Internat. J. Math. & Math. Sci.
Vol. 9 No. 2 (1986) 273-276

GENERALIZED RAMSEY NUMBERS FOR PATHS IN 2-CHROMATIC GRAPHS

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(Received April 26, 1984 and in revised form January 6, 1985)

ABSTRACT. Chung and Liu have defined the d-chromatic Ramsey number as follows. Let $1\le d\le c$ and let $t = \binom{c}{d}$. Let $1,2,\ldots,t$ be the ordered subsets of d colors chosen from c distinct colors. Let $G_1,G_2,\ldots,G_t$ be graphs. The d-chromatic Ramsey number denoted by $r_d^c(G_1,G_2,\ldots G_t)$ is defined as the least number p such that, if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i, the subgraph whose edges are colored in the ith subset of colors contains a G_i. In this paper it is shown that $r_2^3(P_i,P_j,P_k) = [(4k+2j+i-2)/6]$ where $i\le j\le k<r(P_i,P_j)$, r_2^3 stands for a generalized Ramsey number on a 2-colored graph and P_i is a path of order i.

KEY WORDS AND PHRASES. Ramsey Number, Generalized Ramsey Number, d-Chromatic Ramsey number, Colored Graph.

1980 AMS SUBJECT CLASSIFICATION CODES: 05C15, 05C10

1. INTRODUCTION AND NOTATION.

Chung and Liu [1] have defined the d-chromatic Ramsey number as follows. Let $1\le d\le c$ and let $t = \binom{c}{d}$. Let $1,2,\ldots,t$ be the ordered subsets of d colors chosen from c distinct colors. Let $G_1,G_2,\ldots,G_t$ be graphs. The d-chromatic Ramsey number denoted by $r_d^c(G_1,G_2,\ldots,G_t)$ is defined as the least number p such that, if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i, the subgraph whose edges are colored in the ith subset of colors contains a G_i. In this paper the value of $r_2^3(P_i,P_j,P_k)$ is found. Let $P_{i(r,s)}$ and $C_{i(r,s)}$ respectively denote a path or a cycle connecting i nodes whose edges are colored in color r or color s. Let $d_i(x)$ denote the degree of node x in color i. Let $|N_i(x)UN_i(y)|$ denote the number of vertices adjacent to x or y in color i, and $r(P_i,P_j)$ be the least number p such that when the edges of the full graph K_p are colored in colors 1 and 2 contains a $P_{i(1)}$ or $P_{j(2)}$. It is assumed throughout that $2\le i\le j\le k$. Let [i] and {i} respectively denote the largest integer less than or equal to i and the smallest integer greater than or equal to i. A colored graph G is a complete graph whose edges are colored in colors 1, 2, or 3. V(G) and E(G) denote the set of nodes and edges of G and E_i is the set of edges in color i.

2. MAIN RESULT.

First a series of Lemmas are presented which is followed by a bounding theorem for $r_2^3(P_i,P_j,P_k)$ and finally, an example shows that the bound is tight.

Lemma 1:

$$r_2^3(P_i,P_j,P_k) = r(P_i,P_j) \text{ when } i\le 3. \tag{2.1}$$

Proof: It is well known that

$$r(P_i,P_j) = j+[i/2] \text{ when } j\ge i\ge 2. \tag{2.2}$$

In [1] it is shown that

$$r_2^3(G_i,G_j,G_k)\le r(G_i,G_j). \text{ for } i\le j\le k, \tag{2.3}$$

and the equality holds if $k\ge r(G_i,G_j)$.

Lemma 2:

$$r_2^3(P_i,P_j,P_k)\le[(4k+2j+i-2)/6] \tag{2.4}$$

when $i=4$ and $k<r(P_i,P_j)$.

Proof: From (2.2) and (2.3), $k=j$ and $[(4k+2j+i-2)/6]=j+[(i-2)/6]$. Let $j=4$ and G be a colored K_4 with no $P_{4(1,2)}$ as a subgraph which implies that $\exists$ a $x\varepsilon V(G)\ni d_3(x)\ge 2$. If y and z are adjacent to x is color 3 then G has a $P_{4(1,3)}$ or $P_{4(2,3)}$. Assume that

$$r_2^3(P_4,P_{j-1},P_{j-1}) = j-1 \qquad \text{for all } j>4. \tag{2.5}$$

Let G be a colored K_j with no $P_{4(1,2)}$ as a subgraph. If $\exists$ a $x\varepsilon V(G)\ni d_3(x)\ge j-3$, then by (2.5) G-x contains a $P_{[j-1](1,3)}$ or $P_{[j-1](2,3)}$ and so G contains a $P_{j(1,3)}$ or $P_{j(2,3)}$. Hence, let $d_{(1,2)}(x) \ge 3$ $\forall$ $x\varepsilon N(G)$, which implies that G contains a $P_{4(1,2)}$, a contradiction.

