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Generalized resolution for 0–1 linear inequalities *

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We show how the resolution method of theorem proving can be extended to obtain a procedure for solving a fundamental problem of integer programming, that of finding all valid cuts of a set of linear inequalities in 0–1 variables. Resolution generalizes to two cutting plane operations that, when applied repeatedly, generate all strongest possible or "prime" cuts (analogous to prime implications in logic). Every valid cut is then dominated by at least one of the prime cuts. The algorithm is practical when restricted to classes of inequalities within which one can easily tell when one inequality dominates another. We specialize the algorithm to several such classes, including inequalities representing logical clauses, for which it reduces to classical resolution.

Keywords: Propositional logic; cutting planes.

1. Introduction

Resolution is a well-known inference or theorem-proving procedure that can be applied to propositional or predicate logic [5]. In propositional logic it not only determines, in finitely many steps, whether a set of logical clauses is satisfiable, but Quine [12,13] showed over 30 years ago that it can be used to generate all "prime" or "strongest possible" implications of the set (defined below). Any other implication of the set follows on inspection from at least one prime implication. Thus resolution reduces the problem of finding the implications of a set of clauses to that of finding the implications of a single clause, and the latter problem is trivial.

The notion of a prime implication has an obvious analog for 0–1 linear inequalities (i.e., linear inequalities with integer coefficients and variables that take the values 0 and 1). We say that one inequality *implies* or *dominates* another when all the 0–1 points satisfying the former satisfy the latter. We show below that a generalization of resolution can be used to obtain all "prime inequalities" of a given set of 0–1 inequalities. Furthermore, a resolvent can be

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interpreted as a certain type of cutting plane, and we generalize resolution by using a broader class of cutting planes.

In [8] we pursued this idea to generalize resolution to apply to an extended class of logical clauses that assert that at least β of the literals in the clause are true (in ordinary clauses, $\beta = 1$). Here we pursue it further to identify two particular types of cutting planes that, when generated repeatedly, yield all prime inequalities for a given set of 0–1 inequalities.

In this way we reduce the problem of finding the implications of a set of 0–1 inequalities, i.e. the problem of finding all "valid cuts", to that of finding the implications of a single inequality. But in contrast with the case of logical clauses, finding the implications of a single inequality is not a trivial task. Our algorithm is therefore practical only when applied to classes of inequalities for which it is easy to recognize when one single inequality dominates another. In this paper we specialize the algorithm to several such classes: set covering inequalities, for which the algorithm is a trivial operation of deleting absorbed inequalities; set covering inequalities with right hand sides larger than one, for which it generates a well-known type of cutting plane; set packing inequalities, for which it generates another well-known cut; logical clauses, for which it reduces to classical resolution; extended clauses, for which it reduces to the generalized resolution mentioned above; and certain clauses with coefficients in $\{0, \pm 1, \pm 2\}$.

Prime inequalities are in a sense the "strongest" valid inequalities of a set of 0–1 inequalities. Another class of inequalities that might be so described are the facet-defining inequalities of the convex hull of the 0–1 point satisfying the system. But the two concepts are quite different. We will see that a prime inequality may fail to be facet-defining, and a facet-defining inquality may not be prime. A prime inequality may even strictly dominate a facet-defining inequality, in the sense that some points violating the former satisfy the latter.

Since classical resolution has exponential complexity in the worst case [7], the same is true of our procedure.

Sections 2 and 3 are devoted to basic definitions concerning logic and 0–1 inequalities. Section 4 introduces the two types of cutting plane operations we use, and section 5 proves that they generate all prime inequalities. The remaining four sections apply the procedure to the special cases mentioned above.

2. Logical preliminaries

A *clause* in propositional logic is a disjunction of *literals*, which are atomic propositions or their negations. An example is $x_1 \vee \neg x_2$, in which $\vee$ means "or" and $\neg$ means "not". A *truth assignment* assigns "true" or "false" to every atomic proposition x_j. A set S of clauses *logically implies* a clause C if every truth assignment that makes all the clauses in S true makes C true. A clause C

absorbs a clause D when every literal of C occurs in D. Clearly C logically implies D if and only if C absorbs D. C is a *prime implication* of a set S of clauses if it is a strongest possible implication of S; i.e., S implies C but implies no other clause that absorbs C. S is logically equivalent to its set of prime implications (i.e., either set logically implies all the clauses in the other). S is unsatisfiable if and only if it implies the empty clause, which is false by convention. A clause containing a variable and its negation is *tautologous*.

Given two nontautologous clauses, if exactly one variable x_j appears negated in one clause and unnegated in the other, the clauses have as their *resolvent on* x_j the clause consisting of all literals in either *parent* clause except x_j and $\neg x_j$. For instance, the resolvent of $x_1 \vee \neg x_2$ and $x_2 \vee x_3$ is $x_1 \vee x_3$. Quine [12,13] showed that resolution generates all prime implications of a set of nontautologous classes. The problem of checking whether a given clause is a prime implication is NP-complete because the satisfiability problem (checking whether the empty clause is a prime implication) is NP-complete [4].

