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Generalized resolvent estimates for the Stokes system in bounded and unbounded domains

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1. Introduction.

In this paper we investigate the *generalized Stokes resolvent problem* on some domain $\Omega\subseteq\boldsymbol{R}^n$, $n\geqq 2$,

$$\begin{aligned} \lambda u-\Delta u+\nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= g && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where the resolvent parameter λ is contained in the sector

$$S_\varepsilon=\{0\neq z\in\boldsymbol{C}\,;\,|\arg z|<\pi-\varepsilon\},\qquad 0<\varepsilon<\pi\,;$$

$f=(f_1, \cdots, f_n)\in L^q(\Omega)^n$, $1<q<\infty$, is the prescribed force and g is the given divergence of the problem. We are interested in L^q-estimates of the unknown velocity field $u=(u_1, \cdots, u_n)$ and the pressure p. In particular we have the following aims:

—Up to now most research concerns the resolvent problem (1.1) when $\operatorname{div} u=0$; the case $g\neq 0$ seems to be a rather new aspect although there are many important applications (e.g. for treating more general boundary value problems and for using cut-off procedures).

—We include new unbounded domains having noncompact boundary $\partial\Omega$ such as perturbed half spaces.

—We give a self-contained approach to the half space problem which rests on the multiplier technique.

—We assume that the boundary is of class $C^{1,1}$ only and include results for cones in $\boldsymbol{R}^n$, $n\geqq 3$, with opening angle close to π.

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—Solving (1.1) yields an important special class of solutions $u\in W^{2,q}(\Omega)^n \cap W_0^{1,q}(\Omega)^n$ of the divergence problem div $u=g$.

To describe our main results we formulate some assumptions on the domains under consideration and introduce some notations. In this paper, Ω is the $\boldsymbol{R}^n$, the half space

$$\boldsymbol{R}_+^n=\{x=(x', x_n)\in\boldsymbol{R}^n ;\ x_n>0\},$$

a bounded or exterior domain or a domain which is obtained from the half space by a perturbation within a finite region. In Section 3 we also consider the bended half space H_ω defined by $x_n>\omega(x_1, \cdots, x_{n-1})$ where $\|\partial_i\omega\|_\infty$, $i=1, \cdots, n-1$, are sufficiently small. The precise description for $\Omega\neq\boldsymbol{R}^n$ and $\Omega\neq H_\omega$ reads as follows.

ASSUMPTION 1.1. *Let $\Omega\subseteq\boldsymbol{R}^n$, $n\geqq 2$, be a domain with boundary $\partial\Omega\in C^{1,1}$ and suppose one of the following cases:*

(i) *Ω is bounded*

(ii) *Ω is an exterior domain, i.e., a domain having a compact nonempty complement*

(iii) *Ω is a perturbed half space, i.e., there exists some open ball B such that $\Omega\setminus B=\boldsymbol{R}_+^n\setminus B$.*

The last condition (iii) means that Ω behaves like $\boldsymbol{R}_+^n$ for sufficiently large $|x|$. The assumption $\partial\Omega\in C^{1,1}$ means that for each $x\in\partial\Omega$ there exists an open ball B_x centered at x and a function $\omega\in C^{1,1}(G)$ on some domain $G\subseteq\boldsymbol{R}^{n-1}$ such that after a rotation of the Cartesian coordinates, if necessary, the following holds: $y_n>\omega(y')$ for all $(y', y_n)\in\Omega\cap B_x$, $y_n<\omega(y')$ for all $(y', y_n)\in(\boldsymbol{R}^n\setminus\bar{\Omega})\cap B_x$ and $y_n=\omega(y')$ for all $(y', y_n)\in(\partial\Omega)\cap B_x$, where $y'=(y_1, \cdots, y_{n-1})$.

We will use the standard notations $L^q(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)}$ (or $\|\cdot\|_q$ if the underlying domain is known from the context), $L_{loc}^q(\Omega)$, $L_{loc}^q(\bar{\Omega})$ and $W^{1,q}(\Omega)$, $W_0^{1,q}(\Omega)$, $W^{2,q}(\Omega)$, etc. for Sobolev spaces of scalar functions. In particular, $u\in L_{loc}^q(\bar{\Omega})$ where $\bar{\Omega}$ is the closure of Ω means that $u\in L^q(\Omega\cap B)$ for all balls B with $\Omega\cap B\neq\emptyset$. For vector-valued functions in $L^q(\Omega)^n$, etc. we will use the same symbol $\|\cdot\|_q$ for the L^q-norm; more generally $\|(f_1, \cdots, f_m)\|_q=(\sum_{i=1}^m \|f_i\|_q^q)^{1/q}$ for $f_i\in L^q(\Omega)$ or $L^q(\Omega)^n$, etc.. For $1<q<\infty$ let q' denote the dual exponent, i.e., $1/q+1/q'=1$, and let $\langle\cdot, \cdot\rangle$ denote the $L^q-L^{q'}$-pairing of scalar, vector or matrix functions on Ω. If X is a Banach space and X^* its dual space, then we write $[x^*, x]$ for the evaluation of $x^*\in X^*$ in $x\in X$. However for the trace space $W^{1-1/q,q}(\partial\Omega)$ and its dual $W^{-1/q',q'}(\partial\Omega)$ we use $[\cdot, \cdot]_{\partial\Omega}$. Further let $\partial_i=\partial/\partial x_i$, $i=1, \cdots, n$, $\nabla=(\partial_1, \cdots, \partial_n)$, $\Delta=\partial_1^2+\cdots+\partial_n^2$ and $\nabla^2=(\partial_i\partial_j)_{i,j=1}^n$. Finally let

$$\mathcal{D}(\Delta_q)=W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)$$

denote the usual domain of definition of the Laplace operator $\Delta=\Delta_q$ in L^q-space with zero Dirichlet boundary condition.

In this paper we need the *homogeneous Sobolev space* $\hat{W}^{1,q}(\Omega)$ with norm containing only the first order expression $\|\nabla u\|_q=(\|\partial_1 u\|_q^q+\cdots+\|\partial_n u\|_q^q)^{1/q}$. We put

$$\hat{W}^{1,q}(\Omega)=\{u\in L_{loc}^q(\bar{\Omega}):\nabla u\in L^q(\Omega)^n\}$$

with norm $\|u\|_{\hat{W}^{1,q}(\Omega)}=\|\nabla u\|_q$, where we have to identify two elements differing by a constant. If Ω is bounded, $L_{loc}^q(\bar{\Omega})$ may be replaced by $L^q(\Omega)$. If Ω is unbounded, then

$$C_0^\infty(\bar{\Omega})=\{u|_\Omega: u\in C_0^\infty(\boldsymbol{R}^n)\},$$

the space of the restrictions to Ω of all functions $u\in C_0^\infty(\boldsymbol{R}^n)$, is a dense subspace of $\hat{W}^{1,q}(\Omega)$, see Lemma 5.1, i.e., we have

$$\overline{C_0^\infty(\Omega)}^{\|\nabla\cdot\|_q}=\hat{W}^{1,q}(\Omega).$$

Observe that if (u_i) is a Cauchy sequence in $C_0^\infty(\bar{\Omega})$ under $\|\nabla\cdot\|_q$, we know that there are constants c_i, $i\in\boldsymbol{N}$, such that (u_i+c_i) is converging in $L_{loc}^q(\bar{\Omega})$, i.e., in each $L^q(\Omega\cap B)$ where B is any ball.

If Ω is bounded, then we may fix a representative $u\in\hat{W}^{1,q}(\Omega)$ by $\int_\Omega u\,dx=0$. Therefore, setting

$$L_0^q(\Omega)=\left\{u\in L^q(\Omega):\int_\Omega u\,dx=0\right\},$$

we may identify

$$\hat{W}^{1,q}(\Omega)=W^{1,q}(\Omega)\cap L_0^q(\Omega)$$

in this case.

Let

$$\hat{W}^{-1,q}(\Omega)=[\hat{W}^{1,q'}(\Omega)]^*$$

be the dual space of $\hat{W}^{1,q'}(\Omega)$ endowed with the norm

$$\|g\|_{\hat{W}^{-1,q}(\Omega)}=\|g\|_{-1,q}=\sup_{0\neq v\in\hat{W}^{1,q'}(\Omega)}|[g,v]|/\|\nabla v\|_{q'}.$$

If Ω is unbounded, each functional $g\in\hat{W}^{-1,q}(\Omega)$ is determined by its restriction to the dense subspace $C_0^\infty(\bar{\Omega})\subseteq\hat{W}^{1,q'}(\Omega)$, thus we have

$$\|g\|_{-1,q}=\sup_{0\neq v\in C_0^\infty(\bar{\Omega})}|[g,v]|/\|\nabla v\|_{q'}$$

and the further restriction to $C_0^\infty(\Omega)$ yields a well defined distribution in the usual sense. Consider now any $g\in W^{1,q}(\Omega)$ for unbounded Ω. Then the functional

$$\langle g, \cdot \rangle : v \longmapsto \langle g, v \rangle = \int_\Omega gv dx , \qquad v \in C_0^\infty(\bar{\Omega}) ,$$

being identified with g, yields a well defined element in $\hat{W}^{-1,q}(\Omega)$ if and only if it is continuous under $\|\nabla v\|_{q'}$; we write $g \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$ in this case. Thus we set

$$W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$$
$$= \{g \in W^{1,q}(\Omega) : \langle g, \cdot \rangle \text{ continuous on } C_0^\infty(\bar{\Omega}) \text{ under } \|\nabla \cdot\|_{q'}\} .$$

Observe that the test functions $v \in C_0^\infty(\bar{\Omega})$ used here may be nonzero on $\partial\Omega$ and easy examples show that not every $g \in W^{1,q}(\Omega)$ yields a functional in $\hat{W}^{-1,q}(\Omega)$, see the Appendix. Thus the space $\hat{W}^{-1,q}(\Omega)$ should not be confused with the usual space $W^{-1,q}(\Omega) = [W_0^{1,q'}(\Omega)]^*$ which is too large for our purpose, we need the restriction $g \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$ for solving the system (1.1).

However, if Ω is bounded, each $g \in W^{1,q}(\Omega)$ with $\int_\Omega g dx = 0$ yields a functional $\langle g, \cdot \rangle : v \mapsto \langle g, v \rangle$ which is continuous on $\hat{W}^{1,q'}(\Omega)$. Thus we have $W^{1,q}(\Omega) \cap L_0^q(\Omega) \subseteq \hat{W}^{-1,q}(\Omega)$ in this case. See the Appendix for further properties on $\hat{W}^{1,q}(\Omega)$.

The spaces $\hat{W}^{1,q}(\Omega)$ and $\hat{W}^{-1,q}(\Omega)$ are natural for the problem (1.1); $\hat{W}^{1,q}(\Omega)$ is the space for the pressure p and $W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$ is the natural space for the divergence $g = \operatorname{div} u$ in the system (1.1) if Ω is unbounded.

In this paper $c, c_1, c_2, \cdots, C, C_1, C_2, \cdots$ are positive constants which may change from line to line.

The next theorem is our main result. It yields the unique solvability of the system (1.1) for $\lambda \in S_\varepsilon$ as well as a priori estimates. These are valid away from $\lambda = 0$, i.e., for $|\lambda| \geqq \delta > 0$ with a constant C depending on δ which is arbitrarily given. Then we give conditions for the validity of the a priori estimates even near $\lambda = 0$, i.e. the constant C does not depend on δ. Just these properties for small $|\lambda|$ have important consequences later on.

THEOREM 1.2. *Let $1 < q < \infty$, $0 < \varepsilon < \pi$, $\delta > 0$ and let $\Omega \subseteq \boldsymbol{R}^n$, $n \geqq 2$, be a domain satisfying the Assumption 1.1. Then for every $\lambda \in S_\varepsilon$, $f \in L^q(\Omega)^n$ and $g \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$ if Ω is unbounded or $g \in W^{1,q}(\Omega)$ with $\int_\Omega g dx = 0$ if Ω is bounded, there exists a unique solution $(u, p) \in \mathcal{D}(\Delta_q)^n \times \hat{W}^{1,q}(\Omega)$ of the generalized resolvent problem*

$$\lambda u - \Delta u + \nabla p = f , \qquad \operatorname{div} u = g .$$

The solution (u, p) satisfies the a priori estimates

$$\|\lambda u\|_q + \|\nabla^2 u\|_q + \|\nabla p\|_q \leqq C(\|f\|_q + \|\nabla g\|_q + \|\lambda g\|_{-1,q}) \tag{1.2}$$

and

$$\|\lambda u\|_q+\|-\Delta u+\nabla p\|_q \leqq C(\|f\|_q+\|\lambda g\|_{-1,q}), \tag{1.3}$$

where $C=C(\Omega, q, \varepsilon, \delta)>0$ *is a constant and* $|\lambda|\geqq\delta$.
The constant C *in* (1.2) *is independent of* δ *if one of the following conditions is satisfied*:

(i) Ω *is bounded. In this case the term* $\|\nabla^2 u\|_q$ *in* (1.2) *may be replaced by* $\|u\|_{W^{2,q}(\Omega)}$ *and* $\lambda=0$ *is included.*

(ii) Ω *is an exterior domain or a perturbed half space and* $1<q<n/2$, $n\geqq 3$.

Further the constant C *in* (1.3) *is independent of* δ *if one of the following conditions is satisfied*:

(iii) Ω *is bounded.*

(iv) Ω *is an exterior domain or a perturbed half space and* $n/(n-2)<q<\infty$, $n\geqq 3$.

In the whole space $\Omega=\boldsymbol{R}^n$ problem (1.1) has the form

$$\begin{aligned} \lambda u-\Delta u+\nabla p &= f \\ \operatorname{div} u &= g \end{aligned} \tag{1.4}$$

and $\mathscr{D}(\Delta_q)=W^{2,q}(\boldsymbol{R}^n)$. Next we formulate our results for the whole space and half space problem. In these cases there are no restrictions on the validity of the estimates (1.2) and (1.3) for small $|\lambda|$.

THEOREM 1.3 (*whole space and half space*). *Let* $n\geqq 2$, $1<q<\infty$, $0<\varepsilon<\pi$ *and let* $\Omega=\boldsymbol{R}^n$ *or* $\Omega=\boldsymbol{R}^n_+$. *Then for every* $f\in L^q(\Omega)^n$, $g\in W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ *and* $\lambda\in S_\varepsilon$ *there exists a unique solution* $(u, p)\in\mathscr{D}(\Delta_q)^n\times\hat{W}^{1,q}(\Omega)$ *of*

$$\lambda u-\Delta u+\nabla p = f\,, \qquad \operatorname{div} u = g\,.$$

(i) (*estimates*) *The inequalities* (1.2) *and* (1.3) *hold true for all* λ *in* S_ε *with some constant* $C=C(n, q, \varepsilon)>0$.

(ii) (*regularity*) *If for some* $s\in(1, \infty)$ *additionally* $f\in L^s(\Omega)^n$ *and* $g\in W^{1,s}(\Omega)\cap\hat{W}^{-1,s}(\Omega)$, *then* $u\in\mathscr{D}(\Delta_s)^n$ *and* $p\in\hat{W}^{1,s}(\Omega)$.

Taking the limit $\lambda\to 0$ we obtain existence, uniqueness and regularity results for the Stokes system (1.4) or (1.1) in $\Omega=\boldsymbol{R}^n$ or $\boldsymbol{R}^n_+$ when $\lambda=0$; see Corollary 2.6. By a perturbation argument similar results as in Theorem 1.3 are obtained for the *bended half space*

$$H_\omega = \{x=(x', x_n)\in\boldsymbol{R}^n\,;\ x_n>\omega(x')\}$$

with $x'=(x_1, \cdots, x_{n-1})$ where $\omega: \boldsymbol{R}^{n-1}\to\boldsymbol{R}$ is a function in $W^{2,1}_{\mathrm{loc}}(\boldsymbol{R}^{n-1})$ with $\|\partial_i\omega\|_\infty$, $i=1, \cdots, n-1$, sufficiently small; see Theorem 3.1. As a special case we consider the convex or concave cone

$$H_\alpha = \{x=(x', x_n)\in \boldsymbol{R}^n ;\ x_n>\alpha|x'|\},$$

where $\omega(x')=\alpha|x'|=\alpha(x_1^2+\cdots+x_{n-1}^2)^{1/2}$.

COROLLARY 1.4 (*cones*). *Let* $n\geqq 3$, $1<q<n-1$, $0<\varepsilon<\pi$, $\alpha\in\boldsymbol{R}$. *Then there exists a constant* $K=K(n, q, \varepsilon)>0$ *such that if* $|\alpha|\leqq K$, *then for every* $f\in L^q(H_\alpha)^n$, $g\in W^{1,q}(H_\alpha)\cap\hat{W}^{-1,q}(H_\alpha)$ *and* $\lambda\in S_\varepsilon$ *there is a unique solution* $(u, p)\in\mathcal{D}(\Delta_q)^n\times\hat{W}^{1,q}(H_\alpha)$ *of*

$$\lambda u-\Delta u+\nabla p = f\,,\qquad \operatorname{div} u = g\,.$$

Furthermore (u, p) *satisfies the inequality* (1.2) *for all* $\lambda\in S_\varepsilon$ *with some constant* $C=C(\alpha, n, q, \varepsilon)>0$.

There are many references on (1.1) if $g=\operatorname{div} u=0$ and if Ω is a bounded or exterior domain with boundary of class C^2 at least, see [**7**], [**11**], [**18**], [**19**], [**22**], [**25**], [**29**], [**30**]; for results on half spaces see [**21**], [**28**]. If Ω is bounded with $\partial\Omega\in C^{2,\mu}$, $0<\mu<1$, Giga [**18**] proved the estimate (1.2) and Theorem 1.2 (i) in the case $g=0$ by the theory of pseudo differential operators; if Ω is an exterior domain with $\partial\Omega\in C^{2,\mu}$ and $g=0$ he proved the estimate (1.2) only for $\lambda\in S_\varepsilon$, $|\lambda|\geqq\delta>0$. This restriction on $|\lambda|$ could be removed by Borchers-Sohr [**7**] if $n\geqq 3$ (and $g=0$). A completely different proof of the estimate (1.2) if $g=0$ and $\Omega\subseteqq\boldsymbol{R}^3$ is a bounded or exterior domain with $\partial\Omega\in C^2$ has been given by Solonnikov [**25**]; here $0<\varepsilon<\pi/2$ could not be prescribed arbitrarily; see v. Wahl [**29**] for $n>3$. For the generalized Stokes problem with $\lambda=0$ and $g\neq 0$ we refer to [**9**], [**13**], [**16**], [**17**], [**26**], but for the crucial case $\lambda\neq 0$, $g\neq 0$ we only know the references [**12**], [**19**]; in [**19**] some cases $g\neq 0$ are treated in exterior domains while (1.1) in $\boldsymbol{R}^n$, $\boldsymbol{R}_+^n$ and bended half spaces is investigated in [**12**]. Even for the case $g=0$ we have no reference on $\lambda\neq 0$ for cones or perturbed half spaces.

