

# GENERALIZED RESOLUTION AND MINIMUM ABERRATION CRITERIA FOR PLACKETT-BURMAN AND OTHER NONREGULAR FACTORIAL DESIGNS

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*Abstract:* Resolution has been the most widely used criterion for comparing *regular* fractional factorials since it was introduced in 1961 by Box and Hunter. In this paper, we examine how a generalized resolution criterion can be defined and used for assessing *nonregular* fractional factorials, notably Plackett-Burman designs. Our generalization is intended to capture projection properties, complementing that of Webb (1964) whose concept of resolution concerns the estimability of lower order effects under the assumption that higher order effects are negligible. Our generalized resolution provides a fruitful criterion for ranking different designs while Webb's resolution is mainly useful as a classification rule. An additional advantage of our approach is that the idea leads to a natural generalization of minimum aberration. Examples are given to illustrate the usefulness of the new criteria.

*Key words and phrases:* Confounding, estimability, fractional factorial, Hadamard matrix, orthogonality, projection property, word length pattern.

## 1. Introduction

We consider factorial designs with two levels, denoted by  $+$  and  $-$ . Our focus will be on orthogonal factorial designs, although most of the discussion in the paper generalizes to nonorthogonal designs. By orthogonality we mean that the number of  $+$ 's and  $-$ 's in each design column is the same and that for every two design columns the four level combinations  $(++)$ ,  $(+-)$ ,  $(-+)$  and  $(--)$  occur with the same frequency. Orthogonal factorial designs can be broadly classified into two categories: *regular* fractional factorials and *nonregular* fractional factorials. A regular fractional factorial is determined by its defining relation and has a simple aliasing structure in that any two effects are either orthogonal or fully aliased. In contrast, a nonregular fractional factorial exhibits some complex aliasing structure, meaning that there exist effects that are neither orthogonal nor fully aliased. Regular factorials can be constructed for every run size that is a power of 2. Examples of non-regular factorials include Plackett-Burman (1946) designs, which are constructed from Hadamard matrices. If a

Hadamard matrix exists, then its order  $n$  has to be a multiple of 4 (except when  $n = 1, 2$ ), that is,  $n = 4t$  for some integer  $t$ .

Traditionally, nonregular factorials were not advocated because of their complex aliasing structure. However, in the last decade, they have received increasing attention in the literature. Hamada and Wu (1992) showed that for data from designs with complex aliasing, it is possible to detect interaction effects. Lin and Draper (1992) studied the projection properties of some Plackett-Burman designs, and this line of research was further pursued and explored from a different angle by Wang and Wu (1995) under the term “hidden projection”. Cheng (1995) provided some general results on the projection properties of nonregular factorials, and these results cover as special cases some of the computer findings given in Lin and Draper (1992) and in Wang and Wu (1995).

A basic problem in this area remains unsolved, or at least has not been systematically attempted, despite the above important contributions. The problem is how to assess, compare and rank nonregular factorials in a systematic fashion. We propose *generalized* resolution and minimum aberration criteria for this purpose, paralleling the resolution and minimum aberration criteria traditionally used for assessing regular factorials. Our proposed criteria are natural generalizations in that when they are applied to regular factorials, they reduce to the traditional resolution and minimum aberration, respectively. Obviously, our current work is motivated by the ideas explicitly and implicitly exhibited in the aforementioned papers.

We introduce some notation to be used throughout the paper. A factorial design, regular or nonregular, is denoted by  $D$  and is regarded as a set of  $m$  columns  $D = \{d_1, \dots, d_m\}$  or as an  $n \times m$  matrix  $D = (d_{ij})$ , depending on our convenience. For  $1 \leq k \leq m$  and any  $k$ -subset  $s = \{d_{j_1}, \dots, d_{j_k}\}$  of  $D$ , define

$$J_k(s) = J_k(d_{j_1}, \dots, d_{j_k}) = \left| \sum_{i=1}^n d_{ij_1} \cdots d_{ij_k} \right|. \quad (1)$$

Clearly,  $J_1(s) = J_2(s) = 0$  for orthogonal designs. These  $J_k(s)$  values play an instrumental role in our development of generalized resolution and minimum aberration criteria.