A formula in *disjunctive normal form*, such as $x_1 \cdot \neg x_2 \vee x_2 \cdot x_3$, is a disjunction of *terms*, each of which is a conjunction of literals. *Consensus* operates on such formulas in a manner dual to resolution. For instance, the consensual formula for $x_1 \cdot \neg x_2 \vee x_2 \cdot x_3$ is $x_1 \cdot \neg x_2 \vee x_2 \cdot x_3 \vee x_1 \cdot x_3$. A *prime implicant* C of a formula F is a term that is a weakest possible implicant; i.e., C implies F but implies no other term that implies F. The notion of prime implication, rather than prime implicant, is appropriate here, because clauses (and not terms) are naturally expressed as linear 0–1 inequalities. Quine's result is actually the dual of that cited above, since it deals with consensus rather than resolution.

3. Prime inequalities

We will restrict our attention to the class $\mathscr{I}_n$ of 0–1 inequalities $ax \geqslant a_0$ for which $a = (a_1, \ldots, a_n)$ and $a_0, a_1, \ldots, a_n$ are integers. It will sometimes be convenient to write $ax \geqslant a_0$ in the form $ax \geqslant \beta + n(a)$, where $n(a)$ is the sum of the negative components of a, and β is the *degree* of the inequality. For instance, we write $2x_1 - 3x_2 \geqslant -1$ as $2x_1 - 3x_2 \geqslant 2 - 3$, which has degree 2. We suppose without loss of generality that $\beta \geqslant 1$, since otherwise the inequality is, so to speak, tautologous. An inequality $ax \geqslant \beta + n(a)$ is *feasible* (satisfiable) when $\Sigma_j | a_j | \geqslant \beta$. We can make all the coefficients of an inequality nonnegative by *complementing* variables; that is, by replacing varaibles x_j having a negative coefficient with $(1 - \bar{x}_j)$. In the example we get $2x_1 + 3\bar{x}_2 \geqslant 2$. When this is done, the degree is identical to the right-hand side. It is useful to think of the degree as the "true" right hand side, which is "disguised" by the presence of negative coefficients.

It is well known that a nontautologous logical clause, such as $x_1 \vee \neg x_2$, can be written as a 0–1 linear inequality, in this case $x_1 + (1 - x_2) \geqslant 1$ or $x_1 - x_2 \geqslant 1 - 1$. We interpret $x_j = 1$ to mean x_j is true and $x_j = 0$ to mean x_j is false. Note that these *clausal* inequalities (inequalities representing logical clauses) have degree one.

We will state our theorems for $\geqslant$ inequalities, because the latter are related to logical clauses and permit somewhat more natural arguments. But all our results can be restated for $\leqslant$ inequalities by complementing every variable and multiplying every inequality by -1. Thus an inequality $ax \geqslant \beta + n(a)$ becomes $a\bar{x} \leqslant p(a) - \beta$, where $p(a)$ is the sum of the positive components of a, and β is again the degree.

Let us say that the *extension* of a 0–1 linear inequality $ax \geqslant a_0$ is the set of 0–1 vectors x that satisfy it. The extension of a set of 0–1 inequalities is the intersection of the extensions of the inequalities in it. A set is *feasible* if its extension is nonempty. A set (*strictly*) *dominates* an inequality if the extension of the set is a (proper) subset of the extension of the inequality. Two inequalities are *equivalent* if they are coextensive (either dominates the other). For clausal inequalities, feasibility is satisfiability, domination is logical implication, and equivalence is logical equivalence. In general it is not obvious whether two given inequalities are equivalent, or whether one dominates the other (see [2,1]). But there are two useful sufficient conditions for domination.

Let us say that one inequality $ax \geqslant a_0$ *absorbs* another when the latter is the sum of $ax \geqslant a_0$ and zero or more inequalities of the form $x_j \geqslant 0$ (provided $a_j \geqslant 0$), $-x_j \geqslant -1$ (provided $a_j \leqslant 0$), and $0 \geqslant -1$. For instance, $2x_1 - 3x_2 \geqslant 2 - 3$ absorbs $4x_1 - 3x_2 \geqslant 2 - 3$, as well as $4x_1 - 3x_2 \geqslant 1 - 3$ and $4x_1 - 4x_2 \geqslant 1 - 4$. Since the bounds $x_j \geqslant 0$ and $-x_j \geqslant -1$ are valid for 0–1 inequalities, it is clear that an inequality dominates any inequality it absorbs. For clausal inequalities, absorption is logical absorption.

Let us also say that one inequality $ax \geqslant a_0$ *reduces* to another if the latter (the *reduction*) is the sum of $ax \geqslant a_0$ and zero or more inequalities of the form $x_j \geqslant 0$ (provided $a_j < 0$) and $-x_j \geqslant -1$ (provided $a_j > 0$). For instance, $2x_1 - 3x_2 \geqslant 4 - 3$ reduces to $x_1 - 3x_2 \geqslant 3 - 3$, as well as to $x_1 - x_2 \geqslant 1 - 1$, and so on.