Let us consider some consequences of the theorems above. Theorem 1.2 yields a new approach to the divergence problem

$$\begin{aligned} \operatorname{div} u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.5}$$

For a bounded domain Ω with boundary in $C^{0,1}$ Bogovski [**3**], [**4**] constructed a bounded linear operator $R: L_0^q(\Omega)\to W_0^{1,q}(\Omega)^n$ such that $u=Rg$ is a solution of (1.5) satisfying $\|Rg\|_{W^{1,q}(\Omega)}\leqq c\|g\|_q$. Additionally R maps $W_0^{1,q}(\Omega)\cap L_0^q(\Omega)$ into $W_0^{2,q}(\Omega)^n$. See also [**8**], [**30**]. However, although important it has not been proved up to now whether or not this operator R is continuous from $W^{1,q}(\Omega)\cap L_0^q(\Omega)$ to $W^{2,q}(\Omega)^n\cap W_0^{1,q}(\Omega)^n$. Our operator R in Corollary 1.5 below has this property and the a priori estimates are not available up to now from the other approaches to (1.5); for partial results in bounded domains see [**2**]. Setting

$f=0$ and, e.g., $\lambda=1$ in (1.1), Theorems 1.2 and 1.3 yield the following result on (1.5).

COROLLARY 1.5 (*divergence problem*). *Let* $\Omega=\boldsymbol{R}^n$ *or let* $\Omega\subseteqq\boldsymbol{R}^n$, $n\geqq 2$, *be a domain satisfying the Assumption* 1.1. *Further let* $1<q<\infty$. *Then there exists a linear bounded operator* $R: W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)\to\mathcal{D}(\Delta_q)^n$ *if* Ω *is unbounded or* $R: W^{1,q}(\Omega)\cap L_0^q(\Omega)\to\mathcal{D}(\Delta_q)^n$ *if* Ω *is bounded such that* $u=Rg$ *is a solution of* (1.5) *for all* $g\in W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ *or* $g\in W^{1,q}(\Omega)\cap L_0^q(\Omega)$ *respectively*; $u=Rg$ *satisfies the estimates*

$$\|u\|_q\leqq C\|g\|_{-1,q} \quad \text{and} \quad \|u\|_{W^{2,q}(\Omega)}\leqq C(\|\nabla g\|_q+\|g\|_{-1,q}),$$

where $C=C(\Omega, q)>0$ *is a constant.*

Another application of our results concerns the analytic semigroup generated by the Stokes operator—a major tool when solving the instationary Stokes or Navier-Stokes equations. For the definition of the Stokes operator we recall the *Helmholtz projection*

$$P_q: L^q(\Omega)^n \longrightarrow L_\sigma^q(\Omega)$$

from $L^q(\Omega)^n$ onto the subspace $L_\sigma^q(\Omega)=\overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$ where $C_{0,\sigma}^\infty(\Omega)=\{u\in C_0^\infty(\Omega)^n;$ $\operatorname{div} u=0\}$; for the construction of P_q for all classes of domains considered in this paper see [15], [22], [24], [25] and Lemma 5.3. The Helmholtz projection P_q is a bounded linear operator with null space $\nabla\hat{W}^{1,q}(\Omega)=\{\nabla p: p\in\hat{W}^{1,q}(\Omega)\}$ yielding the decomposition

$$f=f_0+\nabla p \quad \text{with} \quad f_0=P_q f, \quad p\in\hat{W}^{1,q}(\Omega),$$

for $f\in L^q(\Omega)^n$. Then the *Stokes operator* A_q with domain of definition $\mathcal{D}(A_q)=\mathcal{D}(\Delta_q)^n\cap L_\sigma^q(\Omega)$ is defined by

$$A_q:\mathcal{D}(A_q)\longrightarrow L_\sigma^q(\Omega), \qquad A_q u=-P_q\Delta u.$$

Considering problem (1.1) when $g=0$ and applying the operator P_q we get that (1.1) is equivalent to

$$(\lambda+A_q)u=f\in L_\sigma^q(\Omega), \qquad u\in\mathcal{D}(A_q). \tag{1.6}$$

COROLLARY 1.6 (*Stokes operator*). *Let* $1<q<\infty$, $0<\varepsilon<\pi$ *and let* $\Omega=\boldsymbol{R}^n$ *or let* $\Omega\subseteqq\boldsymbol{R}^n$, $n\geqq 2$, *be a domain satisfying the Assumption* 1.1. *Then for each* $\lambda\in S_\varepsilon$ *the inverse operator* $(\lambda+A_q)^{-1}$ *exists as a bounded operator on* $L_\sigma^q(\Omega)$.

(i) *Excepting the case that* Ω *is an exterior domain or a perturbed half space in* $\boldsymbol{R}^2$ *the operator estimate*

$$\|(\lambda+A_q)^{-1}\|\leqq C/|\lambda| \qquad \text{for all } \lambda\in S_\varepsilon \tag{1.7}$$

holds true with a constant $C=C(\Omega, q, \varepsilon)>0$. *The same result is obtained* *for a cone* H_α *with* $|\alpha|$ *sufficiently small, if* $n\geqq 3$ *and* $1<q<n-1$.

(ii) *If* $\Omega\subseteq \boldsymbol{R}^2$ *is an exterior domain or a perturbed half space, then*

$$\|(\lambda+A_q)^{-1}\| \leqq C_\delta/|\lambda| \qquad \text{for all } \lambda\in S_\varepsilon,\ |\lambda|\geqq\delta>0\,, \tag{1.8}$$

with $C_\delta=C(\Omega,\ q,\ \varepsilon,\ \delta)>0$.

(iii) A_q *is a closed operator and its dual operator* A_q^* *equals* $A_{q'}$, *where* $1/q+1/q'=1$.

The assertion (1.7) implies that $-A_q$ is the infinitesimal generator of a uniformly bounded analytic semigroup $\{e^{-tA_q}\}_{t\geqq 0}$ and that $e^{-tA_q}u_0\to 0$ as $t\to\infty$ for all $u_0\in L_\sigma^q(\Omega)$. By (1.8) again $-A_q$ generates an analytic semigroup but we do not know whether $\{e^{-tA_q}\}_{t\geqq 0}$ is uniformly bounded for $t\geqq 0$. As mentioned earlier the resolvent estimates (1.7) and (1.8) are known by [7], [18] for bounded and exterior domains with a more regular boundary of class $C^{2,\mu}$, $0<\mu<1$.

REMARK 1.7. We note that the results for perturbed half spaces can be improved. There is a constant C independent of $\delta>0$ such that the a priori estimate (1.2) holds true for $1<q<n$, $n\geqq 2$, and (1.3) for $n/(n-1)<q<\infty$, $n\geqq 2$. Analogously the Stokes operator satisfies (1.7) for $1<q<\infty$, $n\geqq 2$, with $C=C(\Omega,\ q,\ \varepsilon)>0$. For the proof we need some technical extensions in the proof of Lemma 4.2 below by exploiting the zero boundary values of u on the *non-compact* boundary of the perturbed half space.

The organization of this paper is as follows. In Section 2 we consider the resolvent problem (1.1) in the whole space and in the half space where the major tool is the multiplier theorem. The problem for bended half spaces and for cones is investigated in Section 3 by using a perturbation criterion and referring to the half space problem. In Section 4 we use the localization method and perturbation arguments to prove Theorem 1.2 in a series of lemmata, and finally we prove Corollary 1.6. In the Appendix we prove some properties of the spaces $\hat{W}^{1,q}(\Omega)$ and $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ and of the Helmholtz projection.

2. The whole space and the half space.

Consider the generalized resolvent problem (1.1) for $\boldsymbol{R}^n$ and $\boldsymbol{R}_+^n$. We start with the proof of Theorem 1.3 when $\Omega=\boldsymbol{R}^n$; in this case (1.1) has the form (1.4).

PROOF OF THEOREM 1.3 FOR $\Omega=\boldsymbol{R}^n$. Using $\hat{W}^{1,q}(\boldsymbol{R}^n)=\overline{C_0^\infty(\Omega)}^{\|\nabla\cdot\|_q}$, by the well known fundamental solution for Δ and the Calderón-Zygmund theorem we see that the operator

$$-\Delta:\hat{W}^{1,q}(\boldsymbol{R}^n)\longrightarrow\hat{W}^{-1,q}(\boldsymbol{R}^n)\,,\qquad p\mapsto[-\Delta p,\ \cdot\,]\,,$$

where the functional $[-\Delta p, \cdot]$ is defined by

$$[-\Delta p, \varphi] = \langle \nabla p, \nabla \varphi \rangle, \qquad \varphi \in C_0^\infty(\boldsymbol{R}^n),$$

is an isomorphism. In particular the equation $-\Delta p = F \in \hat{W}^{-1,q}(\boldsymbol{R}^n)$ has a unique solution $p \in W_{\mathrm{loc}}^{1,q}(\boldsymbol{R}^n)$ such that $\|\nabla p\|_q \leqq c\|F\|_{-1,q}$ (for the corresponding result for the Stokes equation in $\boldsymbol{R}^n$ see [**13**]). In the first step to solve (1.4) we find a vector field $u_g \in L^q(\boldsymbol{R}^n)^n$ of the divergence equation $\operatorname{div} u_g = g \in \hat{W}^{-1,q}(\boldsymbol{R}^n)$ in the form $u_g = \nabla P$; we choose $P \in \hat{W}^{1,q}(\boldsymbol{R}^n)$ to be the weak solution of $\Delta P = g$. Hence

$$\|u_g\|_q = \|\nabla P\|_q \leqq c\|g\|_{-1,q}.$$

Since $g \in \hat{W}^{1,q}(\boldsymbol{R}^n)$ we easily get that $|\nabla^2 u_g| \in L^q(\boldsymbol{R}^n)$ and $\|\nabla^2 u_g\|_q \leqq c\|\nabla g\|_q$. By an interpolation argument, $\|\sqrt{|\lambda|}\nabla u_g\|_q \leqq c(\|\lambda u_g\|_q + \|\nabla^2 u_g\|_q)$; this inequality also implies that $\|g\|_q$ may be estimated by $\|g\|_{-1,q}$ and $\|\nabla g\|_q$. Next we solve

$$-\Delta p = -\operatorname{div} f + (\lambda - \Delta) g \in \hat{W}^{-1,q}(\boldsymbol{R}^n)$$

and get a unique $p \in \hat{W}^{1,q}(\boldsymbol{R}^n)$ satisfying

$$\|\nabla p\|_q \leqq c(\|(f, \nabla g)\|_q + \|\lambda g\|_{-1,q}).$$

Finally we find a solution $v \in W^{2,q}(\boldsymbol{R}^n)^n$ of the equation

$$(\lambda - \Delta) v = f - (\lambda - \Delta) u_g - \nabla p$$

using Fourier transform. By the multiplier theorem [**27**]

$$\begin{aligned} \|\lambda v\|_q + \|\sqrt{|\lambda|}\nabla v\|_q + \|\nabla^2 v\|_q &\leqq c\|f - (\lambda - \Delta) u_g - \nabla p\|_q \\ &\leqq c(\|(f, \nabla g)\|_q + \|\lambda g\|_{-1,q}). \end{aligned}$$

Then $u = v + v_g$ together with p is a solution of (1.4) satisfying the resolvent estimate (1.2). To prove uniqueness let (u, p) be a solution of the homogeneous Stokes system (1.4). Then $\Delta p = 0$ which yields $\nabla p = 0$. Thus $(\lambda - \Delta) u = 0$. Since $\lambda \notin \boldsymbol{R}_-$, the only solution in $L^q(\boldsymbol{R}^n)^n$ of this equation is $u = 0$.

To prove the estimate (1.3) consider the solution $(u, p) \in \mathscr{D}(\Delta_q)^n \times \hat{W}^{1,q}(\boldsymbol{R}^n)$ of (1.4). Further for given $\tilde{f} \in L^{q'}(\boldsymbol{R}^n)^n$ let $(\tilde{u}, \tilde{p}) \in \mathscr{D}(\Delta_{q'})^n \times \hat{W}^{1,q'}(\boldsymbol{R}^n)$ be a solution of

$$\lambda \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} = \tilde{f}, \qquad \operatorname{div} \tilde{u} = 0 \qquad \text{in } \boldsymbol{R}^n.$$

Then

$$\begin{aligned} \langle u, \tilde{f} \rangle = \langle u, \lambda \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} \rangle &= \langle \lambda u - \Delta u, \tilde{u} \rangle - [g, \tilde{p}] \\ &= \langle f - \nabla p, \tilde{u} \rangle - [g, \tilde{p}] = \langle f, \tilde{u} \rangle - [g, \tilde{p}]. \end{aligned}$$

Using the $L^{q'}$-estimate (1.2) for $(\tilde{u}, \tilde{p})$ we get that

$$|\langle u, \tilde{f}\rangle| \leqq \frac{1}{|\lambda|}(\|f\|_q\|\lambda\tilde{u}\|_{q'}+\|\lambda g\|_{-1,q}\|\nabla\tilde{p}\|_{q'})$$

$$\leqq \frac{c}{|\lambda|}(\|f\|_q+\|\lambda g\|_{-1,q})\|\tilde{f}\|_{q'}.$$

Since $\tilde{f}\in L^{q'}(\boldsymbol{R}^n)^n$ was arbitrary, the estimate (1.3) now follows easily. Thus Theorem 1.3 (i) is proved.

For the proof of the regularity property (ii) let $s\in(1, \infty)$ and assume that additionally $f\in L^s(\boldsymbol{R}^n)^n$ and $g\in W^{1,s}(\boldsymbol{R}^n)\cap\hat{W}^{-1,s}(\boldsymbol{R}^n)$. According to the preceding part we find an additional solution $(\tilde{u}, \tilde{p})\in\mathcal{D}(\Delta_s)^n\times\hat{W}^{1,s}(\boldsymbol{R}^n)$ of (1.4). By our proof the solution (u, p) has an explicit representation in the sense of distributions which only depends on f and g and which is independent of q. Therefore this construction leads to the same pair when q is replaced by s. Thus $u=\tilde{u}$, $\nabla p=\nabla\tilde{p}$ and the assertion (ii) is proved. ∎

REMARK 2.1. (i) Since in the previous proof the condition $g\in L^q(\Omega)$ was not needed we have shown that $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ coincides with $\hat{W}^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ for $\Omega=\boldsymbol{R}^n$. By a simple reflection argument this results also follows for $\Omega=\boldsymbol{R}^n_+$. See Lemma 5.5 for this property in other unbounded domains satisfying the Assumption 1.1.

(ii) Write $f=(f', f_n)$ with $f'=(f_1, \cdots, f_{n-1})$ and analogously $u=(u', u_n)$. Assume that f' and g are even functions with respect to x_n and that f_n is odd in x_n. Then an easy symmetry consideration implies that u' and p are even in x_n while u_n is odd. In particular $u_n(x', 0)=0$ for all $x'=(x_1, \cdots, x_{n-1})\in\boldsymbol{R}^{n-1}$.

PROOF OF THEOREM 1.3 FOR $\Omega=\boldsymbol{R}^n_+$. Using a scaling argument, see e.g. [5], it suffices to consider $\lambda\in S_\varepsilon$, $0<\varepsilon<\pi$, with $|\lambda|=1$. Let us introduce the notion of an even and odd extension of a given function $\varphi:\boldsymbol{R}^n_+\to\boldsymbol{R}$: the even extension φ_e is defined by

$$\varphi_e(x)=\begin{cases}\varphi(x', x_n) & \text{for } x_n>0\\ \varphi(x', -x_n) & \text{for } x_n<0\end{cases}$$

while for the *odd extension* $\varphi_o(x)=-\varphi(x', -x_n)$ if $x_n<0$. Now we proceed with the first four steps of the proof assuming that $|\lambda|=1$.

Let f'_e, g_e denote the even extensions of $f'\in L^q(\boldsymbol{R}^n_+)^{n-1}$ and $g\in W^{1,q}(\boldsymbol{R}^n_+)\cap\hat{W}^{-1,q}(\boldsymbol{R}^n_+)$. Further let f_{no} be the odd extension of the nth component f_n of f. According to Theorem 1.3 let $(U, \nabla P)$ denote the solution of the generalized resolvent equation (1.4) with right-hand side $F=(f'_e, f_{no})\in L^q(\boldsymbol{R}^n)^n$ and $G=g_e\in W^{1,q}(\boldsymbol{R}^n)\cap\hat{W}^{-1,q}(\boldsymbol{R}^n)$. By Remark 2.1 (ii) the nth component U_n of U vanishes on $\Gamma=\partial\boldsymbol{R}^n_+$ while $U'|_\Gamma=\phi'\in W^{2-1/q,q}(\Gamma)^{n-1}$ may be nonzero. Let $\|\cdot\|_{2-1/q,q}$ denote the trace norm in $W^{2-1/q,q}(\Gamma)$. Since $|\lambda|=1$,

$$\begin{aligned}\|\phi'\|_{2-1/q,q} &\leqq C\|U'\|_{W^{2,q}(R^n_+)}\\ &\leqq C(\|(F, \nabla G)\|_{L^q(R^n)}+\|G\|_{\hat{W}^{-1,q}(R^n)})\\ &\leqq C(\|(f, \nabla g)\|_q+\|g\|_{-1,q}).\end{aligned} \tag{2.1}$$

Subtracting $(U, \nabla P)$ the generalized resolvent problem (1.1) is reduced to the problem

$$\begin{aligned}\lambda u-\Delta u+\nabla p &= 0 \quad \text{in } \boldsymbol{R}^n_+\\ \operatorname{div} u &= 0 \quad \text{in } \boldsymbol{R}^n_+\\ u' &= \phi' \quad \text{on } \Gamma\\ u_n &= 0 \quad \text{on } \Gamma.\end{aligned} \tag{2.2}$$

Given $\phi'\in W^{2-1/q,q}(\Gamma)^{n-1}$ and $\lambda\in S_\varepsilon$, $|\lambda|=1$, we prove the existence of a unique solution $(u, \nabla p)\in W^{2,q}(\boldsymbol{R}^n_+)^n\times L^q(\boldsymbol{R}^n_+)^n$ of (2.2); moreover we show that

$$\|(u, \nabla u, \nabla^2 u, \nabla p)\|_q \leqq C\|\phi'\|_{2-1/q,q}. \tag{2.3}$$

Then a combination of (2.1) and of (2.3) will complete the proof of Theorem 1.3 (i).

