Generalized Sampling Theorems in Multiresolution Subspaces

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***Abstract*—It is well known that under very mild conditions on the scaling function, multiresolution subspaces are reproducing kernel Hilbert spaces (RKHS's). This allows for the development of a sampling theory. In this paper, we extend the existing sampling theory for wavelet subspaces in several directions. We consider periodically nonuniform sampling, sampling of a function and its derivatives, oversampling, multiband sampling, and reconstruction from local averages. All these problems are treated in a unified way using the perfect reconstruction (PR) filter bank theory. We give conditions for stable reconstructions in each of these cases. Sampling theorems developed in the past do not allow the scaling function and the synthesizing function to be both compactly supported, except in trivial cases. This restriction no longer applies for the generalizations we study here, due to the existence of FIR PR banks. In fact, with nonuniform sampling, oversampling, and reconstruction from local averages, we can guarantee compactly supported synthesizing functions. Moreover, local averaging schemes have additional nice properties (robustness to the input noise and compression capabilities). We also show that some of the proposed methods can be used for efficient computation of inner products in multiresolution analysis. After this, we extend the sampling theory to random processes. We require autocorrelation functions to belong to some subspace related to wavelet subspaces. It turns out that we cannot recover random processes themselves (unless they are bandlimited) but only their power spectral density functions. We consider both uniform and nonuniform sampling.**

I. Introduction

EVER SINCE Mallat and Meyer [1], [2] came up with the concept of multiresolution analysis (MRA), it has been an interesting field for extension of results obtained in other frameworks. One example is the sampling theory. Originally, the theory was developed for uniform sampling of bandlimited signals [3]. A couple of decades later, the research was concentrated on nonuniform sampling [4]. In the second half of this century, those ideas were extended to random processes [5]. All these results hold true for the class of bandlimited signals. What are other classes of signals from which we can develop similar theory? A more general setting is the class of reproducing kernel Hilbert spaces (RKHS's) [6] (see Appendix A). It turns out that the wavelet subspaces (MRA subspaces) are RKHS's (under very mild restrictions on the scaling function) [7]. The MRA system is specified by the system of increasing closed subspaces $\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots \subset L^2(\mathcal{R})$ with

$$\overline{\bigcup_{m\in\mathcal{Z}} V_m} = L^2(\mathcal{R}) \quad \text{and} \quad \bigcap_{m\in\mathcal{Z}} V_m = \{0\} \tag{1.1}$$

where $\mathcal{Z}$ and $\mathcal{R}$ are the set of integer and real numbers, respectively, and $\{0\}$ is the trivial space, i.e., the space with only the zero element. Furthermore, it is required that there exists a function $\phi(t) \in V_0$ (scaling function) such that $\{2^{k/2}\phi(2^k t - n) | n \in \mathcal{Z}\}$ is a Riesz basis [8] for V_k. The complement subspace of V_k in V_{k+1} is W_k, i.e., $V_{k+1} = V_k \dot{+} W_k$ ($\dot{+}$ stands for a direct sum). For further requirements and a more detailed discussion of MRA, see [9].

In sampling theory, there are two problems with which one has to deal. The first one is that of *uniqueness*. Namely, given a sequence of sampling instants $\{t_n\}$, can we have $\{f(t_n)\} = \{g(t_n)\}$ for some $f(t) \neq g(t)$, where $f, g \in \mathcal{H}$ ($\mathcal{H}$ is the underlying RKHS)? If this cannot happen, we say that $\{t_n\}$ is a sequence of uniqueness for $\mathcal{H}$. The other problem is that of finding a *stable* inversion scheme. This brings to mind the following: Given some sequence of uniqueness $\{t_n\}$, we need to know if it is possible to find synthesizing functions $S_n(t) \in \mathcal{H}$ such that these two things are true: First

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) S_n(t) \quad \forall f(t) \in \mathcal{H} \tag{1.2}$$

and, second, if $\{f(t_n)\}$ is close to $\{g(t_n)\}$, then so is $\Sigma_n f(t_n) S_n(t)$ to $\Sigma_n g(t_n) S_n(t)$ in norms of the corresponding spaces. It is possible that no stable reconstruction exists, even though the uniqueness part is satisfied (see [10]).

As we already mentioned, Walter showed in [7] that V_m are RKHS's under very mild conditions on the decay and regularity of $\phi(t)$. He further showed that a stable reconstruction from samples at $t_n = n$ is possible and constructs the synthesizing functions $S_n(t)$'s. Janssen [11] extended Walter's result to the case of uniform noninteger sampling. Neither of these schemes allows for both $\phi(t)$ and synthesizing functions $S_n(t)$'s to be compactly supported,[1] unless $\phi(t)$ is of a restricted form (characteristic function of [0,1] or its convolution with itself, for example).