When defining prime inequality, it is convenient to specify a set T of inequalities within which it is a strongest possible implication. Thus a 0–1 inequality $I \in T$ is a *prime inequality* for a set S of 0–1 inequalities, with respect to T, if S dominates I but dominates no inequality in T that strictly dominates I. A prime inequality with respect to $\mathscr{I}_n$ is a prime inequality *simpliciter*. Prime clausal inequalities (i.e., prime inequalities with respect to the class of clausal inequalities) are prime implications in the logical sense.

A prime inequality is a "strongest possible" implication in the sense that its extension properly contains that of no other implication in T. But a prime inequality may be equivalent to a number of other (prime) inequalities. This contrasts with prime logical implications, which are in a sense "unique" because distinct clauses necessarily have distinct extensions. It follows that a prime inequality need not in general be the "best" or "tightest" representation of its extension. For instance, if $2x_1 + x_2 \geqslant 1$ is a prime inequality, then so is $x_1 + x_2 \geqslant 1$, which is equivalent to it. The latter might be regarded as a "tighter" representation because of its smaller coefficients. (See [1,2,6,11] for various ways to define and compute a "best" or "tightest" representation.)

A *cut* (or *valid cut*) for a set S of 0–1 linear inequalities is simply a linear inequality that S dominates. (The hyperplane bounding the halfspace defined by a cut is a *cutting plane*.) Thus prime inequalities are cuts of a particular sort. If B is the set of bounds $0 \leqslant x_j \leqslant 1$, a *rank one* cut for $S \cup B$ is the result of taking a nonnegative linear combination of the inequalities in $S \cup B$ and rounding up any nonintegers that result [3].

A particularly "strong" cut is a facet-defining inequality for the convex hull of S's extension. But there is no simple relation between facet-defining and prime inequalities. Obviously a prime inequality may fail to be facet-defining. But, curiously, a facet-defining inequality may fail to be prime, and it may even be strictly dominated by a prime inequality that is equivalent to no facet-defining inequality. For example let $S = \{x_1 + x_2 \geqslant 1,\ x_1 + x_3 \geqslant 1\}$, whose extension is $\{(1, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 0), (1, 1, 1)\}$. The inequality $x_1 + x_2 \geqslant 1$ defines a facet of the convex hull of this extension, but it is not prime because it is strictly dominated by $2x_1 + x_2 + x_3 \geqslant 2$, which S dominates. The latter inequality is satisfied by all points satisfying the former except (0, 1, 0) and is equivalent to no facet-defining inequality.

4. Resolution and diagonal sums

Two general operations on a set S of inequalities suffice, when applied repeatedly, to generate all prime inequalities of the set. One application of either yields a rank one cut with respect to $S \cup B$. One operation is simply resolution. Suppose for instance that each of the two inequalities below is dominated by an inequality in S.

$$x_1 - x_2 \qquad \geqslant 1 - 1 \tag{1}$$

$$x_2 + x_3 \geqslant 1. \tag{2}$$

Then we generate their resolvent,

$$x_1 + x_3 \geqslant 1. \tag{3}$$

Note that (3) is a rank one cut with respect to (1), (2), $x_1 \geqslant 0$, and $x_3 \geqslant 0$. Just take a linear combination in which each receives weight $1/2$ to obtain $x_1 + x_3 \geqslant 1/2$. Rounding up the $1/2$, we have (3). In general the resolvent of two clausal inequalities $ax \geqslant a_0$ and $bx \geqslant b_0$ satisfying $a_j b_j < 0$ for exactly one j is the result of adding the following inequalities, each multiplied by $1/2$, and rouding up the right-hand side: $ax \geqslant a_0$, $bx \geqslant b_0$, $x_j \geqslant 0$ for each j such that $a_j + b_j = 1$, and $-x_j \geqslant -1$ for each j such that $a_j + b_j = -1$. (See [9,10] for further connections between resolution and cutting planes.)

The second type of operation can be illustrated as follows. Suppose that each of the following inequalities is dominated by an inequality in S.

$$x_1 + 5x_2 + 3x_3 + x_4 \geqslant 4, \tag{4}$$

$$2x_1 + 4x_2 + 3x_3 + x_4 \geqslant 4, \tag{5}$$

$$2x_1 + 5x_2 + 2x_3 + x_4 \geqslant 4, \tag{6}$$

$$2x_1 + 5x_2 + 3x_3 \qquad \geqslant 4. \tag{7}$$

Note that each is a reduction of following inequality,

$$2x_1 + 5x_2 + 3x_3 + x_4 \geqslant 5 \tag{8}$$

obtained by reducing one coefficient at a time in a diagonal pattern. By assigning weight $2/10$ to (4), $5/10$ to (5), $3/10$ to (6) and $1/10$ to (7) we obtain the nonnegative linear combination

$$2x_1 + 5x_2 + 3x_3 + x_4 \geqslant 44/10.$$

Rounding up the right-hand side, we obtain (8), which we call the *diagonal sum* of (4)–(7). The diagonal sum is obviously a rank one cut and is hence dominated by the set (4)–(7) and therefore by S.