In the second step of the proof we eliminate u' and the pressure p. Let ∇', Δ' and div' denote the gradient, the Laplacian and the divergence with respect to the $n-1$ variables x' only. Further let $\partial_n=\partial/\partial x_n$. Then apply $\partial_n\operatorname{div}'$ to the first $n-1$ equations of (2.2) and $-\Delta'$ to the nth equation of (2.2); adding both scalar equations and inserting $\partial_n u_n=-\operatorname{div}' u'$ on Γ we are left with the fourth order equation

$$\begin{aligned}-\Delta(\lambda-\Delta)u_n &= 0 \quad \text{in } \boldsymbol{R}^n_+\\ u_n &= 0 \quad \text{in } \Gamma\\ \partial_n u_n &= -\operatorname{div}'\phi' \quad \text{on } \Gamma.\end{aligned} \tag{2.4}$$

We solve (2.4) by introducing the partial Fourier transform $\hat{}=F'$ with respect to the variables x' in the sense of distributions,

$$\hat{u}(\zeta', x_n)=F'u(\zeta', x_n)=\frac{1}{(2\pi)^{(n-1)/2}}\int_{R^{n-1}}u(x', x_n)e^{-i\zeta'\cdot x'}dx', \quad \zeta'\in\boldsymbol{R}^{n-1}.$$

Then (2.4) is transformed into a fourth order ordinary differential equation for $\hat{u}_n(\zeta', x_n)$ with respect to the variable x_n; using the notation $s=|\zeta'|$ for the Euclidean norm of $\zeta'\in\boldsymbol{R}^{n-1}$ we get that

$$\begin{aligned}(s^2-\partial_n^2)(\lambda+s^2-\partial_n^2)\hat{u}_n(\zeta', x_n)&=0 \qquad \text{in } (0, \infty)\\ \hat{u}_n(\zeta', 0)&=0 \\ \partial_n\hat{u}_n(\zeta', 0)&=-i\zeta'\cdot\hat{\phi}'\,.\end{aligned} \tag{2.5}$$

For fixed $0\neq\zeta'\in\boldsymbol{R}^{n-1}$ and $\lambda\in S_\varepsilon$, $|\lambda|=1$, there is a unique bounded solution $\hat{u}_n$ of (2.5) which is a linear combination of e^{-sx_n} and $e^{-\sqrt{\lambda+s^2}x_n}$; to be more precise,

$$\hat{u}_n(\zeta', x_n)=i\zeta'\cdot m_0(s, x_n)\hat{\phi}'$$

where

$$m_0(s, x_n)=(e^{-\sqrt{\lambda+s^2}x_n}-e^{-sx_n})(\sqrt{\lambda+s^2}-s)^{-1}\,. \tag{2.7}$$

In the fourth step we give an explicit representation of the solution (u, p) $=((u', u_n), p)$ of (2.2). Applying div$'$ to the first $n-1$ equations of (2.2) we get

$$\hat{p}(\zeta', x_n)=-\frac{1}{s^2}(\lambda+s^2-\partial_n^2)\partial_n\hat{u}_n \tag{2.8}$$

since $\operatorname{div} u=0$. Defining u' by $\hat{u}'(\zeta', x_n)=(i\zeta'/s^2)\partial_n\hat{u}_n$ will yield a solution of the homogeneous Stokes system but generally $u'|_\Gamma\neq\phi'$. Thus we add a term from the kernel of $\lambda-\Delta$ with vanishing divergence in $\boldsymbol{R}^{n-1}$. A simple calculation yields

$$\begin{aligned}\hat{u}'(\zeta', x_n)&=\frac{i\zeta'}{s^2}\partial_n\hat{u}_n+\Big(I-\frac{\zeta'\zeta'}{s^2}\Big)\hat{h}(\zeta', x_n)\\ &=-\partial_n m_0(s, x_n)\frac{\zeta'\zeta'}{s^2}\hat{\phi}'(\zeta')+\Big(I-\frac{\zeta'\zeta'}{s^2}\Big)\hat{h}(\zeta', x_n)\end{aligned} \tag{2.9}$$

with

$$\hat{h}(\zeta', x_n)=e^{-\sqrt{\lambda+s^2}x_n}\hat{\phi}' \tag{2.10}$$

where I denotes the identity matrix in $\boldsymbol{R}^{n-1,n-1}$ and $\zeta'\zeta'$ the dyadic product of ζ' with itself.

Before proving the $L^q(\boldsymbol{R}_+^n)$-estimates of $(u, \nabla p)$ and of the derivatives of u we recall the multiplier theorem of Hörmander and Michlin and give estimates of the multipliers involved in (2.6), $\cdots$, (2.10).

DEFINITION 2.2. (i) A function $m\in C^\infty(\boldsymbol{R}^{n-1}\setminus\{0\})$ is said to satisfy the *multiplier condition* (M) if there are constants $c_k>0$, $k=0, 1, \cdots, [(n-1)/2]+1$, such that

$$s^k|\nabla'^k m(\zeta')|\leqq c_k \qquad \text{for all } \zeta'\in\boldsymbol{R}^{n-1}\setminus\{0\},$$

where $\nabla'^k=(\partial/\partial\zeta_{i_1}\cdots\partial/\partial\zeta_{i_k})_{i_1,\cdots,i_k=1}^{n-1}$.

(ii) A function $m^*\in C^\infty((\boldsymbol{R}^{n-1}\setminus\{0\})\times\boldsymbol{R}_+)$ satisfies the *multiplier condition* (M^*) if there are constants $c_k>0$, $k=0, 1, \cdots, [(n-1)/2]+1$, and a positive $\delta>0$ such that

$$s^k|\nabla'^k m^*(\zeta', x_n)| \leqq \frac{c_k e^{-\delta s x_n}}{1+x_n} \qquad \text{for all } \zeta' \in \boldsymbol{R}^{n-1} \setminus \{0\} \text{ and } x_n > 0.$$

THEOREM 2.3 (*see e.g.* [27]). *Let* $S(\boldsymbol{R}^{n-1})$, $n \geqq 2$, *denote the Schwartz space of rapidly decreasing functions on* $\boldsymbol{R}^{n-1}$ *and let* $m \in C^\infty(\boldsymbol{R}^{n-1} \setminus \{0\})$ *satisfy the multiplier condition* (M). *Then the operator* T *defined on* $S(\boldsymbol{R}^{n-1})$ *by* $Tf = (F')^{-1}(mF'f)$ *admits an extension to a bounded operator* $T: L^q(\boldsymbol{R}^{n-1}) \to L^q(\boldsymbol{R}^{n-1})$ *for each* $q \in (1, \infty)$. *Furthermore the norm of the operator* T *may be estimated by* $c(q, n) \cdot \max\{c_k; 0 \leqq k \leqq [(n-1)/2]+1\}$.

LEMMA 2.4. *Let* $m \in C^\infty(\boldsymbol{R}^{n-1} \setminus \{0\})$ *satisfy the multiplier condition* (M) *and let* $m^* \in C^\infty((\boldsymbol{R}^{n-1} \setminus \{0\}) \times \boldsymbol{R}_+)$.

(i) *If* m^* *is a function only of* $(s, x_n) \in \boldsymbol{R}_+^2$ *where* $s = |\zeta'|$ *and if* m^* *satisfies the condition*

$$s^k \left| \left(\frac{\partial}{\partial s}\right)^k m^*(s, x_x) \right| \leqq \frac{c_k e^{-\delta s x_n}}{1+x_n}, \qquad s > 0,\ x_n > 0,$$

for some $\delta > 0$ *and with constants* c_k, $k = 0, \cdots, [(n-1)/2]+1$, *then* m^* *satisfies the multiplier condition* (M^*).

(ii) *If* m^* *satisfies* (M^*), *then* $e^{\delta s x_n} m^*(\zeta', x_n)$ *satisfies* (M^*) *for some positive* δ.

(iii) *If* m^* *satisfies* (M^*), *then* mm^* *satisfies* (M^*).

PROOF. (i) is an easy consequence of the chain rule. Note that the same $\delta > 0$ may be used in condition (M^*). The assertion (ii) follows from Leibniz's formula and the trivial estimate $(s x_n)^k \leqq c_k(\varepsilon) e^{\varepsilon s x_n}$ for all $s > 0$, $x_n > 0$ and sufficiently small $\varepsilon > 0$. If m^* satisfies (M^*) with a given $\delta_0 > 0$, then $e^{\delta s x_n} m^*$ satisfies (M^*) for all $\delta \in (0, \delta_0)$. Finally (iii) is a trivial consequence of Leibniz's formula. Obviously the same $\delta > 0$ may be used. ∎

LEMMA 2.5. *Let* $\lambda \in S_\varepsilon$, $0 < \varepsilon < \pi$, *with* $|\lambda| = 1$. *Then the functions* $e^{-\sqrt{\lambda+s^2} x_n}$, $s m_0(s, x_n)$ *and* $\partial_n m_0(s, x_n)$ *satisfy the multiplier condition* (M^*) *with some* $\delta > 0$ *which depends only on* ε.

PROOF. Since $|\arg \lambda| \leqq \pi - \varepsilon$, $\varepsilon > 0$, it is easily seen that $\mathrm{Re}(\sqrt{\lambda+s^2}) \geqq \delta_0(1+s)$ for all $s \geqq 0$ with a constant $\delta_0 = \delta_0(\varepsilon) \in (0, 1)$. Thus $|e^{-\sqrt{\lambda+s^2} x_n}| \leqq e^{-\delta_0 x_n} e^{-\delta_0 s x_n}$. Differentiating we get

$$\frac{\partial}{\partial s}(e^{-\sqrt{\lambda+s^2} x_n}) = \frac{-s x_n}{\sqrt{\lambda+s^2}} e^{-\sqrt{\lambda+s^2} x_n}$$

and more generally $(\partial/\partial s)^k e^{-\sqrt{\lambda+s^2} x_n} = g_k(s, s x_n) e^{-\sqrt{\lambda+s^2} x_n}$ with functions g_k, where

$$s^k |g_k(s, s x_n)| \leqq c_k (1 + (s x_n) + \cdots + (s x_n)^k)$$

with constants c_k, $k=0, \cdots, [(n-1)/2]+1$. Now the assertion for $e^{-\sqrt{\lambda+s^2}x_n}$ follows from the previous observations and from Lemma 2.4 (i). Considering $sm_0(s, x_n)$ we prove in a first step the estimate

$$s|m_0(s, x_n)| \leqq \frac{ce^{-\delta s x_n}}{1+x_n}. \tag{2.11}$$

Obviously $|\sqrt{\lambda+s^2}-s|=|\lambda|/|\sqrt{\lambda+s^2}+s|\geqq a/(1+s)$ with a constant $a=a(\varepsilon)>0$. Thus

$$\begin{aligned} s|m_0(s, x_n)| &\leqq cs(1+s)e^{-\delta_0 s x_n}(e^{(\delta_0-1)s x_n}+e^{-\delta_0 x_n}) \\ &\leqq cs(1+s)e^{-\delta_0 s x_n}. \end{aligned}$$

Using the inequality $sx_n \leqq ce^{(\delta_0-\delta)s x_n}$ we get (2.11) for $\delta\in(0, \delta_0)$ if $0<s\leqq 1$ or if $x_n\geqq 1$. Finally for $0\leqq x_n\leqq 1\leqq s$ the estimate

$$s|m_0(s, x_n)| \leqq cs^2 e^{-s x_n}\left|1-\exp\frac{-\lambda x_n}{s+\sqrt{\lambda+s^2}}\right| \leqq cs^2 e^{-s x_n}\frac{x_n}{s}$$

yields (2.11). To prove (M^*) for $k\geqq 1$ note that

$$\frac{\partial}{\partial s}m_0(s, x_n)=m_0(s, x_n)\left(\frac{1}{\sqrt{\lambda+s^2}}-x_n\right)+\frac{x_n}{\sqrt{\lambda+s^2}}e^{-\sqrt{\lambda+s^2}x_n}$$

and, by induction, that

$$\left(\frac{\partial}{\partial s}\right)^k m_0(s, x_n)=g_k(s, x_n)m_0(s, x_n)+h_k(s, x_n)e^{-\sqrt{\lambda+s^2}x_n}$$

with functions g_k, h_k such that

$$s^k|g_k(s, x_n)|+s^{k+1}|h_k(s, x_n)| \leqq c_k(1+(sx_n)+\cdots+(sx_n)^k).$$

Then the assertion for $sm_0(s, x_n)$ follows from

$$s^k\left(\frac{\partial}{\partial s}\right)^k(sm_0(s, x_n))=(s^k g_k+ks^{k-1}g_{k-1})sm_0+(s^{k+1}h_k+ks^k h_{k-1})e^{-\sqrt{\lambda+s^2}x_n},$$

the previous inequality, (2.11), the estimates of $e^{-\sqrt{\lambda+s^2}x_n}$ and from Lemma 2.4 (i). The third problem is now clear since $\partial_n m_0(s, x_n)=-sm_0(s, x_n)-e^{-\sqrt{\lambda+s^2}x_n}$. ■

Now we proceed with the 5th step of the proof of Theorem 1.3 when $\Omega=\boldsymbol{R}_+^n$. By Lemma 2.4 and Lemma 2.5 the function $i\zeta' m_0(s, x_n)$ in the definition (2.6) of $\hat{u}_n$ is a multiplier satisfying (M^*). Thus for fixed $x_n>0$ Theorem 2.3 implies that $u_n(\cdot, x_n)\in L^q(\boldsymbol{R}^{n-1})$ and that

$$\int_{R^{n-1}}|u_n(x', x_n)|^q dx' \leqq \frac{c}{(1+x_n)^q}\int_{R^{n-1}}|\phi'(x')|^q dx'.$$

Integrating with respect to $x_n\in(0, \infty)$ we get that $u_n\in L^q(\boldsymbol{R}_+^n)$ satisfying $\|u_n\|_q$

$\leqq c\|\phi'\|_{L^q(R^{n-1})}$. Analogously (2.9), (2.10), Lemma 2.4, Lemma 2.5 and Theorem 2.3 imply that $u'\in L^q(\boldsymbol{R}_+^n)^{n-1}$ and that $\|u'\|_q\leqq c\|\phi'\|_{L^q(R^{n-1})}$. Here we use that ζ'/s and $\zeta'\zeta'/s^2$ are multipliers of type (M). In order to estimate derivatives of u we introduce $v(x', x_n)$ as the L^q-solution of

$$(-\delta^2\Delta'-\partial_n^2)v(x', x_n)=0 \quad \text{in } \boldsymbol{R}_+^n$$
$$v(x', 0)=\phi' \quad \text{on } \Gamma.$$

From the theory of the Poisson semigroup (see [1], [27]) it is well known that $\nabla v\in L^q(\boldsymbol{R}_+^n)^{n(n-1)}$, $|\nabla^2 v|\in L^q(\boldsymbol{R}_+^n)$ and that $\|(\nabla v, \nabla^2 v)\|_q\leqq c\|\phi'\|_{2-1/q,q}$. In terms of partial Fourier transform, $\hat{v}(\zeta', x_n)=e^{-\delta s x_n}\hat{\phi}'(\zeta')$. Writing (2.6) in the form $\hat{u}_n(\zeta', x_n)=i\zeta' m_0(s, x_n)e^{\delta s x_n}\hat{v}(\zeta', x_n)$ we get that

$$\widehat{\nabla' u_n}(\zeta', x_n)=i\zeta' m_0(s, x_n)e^{\delta s x_n}\widehat{\nabla' v}(\zeta', x_n),$$
$$\widehat{\partial_n u_n}(\zeta', x_n)=(\partial_n m_0(s, x_n))e^{\delta s x_n}\,\widehat{\operatorname{div}' v}(\zeta', x_n).$$

Thus Theorem 2.3, Lemma 2.4 and Lemma 2.5 imply that $\nabla u_n\in L^q(\boldsymbol{R}_+^n)^n$ and that $\|\nabla u_n\|_q\leqq c\|\phi'\|_{2-1/q,q}$. Analogously we get that $\|(\partial_n\nabla' u_n, \nabla'^2 u_n)\|_q\leqq c\|\phi'\|_{2-1/q,q}$. Concerning $\partial_n^2 u_n$ we use the identity

$$\partial_n^2(i\zeta' m_0(s, x_n))=[s(sm_0+e^{-\sqrt{\lambda+s^2}x_n})+\sqrt{\lambda+s^2}\,e^{-\sqrt{\lambda+s^2}x_n}]i\zeta'$$

and the fact that $i\zeta'/s$ and $\sqrt{\lambda+s^2}/(1+s)$ are multipliers satisfying the condition (M). Then we proceed as in the previous estimates and get that $\|\partial_n^2 u_n\|_q\leqq c\|\phi'\|_{2-1/q,q}$. To estimate first and second order partial derivatives of u' we use the representation (2.9), (2.10) and proceed as before. Thus we proved that $\|(\nabla u, \nabla^2 u)\|_q\leqq c\|\phi'\|_{2-1/q,q}$. Analogously we get an estimate of $\nabla' p\in L^q(\boldsymbol{R}_+^n)^{n-1}$ from (2.8). Finally (2.5) and (2.8) yield $\widehat{\partial_n p}=-(\lambda+s^2-\partial_n^2)\hat{u}_n$; hence $\|\nabla p\|_q\leqq c\|\phi'\|_{2-1/q,q}$. Thus inequality (2.3) is completely proved and combining with (2.1) we get the estimate (1.2).

The proof of the inequality (1.3) is the same as for $\boldsymbol{R}^n$. We only need the solvability of the resolvent problem

$$\begin{aligned} \lambda\tilde{u}-\Delta\tilde{u}+\nabla\tilde{p}&=\tilde{f} \quad &&\text{in } \boldsymbol{R}_+^n\\ \operatorname{div}\tilde{u}&=0 &&\text{in } \boldsymbol{R}_+^n\\ \tilde{u}&=0 &&\text{on } \Gamma \end{aligned} \tag{2.12}$$

for $\tilde{f}\in L^{q'}(\boldsymbol{R}_+^n)^n$ and the a priori estimate (1.2) for $\tilde{u}\in\mathcal{D}(\Delta_{q'})^n$, $\tilde{p}\in\hat{W}^{1,q'}(\boldsymbol{R}_+^n)$.

To prove uniqueness let $(u, p)\in\mathcal{D}(\Delta_q)^n\times\hat{W}^{1,q}(\boldsymbol{R}_+^n)$ be a solution of the generalized resolvent equation (1.1) with right-hand side $f=0$, $g=0$. Then $\tilde{f}=|u|^{q-2}\bar{u}\in L^{q'}(\boldsymbol{R}_+^n)^n$ where $\bar{u}$ means the complex conjugate of u and $\langle\tilde{f}, u\rangle=\int u\tilde{f}\,dx=\|u\|_q^q<\infty$. By the existence part of Theorem 1.3 proved just before we get a solution $(\tilde{u}, \nabla\tilde{p})\in\mathscr{D}(\Delta_{q'})^n\times\hat{W}^{1,q'}(\boldsymbol{R}^n_+)$ of (2.12). Then simple approximation arguments justify the following computation:

$$\begin{aligned}\|u\|_q^q &= \langle\tilde{f}, u\rangle = \langle\lambda\tilde{u}-\Delta\tilde{u}+\nabla\tilde{p}, u\rangle\\ &= \langle\tilde{u}, \lambda u-\Delta u\rangle = \langle\tilde{u}, \lambda u-\Delta u+\nabla p\rangle = 0\,.\end{aligned}$$

Thus $u=0$, and consequently also $\nabla p=0$.