The paper is organized as follows. In Section 2 we introduce generalized resolution, and its projection properties and statistical implications are discussed. By extending the notion of word length pattern, Section 3 introduces the confounding frequency vector of a design, based on which a generalized minimum aberration criterion is defined. Examples are given in Sections 2 and 3 to illustrate how this new set of criteria can be used to assess and compare two different designs.

## 2. Generalized Resolution for Non-regular Factorials

### 2.1. Motivation

The resolution of a regular fractional factorial can be interpreted in two equivalent ways. From the projection viewpoint, a regular factorial has resolution  $r$  if the  $2^{r-1}$  possible level combinations in the projection design onto any  $(r-1)$  factors occur with the same frequency. From the estimability viewpoint, a regular factorial has resolution  $r$  if when  $r$  is odd, the effects involving  $(r-1)/2$  or fewer factors are estimable under the assumption that those involving  $(r+1)/2$  or more factors are negligible, and when  $r$  is even, the effects involving  $(r-2)/2$  or fewer factors are estimable under the assumption that those involving  $(r+2)/2$  or more factors are negligible. For a detailed discussion on the concept of resolution for regular factorials, we refer to Box and Hunter (1961).

The interpretation of the resolution from the estimability viewpoint can be straightforwardly generalized to any factorial design, outside the family of regular fractional factorials. This definition of resolution was given in Webb (1964), and used to construct many useful designs in Rechtschaffner (1967), Srivastava and Chopra (1971), and others. We note that the resolution defined in this way is mainly meant to be a classification rule, and does not provide a fruitful criterion for ranking different designs. For example, when the experimental error is substantial and warrants efficiency consideration, a resolution V design may be less efficient for estimating the main effects than a resolution III design. Such situations can arise when the experimenter wants to estimate all the main effects and two factor interactions before the experiment is conducted, only to find out that no two factor interaction is important once the data are collected and analyzed.

We generalize the concept of resolution from the projection viewpoint so that it can be used to rank different nonregular factorials. A naive approach would be simply to say that a nonregular factorial has generalized resolution  $r$  if the  $2^{r-1}$  possible level combinations in the projection design onto any  $(r-1)$  factors occur with the same frequency. However, this does not yield any new concept and we are paraphrasing the strength of orthogonal arrays. (A two level orthogonal array of strength  $r'$  is a matrix with entries  $+$  and  $-$  such that the  $2^{r'}$  possible level combinations occur in any submatrix with  $r'$  columns with the same frequency.) Our definition of generalized resolution captures the projection properties in a more precise manner and also leads to a generalization of minimum aberration.

### 2.2. Assigning resolution to nonregular factorials

For a design  $D$ , *regular* or *nonregular*, let  $r$  be the smallest integer such that  $\max_{|s|=r} J_r(s) > 0$ , where  $J_k(s)$  is defined in (1) and the maximization is over all

the subsets of  $r$  distinct columns of  $D$ . We define its generalized resolution to be

$$R(D) = r + [1 - \max_{|s|=r} J_r(s)/n]. \quad (2)$$

Clearly,  $r \leq R(D) < r+1$ . For orthogonal designs, we have  $R(D) \geq 3$ . According to this criterion, a design with higher generalized resolution is preferred.