In general a feasible inequality $ax \geqslant \beta + n(a)$ in $\mathscr{I}_n$ is the diagonal sum of the inequalities $a^i x \geqslant \beta - 1 + n(a^i)$ for $i \in J \subset \mathbb{N} = \{1, \ldots, n\}$ when $a_j \neq 0$ for all $j \in J$, $a_j = 0$ for all $j \in \mathbb{N} \setminus J$, and

$$a_j^i = \begin{cases} a_j - 1 & \text{if } j = i \text{ and } a_j > 0, \\ a_j + 1 & \text{if } j = i \text{ and } a_j < 0, \\ a_j & \text{otherwise.} \end{cases} \tag{9}$$

To verify that $ax \geqslant \beta + n(a)$ is a rank one cut (when $n \geqslant 2$), assign each $a^i x \geqslant \beta - 1 + n(a^i)$ weight $|a_i|/(W-1)$, where $W = \Sigma_j |a_j|$. The weighted sum of the inequalities $a^i x \geqslant \beta - 1 + n(a^i)$ is $ax \geqslant (\beta - 1)W/(W-1) + n(a)$. Since feasibility implies $W \geqslant \beta$, we have $\beta - 1 < (\beta - 1)W/(W-1) \leqslant \beta$, so that we obtain the desired $ax \geqslant \beta + n(a)$ after rounding up the right-hand side.

We will use the following algorithm to generate prime inequalities for a feasible set S with respect to T.

ALGORITHM P

Step 0. Set $S' = S$. Remove inequalities from S', if necessary, to ensure that no inequality in S' dominates another.

Step 1. If possible, find clausal inequalities C and D that have a resolvent R that no inequality in S' dominates, such that C and D are each dominated by some inequality in S'. Remove from S' all inequalities that R dominates, and add R to S'.

Step 2. If possible, find inequalities $I_1, \ldots, I_m$ in T that have a diagonal sum I in T that no inequality in S' dominates, such that $I_1, \ldots, I_m$ are each dominated by some inequality in S'. Remove from S' all inequalities that I dominates, and add I to S'.

Step 3. If inequalities were added to S' in either step 1 or step 2, return to step 1. Otherwise stop.

Algorithm P is clearly finite, since there are finitely many nonequivalent inequalities in $\mathscr{I}_n$, and S' contains no pairs of equivalent inequalities, and an inequality is never added to S' once it has been removed.

A set S' of prime inequalities for S is *complete* with respect to T if every prime inequality for S with respect to T is equivalent to some inequality in S'. We will show that algorithm P generates a complete set of prime inequalities with respect to T, provided T is *monotone* in the following sense: T contains all clausal inequalities, and given any inequality $ax \geqslant \beta + n(a)$ in T, T contains all inequalities $a'x \geqslant \beta' + n(a')$ such that $|a'_j| \leqslant |a_j|$ for all j, and $0 \leqslant \beta' \leqslant \beta$. The set of clausal inequalities is obviously monotone, as is $\mathscr{I}_n$.

5. The main result

To prove our main result, namely that algorithm P generates a complete set of prime inequalities with respect to T, we begin with two lemmas. Let the *length* of an inequality be the sum of the absolute values of its coefficients. We say that an inequality is *longest* with respect to a property if it has the property and would lose it if one or more coefficients were increased in absolute value. As usual, S is a set of inequalities.

LEMMA 1

Let $ax \geqslant 1 + n(a)$ be a longest clausal inequality that is dominated by S but by no inequality in S, and suppose that $a_k = 0$ for some k. Then $ax \geqslant 1 + n(a)$ is the resolvent of two inequalities, each of which is dominated by an inequality in S.

Proof

The clausal inequalities $x_k + ax \geqslant 1 + n(a)$ and $-x_k + ax \geqslant 1 + n(a) - 1$ are dominated by S, because they are absorbed and therefore dominated by $ax \geqslant 1 + n(a)$. Since they are longer than $ax \geqslant 1 + n(a)$, each is dominated by some inequality in S. But their resolvent is $ax \geqslant 1 + n(a)$, and the lemma follows. □

The next lemma will form the initial step of an inductive argument.

LEMMA 2

Once algorithm P is applied to S to yield S' for monotone T, any clausal inequality dominated by S is dominated by some inequality in S'.

Proof

Suppose otherwise, and let $ax \geqslant 1 + n(a)$ be a longest clausal inequality in T dominated by S but by no inequality in S'. Since T is monotone and therefore contains all clausal inequalities, $ax \geqslant 1 + n(a)$ is a longest clausal inequality dominated by S but by no inequality in S'. Since variables can be complemented, we suppose without loss of generality that $a \geqslant 0$. We claim that $a_k = 0$ for some k. To see this, note that otherwise the only point x violating $ax \geqslant 1 + n(a)$ is the origin, which means that any clause violated by the origin dominates $ax \geqslant 1 + n(a)$. If the origin violated no clause in S, S would not dominate $ax \geqslant 1 + n(a)$. Therefore the origin violates some clause in S, and this clause dominates $ax \geqslant 1 + n(a)$, contrary to hypothesis. We conclude that some $a_k = 0$. Given this, lemma 1 says that S' contains a resolvent that dominates $ax \geqslant 1 + n(a)$, contrary to hypothesis. □

THEOREM 1

Once algorithm P has been applied to a feasible set S of inequalities to yield S', S' is a complete set of prime inequalities for S with respect to any monotone set T.