The proof of the regularity assertion is completely parallel to the proof for the whole space problem; the solution pair (u, p) only depends on the distributions f and g and does not depend on q. This completes the proof of Theorem 1.2 for $\Omega=\boldsymbol{R}^n_+$. ■

Considering the limit $\lambda\to 0$ we get existence, uniqueness and regularity results for the Stokes system

$$\begin{aligned}-\Delta u+\nabla p &= f \quad \text{in } \Omega\\ \nabla\,\mathrm{div}\, u &= \nabla g \quad \text{in } \Omega\\ u &= 0 \quad \text{on } \partial\Omega \quad \text{if } \Omega=\boldsymbol{R}^n_+\,.\end{aligned} \tag{2.13}$$

COROLLARY 2.6 (*Stokes system for $\boldsymbol{R}^n$, $\boldsymbol{R}^n_+$*). *Let $n\geqq 2$, $1<q<\infty$ and let $\Omega=\boldsymbol{R}^n$ or $\Omega=\boldsymbol{R}^n_+$. Then for every $f\in L^q(\Omega)^n$, $g\in W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$, there exists a solution $(u, p)\in W^{2,q}_{\mathrm{loc}}(\bar{\Omega})^n\times\hat{W}^{1,q}(\Omega)$ of (2.13) with $|\nabla^2 u|\in L^q(\Omega)$ and the following properties:*

(i) *(u, p) satisfies*

$$\|\nabla^2 u\|_q+\|\nabla p\|_q \leqq C(\|f\|_q+\|\nabla g\|_q) \tag{2.14}$$

with some constant $C=C(n, q, \varepsilon)>0$. The pressure p is unique up to a constant and the velocity field u is unique up to a linear polynomial $a+Ax$ where $a\in\boldsymbol{C}^n$, $A\in\boldsymbol{C}^{n,n}$ with trace$(A)=0$, *if $\Omega=\boldsymbol{R}^n$, and up to a linear term $\begin{pmatrix}a'\\0\end{pmatrix}x_n$ where $a'\in\boldsymbol{C}^{n-1}$, if $\Omega=\boldsymbol{R}^n_+$. If $1<q<n$ and $1/n+1/r=1/q$ then we may single out a special solution by the condition $|\nabla u|\in L^r(\Omega)$.*

(ii) *If for some $s\in(1, \infty)$ additionally $f\in L^s(\Omega)^n$ and $g\in W^{1,s}(\Omega)\cap\hat{W}^{-1,s}(\Omega)$, then $|\nabla^2 u|\in L^s(\Omega)$ and $p\in\hat{W}^{1,s}(\Omega)$.*

PROOF. First we consider the uniqueness assertion. Let $(u, p)\in W^{2,q}_{\mathrm{loc}}(\Omega)^n\times\hat{W}^{1,q}(\Omega)$ with $|\nabla^2 u|\in L^q(\Omega)$ be a solution of (2.13) with $f=0$, $g=0$. Then in the case $\Omega=\boldsymbol{R}^n$ we get $\Delta p=0$, and $\nabla p\in L^q(\boldsymbol{R}^n)^n$ implies that $\nabla p=0$. Thus $\Delta u=0$ and $\Delta(\nabla^2 u)=0$ yielding $\nabla^2 u=0$. Hence u is a linear polynomial $a+Ax$ such that $0=\mathrm{div}\, u=\mathrm{trace}\,(A)$. If $1<q<n$, then Sobolev's imbedding theorem yields the existence of a constant $c\in\boldsymbol{R}^{n^2}$ such that $\nabla u-c\in L^r(\boldsymbol{R}^n)^{n^2}$ where

$1/n+1/r=1/q$.

Now let $\Omega=\mathbf{R}_+^n$. Using the notations in the proof of Theorem 1.3 we now obtain the equations (2.4) with $\lambda=0$, $\hat{\phi}'=0$, i.e., $\Delta^2 u_n=0$, $u_n|_\Gamma=0$ and $\partial_n u_n|_\Gamma=0$. To prove that $u_n=0$ we use the partial Fourier transform $\hat{u}_n$ of u_n and get that

$$(s^2-\partial_n^2)(s^2-\partial_n^2)\hat{u}_n(\zeta', x_n)=0, \quad \hat{u}_n(\zeta', 0)=0, \quad \partial_n\hat{u}_n(\zeta', 0)=0$$

which is (2.5) for $\lambda=0$. Then for $s\neq 0$ we conclude that $\hat{u}_n$ is a linear combination of e^{-sx_n} and $x_n e^{-sx_n}$ since $|\nabla^2 u_n|\in L^q(\mathbf{R}_+^n)$. Then the boundary conditions yield $\hat{u}_n(\zeta', x_n)=0$ for $\zeta'\neq 0$, $x_n\geqq 0$. Thus $u_n(x', x_n)$ does not depend on x'. From $|\nabla^2 u_n|\in L^q(\mathbf{R}_+^n)$ we see that $u_n=a+bx_n$ with $a=0$, $b=0$ using again the boundary conditions for $\hat{u}_n$. Hence $u_n=0$. Then $\operatorname{div}' u'=0$, and the equation (2.13) for $f=0$, $\nabla g=0$ yields $\Delta' p=0$. Since $|\nabla p|\in L^q(\mathbf{R}_+^n)$ we conclude that $\nabla' p=0$ and $\nabla p=0$. Then $-\Delta u'=0$ and $u(x', 0)=0$ lead to the desired form $u(x', x_n)=\begin{pmatrix} a' \\ 0 \end{pmatrix} x_n$ with $a'\in \mathbf{C}^{n-1}$; this is proved by the partial Fourier transform as before.

The existence assertion immediately follows from Theorem 1.3 by letting $\lambda\to 0$ for fixed f and g. This is possible since the constant $C=C(n, q, \varepsilon)$ in Theorem 1.3 does not depend on λ. Indeed we may choose any sequence $(\lambda_i)\in S_\varepsilon$ with $\lambda_i\to 0$ as $i\to\infty$ and consider the corresponding solutions $(u_i, p_i)\in \mathcal{D}(\Delta_q)^n\times \hat{W}^{1,q}(\Omega)$ of (1.1) or (1.4). Due to Theorem 1.3 we get $\sup_i\|(\lambda_i u_i, \nabla^2 u_i, \nabla p_i)\|_q<\infty$. Then we may assume, after possibly taking a subsequence, that there are constants c_i and linear polynomials $a_i+A_i x$, $i\in\mathbf{N}$, such that $u_i-(a_i+A_i x)$ and p_i-c_i converge locally in L^q as $i\to\infty$ to some $u\in L^q_{\mathrm{loc}}(\bar{\Omega})^n$ and $p\in L^q_{\mathrm{loc}}(\bar{\Omega})$, respectively, and that $\nabla^2 u_i$, ∇p_i and $\lambda_i u_i$ tend weakly in L^q to $\nabla^2 u$, ∇p and some u^*, respectively. Since $\lambda_i\to 0$, we see that $\nabla^2 u^*=0$ and consequently $u^*=0$ because of $\|u^*\|_q<\infty$. This leads to $-\Delta u+\nabla p=f$, $\nabla \operatorname{div} u=\nabla g$ in the sense of distributions and to (2.14). Since $\operatorname{div} u_i=g$ we conclude that trace (A_i) is convergent as $i\to\infty$; thus we may assume that trace $(A_i)=0$ for $i\in\mathbf{N}$. If $\Omega=\mathbf{R}_+^n$ we get from $u_i|_\Gamma=0$ and the trace theorem that $a_i+A_i x|_\Gamma$ converges locally in $L^q(\mathbf{R}^{n-1})$; therefore we may suppose that $a_i=0$ and $A_i x=\begin{pmatrix} a_i' \\ 0 \end{pmatrix} x_n$ which leads to $u|_\Gamma=0$. If $1<q<n$ we may assume by Sobolev's imbedding theorem that $|\nabla u|\in L^r(\Omega)$ in both cases. This proves the existence of some solution $(u, p)\in W^{2,q}_{\mathrm{loc}}(\bar{\Omega})^n\times \hat{W}^{1,q}(\Omega)$ with (i).

In order to prove the regularity property (ii) let $f\in L^q(\Omega)^n\cap L^s(\Omega)^n$ and $g\in W^{1,q}(\Omega)\cap W^{1,s}(\Omega)\cap \hat{W}^{-1,q}(\Omega)\cap \hat{W}^{-1,s}(\Omega)$. Then the above procedure yields a solution $(\tilde{u}, \tilde{p})\in (W^{2,q}_{\mathrm{loc}}(\bar{\Omega})^n\cap W^{2,s}_{\mathrm{loc}}(\bar{\Omega})^n)\times(\hat{W}^{1,q}(\Omega)\cap \hat{W}^{1,s}(\Omega))$ of (2.13) and $\tilde{u}|_{\partial\Omega}=0$ if $\Omega=\mathbf{R}_+^n$. The uniqueness assertion yields $u=\tilde{u}$ up to a linear expression and

$p=\tilde{p}$ up to a constant. This yields the desired assertion and Corollary 2.6 is proved. ■

3. The bended half space.

Let $\omega: \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ be a Lipschitz continuous function with bounded gradient $\nabla'\omega=(\partial_1, \cdots, \partial_{n-1})\omega$ in $\boldsymbol{R}^{n-1}$, and let H_ω be the bended half space defined by

$$H_\omega = \{x=(x', x_n) \in \boldsymbol{R}^n ; x_n > \omega(x')\}.$$

Considering the generalized resolvent problem (1.1) in H_ω we get the following main result which is only partially contained in Theorem 1.2 and which enables us to consider also cones.

THEOREM 3.1. *Let* $n \geqq 2$, $1<q<\infty$, $0<\varepsilon<\pi$ *and* $\omega \in C^{0,1}(\boldsymbol{R}^{n-1}) \cap W^{2,1}_{\mathrm{loc}}(\boldsymbol{R}^{n-1})$. *Then there are constants* $K=K(n, q, \varepsilon)>0$ *and* $\lambda_0=\lambda_0(\omega, n, q, \varepsilon)>0$ *with the following properties*:

(i) *If* $\|\nabla'\omega\|_\infty \leqq K$ *and* $\|\nabla'^2\omega\|_\infty < \infty$, *then for all* $f \in L^q(H_\omega)^n$, $g \in W^{1,q}(H_\omega) \cap \hat{W}^{-1,q}(H_\omega)$ *and* $\lambda \in S_\varepsilon$ *with* $|\lambda| \geqq \lambda_0$ *there exists a unique solution* $(u, p) \in \mathcal{D}(\Delta_q)^n \times \hat{W}^{1,q}(H_\omega)$ *of* (1.1). *This solution satisfies the a priori estimates* (1.2) *and* (1.3) *with a constant* $C=C(\omega, n, q, \varepsilon)>0$.

(ii) *If additionally* $f \in L^s(H_\omega)^n$, $g \in W^{1,s}(H_\omega) \cap \hat{W}^{-1,s}(H_\omega)$ *for some* $s \in (1, \infty)$ *and if* $\|\nabla'\omega\|_\infty \leqq \min\{K(n, q, \varepsilon), K(n, s, \varepsilon)\}$, $|\lambda| \geqq \max\{\lambda_0(\omega, n, q, \varepsilon), \lambda_0(\omega, n, s, \varepsilon)\}$, *then* $u \in \mathcal{D}(\Delta_s)^n$ *and* $p \in \hat{W}^{1,s}(H_\omega)$.

(iii) *If* $n \geqq 3$, $1<q<n-1$, $\|\nabla'\omega\|_\infty \leqq K$ *and if* $\|\nabla'^2\omega\|_{L^{n-1}(\boldsymbol{R}^{n-1})} \leqq K$ *or* $\||\cdot|\,|\nabla'^2\omega|\|_\infty \leqq K$, *where* $|\cdot|: x' \mapsto |x'|$, *then for all* $f \in L^q(H_\omega)^n$, $g \in W^{1,q}(H_\omega) \cap \hat{W}^{-1,q}(H_\omega)$ *and all* $\lambda \in S_\varepsilon$ *there exists a unique solution* $(u, p) \in \mathcal{D}(\Delta_q)^n \times \hat{W}^{1,q}(H_\omega)$ *of* (1.1). *This solution satisfies the a priori estimate* (1.2) *with a constant* $C=C(\omega, q, n, \varepsilon)>0$.

PROOF. Following [12] the problem in H_ω will be reduced to the half space by elementary transformation and perturbation arguments. Let us introduce the transformation $\phi: H_\omega \to \boldsymbol{R}^n_+$ defined by $\tilde{x}=(\tilde{x}', \tilde{x}_n)=\phi(x)=(x', x_n-\omega(x'))$. Obviously ϕ is a bijection with Jacobian equal to 1. For a function or a vector field u on H_ω we define the function or vector field $\tilde{u}$ on $\boldsymbol{R}^n_+$ by $\tilde{u}(\tilde{x})=u(x)$. Further we use the notations $\tilde{\partial}_i=\partial/\partial\tilde{x}_i$, $i=1, \cdots, n$, $\tilde{\nabla}=(\tilde{\nabla}', \tilde{\partial}_n)$, $\tilde{\Delta}$ and $\widetilde{\mathrm{div}}$ for partial differential operators acting on the variables $\tilde{x} \in \boldsymbol{R}^n_+$. Then we obtain the relations

$$\begin{aligned}
\partial_i u &= (\tilde{\partial}_i-(\partial_i\omega)\tilde{\partial}_n)\tilde{u}, \qquad i=1, \cdots, n-1, \\
\Delta u(x) &= [\tilde{\Delta}+|\nabla'\omega|^2\tilde{\partial}_n^2-2(\nabla'\omega, 0)\cdot(\tilde{\nabla}\tilde{\partial}_n)-(\Delta'\omega)\tilde{\partial}_n]\tilde{u}(\phi(x)), \\
\nabla p(x) &= [\tilde{\nabla}-(\nabla'\omega, 0)\tilde{\partial}_n]\tilde{p}(\phi(x)), \\
\mathrm{div}\, u(x) &= [\widetilde{\mathrm{div}}-(\nabla'\omega, 0)\cdot\tilde{\partial}_n]\tilde{u}(\phi(x))
\end{aligned} \tag{3.1}$$

and a similar formula for $\nabla^2 u(x)$. For $u \in W^{2,q}(H_\omega)$ this leads to

$$\begin{aligned}
\|u\|_{L^q(H_\omega)} &= \|\tilde{u}\|_{L^q(R^n_+)}\,, \\
\|\nabla u\|_{L^q(H_\omega)} &\leqq c(1+\|\nabla'\omega\|_\infty)\|\tilde{\nabla}\tilde{u}\|_{L^q(R^n_+)}\,, \\
\|\nabla^2 u\|_{L^q(H_\omega)} &\leqq c(1+\|\nabla'\omega\|_\infty)^2\|\tilde{\nabla}^2\tilde{u}\|_{L^q(R^n_+)}+c\|(\nabla'^2\omega)\tilde{\partial}_n\tilde{u}\|_{L^q(R^n_+)}\,.
\end{aligned} \tag{3.2}$$

Finally, since for $g \in W^{1,q}(H_\omega) \cap \hat{W}^{-1,q}(H_\omega)$ the identity $\int_{H_\omega} g\varphi dx = \int_{R^n_+} \tilde{g}\tilde{\varphi} d\tilde{x}$ holds for all $\varphi \in C_0^\infty(\overline{H}_\omega)$, we obtain

$$\|g\|_{-1,q} \leqq c\|\tilde{g}\|_{\hat{W}^{-1,q}(R^n_+)}\,.$$

The estimate of $\|(\nabla'^2\omega)\tilde{\partial}_n\tilde{u}\|_{L^q(R^n_+)}$ in (3.2) in terms of $\|\tilde{u}\|_{L^q(R^n_+)}$ and $\|\tilde{\nabla}^2\tilde{u}\|_{L^q(R^n_+)}$ is more complicated. By Sobolev's imbedding theorem [**14**] there is a constant $c>0$ such that for all $\delta>0$

$$\|(\nabla'^2\omega)\tilde{\partial}_n\tilde{u}\|_{L^q(R^n_+)} \leqq \frac{c}{\delta}\|\nabla'^2\omega\|_\infty^2\|\tilde{u}\|_{L^q(R^n_+)}+\delta\|\tilde{\nabla}^2\tilde{u}\|_{L^q(R^n_+)}\,. \tag{3.3}$$

If $1<q<n-1$, $n \geqq 3$, we define $s>q$ by $1/(n-1)+1/s=1/q$ and use for each $\tilde{x}_n>0$ the inequality

$$\|\tilde{\partial}_n\tilde{u}(\cdot,\, \tilde{x}_n)\|_{L^s(R^{n-1})} \leqq c\|\tilde{\nabla}'\tilde{\partial}_n\tilde{u}(\cdot,\, \tilde{x}_n)\|_{L^q(R^{n-1})}\,,$$

the constant c being independent of $\tilde{x}_n>0$. If $|\nabla'^2\omega| \in L^{n-1}(\boldsymbol{R}^{n-1})$, then Hölder's inequality yields

$$\begin{aligned}
\|(\nabla'^2\omega)\tilde{\partial}_n\tilde{u}\|_{L^q(R^n_+)}^q &\leqq c\int_0^\infty d\tilde{x}_n\int_{R^{n-1}} dx'|(\nabla'^2\omega)|^q|\tilde{\partial}_n\tilde{u}(x',\, \tilde{x}_n)|^q \\
&\leqq c\|\nabla'^2\omega\|_{L^{n-1}(R^{n-1})}^q\|\tilde{\nabla}'\tilde{\partial}_n\tilde{u}\|_{L^q(R^n_+)}^q\,.
\end{aligned} \tag{3.4}$$

If however $|\cdot||\nabla'^2\omega| \in L^\infty(\boldsymbol{R}^{n-1})$ we use the weighted inequality

$$\||\cdot|^{-1}\tilde{\partial}_n\tilde{u}(\cdot,\, \tilde{x}_n)\|_{L^q(R^{n-1})} \leqq c\|\tilde{\nabla}'\tilde{\partial}_n\tilde{u}(\cdot,\, \tilde{x}_n)\|_{L^q(R^{n-1})}$$

for each $\tilde{x}_n>0$ and obtain

$$\|(\nabla'^2\omega)\tilde{\partial}_n\tilde{u}\|_{L^q(R^n_+)}^q \leqq c\||\cdot|\tilde{\nabla}'^2\omega\|_\infty^q\|\tilde{\nabla}'\tilde{\partial}_n\tilde{u}\|_{L^q(R^n_+)}^q\,. \tag{3.5}$$

To apply our perturbation argument we introduce the Banach spaces

$$\begin{aligned}
X &= \mathcal{D}(\Delta_q)^n \times \hat{W}^{1,q}(H_\omega)\,, \qquad \|(u,\, p)\|_X = \|(\lambda u,\, \nabla^2 u,\, \nabla p)\|_q \\
Y &= L^q(H_\omega)^n \times (W^{1,q}(H_\omega) \cap \hat{W}^{-1,q}(H_\omega))\,, \\
&\qquad \|(f,\, g)\|_Y = \|(f,\, \nabla g)\|_q + \|\lambda g\|_{-1,q}
\end{aligned}$$

and the operator

$$S_{q,\lambda}: X \longrightarrow Y\,, \qquad S_{q,\lambda}(u,\ p) = (\lambda u - \Delta u + \nabla p,\ -\mathrm{div}\ u)\,.$$

Further let $\tilde{X}$, $\tilde{Y}$, $\|\cdot\|_{\tilde{X}}$, $\|\cdot\|_{\tilde{Y}}$ and $\tilde{S}_{q,\lambda}$ denote the corresponding expressions when H_ω, $u(x)$, ∇, etc. are replaced by $\boldsymbol{R}^n_+$, $\tilde{u}(\tilde{x})$, $\tilde{\nabla}$, etc.. Using (3.1) we get the decomposition

$$S_{q,\lambda}(u,\ p)(x) = \tilde{S}_{q,\lambda}(\tilde{u},\ \tilde{p})(\tilde{x}) + R_q(\tilde{u},\ \tilde{p})(\tilde{x})$$

where the remainder R_q is defined on $\tilde{X}$ by

$$R_q(\tilde{u},\ \tilde{p}) = (-|\nabla'\omega|^2\tilde{\partial}_n^2\tilde{u} + 2(\nabla'\omega,\ 0)\cdot\tilde{\nabla}\tilde{\partial}_n\tilde{u}$$
$$+(\Delta'\omega)\tilde{\partial}_n\tilde{u} - (\nabla'\omega,\ 0)\tilde{\partial}_n\tilde{p},\ (\nabla'\omega,\ 0)\cdot\tilde{\partial}_n\tilde{u})\,.$$

Note that $\tilde{S}_{q,\lambda}$ is an isomorphism from $\tilde{X}$ to $\tilde{Y}$ due to Theorem 1.3. Using (3.3), (3.4) or (3.5) we get that

$$\|R_q(\tilde{u},\ \tilde{p})\|_{\tilde{Y}} = k\|\tilde{S}_{q,\lambda}(\tilde{u},\ \tilde{p})\|_{\tilde{Y}} \qquad \text{with } k<1$$

independent of λ provided that $\|\nabla'\omega\|_\infty \leqq K$ and additionally $|\lambda| \geqq \lambda_0$, $\|\nabla'^2\omega\|_{L^{n-1}(R^{n-1})} \leqq K$ or $\||\cdot|\nabla'^2\omega\|_\infty \leqq K$ for positive constants K and λ_0. Due to Kato's perturbation criterion $\tilde{S}_{q,\lambda}+R_q$ is an isomorphism from $\tilde{X}$ to $\tilde{Y}$ and consequently $S_{q,\lambda}$ is an isomorphism from X to Y. Using again (3.3), (3.4) or (3.5) we obtain the estimate

$$\|(u,\ p)\|_X \leqq c_1\|(\tilde{u},\ \tilde{p})\|_{\tilde{X}} \leqq c_2\|\tilde{S}_{q,\lambda}(\tilde{u},\ \tilde{p})\|_{\tilde{Y}}$$
$$\leqq c_3\|(\tilde{S}_{q,\lambda}+R_q)(\tilde{u},\ \tilde{p})\|_{\tilde{Y}} \leqq c_4\|S_{q,\lambda}(u,\ p)\|_Y$$

with constants c_1, c_2, c_3 and c_4 independent of λ. This proves the a priori estimate (1.2) in (i) and (iii). Finally the estimate (1.3) in (i) follows by duality arguments as explained in Section 2.

