcal{M}'$ by putting $k' = k, l' = l$, and $m' = n' = 0$. Then $\mathcal{M}' \models \psi'$. Since $m' = n' = 0$ we can 'plug in' occurrences of m' and n' in the inequalities corresponding to ψ', at any place we like. But then we may assume that $\mathcal{M}' \models \psi$. Thus $\mathcal{M}' \models \varphi$, and so, by CONSERV and EXT, $\mathcal{M} \models \varphi$. $\dashv$

To characterize the $\uparrow$MON formulas, we need to specify what it is for a proposition letter to occur *positively* (or *negatively*) in a QM-formula. The appropriate inductive definition has the usual clauses for proposition letters and the Boolean connectives, while a positive (negative) occurrence of p in φ is a positive (negative) occurrence of p in $atleast(\varphi, \psi)$, and a positive (negative) occurrence of p in φ is a negative (positive) occurrence of p in

$atleast(\psi, \varphi)$.

THEOREM 5.10. *Let $\varphi(p)$ be a QM-formula that is equivalent to a disjunction Ψ of formulas of the form $(\neg)atleast(\chi_1, \chi_2)$, where χ_1, χ_2 are purely propositional. Then, on finite models, φ is $\uparrow$MON (in p) iff φ is equivalent to a formula in which all occurrences of p are positive.*

Proof. The direction from right to left is similar to the corresponding case in Theorem 4.2. Assume φ is $\uparrow$MON, and let ψ be a disjunct in Ψ, say $\psi \equiv (\neg)atleast(\chi_1, \chi_2)$, where χ_1, χ_2 are disjunctions of conjunctions of literals. Since $\models atleast(A, B) \leftrightarrow atleast(A \wedge \neg B, \neg A \wedge B)$, we may assume that χ_1, χ_2 are mutually exclusive. Moreover, using propositional logic, ψ can be brought into the form

$$(\psi) \qquad (\neg)atleast\big((p \wedge D_1) \vee (\neg p \wedge D_2), (p \wedge D_3) \vee (\neg p \wedge D_4)\big),$$

where D_1, D_2, D_3, D_4 are p-free, and both $\vdash (p \wedge D_1) \wedge (p \wedge D_3) \rightarrow \bot$ and $\vdash (\neg p \wedge D_2) \wedge (\neg p \wedge D_4) \rightarrow \bot$. Now, if ψ has the form $atleast(\chi_1, \chi_2)$ define

$$(\psi') \qquad \begin{aligned} atleast\big(&(p \wedge D_1 \wedge \neg D_3) \vee (D_2 \wedge \neg D_4), \\ &(\neg p \wedge \neg D_2 \wedge D_4) \vee (\neg D_1 \wedge \neg D_2 \wedge D_3 \wedge D_4)\big). \end{aligned}$$

Otherwise, if ψ has the form $\neg atleast(\chi_1, \chi_2)$ define

$$(\psi') \qquad \begin{aligned} \neg atleast\big(&(\neg p \wedge D_2 \wedge \neg D_4) \vee (D_1 \wedge D_2 \wedge \neg D_3 \wedge \neg D_4), \\ &(p \wedge \neg D_1 \wedge D_3) \vee (\neg D_2 \wedge D_3)\big). \end{aligned}$$

Let Ψ' be the result of substituting ψ' for ψ in Ψ (for all ψ). Then all occurrences of p in Ψ are positive. Our claim is that for any $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \Psi'$. To prove this, we use χ_l to denote χ_1 and χ_r to denote χ_2, for a formula $\chi \equiv atleast(\chi_1, \chi_2)$. One direction of the claim is easy. Suppose $\mathcal{M} \models \varphi$, say $\mathcal{M} \models \psi$, for some disjunct ψ in Ψ. Assume also that ψ has the form $atleast(\chi_1, \chi_2)$. Then, since $\vdash \psi_l \rightarrow \psi'_l$ and $\vdash \psi'_r \rightarrow \psi_r$ we immediately have $\mathcal{M} \models \psi'$. To prove the opposite direction we have to do some more work. Assume $\langle W, V' \rangle \models \psi'$, for some disjunct ψ' in Ψ, and assume also that ψ' has the form $atleast(\chi_1, \chi_2)$. Given a valuation V on W we are interested in the number of worlds verifying formulas of the form $(\neg)D_1 \wedge (\neg)D_2 \wedge (\neg)D_3 \wedge (\neg)D_4$. Given such a formula θ_i $(1 \leq i \leq 16)$ the number of worlds verifying $p \wedge \theta_i$ is denoted by x_i, and the number of worlds verifying $\neg p \wedge \theta_i$ is denoted by y_i.