Proof

We will prove that any inequality $ax \geqslant \beta + n(a)$ in T that is dominated by S is dominated by an inequality in S'. The proof is by induction on the degree β. Let $\operatorname{sgn}(\alpha)$ be 1 when $\alpha > 0$, -1 when $\alpha < 0$, and 0 when $\alpha = 0$.

We first suppose $\beta = 1$. It is easy to see that any inequality $ax \geqslant 1 + n(a)$ of degree one is equivalent to the clausal inequality $a'x \geqslant 1 + n(a')$, where $a'_j = \operatorname{sgn}(a_j)$. But since T is monotone, we know from lemma 2 that $a'x \geqslant 1 + n(a')$, and therefore $ax \geqslant 1 + n(a)$, is dominated by an inequality in S'.

We now assume that the theorem is true for all inequalities in T of degree $\beta - 1$ and show that it is true for inequalities in T of degree β. Suppose otherwise. Let $ax \geqslant \beta + n(a)$ be a longest inequality of degree β in T that is dominated by S but by no inequality in S'. For all $i \in \{j \mid a_j \neq 0\} = J$, let a^i be defined by (9). Then $ax \geqslant \beta + n(a)$ is the diagonal sum of the inequalities $a^i x \geqslant (\beta - 1) + n(a^i)$ for $i \in J$. We can make the following statements about $a^i x \geqslant (\beta - 1) + n(a^i)$ for each $i \in J$: (a) it is a reduction of $ax \geqslant \beta + n(a)$ and is therefore dominated by S; (b) it belongs to T, since T is monotone; (c) since it has degree $\beta - 1$, (a), (b) and the induction hypothesis imply that it is dominated by some inequality in S'. But (c), together with step 2 of algorithm P, imply that $ax \geqslant \beta + n(a)$ is also dominated by an inequality in S', contrary to assumption. The theorem follows. □

6. Set covering inequalities

Algorithm P does not specify how to check whether an inequality is dominated by an inequality in S', since this is hard to do in general. But in certain special cases it is easy, and we can specify a simple procedure that checks for domination. The special case we first consider is that when S consists of *set covering* inequalities, in which each coefficient belongs to $\{0, 1\}$ and the right-hand side is equal to one. The following is obvious.

LEMMA 3

One set covering inequality dominates another if and only if the former absorbs the latter. Equivalent set covering inequalities are identical.

To apply theorem 1 we let T be the set of clausal inequalities. But since all coefficients of the inequalities in S are nonnegative and resolvents will never be generated, all prime inequalities with respect to T are set covering inequalities. Also no diagonal sums are performed, and only step 0 of algorithm P is operative. We conclude that the unique complete set of prime set covering inequalities for S is obtained by deleting from S all inequalities absorbed by shorter inequalities in S.

A more interesting problem is to obtain prime inequalities that are set covering inequalities except that they may have an integer right-hand side larger than one, which we may call *extended* set covering inequalities. By generating all such prime inequalities we can clearly find a minimum cardinality cover (i.e., we can minimize $\Sigma_j x_j$ subject to the constraints in S). The cardinality of a minimum cover is just the maximum degree of the prime inequalities generated.

To apply theorem 1 we let T contain all clausal and extended set covering inequalities. T is therefore monotone. Again since no resolutions are performed, all prime inequalities with respect to T are extended set covering inequalities. The following check for domination is easy to verify.

LEMMA 4

An extended set covering inequality $ax \geqslant \beta_a$ dominates another one $bx \geqslant \beta_b$ if and only if $\Sigma_j a_j(1 - b_j) \leqslant \beta_a - \beta_b$.

Now algorithm P reduces to the following.

ALGORITHM P_1 (SET COVERING INEQUALITIES)

Step 0. Let S be a set of set covering inequalities, and set $S' = S$. Remove from S' every inequality that is absorbed by a shorter one in S'.

Step 1. If possible, find a set $J \subset \mathbb{N} = \{1, \ldots, n\}$ of at least 2 indices and, for each $i \in J$, an inequality $a^i x \geqslant \beta - 1 + \gamma_i$ such that

$$\gamma_i \geqslant \sum_{j \in (\mathbb{N} \setminus J) \cup \{i\}} a^i_j.$$

Let the diagonal sum be $ax \geqslant \beta$, where $a_j = 1$ for $j \in J$ and $a_j = 0$ for $j \in \mathbb{N} \setminus J$. If no inequality in S' dominates the diagonal sum, then remove from S' all inequalities dominated by the diagonal sum, and add the diagonal sum to S'.

Step 2. If an inequality was added to S' in step 1, return to step 1. Otherwise stop.