Now suppose that  $D$  is a regular factorial. Then  $J_k(s)$  for a  $k$ -subset  $s = \{d_{j_1}, \dots, d_{j_k}\}$  of  $D$  must equal either 0 or  $n$ , with 0 corresponding to orthogonality and  $n$  to full aliasing. For example,  $J_k(s) = 0$  implies that the main effect  $d_{j_1}$  is orthogonal to the  $(k-1)$  factor interaction  $d_{j_2}, \dots, d_{j_k}$ , while  $J_k(s) = n$  implies that these two effects are fully aliased. If  $J_k(s) = J_k(d_{j_1}, \dots, d_{j_k}) = n$ , we say that these  $k$  columns in  $s$  form a word of length  $k$ . For a regular factorial  $D$ , its generalized resolution is precisely the usual resolution  $r$ , that is  $R(D) = r$ , because in this case  $\max_{|s|=r} J_r(s) > 0$  is equivalent to  $\max_{|s|=r} J_r(s) = n$ .

Let  $p_k(s) = J_k(s)/n$  and  $q_k(s) = 1 - p_k(s)$ . Obviously, we have  $0 \leq p_k(s) \leq 1$  and  $0 \leq q_k(s) \leq 1$ . Note that  $q_k(s)$  provides a measure for the “degree of confounding” among the  $k$  columns in  $s$ , with smaller values of  $q_k(s)$  implying more serious confounding. Now the definition of  $R(D)$  in (2) can be rewritten as

$$R(D) = r + \delta, \quad (3)$$

where

$$\delta = \min_{|s|=r} q_r(s) = 1 - \max_{|s|=r} J_r(s)/n. \quad (4)$$

We show later that  $\delta$  itself has a nice geometric interpretation.

**Example 1.** Generate a 20-run Plackett-Burman design by using

$$(+ + - - + + + + - + - + - - - - + + -)$$

as the first row, shifting this row one place to the right 18 times, and then adding a row of minus signs. Consider three designs  $D_1$ ,  $D_2$  and  $D_3$  with their columns selected from the 20-run Plackett-Burman design, where  $D_1$  consists of columns 1–4,  $D_2$  columns 1–3 and 6, and  $D_3$  columns 1–3 and 16. In fact, these are the three nonequivalent projection designs onto  $k = 4$  dimensions discovered by Lin and Draper (1992). Simple calculation gives  $\max_{|s|=3} J_3(s) = 4$  for  $D_1$ , and thus  $D_1$  has generalized resolution 3.8. For  $D_2$  and  $D_3$ , we have  $R(D_2) = 3.4$  and  $R(D_3) = 3.8$ . Both  $D_1$  and  $D_3$  are better than  $D_2$ . Indeed, projection onto any three columns of  $D_1$  or  $D_3$  contains two copies of a complete  $2^3$  factorial plus one copy of a half replicate of  $2^3$  factorial, while for  $D_2$ , one of the four possible projections onto three columns contains one copy of a complete  $2^3$  factorial plus three copies of a half replicate. The designs  $D_1$  and  $D_3$  have the

same three-dimensional projection properties, and consequently have the same generalized resolution. However, the projection properties of the two designs onto 4 dimensions are different. Generalized resolution cannot distinguish the designs. In Section 3, we will see that the two designs can be discriminated using generalized minimum aberration.

A Hadamard matrix  $H$  of order  $n$  is an  $n \times n$  orthogonal matrix of  $\pm 1$ , that is  $H'H = nE$ , where  $E$  is the identity matrix. A Hadamard matrix can be normalized so that all the entries in the first column equal  $+1$ . Deleting the first column gives a saturated design with  $n$  runs and  $(n - 1)$  columns, henceforth called a Hadamard design. Plackett-Burman designs are special cases of Hadamard designs. From a Hadamard design, we can select  $m$  columns to obtain a design with  $n$  runs and  $m$  columns. The following preliminary results are easily established.