It remains to prove (ii). For this purpose we repeat the previous proof for both exponents q and s and introduce the notations X_q, Y_q and X_s, Y_s in order to distinguish the spaces X, Y for different exponents. Then we consider the intersection $X_q \cap X_s$ with norm $\|\cdot\|_{X_q \cap X_s} = \|\cdot\|_{X_q} + \|\cdot\|_{X_s}$ and correspondingly $Y_q \cap Y_s$. The same perturbation argument as above now yields that the operator

$$(u,\ p) \longmapsto (\lambda u - \Delta u + \nabla p,\ -\mathrm{div}\ u)$$

is an isomorphism from $X_q \cap X_s$ to $Y_q \cap Y_s$. Therefore, for given $f \in L^q(H_\omega)^n \cap L^s(H_\omega)^n$, $g \in W^{1,q}(H_\omega) \cap W^{1,s}(H_\omega) \cap \hat{W}^{-1,q}(H_\omega) \cap \hat{W}^{-1,s}(H_\omega)$ we find a solution $(\tilde{u},\ \tilde{p}) \in (\mathcal{D}(\Delta_q)^n \cap \mathcal{D}(\Delta_s)^n) \times (\hat{W}^{1,q}(H_\omega) \cap \hat{W}^{1,s}(H_\omega))$ of (1.1). Since the above solution $(u,\ p) \in \mathcal{D}(\Delta_q)^n \cap \hat{W}^{1,q}(H_\omega)$ is unique we obtain $u=\tilde{u}$, $p=\tilde{p}$. This proves the desired regularity property (ii). The proof of Theorem 3.1 is complete. ∎

PROOF OF COROLLARY 1.4. For the case H_α, where $\omega(x')=\alpha|x'|$, we use the condition $\||\cdot|\nabla'^2\omega\|_\infty \leqq K$ in Theorem 3.1 (iii). ∎

4. Proof of Theorem 1.2.

In this section let $\Omega\subseteqq \boldsymbol{R}^n$, $n\geqq 2$, be a domain different from $\boldsymbol{R}^n$ and from $\boldsymbol{R}^n_+$ and satisfying the Assumption 1.1. To prove Theorem 1.2 we use the generalized Stokes operator $S_{q,\lambda}$ as in the previous section and prove some preliminary properties. This operator

$$S_{q,\lambda}:(u,\ p)\longmapsto S_{q,\lambda}(u,\ p)=(\lambda u-\Delta u+\nabla p,\ -\operatorname{div} u)$$

is defined on $\mathcal{D}(S_{q,\lambda})=\mathcal{D}(\Delta_q)^n\times\hat{W}^{1,q}(\Omega)\subseteqq L^q(\Omega)^n\times\hat{W}^{1,q}(\Omega)$ with range $\mathcal{R}(S_{q,\lambda})\subseteqq L^q(\Omega)^n\times\hat{W}^{-1,q}(\Omega)$. Furthermore we define the restriction

$$S^0_{q,\lambda}:(u,\ p)\longmapsto S^0_{q,\lambda}(u,\ p)=\lambda u-\Delta u+\nabla p$$

with $\mathcal{D}(S^0_{q,\lambda})=\{(u,\ p)\in\mathcal{D}(S_{q,\lambda}):\operatorname{div} u=0\}$ and $\mathcal{R}(S^0_{q,\lambda})\subseteqq L^q(\Omega)^n$.

LEMMA 4.1. *Let* $1<q<\infty$, $1\in S_\varepsilon$, $0<\varepsilon<\pi$, *let* $(u,\ p)\in\mathcal{D}(S_{q,\lambda})$ *and* $(f,\ -g)=S_{q,\lambda}(u,\ p)$.

(i) *There exists a bounded subdomain* $G\subseteqq\Omega$ *such that*

$$\|(\lambda u,\ \nabla^2 u,\ \nabla p)\|_q\leqq C(\|(f,\ \nabla g)\|_q+\|\lambda g\|_{-1,q}+\|u\|_{W^{1,q}(G)}+\|p\|_{L^q(G)}+\|\lambda u\|_{[W^{1,q'}(G)]^*}) \tag{4.1}$$

holds true with a constant $C=C(\Omega,\ G,\ q,\ \varepsilon)>0$. *Here* $[W^{1,q'}(G)]^*$ *denotes the dual space of* $W^{1,q'}(G)$ *with* $1/q+1/q'=1$. *If* Ω *is bounded the term* $\|\nabla^2 u\|_q$ *on the left-hand side of* (4.1) *may be replaced by* $\|u\|_{W^{2,q}(\Omega)}$.

(ii) *The operator* $S_{q,\lambda}$ *is injective. For a bounded domain even* $S_{q,0}$ *is injective.*

(iii) *The range* $\mathcal{R}(S_{q,\lambda})$ *is dense in* $L^q(\Omega)^n\times\hat{W}^{-1,q}(\Omega)$ *and* $\mathcal{R}(S^0_{q,\lambda})$ *is dense in* $L^q(\Omega)^n$.

The proof of Lemma 4.1 is based on the localization method by which we reduce the problem to the special cases $\boldsymbol{R}^n$, $\boldsymbol{R}^n_+$ and H_ω where ω even has compact support. Next we will explain this procedure.

First suppose that Ω is an exterior domain. According to the Assumption 1.1 we may choose open balls $B_0=B$ and $B_1,\ \cdots,\ B_m\subseteqq\boldsymbol{R}^n$ and nonnegative cut-off functions $\varphi_0,\ \cdots,\ \varphi_m\in C^\infty(\boldsymbol{R}^n)$ with the following properties:

$$\Omega\setminus B_0=\boldsymbol{R}^n\setminus B_0,\qquad \bar{\Omega}\subseteqq(\boldsymbol{R}^n\setminus\bar{B}_0)\cup\bigcup_{j=1}^m B_j, \tag{4.2}$$

$\varphi_0=0$ in a neighbourhood of $\bar{B}_0$, $\varphi_0=1$ outside of some ball B'_0 with $\bar{B}_0\subseteqq B'_0$, $\operatorname{supp}\varphi_j\subseteqq B_j$ for $1\leqq j\leqq m$ and $\sum_{j=0}^m\varphi_j=1$ in Ω. Since $\partial\Omega\in C^{1,1}$ these balls can be chosen in such a way that we find for each $j\in\{1,\ \cdots,\ m\}$ with $B_j\cap\partial\Omega\neq\emptyset$ (after a translation and rotation of Cartesian coordinates depending on j which for simplicity we will suppress in the following) a function $\omega_j \in C^{1,1}(\boldsymbol{R}^{n-1})$ of compact support such that (with $H_j = H_{\omega_j}$)

$$B_j \cap \Omega \subseteq H_j \quad \text{and} \quad B_j \cap \partial\Omega \subseteq \partial H_j .$$

We want to apply Theorem 3.1 (i) and (ii) on H_j for a finite number of exponents $s=s_k$, $1 \leqq k \leqq k(q)$, to be fixed later on in the proof of Lemma 4.1. Therefore we need that

$$\|\nabla' \omega_j\|_\infty \leqq \min\{K(n, q, \varepsilon), K(n, s_1, \varepsilon), \cdots, K(n, s_{k(q)}, \varepsilon)\}. \tag{4.3}$$

This may be easily achieved by choosing a sufficiently large number of balls B_j such that the support of ω_j is sufficiently small. Finally let us assume that $\bar{B}_j \subseteq \Omega$ if $B_j \cap \partial\Omega = \emptyset$, $1 \leqq j \leqq m$. Summarizing we get two types of balls B_j and cut-off functions φ_j:

$$\text{type } \boldsymbol{R}^n: \varphi_0 \text{ and } \varphi_j \text{ if } \bar{B}_j \subseteq \Omega \quad (1 \leqq j \leqq m)$$

$$\text{type } H_\omega: \varphi_j \text{ if } B_j \cap \partial\Omega \neq \emptyset \quad (1 \leqq j \leqq m).$$

If Ω is a perturbed half space, then we get $\Omega \setminus B_0 = \boldsymbol{R}^n_+ \setminus B_0$ leading to a problem of type $\boldsymbol{R}^n_+$. Finally for a bounded domain the ball B_0 has to be omitted and we are left with a finite number of problems of type $\boldsymbol{R}^n$ and H_ω.

Let (u, p) be a solution of the generalized resolvent problem (1.1) and let $\varphi_j \in C^\infty(\boldsymbol{R}^n)$, $0 \leqq j \leqq m$, be a cut-off function. Then $(\varphi_j u, \varphi_j p)$ satisfies the local equations

$$\begin{aligned} \lambda(\varphi_j u) - \Delta(\varphi_j u) + \nabla(\varphi_j p) &= f_j \\ \operatorname{div}(\varphi_j u) &= g_j \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} f_j &= \varphi_j f - 2(\nabla\varphi_j)\nabla u - (\Delta\varphi_j)u + (\nabla\varphi_j)p \\ g_j &= \varphi_j g + (\nabla\varphi_j)\cdot u . \end{aligned} \tag{4.5}$$

The equation (4.4) may be considered as a generalized resolvent equation for $(\varphi_j u, \varphi_j p)$ on $\boldsymbol{R}^n$, $\boldsymbol{R}^n_+$ or H_ω depending on the type of the function φ_j.

PROOF OF LEMMA 4.1. (i) To prove the a priori estimate (4.1) let $(u, p) \in \mathcal{D}(\Delta_q)^n \times \hat{W}^{1,q}(\Omega)$ and $(f, -g) = S_{q,\lambda}(u, p)$. Using the partition of unity $\{\varphi_j\}_{j=0}^m$ as above we consider the local equations (4.4) if φ_j is of type $\boldsymbol{R}^n$ or $\boldsymbol{R}^n_+$, but

$$\begin{aligned} (\lambda+\lambda_0)(\varphi_j u) - \Delta(\varphi_j u) + \nabla(\varphi_j p) &= f_j + \lambda_0 \varphi_j u \\ \operatorname{div}(\varphi_j u) &= g_j \end{aligned} \tag{4.6}$$

if φ_j is of type H_ω. Here $\lambda_0 > 0$ is chosen sufficiently large such that Theorem 3.1 (i) may be applied. If Ω is unbounded, then let $G \subseteq \Omega$ be a bounded subdomain containing $(\operatorname{supp}\nabla\varphi_j)\cap\Omega$, $j=0, \cdots, m$. However for a bounded domain Ω let $G=\Omega$. Now we apply the a priori estimate (1.2) to the whole space, half space or bended half space problems defined in (4.4), (4.6). For the estimate of the right-hand sides f_j, $f_j+\lambda_0\varphi_j u$ and g_j we only mention how to deal with $g_j=\operatorname{div}(\varphi_j u)\in\hat{W}^{-1,q}(\boldsymbol{R}_+^n)$ if φ_j is of type $\boldsymbol{R}_+^n$; the other estimates are easy. For $\Psi\in C_0^\infty(\bar{\boldsymbol{R}}_+^n)$ let $\tilde{\Psi}=\Psi-h(\Psi)$ where $h(\Psi)=\int_{G_j}\Psi dx/|G_j|$ and $G_j=\operatorname{supp}\nabla\varphi_j$. Since $\int_{G_j}\tilde{\Psi}dx=0$ we get by Poincaré's inequality $\|\nabla(\varphi_j\tilde{\Psi})\|_{q',\Omega}\leqq c_1\|\nabla\Psi\|_{q',\boldsymbol{R}_+^n}$ and $\|(\nabla\varphi_j)\tilde{\Psi}\|_{W^{1,q'}(G)}\leqq c_2\|\nabla\Psi\|_{q',\boldsymbol{R}_+^n}$. Using the identity

$$\begin{aligned}\langle g_j, \Psi\rangle &= \langle\operatorname{div}(u\varphi_j), \Psi\rangle\\ &= -\langle u\varphi_j, \nabla\tilde{\Psi}\rangle = -\langle u, \nabla(\varphi_j\tilde{\Psi})\rangle+\langle u, (\nabla\varphi_j)\tilde{\Psi}\rangle\end{aligned}$$

we conclude that

$$\begin{aligned}\|g_j\|_{\hat{W}^{-1,q}(\boldsymbol{R}_+^n)} &= \sup\left\{\frac{|\langle g_j, \Psi\rangle|}{\|\nabla\Psi\|_{q',\boldsymbol{R}_+^n}} : 0\neq\Psi\in C_0^\infty(\bar{\boldsymbol{R}}_+^n)\right\}\\ &\leqq c_1\sup\left\{\frac{|\langle u, \nabla v\rangle|}{\|\nabla v\|_{q',\Omega}} : 0\neq v\in\hat{W}^{1,q'}(\Omega)\right\}+c_2\|u\|_{[W^{1,q'}(G)]^*}\\ &= c_1\|g\|_{\hat{W}^{-1,q}(\Omega)}+c_2\|u\|_{[W^{1,q'}(G)]^*}.\end{aligned}$$

Consequently $\|\lambda g_j\|_{\hat{W}^{-1,q}(\boldsymbol{R}_+^n)}\leqq c(\|\lambda g\|_{\hat{W}^{-1,q}(\Omega)}+\|\lambda u\|_{[W^{1,q'}(G)]^*})$. Summing up the obtained inequalities for $j=0, \cdots, m$ ($j>0$ if Ω is bounded) we get (4.1). If Ω is bounded the well known estimate $\|u\|_{W^{2,q}(\Omega)}\leqq c\|\nabla^2 u\|_{L^q(\Omega)}$ for $u\in\mathcal{D}(\Delta_q)^n$ yields the additional remark in (i).

(ii) To prove the injectivity of the operator $S_{q,\lambda}$ let $(u, p)\in\mathcal{D}(\Delta_q)^n\times\hat{W}^{1,q}(\Omega)$ and $S_{q,\lambda}(u, p)=0$. If $q=2$ we take the scalar product in $L^2(\Omega)^n$ of $\lambda u-\Delta u+\nabla p=0$ with u, use integration by parts and conclude that $u=0$, $\nabla p=0$ since $\operatorname{div} u=0$. For $q\neq 2$ we will show in a finite number of steps that $u\in\mathcal{D}(\Delta_2)^n$ and $p\in\hat{W}^{1,2}(\Omega)$ which again leads to $u=0$, $\nabla p=0$.

First let $q>2$. If φ_j has the type $\boldsymbol{R}^n$, we consider the local equations (4.4), (4.5) with $f=0$, $g=0$ and use the compactness of $\operatorname{supp}\nabla\varphi_j$ in order to get that $f_j\in L^2(\boldsymbol{R}^n)^n$, $g_j\in W^{1,2}(\boldsymbol{R}^n)\cap\hat{W}^{-1,2}(\boldsymbol{R}^n)$. Then the regularity assertion in Theorem 1.3 (ii) yields $(\varphi_j u, \varphi_j p)\in W^{2,2}(\boldsymbol{R}^n)^n\times\hat{W}^{1,2}(\boldsymbol{R}^n)$. For a cut-off function φ_j of type $\boldsymbol{R}_+^n$ we proceed in an analogous way, and if φ_j has the type H_ω we use (4.6), the compactness of $\operatorname{supp}\varphi_j$ and the regularity assertion of Theorem 3.1 (ii). Thus $(\varphi_j u, \varphi_j p)\in W^{2,2}(H_j)^n\times\hat{W}^{1,2}(H_j)$. Summarizing we get $u\in\mathcal{D}(\Delta_2)^n$ and $p\in\hat{W}^{1,2}(\Omega)$.

If $1<q<2$ we define $s_1>q$ by $1/n+1/s_1=1/q$. Let φ_j be a cut-off function of type H_ω. By Sobolev's imbedding theorem, (4.3) and Theorem 3.1 (ii), we obtain that $f_j\in L^{s_1}(H_j)^n$, $g_j\in W^{1,s_1}(H_j)\cap\hat{W}^{-1,s_1}(H_j)$ and $(\varphi_j u, \varphi_j p)\in\mathcal{D}(\Delta_{s_1})^n\times$

$\hat{W}^{1,s_1}(H_j)$. A similar result holds true if φ_j is of type $\boldsymbol{R}^n$ or $\boldsymbol{R}^n_+$ applying Theorem 1.3 (ii). Thus $(u, p)\in\mathcal{D}(\Delta_{s_1})^n\times\hat{W}^{1,s_1}(\Omega)$. If $s_1<2$ we repeat this procedure a finite number of times getting exponents $q<s_1<\cdots<s_{k(q)}$ with $s_{k(q)}\geqq 2$. Thus the problem is reduced to the case $q=2$ and we get that $u=0$, $\nabla p=0$.

(iii) To show that $\mathcal{R}(S^0_{q,\lambda})$ is dense in $L^q(\Omega)^n$ we start with the case $q=2$. Using the scalar product and Riesz's representation theorem we easily get that each $f\in L^2(\Omega)^n$ has the unique decomposition $(\lambda-\Delta)u+\nabla p=f$ in the sense of distributions with $p\in\hat{W}^{1,2}(\Omega)$ and $u\in W^{1,2}_0(\Omega)^n$ such that $\Delta u\in L^2(\Omega)^n$ and $\operatorname{div} u=0$. To show that $u\in\mathcal{D}(\Delta_2)^n$ we consider the local equations (4.4) or (4.6) where $g=0$ in (4.5). Obviously $f_j\in L^2(H_j)^n$, $g_j\in W^{1,2}(H_j)\cap\hat{W}^{-1,2}(H_j)$ and consequently $(\varphi_j u, \varphi_j p)\in W^{2,2}(H_j)^n\times\hat{W}^{-1,2}(H_j)$ due to Theorem 3.1 if φ_j is of type H_ω. Analogous results hold true if φ_j is of type $\boldsymbol{R}^n$ or $\boldsymbol{R}^n_+$. Summarizing we conclude that $(u, p)\in\mathcal{D}(S^0_{2,\lambda})$. Thus even $\mathcal{R}(S^0_{2,\lambda})=L^2(\Omega)^n$. If $q\neq 2$ let f be an element of the dense subspace $L^2(\Omega)^n\cap L^q(\Omega)^n$ of $L^q(\Omega)^n$. By the previous step there is a unique solution $(u, p)\in\mathcal{D}(S^0_{2,\lambda})$ of the equation $S^0_{2,\lambda}(u, p)=f$. Repeating the regularity arguments of part (i) (with q replaced by 2) we get that $(u, p)\in\mathcal{D}(S^0_{q,\lambda})$. Thus $\mathcal{R}(S^0_{q,\lambda})$ is dense in $L^q(\Omega)^n$.