Lemma 3: Let $k\ge 3$ and $j = k-2[(k+4)/6]$. Then

$$r_2^3(P_j,P_j,P_k)\le k-1.$$

Proof: Let s be the least non-negative integer $\ni$ $s \equiv k(\text{mod } 6)$.
It is easily shown that

$$r(P_j,P_j) = j+[j/2]-1 = K-1-[s/2]. \tag{2.6}$$

From (2.3) and (2.6) the lemma follows.

Lemma 4: Let $k\ge 3$ and $\ell = k+[(k-2)/6]$. Let G be a colored K_ℓ. If G contains a $C_{[k-1](1,2)}$, $C_{[k-1](1,3)}$, or $C_{[k-1](2,3)}$, then G contains a $P_{k(1,2)}$, $P_{k(1,3)}$, or $P_{k(2,3)}$, respectively.

Proof: Without loss of generality assume that G contains a $C_{[k-1](2,3)}$ but not $P_{k(2,3)}$ which implies that the $[(k+4)/6]$ vertices of G not in $C_{[k-1](2,3)}$ are adjacent in color 1 to C_{k-1}. By Lemma 3, the subgraph generated by nodes of C_{k-1} contains a $P_{j(1,2)}$ or $P_{j(1,3)}$ where $j = k-2[(k+4)/6]$. Without loss of generality assume that $P_{j(1,2)}$ is present and let x be one of its end vertices. Consider the remaining $2[(k+4)/6]-1$ vertices of C_{k-1}. Since there exists $[(k+4)/6]$ vertices not in C_{k-1}, but adjacent to every vertex of C_{k-1} in color 1, there exists a path P with $2[(k+4)/6]$ vertices in color 1, vertex disjoint from $P_{j(1,2)}$ referred above.

This path P has an end vertex adjacent to x in color 1 and hence G contains $P_{k(1,2)}$ as a subgraph.

Theorem 1:

$$r_2^3(P_i,P_j,P_k) \leq [(4k+2j+i-2)/6] \tag{2.7}$$

when $k<j+[i/2]-1 = r(P_i,P_j)$.

Proof: If $i\leq 4$, the theorem follows from Lemma 2. The rest of the proof is divided into three main cases. Assume that the theorem holds when $i'<i, j'<j$ or $k'<k$ where $i\geq 5$. Define $\ell = [(4k+2j+i-2)/6]$. Let G be a colored K_ℓ.

Case 1: Let $i = j = k$. Without loss of generality let $x_1 \varepsilon V(G)$ be ∋

$$n = d_1(x_1) \geq d_i(x) \tag{2.8}$$

For $i = 2,3$ and $x\varepsilon V(G)$ and $2 \leq n$.

Consider $G-x_1$. By the induction hypothesis,

$$r_2^3(P_{i-2},P_{i-2},P_i) \leq i-2+[(i+4)/6]$$

which implies that $G-x_1$ contains a $P_{[i-2](1,2)}$, $P_{[i-2](1,3)}$ or $P_{i(2,3)}$.

Without loss of generality assume that $G-x_1$ has $P_{[i-2](1,2)}$ and denote this path by $P = (y,\dots,z)$.

Case 1.1: Let $(x_1,y)\varepsilon E_1$ and $(x_1,z)\varepsilon E_3$, since otherwise the proof follows from Lemma 4. If $(x_1,u)\varepsilon E-E(P)$ and $(x_1,u)\varepsilon E_1$ then G has a $P_{i(1,2)}$. Thus X_1 is adjacent to n vertices of V(P) in color 1. Let $v \neq y$ be ∋ $(x_1,v)\varepsilon E_1$ and $v\varepsilon V(P)$. Let u be adjacent to v in P on the segment from v to y. Let $(y,u)\notin E_3$. Then the existence of cycle $(x_1,y,\dots,u,z,\dots,v,x_1)$, by Lemma 4 implies the existence of $P_{i(1,2)}$ completing the proof.

Suppose we let $(z,u)\varepsilon E_3$. Let $f\varepsilon$ V(G)-V(P). If $(z,f)\varepsilon E_1 \cup E_2$, then G has a $P_{i(1,2)}$ and hence let $(z,f)\varepsilon E_3$ for all f. Since $V(G)-V(P) = [(i+4)/6]+1$, $d_3(z)\geq[(i+4)/6]+1+n-1$. Since $d_3(z)\leq n$, $[(i+4)/6] = 0$ contradicting $i\geq 5$.