**Proposition 1.** *Let  $D$  be a design with its  $m$  columns selected from a Hadamard design of size  $n$ .*

- (i) *If  $m \geq n/2 + 1$ , then  $3 \leq R(D) < 4$ .*
- (ii) *If  $n$  is not a multiple of 8, then  $3 < R(D) < 4$  for any  $m \geq 3$ .*
- (iii) *If  $n$  is a multiple of 8 and  $m \leq n/2$ , a design  $D$  can be constructed such that  $R(D) \geq 4$ .*

**Proof.** The assertion in (i) is obvious. Quoting a result from Deng, Lin and Wang (1994) we have  $\max_{|s|=3} J_3(s) = 4 + 8j$  for a nonnegative integer  $j$  when  $n$  is not a multiple of 8. This shows that  $\max_{|s|=3} J_3(s)$  is always larger than 0 when  $n$  is not a multiple of 8, which proves that  $R(D) < 4$ . Using Lemma 2.2 of Cheng (1995), there does not exist an  $s$  with three columns satisfying  $J_3(s) = n$  because  $n$  is not a multiple of 8. This shows that  $R(D) > 3$ . To see why (iii) is true, note that if  $n$  is a multiple of 8, we can use the foldover method to construct a design with  $m = n/2$  columns, which has generalized resolution at least 4.

From Proposition 1, the generalized resolution of a design having run size  $n = 12, 20, 28, \dots$ , with its  $m(\geq 3)$  columns selected from a Hadamard design is strictly between 3 and 4. On the other hand, it is possible to achieve a larger generalized resolution for designs of run size  $n = 16, 24, 32, \dots$ , with  $m \leq n/2$ . The generalized resolution criterion is a useful tool to compare designs of the *same* size. Increasing the run size ( $n$ ), in general, will increase the design efficiency. Increasing the number of columns ( $m$ ), in general, will decrease the design resolution.

**Example 2.** Consider a 24-run design  $D$  with 12 columns, obtained by folding over the 12 run Plackett-Burman design, obtained by using

$$(+ + - + + + - - - + -)$$

as the first row, cyclically shifting this row one place to the right 10 times, and then adding a row of minus signs. That is,  $D = (H'_{12}, -H'_{12})'$  where  $H_{12}$  is the Hadamard matrix given by adding the all +’s column to the 12-run Plackett-Burman design, and  $H'_{12}$  is its transpose. Obviously,  $R(D) \geq 4$ . It can easily be shown or verified on a computer that  $\max_{|s|=4} J_4(s) = 8$ . Therefore the design has generalized resolution  $R(D) = 4 + (1 - 8/24) = 4.67$ .

**Note 1.** Generalized resolution is also useful for assessing nonorthogonal designs. For example, in the situation  $n = 2 \pmod{4}$ , no orthogonal design exists. Assume  $D$  is a nonorthogonal design with each of its columns having the same number of +’s and -’s. Then its generalized resolution is  $R(D) = 2 + [1 - \max_{|s|=2} J_2(s)/n]$ . In this case the generalized resolution is equivalent to the  $\max_{i < j} s_{ij}^2$  criterion well known in the context of supersaturated designs. For example, see Lin (1995).

### 2.3. Projection properties

A regular factorial design of resolution  $r$  is an orthogonal array of strength  $r - 1$ . When such a design is projected onto any  $r - 1$  columns, the  $2^{r-1}$  level combinations occur with the same frequency. For a nonregular factorial, the generalized resolution newly defined has a similar geometric interpretation. To be specific, suppose a nonregular factorial  $D$  has generalized resolution  $r \leq R(D) < r + 1$ . Then when it is projected onto any  $r - 1$  columns, the  $2^{r-1}$  level combinations also occur with the same frequency. In addition, the projection properties onto  $r$  dimensions are determined by the value of  $R(D)$ . The closer  $R(D)$  is to  $r + 1$ , the better the projection properties. These statements are supported by the following proposition.