$\#p$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
D_1	0	0	0	0	0	0	0	0
D_2	0	0	0	0	1	1	1	1
D_3	0	0	1	1	0	0	1	1
D_4	0	1	0	1	0	1	0	1
$\#\neg p$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$\cdots$

x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	$\#p$
1	1	1	1	1	1	1	1	D_1
0	0	0	0	1	1	1	1	D_2
0	0	1	1	0	0	1	1	D_3
0	1	0	1	0	1	0	1	D_4
y_9	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}	$\#\neg p$

As is easily computed, ψ is true under some valuation V on W iff the following inequality is satisfied in $\langle W, V \rangle$:

$$x_9 + x_{10} + x_{13} + x_{14} + y_5 + y_7 + y_{13} + y_{15} \geq$$
$$x_3 + x_4 + x_7 + x_8 + y_2 + y_4 + y_{10} + y_{12}. \tag{7}$$

Consider the following inequality:

$$x_9 + x_{10} + x_{14} + (x_5 + y_5) + (x_7 + y_7) + (x_{13} + y_{13}) + (x_{15} + y_{15}) \geq$$
$$y_2 + (x_4 + y_4) + y_{10} + y_{12}. \tag{8}$$

We leave it to the reader to check that for any V on W, $\langle W, V \rangle \models \psi'$ iff $\langle W, V \rangle$ satisfies (8).

Let V be a valuation for ψ' on W such that $V(p) \subseteq V'(p)$ is minimal, while $V(q) = V'(q)$ for $q \not\equiv p$. Then, in $\langle W, V \rangle$, we have that

$$x_3 = x_5 = x_7 = x_8 = x_{13} = x_{15} = 0. \tag{9}$$

This is trivial for x_5, x_7, x_{13}, x_{15}. Consider for example x_5; if $x_5 > 0$, transfer all elements in $V(p \wedge \neg D_1 \wedge D_2 \wedge \neg D_3 \wedge \neg D_4)$ to $V(\neg p \wedge \neg D_1 \wedge D_2 \wedge \neg D_3 \wedge \neg D_4)$ to obtain V''. Then $x_5 + y_5$ in $\langle W, V'' \rangle$ equals $x_5 + y_5$ in $\langle W, V \rangle$, while the other quantities x_i, y_i occurring in (8) remain unchanged, i.e., $\langle W, V'' \rangle$ is also a model for ψ', while $V''(p) \subsetneq V(p)$ – a contradiction. Next, x_3, x_8 also equal 0 since neither x_3, y_3 nor x_8, y_8 occur in (8). So any elements in the slot corresponding to x_3 (x_8) may be transferred to the slot corresponding to y_3 (y_8) without changing the truth-value of (8).

Applying (9) to (8) we see that in $\langle W, V \rangle$ the following inequality must be satisfied:

$$x_9 + x_{10} + x_{13} + x_{14} + (0 + y_5) + (0 + y_7) + (0 + y_{13}) + (0 + y_{15}) \geq$$
$$x_3 + x_4 + x_7 + x_8 + y_2 + + y_4 + y_{10} + y_{12}.$$

Hence, $\langle W, V \rangle \models \psi$. By the monotonicity of φ this implies $\langle W, V' \rangle \models \varphi$ – as required. $\dashv$

Since, in 5.10, we have restricted ourselves to QM-formulas that are equivalent to disjunctions of formulas of the form $(\neg)atleast(\chi_1, \chi_2)$, with χ_1, χ_2 purely propositional, we have only proved a 'partial' Lyndon Theorem there; to prove a Lyndon Theorem for the full language we would have to consider disjunctions of conjunctions of formulas of the above form (this is because of 5.6). We believe that there is indeed a Lyndon Theorem for the full language of QM. However, we doubt whether the method we used in 5.10 to prove a partial Lyndon Theorem would be the most efficient way to obtain the more general result.