For instance, the following extended set covering inequalities,

$$\begin{aligned} x_2 \phantom{{}+x_3} + x_4 &\geqslant 2, \\ x_1 + x_3 \phantom{{}+x_4} &\geqslant 1, \\ x_1 + x_2 + x_4 &\geqslant 2, \end{aligned} \tag{10}$$

respectively dominate the following inequalities,

$$\begin{aligned} x_2 + x_3 &\geqslant 1, \\ x_1 + x_3 &\geqslant 1, \\ x_1 + x_2 &\geqslant 1, \end{aligned}$$

which have the diagonal sum,

$$x_1 + x_2 + x_3 \geqslant 2. \tag{11}$$

Therefore if S' contains inequalities (10), algorithm P_1 generates (11). This type of cut is commonplace in cutting plane theory.

7. Set packing inequalities

Set packing inequalities have the form $ax \leqslant 1$, where each $a_j \in \{0, 1\}$. After complementing variables the inequality becomes $a\bar{x} \geqslant p(a) - 1$. Although set packing inequalities appear very similar to set covering inequalities, they are not, because their degree is $p(a) - 1$ rather than 1.

Absorption for set packing inequalities corresponds to reduction for extended set covering inequalities. If $ax \leqslant 1$ "absorbs" $bx \leqslant 1$ (i.e., $a \geqslant b$), then the former corresponds to an extended set covering inequality that reduces to one corresponding to the latter.

Again resolution does not apply. Fortuitously, diagonal summation is quite simple, since the diagonal sum of set packing inequalities is a set packing inequality. Thus we can compute prime inequalities for a set S of set packing inequalities without introducing "extended" set packing inequalities. Algorithm P simplifies to the following.

ALGORITHM P_2 (SET PACKING INEQUALITIES)

Step 0. Let S be a set of set packing inequalities, and set $S' = S$. Remove from S' every inequality that is absorbed by a longer one in S'.

Step 1. If possible, find a set $J \subset \mathbb{N} = \{1, \ldots, n\}$ of at least 2 indices and, for each $i \in J$, an inequality $a^i x \leqslant 1$ in S' such that $a^i_j = 1$ for all $j \in J \setminus \{i\}$. (We can require without loss of generality that $a^i_i = 0$.) Let the diagonal sum be $ax \leqslant 1$, where $a_j = 1$ for $j \in J$ and $a_j = 0$ for $j \in \mathbb{N} \setminus J$. If no inequality in S' dominates the diagonal sum, then remove from S' all inequalities dominated by the diagonal sum, and add the diagonal sum to S'.

Step 2. If an inequality was added to S' in step 1, return to step 1. Otherwise stop.

For instance, the following set packing inequalities,

$$\begin{aligned} x_2 + x_3 + x_4 &\leqslant 1, \\ x_1 \qquad x_3 \qquad &\leqslant 1, \\ x_1 + x_2 \qquad + x_4 &\leqslant 1, \end{aligned} \tag{12}$$

respectively absorb the following inequalities,

$$\begin{aligned} x_2 + x_3 &\leqslant 1, \\ x_1 \qquad + x_3 &\leqslant 1, \\ x_1 + x_2 \qquad &\leqslant 1, \end{aligned}$$

which have the diagonal sum,

$$x_1 + x_2 + x_3 \leqslant 1. \tag{13}$$

Therefore if S' contains inequalities (12), algorithm P_2 generates (13). This is a well-known cut for set packing problems.

The cardinality of the maximum set packing (maximum of $\Sigma_j x_j$ subject to S) is the minimum of $n - p(a) + 1$ over all prime set packing inequalities $ax \leqslant 1$.

8. Logical clauses

Since clausal inequalities have degree one, we can compute prime clausal inequalities (i.e., prime implications) without recourse to diagonal sums. Only resolution applies, so that algorithm P reduces to the classical resolution algorithm.

We can also specialize the algorithm to inequalities that are clausal except that the degree β may be larger than 1. We call these *extended clauses*, which assert that at least β of the literals listed are true. Domination between extended clauses is characterized by the following, which we prove in [8].

LEMMA 5

One extended clause dominates another if and only if the former reduces to an extended clause that absorbs the latter. Equivalent extended clauses are identical.

From lemma 5 we have,

LEMMA 6

An extended clause $ax \geqslant \beta_a + n(a)$ dominates another one $bx \geqslant \beta_b + n(b)$ if and only if $\Sigma_j |a_j|[1-(a_j b_j)^+] \leqslant \beta_a - \beta_b$, where $\alpha^+ = \alpha$ if $\alpha > 0$ and $\alpha^+ = 0$ otherwise.

Due to lemma 6, algorithm P becomes the specialized algorithm below. In [8] we call essentially the same procedure *generalized resolution*. We state the procedure somewhat differently here because of our use of lemma 6.

ALGORITHM P_3 (GENERALIZED RESOLUTION)

Step 0. Set $S' = S$. Remove inequalities from S', if necessary, to ensure that no inequality in S' dominates another.