Finally we show the density of $\mathcal{R}(S_{q,\lambda})$ in $L^q(\Omega)^n\times\hat{W}^{-1,q}(\Omega)$. Let $(f', g')\in L^{q'}(\Omega)^n\times\hat{W}^{1,q'}(\Omega)$, the dual space of $L^q(\Omega)^n\times\hat{W}^{-1,q}(\Omega)$, and suppose that

$$[S_{q,\lambda}(u, p), (f', g')] = \langle\lambda u-\Delta u+\nabla p, f'\rangle-[\operatorname{div} u, g'] = 0$$

for all $(u, p)\in\mathcal{D}(S_{q,\lambda})$. In particular $\langle f, f'\rangle=0$ for $f\in\mathcal{R}(S^0_{q,\lambda})$ yielding $f'=0$ due to the density of $\mathcal{R}(S^0_{q,\lambda})$ in $L^q(\Omega)^n$. Thus $0=-[\operatorname{div} u, g']=\langle u, \nabla g'\rangle$ for all $u\in C_0^\infty(\Omega)^n\subseteqq\mathcal{D}(\Delta_q)^n$. Hence also $g'=0$ in $\hat{W}^{1,q'}(\Omega)$. Now the proof of Lemma 4.1 is complete. ■

Next we show that we may omit the last three terms on the right-hand side of the estimate (4.1) thus preparing the proof of inequality (1.2) and of Theorem 1.2 (i), (ii).

LEMMA 4.2. *Let* $1<q<\infty$, $0<\varepsilon<\pi$, $\lambda\in S_\varepsilon$, *let* $(u, p)\in\mathcal{D}(S_{q,\lambda})$ *and* $(f, -g)=S_{q,\lambda}(u, p)$.

(i) *If* $|\lambda|\geqq\delta>0$, *then* (u, p) *satisfies* (1.2) *with* $C=C(\Omega, q, \varepsilon, \delta)>0$.

(ii) *If* Ω *is bounded, then* (u, p) *satisfies* (1.2) *with* $C=C(\Omega, q, \varepsilon)>0$; *here* $\lambda=0$ *is admitted.*

(iii) *If* Ω *is an exterior domain or a perturbed half space and* $1<q<n/2$, $n\geqq 3$, *then* (u, p) *satisfies* (1.2) *with* $C=C(\Omega, q, \varepsilon)>0$.

PROOF. Assume that under the assumptions (i), (ii) or (iii) the inequality (1.2) is not true. Then we find sequences $(u_j, p_j)\in\mathcal{D}(\Delta_q)^n\times\hat{W}^{1,q}(\Omega)$ and $\lambda_j\in S_\varepsilon$ or ($\lambda_j\in S_\varepsilon\cup\{0\}$ if Ω is bounded), $j\in\boldsymbol{N}$, such that

$$\|\lambda_j u_j\|_q + \|\nabla^2 u_j\|_q + \|\nabla p_j\|_q = 1 \qquad \text{for all } j \in \boldsymbol{N}, \tag{4.7}$$

$$\|f_j\|_q + \|\nabla g_j\|_q + \|\lambda g_j\|_{-1,q} \longrightarrow 0 \qquad \text{as } j \to \infty \tag{4.8}$$

where $(f_j, -g_j) = S_{q,\lambda_j}(u_j, p_j)$. We may suppose that $\int_G p_j dx = 0$ for all $j \in \boldsymbol{N}$ with G as in the proof of Lemma 4.1. Without loss of generality we may also assume that (λ_j) converges to some $\lambda \in \bar{S}_\varepsilon \cup \{\infty\}$ as $j \to \infty$. Since $(\lambda_j u_j)$, $(\nabla^2 u_j)$ and (∇p_j) are bounded sequences in $L^q(\Omega)^n$, $L^q(\Omega)^{n^3}$ and $L^q(\Omega)^n$, respectively, we finally suppose the weak convergence

$$\lambda_j u_j \rightharpoonup v, \quad \nabla^2 u_j \rightharpoonup \nabla^2 \hat{u}, \quad \nabla p_j \rightharpoonup \nabla p \qquad \text{as } j \to \infty \tag{4.9}$$

with some $v \in L^q(\Omega)^n$, some $\hat{u} \in W^{2,q}_{loc}(\Omega)^n$ such that $\nabla^2 \hat{u} \in L^q(\Omega)^{n^3}$, being uniquely determined only up to a linear polynomial, and some $p \in \hat{W}^{1,q}(\Omega)$. Furthermore due to (4.8), we obtain

$$\begin{gathered} v - \Delta \hat{u} + \nabla p = 0, \qquad \nabla \operatorname{div} \hat{u} = 0, \qquad \operatorname{div} v = 0 \quad \text{in } \Omega \\ \text{and} \quad v \cdot N = 0 \qquad \text{on } \partial\Omega, \end{gathered} \tag{4.10}$$

where the latter property is understood in the sense of the trace theorem and N denotes the outward normal vector on $\partial\Omega$. Finally in the bounded domain G we may use the obvious compact imbeddings $W^{2,q}(G) \subseteq W^{1,q}(G)$, $\hat{W}^{1,q}(G) \subseteq L^q_0(G)$ and $L^q(G) \subseteq [W^{1,q'}(G)]^*$. In particular we conclude that the restrictions to G of the sequences $(\lambda_j u_j)$ and (p_j) converge strongly in the spaces $[W^{1,q'}(G)]^*$ and $L^q_0(G)$, respectively. Now we distinguish the three cases $\lambda \neq 0$, $\lambda = \infty$ and $\lambda = 0$.

Let $\lambda_j \to \lambda \neq 0$. Then (4.7) implies that (u_j) is a bounded sequence in $W^{2,q}(\Omega)^n$. Now we get the existence of some $u \in \mathscr{D}(\Delta_q)^n$ such that $u_j \rightharpoonup u$, $\nabla u_j \rightharpoonup \nabla u$ in $L^q(\Omega)$, $\lambda u = v$ and $\nabla^2 u = \nabla^2 \hat{u}$. Hence $S_{q,\lambda}(u, p) = 0$ by (4.10) and $u = 0$, $\nabla p = 0$ due to Lemma 4.1. It follows $p = 0$ since $\int_G p dx = 0$. Finally the compact imbedding $W^{2,q}(G) \subseteq W^{1,q}(G)$ implies that $u_j \to u$ in $W^{1,q}(G)$. Then (4.1) applied to $\lambda_j u_j$ and p_j together with (4.7) and (4.8) yields the contradiction

$$1 \leq c(\|u\|_{W^{1,q}(G)^n} + \|p\|_{L^q(G)} + \|v\|_{[W^{1,q'}(G)]^*})$$

since $u = v = 0$ and $p = 0$.

Let $\lambda_j \to \infty$. Here (4.7) yields $u_j \to 0$ in $L^q(\Omega)^n$ and consequently $\Delta \hat{u} = 0$. Thus (4.10) defines the uniquely determined Helmholtz decomposition of the zero vector field — for details see Lemma 5.3 in the Appendix. Hence $v = 0$, $\nabla p = 0$ leading together with $u_j \to u$ in $W^{1,q}(G)^n$ to the same contradiction as in the previous case. Now (i) is proved.

To prove our Lemma in the cases (ii) or (iii) it remains to consider $\lambda = 0$.

If Ω is a bounded domain, note that in Lemma 4.1 (i) the term $\|\nabla^2 u\|_q$ on the left-hand side of (4.1) may be replaced by $\|u\|_{W^{2,q}(\Omega)}$. Then the proof is completely parallel to above since here $S_{q,0}(u, p)=0$ and $S_{q,0}$ is injective for a bounded domain.

Next let $1<q<n/2$, $n\geqq 3$, and $\Omega\subsetneqq R^n$ be an exterior domain or a perturbed half space. Further let r, s be defined by $1/n+1/r=1/q$ and $1/n+1/s=1/r$. Using Sobolev's imbedding theorem and (4.7) we obtain

$$\|u_j\|_s \leqq c_1\|\nabla u_j\|_r \leqq c_2\|\nabla^2 u_j\|_q \leqq c_3 < \infty .$$

Thus we may assume in addition to (4.9) that

$$u_j \rightharpoonup u \text{ in } L^s(\Omega)^n, \qquad \nabla u_j \rightharpoonup \nabla u \text{ in } L^r(\Omega)^{n^2}, \qquad \nabla^2 u_j \rightharpoonup \nabla^2 u \text{ in } L^q(\Omega)^{n^3}$$

for some $u\in W^{2,q}_{loc}(\Omega)^n$ with $u=0$ on $\partial\Omega$. Since $\lambda_j\to 0$ as $j\to\infty$ we also get that $v=\lambda u=0$. Hence (u, p) solves the Stokes equation

$$-\Delta u+\nabla p = 0, \qquad \operatorname{div} u = 0 \quad \text{in } \Omega$$

with the boundary condition $u=0$ on $\partial\Omega$. Now we argue as in the proof of Lemma 4.1 (ii) but using the regularity assertion of Corollary 2.6 (ii) rather than Theorem 1.3 (ii). After a finite number of steps we arrive at the regularity results

$$\begin{aligned} \|(\nabla^2 u, \nabla p)\|_\alpha < \infty \qquad & \text{for } \alpha\in(1, \infty), \\ \|\nabla u\|_\beta < \infty \qquad & \text{for } \beta\in(n/(n-1), \infty), \end{aligned} \tag{4.11}$$

and

$$\|u\|_\gamma < \infty \qquad \text{for } \gamma\in(n/(n-2), \infty).$$

Here the restrictions on β, γ are caused by the above restrictions on q, r, s. Since $n\geqq 3$ we may define α by $1/n+1/2=1/\alpha$ which leads to $p-C\in L^2(\Omega)$ with some constant C. Further we may choose $\beta=2$ and $\gamma=(1/2-1/n)^{-1}$. Then it is easy to see that there is a sequence $u_k\in C_0^\infty(\Omega)^n$, $k\in N$, with $\|\nabla u-\nabla u_k\|_2\to 0$ as $k\to\infty$. Thus

$$0 = \langle -\Delta u+\nabla p,\ \bar{u}_k\rangle = \langle \nabla u,\ \nabla\bar{u}_k\rangle - \langle p-C,\ \operatorname{div}\bar{u}_k\rangle$$

converges to $\|\nabla u\|_2^2-\langle p-C, 0\rangle=\|\nabla u\|_2^2$ as $k\to\infty$, $\bar{u}_k$ being the conjugate complex value of u_k. Consequently $\nabla u=0$ yielding $u=0$ since $u=0$ on $\partial\Omega$. Furthermore $\nabla p=0$. This will lead to a contradiction in the same way as in the preceding steps and Lemma 4.2 is proved. ■

LEMMA 4.3. *Let* $1<q<\infty$, $0<\varepsilon<\pi$, $\lambda\in S_\varepsilon$, *let* $(u, p)\in\mathcal{D}(S_{q,\lambda})$ *and* $(f, -g) =S_{q,\lambda}(u, p)$.

(i) *If* $|\lambda|\geqq\delta>0$ *then* (u, p) *satisfies* (1.3) *with* $C=C(\Omega, q, \varepsilon, \delta)>0$.

(ii) *If* Ω *is bounded, then* (u, p) *satisfies* (1.3) *with* $C=C(\Omega, q, \varepsilon)>0$; *here*

$\lambda=0$ *is admitted.*

(iii) *If* Ω *is an exterior domain or a perturbed half space and* $q>n/(n-2)$, $n\geqq 3$, *then* (u, p) *satisfies* (1.3) *with* $C=C(\Omega, q, \varepsilon)>0$.

PROOF. The proof is based on a duality argument which we already used in the proof of Theorem 1.3 for $\Omega=\boldsymbol{R}^n$; for this argument we only need the density of $\mathcal{R}(S_{q',\lambda}^0)$ in $L^{q'}(\Omega)^n$, see Lemma 4.1 (iii), and the a priori estimate (1.2) for the dual exponent $q'=q/(q-1)$. Thus the assertions (i), (ii) and (iii) are a consequence of (i), (ii) and (iii) in Lemma 4.2. ■

The next lemma yields a further information on $\mathcal{R}(S_{q,\lambda})$.

LEMMA 4.4. *Let* $1<q<\infty$ *and* $\lambda\in S_\varepsilon$, $0<\varepsilon<\pi$. *Then* $\mathcal{R}(S_{q,\lambda}^0)=L^q(\Omega)^n$ *and* $\mathcal{R}(S_{q,\lambda})=L^q(\Omega)^n\times\text{div}\,\mathcal{D}(\Delta_q)^n$ *where* $\text{div}\,\mathcal{D}(\Delta_q)^n=\{\text{div}\,u : u\in\mathcal{D}(\Delta_q)^n\}$.

PROOF. Due to the a priori estimate (1.2), see Lemma 4.2 (i), the operator $S_{q,\lambda}^0$ has a closed range which is dense in $L^q(\Omega)^n$ by Lemma 4.1 (iii). Thus $\mathcal{R}(S_{q,\lambda}^0)=L^q(\Omega)^n$. To prove the second assertion let $(f, g)\in L^q(\Omega)^n\times\text{div}\,\mathcal{D}(\Delta_q)^n$. Thus there is some $u_0\in\mathcal{D}(\Delta_q)^n$ with $g=\text{div}\,u_0$, and in particular we get $g\in\hat{W}^{-1,q}(\Omega)$. By the first assertion there exists some $(u_1, p)\in\mathcal{D}(\Delta_q)^n\times\hat{W}^{1,q}(\Omega)$ with

$$\lambda u_1-\Delta u_1+\nabla p = f-(\lambda u_0-\Delta u_0), \qquad \text{div}\,u_1 = 0 \quad \text{in } \Omega .$$

Then $(u_0+u_1, p)\in\mathcal{D}(S_{q,\lambda})$ and $S_{q,\lambda}(u_0+u_1, p)=(f, -g)$. This proves the second assertion.

Summarizing the results of Lemma 4.1, 4.2, 4.3, 4.4 and using Lemma 5.5 in Appendix we see that the proof of Theorem 1.2 is complete. For applications to the Stokes operator we need further properties of the operator $S_{q,\lambda}$ and its adjoint $S_{q,\lambda}^*$. Let $\lambda-\Delta_q$ denote the operator $\lambda-\Delta$ with domain of definition $\mathcal{D}(\Delta_q)=W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)\subseteqq L^q(\Omega)$ and range in $L^q(\Omega)$.

COROLLARY 4.5. *Let* $1<q<\infty$, $0<\varepsilon<\pi$, $\lambda\in S_\varepsilon$ *and let* $\Omega\subseteqq\boldsymbol{R}^n$, $n\geqq 2$, *be a domain satisfying Assumption* 1.1.

(i) *The operator* $\lambda-\Delta_q$ *is closed and* $(\lambda-\Delta_q)^*=\lambda-\Delta_{q'}$. *Further* $\lambda-\Delta_q : \mathcal{D}(\Delta_q)\to L^q(\Omega)$ *is bijective and its inverse satisfies the inequality*

$$\|(\lambda-\Delta_q)^{-1}\| \leqq C/|\lambda| \qquad \text{for all } \lambda\in S_\varepsilon ,$$

where $C=C(\Omega, q, \varepsilon)>0$ *is independent of* λ *except for the case that* Ω *is an exterior domain or a perturbed half space in* $\boldsymbol{R}^2$; *in this case* $C=C(\Omega, q, \varepsilon, \delta)>0$ *for* $\lambda\in S_\varepsilon$, $|\lambda|\geqq\delta>0$.

(ii) *The operator* $S_{q,\lambda}$ *is closed and* $S_{q,\lambda}^*=S_{q',\lambda}$.

PROOF. Assertion (i) is well known for bounded and exterior domains.

However, for all domains satisfying Assumption 1.1 we can give the following proof of (i). First we show the estimate

$$\|\lambda u\|_q + \|\Delta u\|_q \leqq C(\Omega, q, \varepsilon, \delta)\|(\lambda - \Delta)u\|_q$$

for $u \in \mathcal{D}(\Delta_q)$, $\lambda \in S_\varepsilon$, $|\lambda| \geqq \delta > 0$, in the same way as in Lemma 4.2 (i) for $S_{q,\lambda}$. It follows that $\lambda - \Delta_q$ is an injective and closed operator with closed range. As in Lemma 4.1 (iii) we see that the range $\mathcal{R}(\lambda - \Delta_q)$ is dense in $L^q(\Omega)$. Consequently $\lambda - \Delta_q$ is a bijection and $(\lambda - \Delta_q)^* = \lambda - \Delta_{q'}$ due to the closed range theorem. Further we obtain the corresponding results as in Lemma 4.2 (ii) and (iii). Finally an interpolation argument as in the proof of Corollary 4.6 below implies that $C = C(\Omega, q, \varepsilon, \delta)$ above does not depend on δ if Ω is not an exterior domain or a perturbed half space in $\boldsymbol{R}^2$; this is caused by the restriction $n \geqq 3$ in Lemma 4.2 (iii).