**Proposition 2.** *Let  $D = (d_1, \dots, d_m)$  be an orthogonal array of strength  $r - 1$  and of size  $n$ .*

- (i) *A subset  $s$  of  $D$  with  $r$  columns is an orthogonal array of strength  $r$  if and only if  $J_r(s) = 0$ .*
- (ii) *More generally, a subset  $s$  of  $D$  with  $r$  columns contains  $[n - J_r(s)]/2^r$  copies of a complete  $2^r$  factorial plus  $J_r(s)/2^{r-1}$  copies of a half replicate of  $2^r$  factorial.*

The results in Proposition 2 are essentially in Cheng (1995). We note that part (ii) of Proposition 2 is more explicit than that given by Cheng (1995). In the discussion following his Theorem 2.1, Cheng (1995) stated that, in our notation,  $s$  contains copies of a complete  $2^r$  factorial plus copies of a half replicate and did not give the explicit numbers of copies in both cases. Part (ii) asserts that the  $n$  runs of design  $s$  can be split into two portions: one portion containing  $n - J_r(s)$  runs is given by copies of a complete  $2^r$  factorial, the other containing  $J_r(s)$  runs is given by copies of a half replicate of  $2^r$  factorial. Note that the proportion of

runs from the complete factorial portion is  $[n - J_r(s)]/n = 1 - J_r(s)/n$ . Here we record some interesting facts. We know that  $n$  is a multiple of  $2^{r-1}$  since  $D$  in Proposition 2 is an orthogonal array of strength  $r - 1$ . Part (ii) of Proposition 2 shows that  $J_r(s)$  is a multiple of  $2^{r-1}$  and  $n - J_r(s)$  is a multiple of  $2^r$ .

Now suppose that  $D$  is a fractional factorial with  $r \leq R(D) < r + 1$ , so  $\max_{|s|=k} J_k(s) = 0$  for  $k = 1, \dots, r - 1$ , and  $\max_{|s|=r} J_r(s) > 0$ . Now because  $\max_{|s|=k} J_k(s) = 0$  for  $k = 1, \dots, r - 1$ , we conclude that  $D$  is an orthogonal array of strength  $r - 1$  by applying Proposition 2(i) recursively. For a subset  $s$  of  $r$  columns, Proposition 2(ii) states that it contains  $[n - J_r(s)]/2^r$  copies of a complete  $2^r$  factorial plus  $J_r(s)/2^{r-1}$  copies of a half replicate. The proportion of runs from the complete factorial portion is given by  $1 - J_r(s)/n$ . We thus arrive at the conclusion that if  $R(D) = r + \delta$  as in (3), with  $\delta = 1 - \max_{|s|=r} J_r(s)/n$  as in (4), then the proportion of runs from the complete factorial portion is at least  $\delta$ . If  $\delta > 0$ , then projections onto any  $r$  factors contain at least one copy of a complete  $2^r$  factorial. Thus the *projectivity*, as defined in Box and Tyssedal (1996), of the design is  $r$ . We see that generalized resolution provides a more precise description of the projection properties than projectivity.

As an application of these results, let us look at the two designs  $D_1$  and  $D_2$  in Example 1. We find  $R(D_1) = 3.8$ , and therefore in the projection design onto any three factors,  $20 \times 80\% = 16$  out of 20 runs ( $\delta = 0.8$ ) are from the complete factorial portion. In this case, all the four  $J_3$  values are equal to 4. We have  $R(D_2) = 3.4$ , and thus in the projection design onto *some* three factors,  $20 \times 40\% = 8$  out of 20 runs are from the complete factorial portion. Since only one  $J_3(s)$  is 12 and the other  $J_3(s)$  values are 4, for the other projections onto three factors there are 16 runs from the complete factorial portion.

#### 2.4. Statistical justification of generalized resolution

We first note the following simple fact. If  $r \leq R(D) < r + 1$  then, as discussed in Section 2.3,  $D$  is an orthogonal array of strength  $r - 1$ . Therefore generalized resolution has the same implication in estimability of factorial effects as resolution does for regular factorials.

The purpose of this section is to provide some additional statistical justification for generalized resolution. To simplify the discussion, we consider the following scenario.