Step 1. If possible, find $k \in \mathbb{N} = \{1, \dots, n\}$ and inequalities $ax \geqslant \beta + \gamma_a + n(a)$ and $bx \geqslant \beta + \gamma_b + n(b)$ in S' (with $\beta \geqslant 1$, $\gamma_a \geqslant 0$, $\gamma_b \geqslant 0$) such that $a_k b_k < 0$ and $\gamma_a + \gamma_b = \Sigma_j(-a_j b_j)^+ - 1$. Let the resolvent $cx \geqslant \beta + n(c)$ be given by $c_k = 0$ and by $c_j = \mathrm{sgn}(a_j + b_j)$ for all $j \neq k$ for which $a_j b_j \geqslant 0$. When $a_j b_j < 0$ (and $j \neq k$), set c_j to 1 or -1 in any fashion that satisfies

$$\sum_{\substack{j \\ a_j b_j < 0}} (-a_j c_j)^+ = \gamma_a \quad \text{and} \quad \sum_{\substack{j \\ a_j b_j < 0}} (-b_j c_j)^+ = \gamma_b.$$

If no inequality in S' dominates the resolvent, then remove from S' all inequalities dominated by the resolvent, and add the resolvent to S'.

Step 2. If possible, find a set $J = \{i_1, \dots, i_m\} \subset \mathbb{N}$ of indices and define a β-clause $cx \geqslant \beta + 1 + n(c)$ with $c_j = 0$ for all $j \in \mathbb{N} \backslash J$, such that S' contains for each $i \in J$ a β-clause $a^i x \geqslant \beta + \gamma_i + n(a^i)$ for which $\gamma_i \geqslant 0$ and $\Sigma_{j \in \mathbb{N}} |a^i_j| - \Sigma_{j \in J \backslash \{i\}} (a^i_j c_j)^+ \leqslant \gamma_i$. If no inequality in S' dominates the diagonal sum $cx \geqslant \beta + 1 + n(c)$, then remove from S' all inequalities dominated by the diagonal sum, and add the diagonal sum to S'.

Step 3. If an inequality was added to S' in step 1 or step 2, return to step 1. Otherwise stop.

Actually, algorithm P adds $cx \geqslant \beta + n(c)$ to S' in step 1 only when $\beta = 1$, since the "resolvent" $cx \geqslant \beta + n(c)$ can be a resolvent in the ordinary sense only when $\beta = 1$. But by permitting larger β in algorithm P_3 (as in [8]) we can reduce the number of steps required. To illustrate the case when $\beta = 2$, suppose S' contains the inequalities,

$$\begin{aligned} x_1 + x_2 \quad\quad - x_4 + x_5 - x_6 - x_7 &\geqslant 2 + 2 - 3, \\ -x_1 \quad\quad - x_3 - x_4 - x_5 + x_6 + x_7 &\geqslant 2 + 1 - 4. \end{aligned}$$

If we pick $k = 1$, they respectively dominate,

$$\begin{aligned} x_1 + x_2 \quad\quad - x_4 \quad\quad\quad\quad - x_7 &\geqslant 2 - 2, \\ -x_1 \quad\quad - x_3 - x_4 - x_5 + x_6 \quad\quad &\geqslant 2 - 4, \end{aligned}$$

which have the "resolvent",

$$x_2 - x_3 - x_4 - x_5 + x_6 - x_7 \geqslant 2 - 2. \tag{14}$$

So we add (14) to S'.

To illustrate step 2, suppose that S' contains the inequalities,

$$\begin{aligned} x_1 - x_2 + x_3 + x_4 - x_5 &\geqslant 1 + 3 - 2, \\ x_2 + x_3 - x_4 - x_5 &\geqslant 1 + 2 - 2, \\ x_4 - x_5 &\geqslant 1 + 1 - 1. \end{aligned}$$

If we choose $J = \{3, 4, 5\}$, these respectively dominate

$$\begin{aligned} x_4 - x_5 &\geqslant 1 - 1, \\ x_3 \quad\quad - x_5 &\geqslant 1 - 1, \\ x_3 + x_4 \quad\quad &\geqslant 1. \end{aligned}$$

Thus if we let the diagonal sum $cx \geqslant \beta + 1 + n(c)$ be,

$$x_3 + x_4 - x_5 \geqslant 1 + 1 - 1, \tag{15}$$

the conditions of step 2 are satisfied, and we add (15) to S'.

9. First and second degree inequalities with coefficients in $\{0, \pm 1, \pm 2\}$

We now enlarge the set T to the set of all first and second degree inequalities in $\mathscr{I}_n$ with coefficients in $\{0, \pm 1, \pm 2\}$. Let us call this set T_2.

LEMMA 7

An inequality A (which we write $ax \geqslant \beta_a + n(a)$ with $a \geqslant 0$) in T_2 dominates an inequality B (which we write $bx \geqslant \beta_b + n(b)$) in T_2 if and only if the following are true:

(i) If $\beta_a = \beta_b = 1$, $b_j > 0$ whenever $a_j > 0$.
(ii) If $\beta_a = 1$ and $\beta_b = 2$, $b_j = 2$ whenever $a_j > 0$.
(iii) If $\beta_a = 2$ and $\beta_b = 1$, then $b_j = 0$ for at most one j for which $a_j = 1$, and $b_j > 0$ for any other j for which $a_j > 0$.
(iv) If $\beta_a = \beta_b = 2$, then $b_j = 2$ whenever $a_j = 2$, and $b_j > 0$ whenever $a_j = 1$.