(ii) To show that $S_{q,\lambda}$ is closed let $(u_j, p_j) \in \mathcal{D}(S_{q,\lambda})$, $j \in \boldsymbol{N}$, be convergent in $L^q(\Omega)^n \times \hat{W}^{1,q}(\Omega)$ to some $(u, p) \in L^q(\Omega)^n \times \hat{W}^{1,q}(\Omega)$ and let $(f_j, -g_j) = S_{q,\lambda}(u_j, p_j)$ converge to $(f, -g) \in L^q(\Omega)^n \times \hat{W}^{-1,q}(\Omega)$ as $j \to \infty$. Then $-\Delta u_j = f_j - \lambda u_j - \nabla p_j$ converges to $f - \lambda u - \nabla p$ in $L^q(\Omega)^n$ as $j \to \infty$. Since $u_j \in \mathcal{D}(\Delta_q)^n$ and $\lambda - \Delta_q$ is closed due to part (i) we conclude that $u \in \mathcal{D}(\Delta_q)^n$, $(u, p) \in \mathcal{D}(S_{q,\lambda})$ and that $S_{q,\lambda}(u, p) = (f, -g)$. Hence $S_{q,\lambda}$ is closed. Let $(u, p) \in \mathcal{D}(S_{q,\lambda})$ and $(\tilde{u}, \tilde{p}) \in \mathcal{D}(S_{q',\lambda})$. Then a simple approximation argument and integration by parts yield

$$\begin{aligned}[S_{q,\lambda}(u, p), (\tilde{u}, \tilde{p})] &= \langle \lambda u - \Delta u + \nabla p, \tilde{u}\rangle - [\operatorname{div} u, \tilde{p}] \\ &= \langle u, \lambda\tilde{u} - \Delta\tilde{u} + \nabla\tilde{p}\rangle - [p, \operatorname{div} \tilde{u}] \\ &= [(u, p), S_{q',\lambda}(\tilde{u}, \tilde{p})] .\end{aligned}$$

Hence $S_{q,\lambda} \subseteqq S^*_{q',\lambda}$. To prove the other inclusion let $(u, p) \in \mathcal{D}(S^*_{q',\lambda}) \subseteqq [L^{q'}(\Omega)^n \times \hat{W}^{-1,q'}(\Omega)]^* = L^q(\Omega)^n \times \hat{W}^{1,q}(\Omega)$. Then by definition the mapping

$$\begin{aligned}(\tilde{u}, \tilde{p}) \longmapsto [S_{q',\lambda}(\tilde{u}, \tilde{p}), (u, p)] &= \langle \lambda\tilde{u} - \Delta\tilde{u} + \nabla\tilde{p}, u\rangle - [\operatorname{div} \tilde{u}, p] \\ &= \langle \lambda\tilde{u} - \Delta\tilde{u} + \nabla\tilde{p}, u\rangle + \langle \tilde{u}, \nabla p\rangle\end{aligned}$$

defined for $(\tilde{u}, \tilde{p}) \in \mathcal{D}(S_{q',\lambda})$ has a continuous extension to all of $L^{q'}(\Omega)^n \times \hat{W}^{1,q'}(\Omega)$. Since $u, \nabla p \in L^q(\Omega)^n$ and $\tilde{u}, \nabla\tilde{p} \in L^{q'}(\Omega)^n$ we get that also the mapping $\tilde{u} \mapsto \langle(\lambda - \Delta)\tilde{u}, u\rangle$ on $\mathcal{D}(\Delta_{q'})^n$ has a continuous extension to $L^{q'}(\Omega)^n$. Thus $u \in \mathcal{D}((\lambda - \Delta_{q'})^*)^n = \mathcal{D}(\lambda - \Delta_q)^n = \mathcal{D}(\Delta_q)^n$ and $(u, p) \in \mathcal{D}(S_{q,\lambda})$. Consequently $S_{q,\lambda} = S^*_{q,\lambda}$. ■

It is not difficult to see that $S_{q,\lambda}$ is not a surjective operator. Let us discuss some consequences of our results for the Stokes operator where we restrict ourselves to solenoidal vector fields. First we consider the Stokes resolvent problem

$$\begin{aligned} \lambda u-\Delta u+\nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega . \end{aligned} \tag{4.12}$$

COROLLARY 4.6. *Let* $1<q<\infty$, $0<\varepsilon<\pi$, $\lambda\in S_\varepsilon$ *and let* $\Omega\subseteqq \boldsymbol{R}^n$, $n\geqq 2$, *be a domain satisfying the Assumption* 1.1. *Then for all* $f\in L^q(\Omega^n)$ *there is a unique solution* $(u, p)\in\mathcal{D}(\Delta_q)^n\times\hat{W}^{1,q}(\Omega)$ *of* (4.12) *and*

$$\|\lambda u\|_q+\|-\Delta u+\nabla p\|_q \leqq C\|f\|_q \tag{4.13}$$

with some constant $C=C(\Omega, q, \varepsilon)>0$ *excepting* Ω *is an exterior domain or a perturbed half space in* $\boldsymbol{R}^2$ *where* (4.13) *holds for all* $\lambda\in S_\varepsilon$, $|\lambda|\geqq\delta>0$ *with* $C=C(\Omega, q, \varepsilon, \delta)>0$.

PROOF. By Theorem 1.2 (i) we get the unique solvability of (4.12) thus defining a linear operator $T_{q,\lambda}: L^q(\Omega)^n\to L^q(\Omega)^n$ with $u=T_{q,\lambda}f$. Due to Lemma 4.2 and 4.3 the operator $T_{q,\lambda}$ is bounded with norm $\|T_{q,\lambda}\|\leqq C/|\lambda|$ where $C=C(\Omega, q, \varepsilon, \delta)$. However we get $C=C(\Omega, q, \varepsilon)$ for a bounded domain, for an exterior domain and for a perturbed half space if $1<q<n/2$ or $n/(n-2)<q<\infty$ $(n\geqq 3)$. Then the Riesz-Thorin interpolation theorem, see [23], [27], proves the assertion. ■

Finally we prove Corollary 1.6 concerning the Stokes operator $A_q=-P_q\Delta$; for the definition of the Helmholtz projection P_q and the space $L^q_\sigma(\Omega)$ see Introduction and Appendix.

PROOF OF COROLLARY 1.6. Solving the resolvent equation $(\lambda+A_q)u=f$, $f\in L^q_\sigma(\Omega)$, may be reduced to (4.12). Then the estimate $\|(\lambda+A_q)^{-1}\|\leqq C/|\lambda|$ follows from (4.13). In particular $(\lambda+A_q)^{-1}$ is closed. To show that $A_q^*=A_{q'}$ let $f\in L^q_\sigma(\Omega)$ and $\tilde{f}\in L^{q'}_\sigma(\Omega)$. Due to Corollary 4.5 define $(u, p)\in\mathcal{D}(S^0_{q,\lambda})$ and $(\tilde{u}, \tilde{p})\in\mathcal{D}(S^0_{q',\lambda})$ by $f=(\lambda-\Delta)u+\nabla p$ and $\tilde{f}=(\lambda-\Delta)\tilde{u}+\nabla\tilde{p}$. Then we see that

$$\langle(\lambda+A_q)^{-1}f, \tilde{f}\rangle=\langle f, (\lambda+A_{q'})^{-1}\tilde{f}\rangle$$

proving $[(\lambda+A_q)^{-1}]^*=(\lambda+A_{q'})^{-1}$ and $A_q^*=A_{q'}$. ■

5. Appendix.

First we discuss the space $\hat{W}^{1,q}(\Omega)=\{u\in L^q_{\mathrm{loc}}(\bar{\Omega}): \nabla u\in L^q(\Omega)^n\}$, $1<q<\infty$, with norm $\|u\|_{\hat{W}^{1,q}(\Omega)}=\|\nabla u\|_q$ where we have to identify two elements which differ by a constant. If Ω is bounded, $L^q_{\mathrm{loc}}(\bar{\Omega})$ may be replaced by $L^q(\Omega)$. Considering any Cauchy sequence (u_i) in this space it is easy to see that we can choose constants c_i, $i\in\boldsymbol{N}$, such that (u_i+c_i) converges in $L^q_{\mathrm{loc}}(\bar{\Omega})$ to some $u\in$

$L^q_{\mathrm{loc}}(\bar{\Omega})$ and ∇u_i converges to $\nabla u \in L^q(\Omega)^n$ as $i \to \infty$. Thus $\hat{W}^{1,q}(\Omega)$ is a Banach space which is even reflexive since it is isometric to the closed subspace $\{\nabla u \in L^q(\Omega)^n : u \in L^q_{\mathrm{loc}}(\bar{\Omega})\}$ of $L^q(\Omega)^n$.

Consider the dual space $\hat{W}^{-1,q}(\Omega) = [\hat{W}^{1,q'}(\Omega)]^*$ of $\hat{W}^{1,q'}(\Omega)$ where $1/q + 1/q' = 1$. Let $g : v \mapsto [g, v]$ be any element of $\hat{W}^{-1,q}(\Omega)$, i.e., any functional on $\hat{W}^{1,q'}(\Omega)$ being continuous under $\|\nabla \cdot\|_{q'}$. Using the isometric imbedding above and the Hahn-Banach theorem we see that there exists some $u \in L^q(\Omega)^n$ such that $[g, v] = \langle -u, \nabla v\rangle$ for all $v \in \hat{W}^{1,q'}(\Omega)$ and $\|u\|_q = \|g\|_{-1,q}$. The reflexivity of $\hat{W}^{1,q}(\Omega)$ yields

$$[\hat{W}^{1,q}(\Omega)]^{**} = [\hat{W}^{-1,q'}(\Omega)]^* = \hat{W}^{1,q}(\Omega) .$$

Next we prove that $C_0^\infty(\bar{\Omega}) = \{u|_\Omega : u \in C_0^\infty(\boldsymbol{R}^n)\}$ is a dense subspace of $\hat{W}^{1,q}(\Omega)$.

LEMMA 5.1. *Let $1 < q < \infty$ and let $\Omega = \boldsymbol{R}^n$ or let $\Omega \subsetneqq \boldsymbol{R}^n$, $n \geqq 2$, be an unbounded domain satisfying the Assumption* 1.1. *Then for each $u \in \hat{W}^{1,q}(\boldsymbol{R}^n)$ there exists a sequence $u_j \in C_0^\infty(\bar{\Omega})$, $j = 1, 2, \cdots$, with $\lim_{j\to\infty} \|\nabla u - \nabla u_j\|_{L^q(\Omega)} = 0$. Therefore $\overline{C_0^\infty(\bar{\Omega})}^{\|\nabla\cdot\|_q} = \hat{W}^{1,q}(\Omega)$.*

PROOF. First we consider the case $\Omega = \boldsymbol{R}^n$. Here we use the well known Helmholtz projection $P_q : L^q(\boldsymbol{R}^n) \to L^q_\sigma(\boldsymbol{R}^n)$, see Lemma 5.3 for details. It is sufficient to show that $C_0^\infty(\boldsymbol{R}^n)$ is dense in $\hat{W}^{1,q}(\boldsymbol{R}^n)$ under the norm $\|\nabla \cdot\|_q$. Due to the Hahn-Banach theorem, each linear continuous functional $F : u \mapsto [F, u]$ defined on $\hat{W}^{1,q}(\boldsymbol{R}^n)$ has the form $[F, u] = \langle \tilde{F}, \nabla u\rangle$ with some $\tilde{F} \in L^{q'}(\boldsymbol{R}^n)^n$. Supposing $[F, u] = 0$ for all $u \in C_0^\infty(\boldsymbol{R}^n)$ yields

$$0 = \langle \tilde{F}, \nabla u\rangle = \langle \tilde{F}_0 + \nabla p, \nabla u\rangle = \langle \nabla p, \nabla u\rangle = -[\Delta p, u]$$

for all $u \in C_0^\infty(\boldsymbol{R}^n)$ with $\tilde{F}_0 = P_{q'}\tilde{F} \in L^{q'}_\sigma(\boldsymbol{R}^n)$, $\nabla p \in L^{q'}(\boldsymbol{R}^n)^n$. From Weyl's lemma we get that p and therefore ∇p are harmonic on $\boldsymbol{R}^n$. Since $\nabla p \in L^{q'}(\boldsymbol{R}^n)^n$ we conclude that $\nabla p = 0$. Using $\tilde{F}_0 \in L^{q'}_\sigma(\boldsymbol{R}^n) = \overline{C^\infty_{0,\sigma}(\boldsymbol{R}^n)}^{\|\cdot\|_{q'}}$ we see that $[F, u] = \langle \tilde{F}, \nabla u\rangle = \langle \tilde{F}_0, \nabla u\rangle = 0$ even for all $u \in \hat{W}^{1,q}(\boldsymbol{R}^n)$. This proves the assertion for $\Omega = \boldsymbol{R}^n$.

Consider now the case $\Omega \neq \boldsymbol{R}^n$ and let $u \in \hat{W}^{1,q}(\Omega)$. Then we can construct an extension $\tilde{u} \in \hat{W}^{1,q}(\boldsymbol{R}^n)$ of u such that $\tilde{u}|_\Omega = u$. If $\Omega = \boldsymbol{R}^n_+$ we define $\tilde{u}$ to be the even extension of u. For the bended half space H_ω we use the transformation to the half space as in Section 3 to get this extension. In the general case we use the cut-off functions $\varphi_0, \cdots, \varphi_m$ as in Section 4 with $1 = \sum_{i=0}^m \varphi_i$ on Ω, write $u = \sum_{i=0}^m u_i$ with $u_i = \varphi_i u$, choose extensions $\tilde{u}_i$ as above and put $\tilde{u} = \sum_{i=0}^m \tilde{u}_i$. Given the extension $\tilde{u} \in \hat{W}^{1,q}(\boldsymbol{R}^n)$ of $u \in \hat{W}^{1,q}(\Omega)$ we find $u_j \in C_0^\infty(\boldsymbol{R}^n)$ with $\|\nabla \tilde{u} - \nabla u_j\|_{q,\boldsymbol{R}^n} \to 0$ as $j \to \infty$. Then $u_j|_\Omega \in C_0^\infty(\bar{\Omega})$, $j \in \boldsymbol{N}$, defines an approximating sequence of u. ∎

We consider a special subset of $\hat{W}^{-1,q}(\Omega)$ by identifying $g \in L^1_{\mathrm{loc}}(\bar{\Omega})$ with the functional

$$\langle g, \cdot\rangle : v \longmapsto \langle g, v\rangle = \int_\Omega gv\,dx\,, \qquad v\in C_0^\infty(\bar{\Omega})\,.$$

In the next lemma we give some sufficient conditions under which $\langle g, \cdot\rangle$ is continuous with respect to $\|\nabla\cdot\|_{q'}$; then it extends to a well defined element of $\hat{W}^{-1,q}(\Omega)$. The following example shows that even for $g\in C_0^\infty(\Omega)$ this is not always true. Consider an unbounded domain Ω, let $\int_\Omega g\,dx\neq 0$ and $q'\geqq n$, i.e., $q\leqq n/(n-1)$. Then we know using a cut-off procedure that there exist $v_i\in C_0^\infty(\boldsymbol{R}^n)$, $i\in\boldsymbol{N}$, with $\lim_{i\to\infty}\|\nabla v_i\|_{q',\boldsymbol{R}^n}=0$ and $\lim_{i\to\infty}\langle g, v_i\rangle=\int_\Omega g\,dx\neq 0$. Thus $\langle g, \cdot\rangle$ is not continuous under $\|\nabla\cdot\|_{q'}$. In particular we see that $W^{1,q}(\Omega)$ is not contained in $\hat{W}^{-1,q}(\Omega)$ if Ω is unbounded and that the condition

$$g\in W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)=\{g\in W^{1,q}(\Omega) : \langle g, \cdot\rangle \text{ continuous under } \|\nabla\cdot\|_{q'}\}$$

is a strong restriction on the divergence g in Theorem 1.2.

LEMMA 5.2. *Let $\Omega=\boldsymbol{R}^n$ or let $\Omega\subseteqq\boldsymbol{R}^n$, $n\geqq 2$, be an unbounded domain satisfying Assumption* 1.1 *and let $g\in L_{\mathrm{loc}}^q(\bar{\Omega})$. Then each of the following conditions* (i), (ii) *or* (iii) *implies that the functional*

$$\langle g, \cdot\rangle : v \longmapsto \langle g, v\rangle = \int_\Omega gv\,dx\,, \qquad v\in C_0^\infty(\bar{\Omega})\,,$$

is continuous under the norm $\|\nabla v\|_{q'}$.

(i) $1<q<\infty$, $g\in L^q(\Omega)$ *with compact support in $\bar{\Omega}$ and* $\int_\Omega g\,dx=0$;

(ii) $q>n/(n-1)$, $g\in L^s(\Omega)$ *where s is defined by* $1/n+1/q=1/s$;

(iii) $q>n/(n-1)$, $|\cdot-x_0|g(\cdot)\in L^q(\Omega)$ *for some* $x_0\in\boldsymbol{R}^n$.

PROOF. To prove the assertion if (i) is satisfied we choose some ball $B'\subseteqq\boldsymbol{R}^n$ with $\operatorname{supp} g\subseteqq B=B'\cap\bar{\Omega}$. Then Poincaré's inequality yields $\int_B|v-v_B|^{q'}dx\leqq c\int_B|\nabla v|^{q'}dx$ where $v\in C_0^\infty(\bar{\Omega})$ and $v_B=\int_B v\,dx/|B|$. This leads to $\left|\int_\Omega gv\,dx\right|=\left|\int_B g(v-v_B)dx\right|\leqq c\|g\|_q\|\nabla v\|_{q'}$ which proves the assertion. If (ii) is satisfied we have $1<q'<n$, and by Sobolev's inequality it follows that $\int\left|_\Omega gv\,dx\right|\leqq\|g\|_s\|v\|_{s'}\leqq c\|g\|\,\|\nabla v\|_{q'}$ where $1/n+1/s'=1/q'$, $1/s+1/s'=1$. In case (iii) we use the weighted inequality $\int_\Omega(|v(x)|^{q'}/|x-x_0|^{q'})dx\leqq c\int_\Omega|\nabla v|^{q'}dx$ and get that

$$\left|\int_\Omega gv\,dx\right|=\left|\int_\Omega g(x)|x-x_0|\frac{v}{|x-x_0|}dx\right|\leqq c\|g|\cdot-x_0|\|_q\|\nabla v\|_{q'}\,.$$

This proves Lemma 5.2. ■

In this paper we need some facts on the Helmholtz decomposition which are well known at least for the whole space, the half space and bounded or exterior domains with sufficiently smooth boundary [**14**], [**21**], [**22**], [**24**], [**25**]. Here we sketch a rather elementary proof based on the localization method of Section 4, see [**24**] for details. Recall that $L_\sigma^q(\Omega)=\overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$ where $C_{0,\sigma}^\infty(\Omega)=\{u\in C_0^\infty(\Omega)^n : \operatorname{div} u=0\}$.

LEMMA 5.3. *Let $1<q<\infty$ and let $\Omega=\boldsymbol{R}^n$ or let $\Omega\subseteqq\boldsymbol{R}^n$, $n\geqq 2$, be a domain satisfying Assumption* 1.1. *Then there exists a linear bounded projection operator P_q from $L^q(\Omega)^n$ onto $L_\sigma^q(\Omega)$ with null space $N(P_q)=\{\nabla p\in L^q(\Omega)^n : p\in L_{\mathrm{loc}}^q\}$. In particular, each $f\in L^q(\Omega)^n$ has a unique decomposition $f=f_0+\nabla p$ with $f_0=P_q f\in L_\sigma^q(\Omega)$, $\nabla p\in N(P_q)$ and $\|f_0\|_q+\|\nabla p\|_q\leqq c\|f\|_q$ where $c=c(\Omega, q)>0$. Furthermore $L_\sigma^q(\Omega)^*$ may be identified with $L_\sigma^{q'}(\Omega)$ and we get $P_q^*=P_{q'}$ where $1/q+1/q'=1$.*

PROOF. The existence of the Helmholtz projection P_q follows from the unique solvability of the Neumann problem

$$\Delta p = \operatorname{div} f, \qquad N\cdot(f-\nabla p)|_{\partial\Omega}=0 \tag{5.1}$$

with $\nabla p\in N(P_q)$ for given $f\in L^q(\Omega)^n$. Here N denotes the exterior normal vector on $\partial\Omega$ and the last condition is understood in the sense of the trace lemma which is based on Gauss's integral theorem, see [**15**], [**24**]. The solution theory for (5.1) rests on the variational inequality [**24**]

$$\|\nabla p\|_q \leqq c \sup_{0\neq v\in\hat{W}^{1,q'}(\Omega)} |\langle\nabla p, \nabla v\rangle|/\|\nabla v\|_{q'} \tag{5.2}$$

for all $p\in\hat{W}^{1,q}(\Omega)$. For $\Omega=\boldsymbol{R}^n$ the inequality (5.2) follows from the Calderón-Zygmund estimate; see the proof of Theorem 1.3. The case $\Omega=\boldsymbol{R}_+^n$ can easily be reduced to $\boldsymbol{R}^n$ by the reflection principle. For the bended half space H_ω (5.2) follows by using the same transformation and perturbation argument as in Section 3; here we only need that $\|\nabla'\omega\|_\infty$ is sufficiently small, see [**24**]. In the general case we use the same localization method as in Section 4 with the cut-off functions φ_j; here the local equations are of the form

$$\Delta(\varphi_j p) = \varphi_j \operatorname{div} f+2(\nabla\varphi_j)(\nabla p)+(\Delta\varphi_j)p \tag{5.3}$$

for $j=0, 1, \cdots, m$. Just as in Lemma 4.1 we first get the weaker estimate

$$\|\nabla p\|_q \leqq c_1\Big(\sup_{0\neq v\in\hat{W}^{1,q'}(\Omega)} |\langle\nabla p, \nabla v\rangle|/\|\nabla v\|_{q'}\Big) \tag{5.4}$$

$$+c_2(\|p\|_{L^q(G)}+\|\nabla p\|_{[W^{1,q'}(G)]^*})$$

with two additional terms on the right; here G is a bounded domain as in (4.1). Then the compactness argument in the proof of Lemma 4.2 shows that the last two terms in (5.4) may be omitted; this leads to (5.2). Note that the regularity assumption $\partial\Omega\in C^1$ is sufficient for proving (5.4).