5.4. *Other Higher Order Quantifiers*

Just like the systems $QUANT$ and $QUANT_k$ did not determine the sets of operators $\{ M_n : n \in \mathbb{N} \}$ and $\{ M_0, M_k \}$, respectively, QM does not determine $atleast$; QM also axiomatizes the complete modal theory of the operator *there are at least as many R-successors satisfying X as there are satisfying Y*, where R is an equivalence relation. And by 5.2, QM also axiomatizes the modal theory of the probabilistic quantifier '*X is at least as likely as Y*', where the underlying probability measure is not based upon statistic bearings but interpreted 'subjectively' (see Gärdenfors 1975 for a brief explanation of how the latter is accounted for by our axioms $C5$ and $C6$).

When added to first order logic the quantifiers *most* and *more* yield languages that are not equivalent as far as their expressive power is concerned (Westerståhl 1989). However, on top of $S5$, the three quantifiers *atleast, more,* and *most* (considered as modal operators) all yield the same language in this respect. Given the abbreviations introduced at the start of Section 5.1, to establish this claim it suffices to show that *atleast* can be defined in terms of *more* (which can be done as follows: $atleast(\varphi, \psi) \leftrightarrow \neg more(\psi, \varphi)$), and that it can also be defined in terms of *most*. On finite models the latter is indeed possible; if $\langle W, V \rangle$ is such a model, then

$$atleast(\varphi, \psi) \leftrightarrow |V(\varphi)| \geq |V(\psi)|$$

$$\leftrightarrow \neg \big(|V(\psi)| > |V(\varphi)|\big)$$
$$\leftrightarrow \neg \big(|V(\psi \wedge \neg\varphi)| > |V(\varphi \wedge \neg\psi)|\big)$$
$$\leftrightarrow \neg most(\psi \oplus \varphi, \psi),$$

where $X \oplus Y$ is the symmetric difference of X and Y. This equivalence implies, of course, that the modal languages with *atleast* and *most*, respectively, are equally expressive on finite models (but, as one of the anonymous referees pointed out, these modal languages are not equally expressive on infinite models). Finally, from the above observations it follows that we can extract complete axiomatizations (for validity on finite models) for the modal operators *more* and *most* from the complete axiomatization we have given for *atleast*.

A natural extension of QM and its language arises when we consider *atleast* not in isolation, but together with one or more operators $atleast_n$ ($n > 0$), where $atleast_n(\varphi, \psi)$ is interpreted as 'there are at least n times as many φs as ψs'. Here, we want to elaborate a bit on a possible axiomatization QM_2 for the modal language with *atleast* and $atleast_2$. QM_2 should at least contain the system QM (for *atleast*), and also axioms corresponding to those in 2.6 to ensure that we have a decent normal form theorem. These normal forms are disjunctions of conjunctions of the form

$$\psi \wedge (\neg)atleast(\psi_1, \chi_1) \wedge \ldots \wedge (\neg)atleast(\psi_n, \chi_n) \wedge$$
$$(\neg)atleast_2(\psi_{n+1}, \chi_{n+1}) \wedge \ldots \wedge (\neg)atleast_2(\psi_{n+m}, \chi_{n+m}),$$

where ψ, ψ_i, χ_i are purely propositional. Such normal forms suggest a natural reduction of QM_2-provability to provability in QM. Replace each conjunct $atleast_2(\psi, \chi)$ by $\big(equal(p, \chi) \wedge L_0 \neg(p \wedge \chi) \wedge atleast(\psi, p \vee \chi)\big)$, where p is a proposition letter not occurring in ψ, χ. Similarly, formulas of the form $\neg atleast_2(\psi, \chi)$ should be replaced by

$$\neg\Big(atleast(\neg\chi, \chi) \wedge (L_0 \neg(p \wedge \chi) \wedge equal(p, \chi) \to atleast(\psi, p \vee \chi))\Big),$$

where p is a proposition letter not occurring in ψ, χ. To get this reduction to work we should have two additional derivation rules (either derived from the axioms, or explicitly added) that amount to

(R^+) if for all proposition letters p we have
$\vdash \big(equal(p, \chi) \wedge L_0 \neg(p \wedge \chi) \wedge atleast(\psi, p \vee \chi)\big) \to \delta$,
then $\vdash atleast_2(\psi, \chi) \to \delta$,

and

(R^-) if for all proposition letters p we have
$\vdash \delta \to \big(atleast(\neg\chi, \chi) \wedge (L_0(p \wedge \chi) \wedge equal(q, \chi) \to$
$atleast(\psi, q \vee \chi))\big)$,
then $\vdash \delta \to atleast_2(\psi, \chi)$.