Proof

We will say $J \subset \{1, \dots, n\}$ is a *roof set* (inspired by the notion of *roof point* in [2]) of an inequality $ax \geqslant a_0$ with $a \geqslant 0$ if $\hat{x}$, given by $\hat{x}_j = 1$ if $j \in J$ and $\hat{x}_j = 0$ otherwise, satisfies $a\hat{x} \geqslant a_0$, but $x \leqslant \hat{x}$ and $x \neq \hat{x}$ imply that x violates $ax \geqslant a_0$. Let a set $J \subset \{1, \dots, n\}$ be a *satisfaction set* for an inequality if setting $x_j = 1$ for each $j \in J$ satisfies the inequality for any set of values assigned the remaining x_j's. It is not hard to show that A dominates B if and only if every roof set of A is a satisfaction set of B. We consider the four cases separately.

(i) Here the roof sets of A are the singletons $\{j\}$ with $a_j > 0$. These are satisfaction sets of B if and only if each corresponding b_j is 1 or 2.

(ii) The roof sets of A are the same as in (i). They are satisfaction sets of B if and only if each corresponding $b_j = 2$.

(iii) The roof sets of A are singletons $\{j\}$ with $a_j = 2$ and doubletons $\{j, k\}$ with $a_j = a_k = 1$. The former are satisfaction sets of B if and only if the corresponding b_j's are 1 or 2. The latter are satisfaction sets if and only if at most one of the corresponding b_j's and b_k's is 0, and the others are 1 or 2.

(iv) The roof sets of A are the same as in (iii). The singletons are satisfaction sets of B if and only if the corresponding b_j's are 2. The doubletons are satisfaction sets if and only if the corresponding b_j's and b_k's are 1 or 2. □

Algorithm P specializes as follows.

ALGORITHM P_4

Step 0. Set $S' = S$. Remove inequalities from S', if necessary, to ensure that no inequality in S' dominates another.

Step 1. If possible, find inequalities $ax \geqslant \beta_a + n(a)$, $bx \geqslant \beta_b + n(b)$ in S' and indices $k \in \mathbb{N} = \{1, \dots, n\}$ and s, $t \in \mathbb{N} \cup \{0\}$ such that (a) $a_k b_k < 0$; (b) $s > 0$ only if $\beta_a = 2$ and $|a_s| = 1$; (c) $t > 0$ only if $\beta_b = 2$ and $|b_t| = 1$; and (d) for all $j \in \mathbb{N}$, $a_j b_j < 0$ only if $j \in \{k, s, t\}$. Let the resolvent be $cx \geqslant 1 + n(c)$ to S', where $c_k = 0$, $c_s = \operatorname{sgn}(b_s)$ if $t \neq s > 0$, $c_t = \operatorname{sgn}(a_t)$ if $s \neq t > 0$, and for all $j \in \mathbb{N} \backslash \{k, s, t\}$, $c_j = \operatorname{sgn}(a_j + b_j)$. If no inequality in S' dominates the resolvent, then remove from S' all inequalities dominated by the resolvent, and add the resolvent to S'.

Step 2. If possible, find 2-clause $cx \geqslant 2 + n(c)$, a set $J \subset \mathbb{N}$ of indices, an inequality $a^i x \geqslant \beta_i + n(a^i)$ in S' for each $i \in J$, and an index $s_i \in \mathbb{N} \cup \{0\}$ for each $i \in J$, such that $s > 0$ only if (a) $|a^i_s| = 1$ and $\beta_i = 2$, (b) $a^i_j = 0$ for all

$j \in \mathbb{N}\backslash(J \cup \{s_i\})$, and (c) $a_j c_j \geqslant 0$ for all $j \in \mathbb{N}\backslash\{s_i\}$. If no inequality in S' dominates the diagonal sum $cx \geqslant 2 + n(c)$, then remove from S' all inequalities dominated by the diagonal sum, and add the diagonal sum to S'.

Step 3. If an inequality was added to S' in step 1 or step 2, return to step 1. Otherwise stop.

To illustrate step 1, suppose that S' contains the inequalities

$$\begin{aligned} 2x_1 - 2x_2 - x_3 \qquad &\geqslant 2-3,\\ -x_1 \qquad + 2x_3 - x_4 &\geqslant 2-2. \end{aligned}$$

They respectively dominate

$$\begin{aligned} x_1 - x_2 \qquad &\geqslant 1-1,\\ -x_1 \qquad + x_3 &\geqslant 1-1, \end{aligned}$$

which have the resolvent

$$-x_2 + x_3 \geqslant 1-1.$$

In step 2 we may suppose that S' contains the inequalities,

$$\begin{aligned} -2x_2 + 2x_3 - x_4 &\geqslant 2-3,\\ 2x_1 \qquad + x_3 \qquad &\geqslant 1,\\ x_1 - x_2 - x_3 \qquad &\geqslant 2-2. \end{aligned}$$

These respectively dominate,

$$\begin{aligned} -x_2 + x_3 &\geqslant 1-1,\\ x_1 \qquad + x_3 &\geqslant 1,\\ x_1 - x_2 \qquad &\geqslant 1-1, \end{aligned}$$

which have the diagonal sum,

$$x_1 - x_2 + x_3 \geqslant 2-1.$$

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