Suppose that we are mainly interested in estimating the main effects in an experiment but cannot afford to use a design of generalized resolution 4 or higher. Also suppose that from previous knowledge, we can safely assume that 3-factor or higher order interactions do not exist, but suspect that some two factor interactions (2fi's) may not be negligible. In this case, a good design should be able to satisfactorily answer one or more of the following questions.

- (i) Suppose we are not interested in estimating these 2fi's. How does their presence affect the estimation of the main effects?
- (ii) Does our design allow estimation of these 2fi's if they are of interest?
- (iii) If our design allows estimation of these 2fi's, does it have high efficiency?

In this paper, we focus on (i), and show that a design of maximum generalized resolution minimizes the contamination of these 2fi's on the estimation of the main effects, in the sense to be given below. We are currently exploring how generalized resolution leads to satisfactory answers to questions (ii) and (iii), and hope to report on this in the near future.

Back to (i). Consider a factorial design of generalized resolution  $3 \leq R(D) < 4$  and suppose its  $m$  columns are  $d_1, \dots, d_m$ . Suppose the true model is

$$y_i = \beta_0 + \sum_{j=1}^m \beta_j d_{ij} + \sum_{k<l}^m \beta_{kl} d_{ik} d_{il} + \epsilon_i,$$

while the fitted model is

$$y_i = \beta_0 + \sum_{j=1}^m \beta_j d_{ij} + \epsilon_i.$$

It is easily seen that the least squares estimates of the main effects  $\beta_1, \dots, \beta_m$  from the fitted model have expectation (taken under the true model)

$$E(\hat{\beta}_j) = \beta_j + n^{-1} \sum_{k<l} I_3(d_j, d_k, d_l) \beta_{kl}$$

for  $j = 1, \dots, m$ , where  $I_3(d_j, d_k, d_l) = \sum_{i=1}^n d_{ij} d_{ik} d_{il}$ . Clearly,  $J_3(d_j, d_k, d_l) = |I_3(d_j, d_k, d_l)|$ . One way to minimize the biases in estimating  $\beta_j$ 's due to the presence of  $\beta_{kl}$  is to minimize  $\max_{j<k<l} J_3(d_j, d_k, d_l)$ , which is equivalent to maximizing the generalized resolution as defined in (2). Therefore, a design of maximum generalized resolution minimizes the biases for estimating the main effects through minimizing  $\max_{j<k<l} J_3(d_j, d_k, d_l)$ , the maximum coefficient in the bias terms.

Finally, we note that our argument for minimizing biases is similar to that in Box and Draper (1959).

### 3. Generalized Minimum Aberration

#### 3.1. The idea

Obviously, the defining relation of a regular factorial  $D$  is the collection of subsets  $s$  of columns such that  $J_k(s) = n$ , for  $k = 3, \dots, m$ . If  $J_k(s) = n$ , the  $k$  columns in  $s$  form a word of length  $k$  in the defining relation. Let  $A_k(D)$  be the

number of words of length  $k$  in the defining relation. The word length pattern of design  $D$  is the vector  $W(D) = (A_3(D), \dots, A_m(D))$ .

Two regular factorials  $D_1$  and  $D_2$  of the same resolution can be distinguished using the minimum aberration criterion. This is done as follows. Suppose  $W(D_1) = (A_3(D_1), \dots, A_m(D_1))$  and  $W(D_2) = (A_3(D_2), \dots, A_m(D_2))$  are the word length patterns of  $D_1$  and  $D_2$ , respectively. If both designs are of the same resolution  $r$ , then both  $A_r(D_1)$  and  $A_r(D_2)$  take on positive values. If  $A_r(D_1) < A_r(D_2)$ ,  $D_1$  has a smaller number of words of length  $r$ , and hence is preferred. If  $A_r(D_1) = A_r(D_2)$ , we proceed to compare  $A_{r+1}(D_1)$  and  $A_{r+1}(D_2)$ . If  $A_{r+1}(D_1) < A_{r+1}(D_2)$ ,  $D_1$  is preferred. Otherwise, the process is continued until the two designs can be distinguished. For results on minimum aberration designs, we refer to Fries and Hunter (1980), Franklin (1984), Chen and Wu (1991), Chen (1992), Tang and Wu (1996), Chen and Hedayat (1996) and Cheng, Steinberg and Sun (1999).