By a standard argument we conclude from (5.2) that $p\mapsto\langle\nabla p, \nabla\cdot\rangle$ defines an isomorphism from $\hat{W}^{1,q}(\Omega)$ onto $[\hat{W}^{1,q'}(\Omega)]^*$. Thus for given $f\in L^q(\Omega)^n$ there exists a unique $p\in\hat{W}^{1,q}(\Omega)$ with $\langle\nabla p, \nabla v\rangle=\langle f, \nabla v\rangle$ for all $v\in\hat{W}^{1,q'}(\Omega)$. Then $f-\nabla p\in X_q\equiv\{u\in L^q(\Omega)^n : \langle u, \nabla v\rangle=0$ for all $v\in\hat{W}^{1,q'}(\Omega)\}$ and we get the direct decomposition $f=f_0+\nabla p$ with $f_0\in X_q$. Observing the symmetry property $\langle\nabla p, \nabla v\rangle=\langle\nabla v, \nabla p\rangle$ we see that the dual space X_q^* of X_q coincides with $X_{q'}$. Next we show that $C_{0,\sigma}^\infty(\Omega)$ is dense in X_q. Indeed, consider any $h\in X_{q'}$ with $\langle w, h\rangle=0$ for all $\omega\in C_{0,\sigma}^\infty(\Omega)$. Then we conclude that $h=\nabla\Psi$ with some $\Psi\in L_{\mathrm{loc}}^{q'}(\bar{\Omega})$ by de Rham's well known argument [**10**]. Therefore $\langle w, \nabla\phi\rangle=0$ even for all $w\in X_q$. In this case we can replace de Rham's argument by the following elementary consideration given in [**24**]. Let $\langle w, h\rangle=0$ for all $w\in C_{0,\sigma}^\infty(\Omega)$ as above. Then a mollification procedure yields a sequence h_i, $i\in\boldsymbol{N}$, of smooth functions such that the line integral $\int_\Gamma h_i ds=0$ for each piecewise smooth closed curve Γ in Ω. By a classical argument $h_i=\nabla\Psi_i$, and letting $i\to\infty$ we get $h=\nabla\Psi$ as above. So we conclude that $X_q=L_\sigma^q(\Omega)$, $L_\sigma^q(\Omega)^*=L_\sigma^{q'}(\Omega)$ and $P_q^*=P_{q'}$. This proves Lemma 5.3. ∎

The next lemma yields a regularity property of the Helmholtz decomposition which is needed to characterize the space $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$.

LEMMA 5.4. *Let $1<q<\infty$ and let $\Omega=\boldsymbol{R}^n$ or let $\Omega\subseteqq\boldsymbol{R}^n$, $n\geqq 2$, be a domain satisfying Assumption 1.1. Suppose $f\in L^q(\Omega)^n$, $\nabla\operatorname{div} f\in L^q(\Omega)^n$ and $N\cdot f|_{\partial\Omega}=0$ if $\partial\Omega\neq\emptyset$ and consider the Helmholtz decomposition $f=f_0+\nabla p$ with $f_0\in L_\sigma^q(\Omega)$ and $\nabla p\in L^q(\Omega)^n$, $p\in L_{\mathrm{loc}}^q(\bar{\Omega})$. Then $\nabla^2 p\in L^q(\Omega)^{n^2}$ and $\operatorname{div} f\in L^q(\Omega)$.*

PROOF. First note that $N\cdot f|_{\partial\Omega}$ well defined by the trace theorem since $f\in L^q(\Omega)^n$ and $\operatorname{div} f\in L_{\mathrm{loc}}^q(\bar{\Omega})$ which follows from $\nabla\operatorname{div} f\in L^q(\Omega)^n$; see [**15**], [**24**]. For the proof of the lemma we use the local equations (5.3) and the same notations as in Section 4. Let φ be any of the cut-off functions φ_j, $j=0, \cdots, m$, with compact support and suppose that the local equation (5.3) is an equation on some bended half space H_ω. Then we get $\hat{f}=\varphi\operatorname{div} f+2(\nabla\varphi)(\nabla p)+(\Delta\varphi)p\in L^q(H_\omega)$ for the right hand side in (5.3), and $p_\varphi=\varphi p$ is a weak solution of the Neumann problem

$$\Delta p_\varphi=\hat{f}\quad\text{in } H_\omega\,,\qquad N\cdot\nabla p_\varphi|_{\partial H_\omega}=N\cdot(\nabla\varphi)p|_{\partial H_\omega}\,.$$

Using $\partial\Omega\in C^{1,1}$, $\hat{f}\in L^q(H_\omega)$ and the compactness of $\operatorname{supp}\varphi$ we will show by a well known procedure of elliptic regularity theory that even $\|\nabla^2 p_\varphi\|_q<\infty$; the details are explained as follows.

A calculation shows that this Neumann problem on H_ω has the variational formulation

$$\langle \nabla p_\varphi, \nabla v\rangle = [F, v], \qquad v\in \hat{W}^{1,q'}(H_\omega),$$

where

$$[F, v] = \langle(\nabla\varphi)p, \nabla v\rangle - \langle\nabla p, (\nabla\varphi)v\rangle + \langle\varphi f, \nabla v\rangle + \langle(\nabla\varphi)f, v\rangle .$$

Carrying out the transformation $\tilde{p}_\varphi(\tilde{x})=p_\varphi(x)$ with $\tilde{x}=(\tilde{x}', \tilde{x}_n)=(x', x_n-\omega(x'))$ as in Section 3 we get on $\boldsymbol{R}_+^n$ the variational problem

$$\langle\tilde{\nabla}\tilde{p}_\varphi, \tilde{\nabla}\tilde{v}\rangle + B(\nabla'\omega, \tilde{\nabla}\tilde{p}_\varphi, \tilde{\nabla}\tilde{v}) = G(\nabla'\omega, \tilde{v}), \qquad \tilde{v}\in \hat{W}^{1,q'}(\boldsymbol{R}_+^n),$$

where $B(\nabla'\omega, \tilde{\nabla}\tilde{p}_\varphi, \tilde{\nabla}\tilde{v})$ and $G(\nabla'\omega, \tilde{v})$ are determined by the equations above. Since in particular $\partial\Omega\in C^1$ we may assume that $\|\nabla'\omega\|_\infty$ is sufficiently small and by (5.2) for $\boldsymbol{R}_+^n$ and Kato's perturbation argument as in Section 3 we obtain the unique solvability of the last variational problem. Next we replace $\tilde{v}$ by a tangential derivative $\tilde{\partial}_i\tilde{v}$, $i=1, \cdots, n-1$, and, suppressing a mollification with respect to $\tilde{x}'$, we obtain an equation of the form

$$\langle\tilde{\nabla}(\tilde{\partial}_i\tilde{p}_\varphi), \tilde{\nabla}\tilde{v}\rangle + B(\nabla'\omega, \tilde{\nabla}(\tilde{\partial}_i\tilde{p}_\varphi), \tilde{\nabla}\tilde{v}) = \hat{G}(\nabla'\omega, \nabla'(\partial_i\omega), \tilde{v}), \quad \tilde{v}\in \hat{W}^{1,q'}(\boldsymbol{R}_+^n),$$

with some expression $\hat{G}(\nabla'\omega, \nabla'(\partial_i\omega), \tilde{v})$. Since $\partial\Omega\in C^{1,1}$ we may suppose that $\|\nabla'^2\omega\|_\infty<\infty$. Then the assumption $\nabla\operatorname{div} f\in L^q(\Omega)^n$ yields the unique solvability of the latter problem. Hence $\|\tilde{\nabla}\tilde{\partial}_i\tilde{p}_\varphi\|_q<\infty$ and $\|\tilde{\nabla}^2\tilde{p}_\varphi\|_q<\infty$.

The same result is obtained if (5.3) is an equation on $\boldsymbol{R}^n$ with compact $\operatorname{supp}\varphi$. Consequently $\nabla^2 p\in L_{\mathrm{loc}}^q(\bar{\Omega})^{n^2}$. To prove $\nabla^2 p\in L^q(\Omega)^{n^2}$ we must consider the equation (5.3) on $\boldsymbol{R}^n$ or $\boldsymbol{R}_+^n$ with some cut-off function $\varphi=\varphi_0$ as in Section 4 where $\operatorname{supp}\varphi$ is not bounded. If (5.3) is an equation on $\boldsymbol{R}^n$ let $\hat{p}_\varphi=\nabla(\varphi p)$ such that $\Delta\hat{p}_\varphi=\nabla\hat{f}$. Since $\operatorname{supp}(\nabla\varphi)$ is bounded and $\nabla^2 p\in L_{\mathrm{loc}}^q(\bar{\Omega})^{n^2}$ we obtain $\nabla\hat{f}\in L^q(\boldsymbol{R}^n)^n$ and $\hat{p}_\varphi\in L^q(\boldsymbol{R}^n)^n$. Then by the Calderón-Zygmund theorem we get that even $\nabla^2\hat{p}_\varphi\in L^q(\boldsymbol{R}^n)^{n^3}$ and by interpolation we see that $\nabla\hat{p}_\varphi\in L^q(\boldsymbol{R}^n)^{n^2}$. The same conclusion holds if (5.3) is an equation on $\boldsymbol{R}_+^n$; this case can be reduced to $\boldsymbol{R}^n$ by the reflection principle. For this purpose we write $\nabla(\varphi p)=(\nabla'(\varphi p), \partial_n(\varphi p))$, denote by $\hat{p}_\varphi$ the even extension of $\nabla'(\varphi p)$ or the odd extension of $\partial_n(\varphi p)$ from $\boldsymbol{R}_+^n$ to $\boldsymbol{R}^n$ and argue as above. Thus we obtain the assertion $\nabla^2 p\in L^q(\Omega)^{n^2}$ and $\Delta p=\operatorname{div} f\in L^q(\Omega)$. Now Lemma 5.4 is proved. ∎

In the next lemma we give a characterization of the space $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ for unbounded domains.

LEMMA 5.5. *Let $1<q<\infty$ and let $\Omega=\boldsymbol{R}^n$ or let $\Omega\subseteq\boldsymbol{R}^n$, $n\geqq 2$, be an unbounded domain satisfying Assumption* 1.1. *Then*

(i) $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega) = \{\operatorname{div} u\in W^{1,q}(\Omega) : u\in L^q(\Omega)^n,\ N\cdot u|_{\partial\Omega}=0\}$,

$$\|\operatorname{div} u\|_{\hat{W}^{-1,q}(\Omega)} = \inf_{v\in L_\sigma^q(\Omega)} \|u+v\|_q,$$

(ii) $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)=\hat{W}^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$,

(iii) $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)=\operatorname{div}\mathcal{D}(\Delta_q)^n$

where $\operatorname{div}\mathcal{D}(\Delta_q)^n=\{\operatorname{div} u: u\in W^{2,q}(\Omega)^n\cap W_0^{1,q}(\Omega)^n\}$.

PROOF. (i) Let $g\in W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$. As we already mentioned in the beginning of Appendix we find some $u\in L^q(\Omega)^n$ such that $\langle g, v\rangle=-\langle u, \nabla v\rangle$ holds for all $v\in C_0^\infty(\bar{\Omega})$ and $\|g\|_{-1,q}=\|u\|_q$. Setting in particular $v\in C_0^\infty(\Omega)$ we get $g=\operatorname{div} u$ and choosing $v|_{\partial\Omega}\neq 0$ we see from Gauss's integral theorem that $\langle g, v\rangle=\int_{\partial\Omega}(N\cdot u)v\,do-\langle u, \nabla v\rangle$ and $\int_{\partial\Omega}(N\cdot u)v\,do=0$ which leads to $N\cdot u|_{\partial\Omega}=0$; observe that this trace is well-defined since $u\in L^q(\Omega)^n$, $\operatorname{div} u\in L^q(\Omega)$. This proves (i).

(ii) Let $g\in\hat{W}^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$. Now we only know that $g\in L^q_{\mathrm{loc}}(\bar{\Omega})$, $\nabla g\in L^q(\Omega^n)$ and that $v\mapsto\langle g, v\rangle$ is continuous under $\|\nabla v\|_{q'}$. As above we find some $u\in L^q(\Omega)^n$ with $g=\operatorname{div} u$ and $N\cdot u|_{\partial\Omega}=0$. Using $\nabla\operatorname{div} u\in L^q(\Omega)^n$ we get from Lemma 5.4 that $\operatorname{div} u\in L^q(\Omega)$ and therefore $g\in W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$.

(iii) Consider the space $Y_q\equiv\{(\nabla\operatorname{div} u, u)\in L^q(\Omega)^n\times L^q(\Omega)^n: N\cdot u|_{\partial\Omega}=0\}$ equipped with the norm $\|\nabla\operatorname{div} u\|_q+\|u\|_q$. From (i), (ii) we conclude that $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ with the norm $\|\nabla g\|_q+\|g\|_{-1,q}$ is isometric to the quotient space Y_q/N_0 of Y_q modulo the subspace $N_0=\{(0, u): u\in L^q_\sigma(\Omega)\}$. The estimate (1.2) shows that $\operatorname{div}\mathcal{D}(\Delta_q)^n$ is a closed subspace of $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$. Therefore it remains to prove that $\operatorname{div}\mathcal{D}(\Delta_q)^n$ is dense in $W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ under $\|\nabla g\|_q+\|g\|_{-1,q}$. For this purpose it is sufficient to show that the space $\tilde{Y}_q=\{(\nabla\operatorname{div} u, u): u\in W^{2,q}(\Omega)^n\cap W_0^{1,q}(\Omega)^n\}$ is dense in Y_q. To prove the last assertion we consider any continuous linear functional on Y_q vanishing on $\tilde{Y}_q$. Due to the Hahn-Banach theorem we find $F, H\in L^{q'}(\Omega)^n$ such that $\langle\nabla\operatorname{div} u, F\rangle+\langle u, H\rangle=0$ for all $u\in W^{2,q}(\Omega)^n\cap W_0^{1,q}(\Omega)^n$. Setting in particular $u\in C_0^\infty(\Omega)^n$ we conclude that $\nabla\operatorname{div} F=-H$ in the sense of distributions. Considering the local equations (4.4) and admissible functions g in the main theorems on $\boldsymbol{R}^n$, $\boldsymbol{R}^n_+$ and H_ω we see that $\operatorname{div} u|_{\partial\Omega}$ takes on all values $v|_{\partial\Omega}$, $v\in C_0^\infty(\bar{\Omega})$, if u varies in $W^{2,q}(\Omega)^n\cap W_0^{1,q}(\Omega)^n$. Therefore we conclude in the same way as above in (i) that $N\cdot F|_{\partial\Omega}=0$. Now we use a cut-off function $\Psi\in C_0^\infty(\boldsymbol{R}^n)$ with $0\leqq\Psi\leqq 1$, $\Psi(x)=1$ for $|x|\leqq 1$ and $\Psi(x)=0$ for $|x|\geqq 2$ and put $\Psi_k(x)=\Psi(x/k)$, $k\in\boldsymbol{N}$. From Lemma 5.4 we get that $\operatorname{div} F\in L^{q'}(\Omega)$. Since $\|\nabla\Psi_k\|_\infty\leqq c/k$ we see that $\|(\nabla\Psi_k)\operatorname{div} u\|_q\to 0$ and $\|(\nabla\Psi_k)\operatorname{div} F\|_{q'}\to 0$ as $k\to\infty$. This yields

$$\begin{aligned}\langle\nabla\operatorname{div} u, F\rangle+\langle u, H\rangle &= \lim_{k\to\infty}(\langle\Psi_k\nabla\operatorname{div} u, F\rangle+\langle u, H\rangle)\\ &= \lim_{k\to\infty}(\langle\nabla(\Psi_k\operatorname{div} u), F\rangle+\langle u, H\rangle)\\ &= \lim_{k\to\infty}(-\langle\operatorname{div} u, \Psi_k\operatorname{div} F\rangle+\langle u, H\rangle)\\ &= \lim_{k\to\infty}(\langle u, \Psi_k\nabla\operatorname{div} F\rangle+\langle u, H\rangle)\\ &= \langle u, \nabla\operatorname{div} F+H\rangle=0\end{aligned}$$

for all $u \in L^q(\Omega)^n$ with $\nabla \operatorname{div} u \in L^q(\Omega)^n$, $N \cdot u|_{\partial\Omega}=0$. This proves the density of $\tilde{Y}_q$ in Y_q and Lemma 5.5 is proved. ∎

We conclude this appendix with a regularity result concerning two different exponents q and s as used in Sections 2 and 3. By considering first the cases $\boldsymbol{R}^n$, $\boldsymbol{R}^n_+$ and H_ω and then the local equations (4.4) or (5.3) we get the following

LEMMA 5.6. *Let* $1<q, s<\infty$ *and let* $\Omega=\boldsymbol{R}^n$ *or* $\Omega \subseteq \boldsymbol{R}^n$, $n \geqq 2$, *be an unbounded domain satisfying Assumption* 1.1.

(i) *Let* $0<\varepsilon<\pi$, $\lambda \in S_\varepsilon$, $f \in L^q(\Omega)^n \cap L^s(\Omega)^n$ *and* $g \in W^{1,q}(\Omega) \cap W^{1,s}(\Omega) \cap \hat{W}^{-1,q}(\Omega) \cap \hat{W}^{-1,s}(\Omega)$. *Then there exists a unique solution* $(u, p) \in (\mathcal{D}(\Delta_q)^n \cap \mathcal{D}(\Delta_s)^n) \times (\hat{W}^{1,q}(\Omega) \cap \hat{W}^{1,s}(\Omega))$ *of* $\lambda u - \Delta u + \nabla p = f$, $\operatorname{div} u = g$.

(ii) *Let* $f \in L^q(\Omega)^n \cap L^s(\Omega)^n$. *Then there exists a unique decomposition* $f = f_0 + \nabla p$ *with* $f_0 \in L^q_\sigma(\Omega) \cap L^s_\sigma(\Omega)$ *and* $\nabla p \in L^q(\Omega)^n \cap L^s(\Omega)^n$.

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