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[1] In the future, whenever we talk about compactly supported synthesizing functions, we assume that $\phi(t)$ is compactly supported as well. Cases when $S_n(t)$'s are compactly supported at the expense of not having compactly supported $\phi(t)$ are not considered.

A. Aims of the Paper

In this paper, we extend Walter's work in several directions. We extend it to the following:

1) periodically nonuniform sampling;
2) reconstruction from local averages;
3) oversampling;
4) reconstruction from undersampled functions and their derivatives;
5) multiband or multiscale sampling;
6) uniform and nonuniform sampling of WSS random processes.

One of the motivations for these new schemes is the desire to achieve compactly supported synthesizing functions. Periodically nonuniform sampling can guarantee compactly supported $S_n(t)$'s under some restrictions, as we explain in Section II-C. In order to overcome those restrictions of periodically nonuniform sampling, we introduce local averaging. This scheme can guarantee compact supports for $S_n(t)$'s under milder constraints. Local averaging offers some additional advantages. It has good noise sensitivity properties and some compression capabilities, as we show in Section II-D. At the expense of a slightly higher sampling rate, oversampling can always guarantee compactly supported $S_n(t)$'s. This is shown in Section III-A. There are situations where besides $f(t)$, its derivative is available as well. In these cases, we can reconstruct $f(t)$ from samples of $f(t)$ and $f'(t)$, at half the usual rate. In Section III-B, we show how this can be done. If it is known that $f(t)$ belongs to some subspace of V_m, this additional information can be used to sample $f(t)$ at a lower rate (sampling of bandpass signals, for example). This problem is treated in Section III-C. An application of some of the above-mentioned methods to efficient computation of inner products in MRA subspaces is explained in Section III-D (see also the next subsection). Finally, in Section III-E, we analyze what happens if we make errors in sampling times, namely, if instead of sampling at $\{t_n\}$, we sample at $\{t'_n\}$. We will show that this error can be bounded in terms of $\delta = \sup_{n\in\mathcal{Z}} |t_n - t'_n|$.

All the above problems are embedded in the framework of *multirate filter banks*. We give sufficient conditions for the existence of stable reconstruction schemes and explicitly derive expressions for the synthesizing functions. The theory of FIR filter banks is used to obtain compactly supported synthesizing functions.

In Section IV, Walter's idea is extended to random processes. Things are little different now. First, we have to specify a class of random processes for which we want to develop the theory. So far, mainly wide sense stationary (WSS) random processes were considered [5]. The autocorrelation function of a random process $\{f(t), -\infty < t < \infty\}$ is defined as

$$R_{ff}(t,\tau) = E[f(t+\tau)f^*(t)] \tag{1.3}$$

where * denotes complex conjugation and $E[\cdot]$ statistical expectation. When $R_{ff}(t,\tau)$ does not depend on t, we call it a WSS random process. We assume that $R_{ff}(\tau)$ is the inverse Fourier transform of the power spectral density (PSD) function $\mathcal{S}_{ff}(\omega)$. In Section IV, we will use assumptions that will ensure that the Fourier transform of $R_{ff}(\tau)$ exists (for the relationship in a general case, see [12]). Now, we can characterize a random process in terms of its autocorrelation function. For example, a random process is bandlimited if its autocorrelation function is bandlimited. In this paper, we consider WSS random processes whose autocorrelation functions belong to some space related to wavelet subspaces. The problem of reconstruction has two meanings now. First, we can talk about the reconstruction of a random process itself, i.e., existence of functions $S_n(t)$ such that $f(t) = \Sigma_n\ f(t_n)S_n(t)$ in the MS sense, i.e.

$$\lim_{N\to\infty} E[|f(t) - \Sigma_{n=-N}^{N}\ f(t_n)S_n(t)|^2] = 0.$$

The other interpretation is a reconstruction of the PSD function $\mathcal{S}_{ff}(\omega)$. We show that a random process itself cannot be reconstructed if the synthesizing functions are assumed to be integer shifts of one function, unless, of course, the process is bandlimited. However, the PSD function can be reconstructed, and we will show how. This is done for uniform sampling in Section IV-A. Nonuniform sampling of WSS random processes is considered in Section IV-B. Deterministic nonuniform sampling of a WSS random process does not give a WSS discrete parameter random process. We introduce randomness into the sampling times (jitter) to take care of this problem.

B. The New Results in the Perspective of Earlier Work

An actual implementation of the MRA requires computation of the inner products $c_n = c_{0,n} = (f(t), \phi(t-n))$, which is computationally rather involved. Mallat proposed a method that gives an approximation of $c_{0,n}$ by highly oversampling $f(t)$. Daubechies suggested another method (another interpretation of Walter's theorem) that computes $c_{0,n}$ exactly but involves convolutions with IIR filters. Shensa [13] proposed a compromise between the above two methods. It has moderate complexity and nonzero error.