Case 1.2: Let (x_1,y) and (x_1,z) be in E_3. Let x_1 be adjacent to n_1 vertices of V(P) and n_2 vertices of V(G)-V(P) in color 1 where $n_1,n_2\geq 0$. Let $v\varepsilon V(P)$ be such that $(x_1,v)\varepsilon E_1$. Let u be a vertex adjacent to v on the segment $(y,\dots,v)$. By an argument similar to that used in Case 1.1, it can be shown that if $(z,u)\notin E_3$ the proof follows from Lemma 3. For the other case, let z be adjacent to at least n_1 vertices of V(P) in color 3. For $w\varepsilon V(G)-V(P)$ if $(x,w)\varepsilon E_1$ and $(z,w)\varepsilon E_1 \cup E_2$ the theorem follows. So z is adjacent to at least n_2 vertices of V(G)-V(P) in color 3. So $d_3(z)\geq n_1+n_2+1$, contradicting (2.8).

Case 2: Let $i = j < k$. If x_1 is ∋ $d_1(x_1) \geq d_i(x)$ for $i = 1,2,3$ and $x\varepsilon V(G)$ then $r_2^3(P_{i-2},P_{i-2},P_k) \leq \ell-1$ and Case 1 applies. Hence, without loss of generality assume that $d_2(x_2) \geq d_i(x)$ for $i = 1,2,3$ and $x\varepsilon V(G)$. Consider $G-x_2$. By induction hypothesis $r_2^3(P_{i-2},P_j,P_{k-1}) \leq \ell-1$ and hence $G-x_2$ contains $P_{[i-2](1,2)}$, $P_{j(1,3)}$ or $P_{[k-1](2,3)}$. If $P_{[i-2](1,2)}$ is present the proof is similar to Case 1. Let $G-x_2$ contain $P_{[k-1](2,3)}$. If z is an end vertex of this path then by arguments similar to Case 1 a contradiction, $d_1(z)$ $d_2(x_2)$ is derived thus proving the theorem. The theorem again follows if $P_{j(1,3)}$ is a subgraph of $G-x_2$.

Case 3: Let $i < j \leq k < j+[i/2]-1$. Let $x \in V(G)$ be such that $d_1(x)+d_2(x) \leq d_1(y)+d_2(y)$ for $y \in V(G)$. By induction hypothesis $r_2^3(P_i,P_{j-1},P_{k-1}) \leq \ell-1$, G-x contains a $P_{i(1,2)}$, $P_{[j-1](1,3)}$ or $P_{[k-1](2,3)}$. The case is not obvious, if one of the latter two paths is present. If $d_1(x)+d_2(x) \leq [j/2]$, then $d_3(x) \geq \{k/2\}$ so that x is adjacent to more than half the vertices of the graph and hence of the path under consideration in color 3. Therefore, G contains a $P_{j(1,3)}$ or $P_{k(2,3)}$. If $d_1(x)+d_2(x) \geq \{j/2\}$ and if $<E_1 U E_2>$ is connected, it is a standard result that G contains a $P_{\ell(1,2)}$, $\ell \geq 2\{j/2\}$ and hence G has a $P_i(1,2)$. However, if $<E_1 U E_2>$ is disconnected, it contains at least two components, each of which is of order $\{j/2\}$ or greater and hence G contains a $P_{j(3)}$.

Theorem 2:

$$r_2^3(P_i,P_j,P_k) > [(4k+2j+i-2)/6]-1.$$

Proof: Let $G = k_{\ell-1}$, where $\ell = [(4k+2j+i-2)/6]$. Let X, Y, Z be pairwise disjoint subgraphs of G such that $|X| = \{(2k+j-i-1)/3\}$, $|Y| = [(2j-2K+i-2)/6]$ and $|Z| = [(k-j+i-2)/3]$. It can be verified that $V(G) = |X| + |Y| + |Z|$. Color the edges of G as follows. Color the edges of X using color 3, edges of Y using color 1, edges of Z using color 2, edges between X and Y using color 1, edges between X and Z using color 2, and edges between Y and Z using color 1. It can be shown that $|X| + |Z| = k-1$ which rules out the existence of $P_{k(2,3)}$. Similarly $2|Y| + |X| \leq j-1$ ruling out $P_{j(1,3)}$ and $2|Y|+2|Z|+1 \leq i-1$ ruling out $P_{i(1,2)}$.

Theorem 3:

$$r_2^3(P_i,P_j,P_k) = [(4k+2j+i-2)/6] \text{ when } k < r(P_i,P_j) = j + [\tfrac{i}{2}] - 1$$

and

$$r_2^3(P_i,P_j,P_k) = r(P_i,P_j) \text{ when } k \geq r(P_i,P_j) = j + [\tfrac{i}{2}] -1.$$

Proof: Follows from (2.3) and Theorems 1 and 2.

ACKNOWLEDGEMENT. This work is based on the doctoral dissertation of the first author (R. Meenakshi) done at the Memphis State University under the guidance of Professor R. J. Faudree and Professor R. H. Schelp and many thanks are due to them.

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