The same idea can be applied to nonregular factorials. Suppose two nonregular factorials  $D_1$  and  $D_2$  have the same generalized resolution  $r \leq R(D_1) = R(D_2) < (r + 1)$ , which implies that the values of  $\max_{|s|=r} J_r(s)$  for the two designs are the same. Clearly, if in  $D_1$  the frequency of combinations of  $r$  distinct columns that attain  $\max_{|s|=r} J_r(s)$  is lower than that in  $D_2$ ,  $D_1$  is preferred. If the two frequencies are the same, we proceed to compare the second largest  $J_r(s)$  values of the two designs. If the second largest  $J_r(s)$  value of  $D_1$  is smaller,  $D_1$  is preferred. If they are equal, we compare the frequencies that give rise to this same second largest  $J_r(s)$  value. The process is continued until the two designs can be distinguished.

The idea can be formally developed using what we call *confounding frequency vectors*, which are natural generalizations of word length patterns.

### 3.2. Generalized aberration criterion

Before giving a definition of a confounding frequency vector, we want to know the possible values of  $J_k(s)$ . The following proposition provides an answer to the question.

**Proposition 3.** *For any  $k$  columns of an orthogonal factorial design, the value of  $J_k(s)$  must be a multiple of 4.*

**Proof.** The proposition is proved by induction. Obviously it holds for  $k = 1, 2$ . Now, suppose it is true for  $k$  columns. Consider  $J_{k+1}(d_{j_1}, \dots, d_{j_{(k+1)}})$  for  $(k + 1)$  columns  $d_{j_1}, \dots, d_{j_{(k+1)}}$ . Let  $d$  denote the column given by the componentwise product of  $d_{j_2}, \dots, d_{j_{(k+1)}}$ , that is,  $d = d_{j_2}, \dots, d_{j_{(k+1)}}$ . Because the proposition is true for  $k$ , we have  $\sum_{i=1}^n d_i = 4t_1$  for an integer  $t_1$ . Let  $n = 4t$ . Then there are  $2t$   $+$ 's and  $-$ 's in column  $d_{j_1}$ , and the numbers of  $+$ 's and  $-$ 's in column  $d$  are

$2t + 2t_1$  and  $2t - 2t_1$ , respectively. Now consider the  $n \times 2$  matrix  $(d_{j_1}, d)$ . Let  $u$  be the number of  $(+, +)$  pairs and note that the numbers of  $(+, -)$ ,  $(-, +)$ , and  $(-, -)$  pairs are  $2t - u$ ,  $(2t + 2t_1) - u$ , and  $(2t - 2t_1) - (2t - u)$ , respectively. This shows that

$$\sum_{i=1}^n d_{ij_1} d_i = u - (2t - u) - [(2t + 2t_1) - u] + [(2t - 2t_1) - (2t - u)] = 4(u - t - t_1).$$

Therefore,  $J_{k+1}(d_{j_1}, \dots, d_{j_{k+1}}) = |\sum_{i=1}^n d_{ij_1} d_i| = 4|u - t - t_1|$ . This completes the proof.