All of the above-mentioned methods involve sampling of signals in V_0. Since our work is about sampling in MRA subspaces, we apply some of the results obtained in Sections II and III to the problem of computation of $c_{0,n}$'s. In Section III-D, we give a qualitative comparison of the new and existing methods in terms of complexity, sampling rate, and approximation error. While all our methods have zero error and pretty low complexities and sampling rates, periodically nonuniform sampling scheme achieves *zero* error at the *minimal* rate with FIR filters (*lowest computational complexity*).

C. Notations and Conventions

1) In all the integrals, the integration is over $(-\infty, \infty)$, unless explicitly indicated.
2) When $R_{ff}(t,\tau)$ is a periodic function of t with period T (and if the same is true for the mean of f), we say that f is a cyclo widesense stationary random process $(\mathrm{CWSS})_T$ [14]. Then, one usually defines the autocorrelation function of this $(\mathrm{CWSS})_T$ process as the

time average of $R_{ff}(t,\tau)$ and denotes it by $R_{ff}(\tau)$, i.e.

$$R_{ff}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} R_{ff}(t,\tau)\,dt. \tag{1.4}$$

3) The Fourier transform operation and its inverse are denoted by $\mathcal{FT}$ and $\mathcal{FT}^{-1}$, respectively.
4) A set of functions $\{f_n(t)\}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if $\{f_n(t)\}$ is complete in $\mathcal{H}$ and if there exist constants A and $B, 0 < A \le B < \infty$ such that

$$A\|c_n\|_2^2 \le \left\|\sum_n c_n f_n(t)\right\|_2^2 \le B\|c_n\|_2^2$$

for any $\{c_n\} \in l^2$ (see [8] for further properties).
5) For the reader's convenience, some frequently used definitions and theorems from mathematical analysis are reviewed in Appendix A.

II. Discrete Representations of Deterministic Signals

In this section, we consider different discrete representations of functions in MRA subspaces V_m. We work in V_0 only since all the relevant properties are independent of the scale (see [7]). Other V_k and W_k subspaces will be considered in the case of multiband sampling (see the next section). Let us first state our basic assumptions and make some preliminary derivations.

A. Assumptions and Preliminary Derivations

We assume that $\{\phi(t-n)\}$ forms a Riesz basis for $V_0 \subset L^2(\mathcal{R})$. In order to show that V_0 is a RKHS, Walter assumes that $\phi(t)$ is continuous and that it decays faster than $1/|t|$ for large t, i.e., there exists $C > 0$ such that $|\phi(t)| < C/(1+|t|)^{1+\epsilon}$ for some $\epsilon > 0$ (see [7]). Janssen derives his result under weaker assumptions, namely, that $\phi(t)$ is bounded and that

$$\sum_n |\phi(t-n)| < C_\phi \tag{2.1}$$

converges uniformly on [0,1]. Note that this assures us that $\phi(t) \in L^1(\mathcal{R})$ and $\{\phi(t-n)\} \in l^1 \subset l^2$ for all t. In addition, the Fourier transform of $\phi(t)$ is well defined and is a continuous function (see [15]).

Since $\{\phi(t-n)\}$ is a Riesz basis for V_0, then for any $f(t) \in V_0$, there exists a unique sequence $\{c_n\} \in l^2$ such that

$$f(t) = \sum_n c_n \phi(t-n). \tag{2.2}$$

If the sampling times are $t_n = n + u_n$, then

$$f(t_n) = \sum_k c_k \phi(t_n - k) = \sum_k c_k \phi(u_n + n - k). \tag{2.3}$$

Let

$$\Phi_{u_n}(e^{j\omega}) = \sum_k \phi(u_n + k)e^{-jk\omega} \tag{2.4}$$

be the l^1-Fourier transform of $\{\phi(u_n+k)\}$.[2] Note that $\Phi(e^{j\omega})$ is a bounded input bounded output (BIBO) stable filter because of (2.1). In the rest of the paper, we will frequently use the following lemma.

Lemma 2.1: The samples $f(t_n)$ can be written as follows:

$$f(t_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(e^{j\omega})\Phi_{u_n}(e^{j\omega})e^{jn\omega}\,d\omega. \tag{2.5}$$

Proof: If we substitute

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(e^{j\omega})e^{jk\omega}\,d\omega$$

in (2.3), we have

$$\begin{aligned} f(t_n) &= \frac{1}{2\pi}\sum_k \int_{-\pi}^{\pi} C(e^{j\omega})e^{jk\omega}\phi(u_n+n-k)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} C(e^{j\omega})\Phi_{u_n}(e^{j\omega})e^{jn\omega}\,d\omega. \end{aligned} \tag{2.6}$$