All in all, assuming that QM_2 contains R^+ and R^- we get the following reduction of provability in QM_2 to provability in QM. Assume φ is consistent in QM_2; we may assume that φ is in NF. Thus, one of the disjuncts φ' in φ is consistent in QM_2. Using R^+ and R^- one can find a formula $\varphi'' \in Form_\geq$ such that φ'' is consistent in QM iff φ' is consistent in QM_2. Now apply 5.3 to find a model for φ''. It is easily verified that this model is also a model for the original formula φ.

6. FURTHER DIRECTIONS; CONCLUDING REMARKS

One might remark that many of our results do not seem to depend on our *modal* point of view. As a first reaction to this remark the authors of this paper would probably agree. But then, after some thought, we would say that we do not claim that modal logic is the answer to all questions in generalized quantifier theory. What we have attempted to do in this paper is to mix the modal and quantifier tradition, and explore some connections between the two, starting from the observation that at a basic level the two traditions share some essential features: they are both variable free formalisms whose model theory *is* Venn Diagrams. This cross-fertilization has brought a number of questions and techniques familiar from the theory of generalized quantifiers to modal logic; this gave rise to several non-trivial results. Conversely, we have been able to use well-understood facts and tools from modal logic to obtain some non-trivial results in generalized quantifier theory.

We think that two of the main features of the modal languages used in this paper are the following. First, in these modal languages complex, non-constructive standard proofs can be replaced by simple, effective manipulations of syntactic objects to obtain results like e.g., a Lyndon Theorem (cf. 4.2, 5.10). Secondly, our semantic (Boolean) intuitions about quantifiers translate more or less directly into syntactic intuitions about modal formulas; as a result both old and new results connecting semantic constraints and special syntactic forms can easily be obtained (cf. 4.1, 4.2, 4.4, 5.10).

Several specific open problems have already been stated in this paper. At this point we want to suggest some general issues that we think are worth further investigations.

First, there are a lot of higher order quantifiers whose modal (or sometimes even Boolean) theory is still pretty much *terra incognita*. Besides the ones mentioned in Section 5.4 these include probabilistic quantifiers like *almost all*, and cardinality quantifiers like *more than κ Xs are Ys* ($\kappa \geq \omega$).

With these and other quantifiers considered in earlier sections of this paper

the precise nature of the individuals constituting our universes of discourse is irrelevant. A natural example of a sentence outside the scope of this extensional point of view is *three boys eat four apples*. To give a modal analysis of the quantifier patterns involved here one may have to move back to the more traditional approach to modal logic where the domain is structured by some relation R. E.g., one way to handle the above sentence would be to add to $QUANT$ operators N_n interpreted as the original graded modalities, i.e., $\mathcal{M}, w \models N_n \varphi$ iff more than n R-successors of w satisfy φ. In such a calculus the above sentence may be represented as $M!_3(B \wedge N!_4 A)$ – this representation has all the readings of the original sentence.

Another reason why one may want to have structured universes of discourse arises when one gives the operators considered in this paper a temporal interpretation as quantifiers over temporal entities. In such an interpretation one could add operators to structure the temporal domain to obtain one's favorite ordering. This would allow one to express such statements as 'it will be the case at least twice that there have been exactly three occasions at which φ held'.

Finally, in Atzeni *et al.* (1988) a complete, but very restricted system for talking about set containment is studied. This system deals with statements of the form

$$(\neg)Q_1 X_1 Y_1, \ldots, (\neg)Q_n X_n Y_n / (\neg)Q_{n+1} X_{n+1} Y_{n+1},$$

where $Q_i \in \{\, all, some \,\}$ and the X_is and Y_is have no structure except maybe a negation sign. Thus, given that we also have a syllogistic, Boolean and modal analysis of *all* and *some*, there is a whole hierarchy of systems for dealing with these quantifiers. We think it may be well worth the effort to study this hierarchy more systematically, and to set up and study similar hierarchies for other pairs of dual quantifiers.

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