Let  $D$  be an orthogonal factorial design of size  $n$  and of  $m$  columns, with  $n = 4t$ . Let  $f_{kj}$  be the frequency of  $k$  column combinations that give  $J_k(s) = 4(t + 1 - j)$ , for  $j = 1, \dots, t, t + 1$ . Since  $\sum_{j=1}^{t+1} f_{kj} = \binom{m}{k}$ , it is sufficient to consider  $f_{kj}$  for  $j = 1, \dots, t$ . As  $f_{1j} = f_{2j} = 0$  for orthogonal designs, this reduces to consideration of  $f_{kj}$  for  $k \geq 3$ . The *confounding frequency vector* of  $D$  is defined to be the vector

$$F = [(f_{31}, \dots, f_{3t}); (f_{41}, \dots, f_{4t}); \dots; (f_{m1}, \dots, f_{mt})]$$

of length  $(m - 2)t$ . This vector provides essential information on how the effects are confounded, in the way the word length pattern reflects on a regular factorial design. The former is in fact a natural extension of the latter, as we can see that for a regular factorial design,  $f_{kj} = 0$  for  $j \geq 2$  and the reduced vector  $(f_{31}, f_{41}, \dots, f_{m1})$  is exactly the word length pattern of the design.

**Note 2.** A general definition of confounding frequency vectors can be given for any factorial design. Note that  $J_k(s)$  is always an integer satisfying  $0 \leq J_k(s) \leq n$ . For any factorial design  $D$ , not necessarily orthogonal, let  $f_{kj}$  be the frequency of  $k$  column combinations that give  $J_k(s) = (n + 1 - j)$  for  $j = 1, \dots, n$ . Then we define the confounding frequency vector of this design as the vector

$$F = [(f_{11}, \dots, f_{1n}); (f_{21}, \dots, f_{2n}); \dots; (f_{m1}, \dots, f_{mn})]$$

of length  $nm$ .

The idea given in Section 3.1 can now be formalized using confounding frequency vectors. Let  $f_l(D_1)$  and  $f_l(D_2)$  be the  $l$ th entries in the confounding frequency vectors of two designs  $D_1$  and  $D_2$ ,  $l = 1, \dots, (m - 2)t$ . Let  $l$  be the smallest integer such that  $f_l(D_1) \neq f_l(D_2)$ . If  $f_l(D_1) < f_l(D_2)$ , we say  $D_1$  has less generalized aberration than  $D_2$ . If no design has less generalized aberration than  $D_1$ , then  $D_1$  is said to have minimum generalized aberration. Clearly, this criterion reduces to the usual minimum aberration for regular factorial designs. Some examples are now in order.

**Example 3.** Consider the two 20-run designs  $D_1$  and  $D_3$  discussed in Example 1. Recall that they have the same generalized resolution  $R(D_1) = R(D_3) = 3.8$ . Their confounding frequency vectors are given by

$$F(D_1) = [(0, 0, 0, 0, 4)_3; (0, 0, 0, 0, 1)_4], \quad F(D_3) = [(0, 0, 0, 0, 4)_3; (0, 0, 1, 0, 0)_4],$$

where the subscripts 3, 4 are used to indicate the groups of frequencies, with group  $k$  representing the group of frequencies given by the subsets of  $k$  columns. So  $D_1$  has less generalized aberration than  $D_3$ . Among the three designs  $D_1$ ,  $D_2$  and  $D_3$  given in Example 1,  $D_1$  is the best according to our generalized aberration criterion. This conclusion is consistent with that of Wang and Wu (1995).

**Example 4.** Consider two designs  $D_1$  and  $D_2$  with their columns selected from the 12-run Plackett-Burman design as given in Example 2, where  $D_1$  consists of columns 1–4, and 10, and  $D_2$  consists of columns 1–5. According to Lin and Draper (1992), these are the only two nonequivalent projection designs onto five factors. The two designs have the same generalized resolution  $R(D_1) = R(D_2) = 3.67$ . The confounding frequency vectors of  $D_1$  and  $D_2$  are given by

$$F(D_1) = [(0, 0, 10)_3; (0, 0, 5)_4; (0, 1, 0)_5], \quad F(D_2) = [(0, 0, 10)_3; (0, 0, 5)_4; (0, 0, 0)_5].$$

Clearly,  $D_2$  has less generalized aberration than  $D_1$ , and hence is a “better” design, a conclusion also reached by Wang and Wu (1995) in their study of “hidden projection properties” of the two designs.

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