The order of integration and summation can be interchanged because

$$\sum_k \int_{-\pi}^{\pi} |C(e^{j\omega})\phi(u_n+n-k)|\,d\omega < \infty$$

(remember that $C(e^{j\omega}) \in L^2[-\pi,\pi] \subset L^1[-\pi,\pi]$ and $\{\phi(u_n+k)\} \in l^2$).[3]. ♢

Let us say a few words about the type of convergence in (2.2). Riesz basis property guarantees L^2-convergence of the sum in (2.2) so that $f(t)$ is determined only almost everywhere (a.e.). Even though sampling of functions defined only a.e. is meaningless, it is a well-defined operation in our case because the sum in (2.2) converges uniformly on $\mathcal{R}$. To see this, use (2.1) and (2.2) to get

$$|f(t)| \le \left(\sum_k |\phi(t-k)|^2\right)^{1/2} \|c_k\|_2 < C\|f\|_2 \tag{2.7}$$

where the second inequality is obtained using the definition of a Riesz basis. This relationship transforms L^2-convergence into uniform convergence. Therefore, sampling of $f(t)$, as given by (2.3), is well defined. In the rest of the paper, we will consider the pointwise convergence of (2.2) and no longer worry about this. Our main concern will be to get sequences $\{c_n\}$ from $\{f(t_n)\}$ in a stable way. The next subsection is a review of Walter's and Janssen's work.

B. Review of Uniform Sampling in Wavelet Subspaces

We will use an abstract setting for the sampling theory. It offers us a unified approach to all the problems in this and the following section. Therefore, let us first explain this approach. The idea of sampling in wavelet subspaces is to find an invertible map between $\{c_n\}$ and $\{f(t_n)\}$ in (2.3). More generally, we want to find invertible maps between $\{c_n\}$ and some other discrete representations of functions in V_0. Let us define this map.

[2] In our notation, $\mathcal{F}(\omega)$ is the Fourier transform of a function $f(t)$ in $L^1(\mathcal{R})$ or $L^2(\mathcal{R})$, whereas $F(e^{j\omega})$ is the Fourier transform of a sequence $\{f_n\}$ from l^1 or l^2..

[3] This argument for the interchangeability of integrals and/or sums will be often used without explicitly stating it.

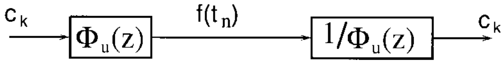

Fig. 1. Interpretation of the uniform sampling result in terms of digital filtering and inverse filtering.

Definition 2.1: By $\mathcal{T}$, we denote an operator from l^2 into l^2. It maps sequence $\{c_n\}$ to $\{d_n\}$, where $\{d_n\}$ is some discrete representation of $f = \Sigma_n\ c_n\phi(t-n)$.

When we talk about sampling theorems, we have $\{d_n\} = \{f(t_n)\}$, whereas in Section II-D, $\{d_n\}$ will be a sequence of local averages. Because of the Fourier transform isomorphism between l^2 and $L^2[-\pi,\pi]$, we can think of $\mathcal{T}$ as a map of $L^2[-\pi,\pi]$ into itself. In this basis, $\mathcal{T}$ is just a multiplication operator (from (2.5), action of $\mathcal{T}$ is multiplication by $\Phi_u(e^{j\omega})$ in the case of uniform sampling). The following theorem is borrowed from [11] just for the completeness of the presentation (the proof can be found in [11]).

Theorem 2.1 (Janssen): Let $t_n = n+u$ for some $u \in [0,1)$. Operator $\mathcal{T}$: $\{c_n\} \to \{f(t_n)\}$ maps l^2 into l^2 for any $f \in V_0$. Furthermore, $\mathcal{T}^{-1}$ is bounded if $\Phi_u(e^{j\omega}) \neq 0$ for all $\omega \in [-\pi,\pi]$.

This theorem can be visualized as in Fig. 1. Therefore, we see that this sampling theorem has a rather simple engineering interpretation in terms of digital filters.[4]

Remarks:

1) There are some interesting connections to regularity theory. Notice that the condition from the above theorem, in terms of [11], is that the Zak transform of $\phi(t), (\mathcal{ZT}\phi)(u,\omega) \neq 0$ a.e. This condition is the same as Rioul's condition [17] for the optimality of his regularity estimates. Therefore, if there is a $u \in [0,1)$ such that $(\mathcal{ZT}\phi)(u,\omega) \neq 0$, then Rioul's regularity estimates are optimal, and $f(t)$ can be reconstructed from $\{f(n+u)\}$.
2) The synthesizing functions $S_n(t)$ have been derived in [7] and [11]. They have the following shift property
$$S_n(t) = S(t-n) \tag{2.8}$$
where
$$S(t) = \sum_k \phi_k^{-u}\phi(t-k) \quad \text{and}$$
$$\{\phi_n^{-u}\} = \mathcal{FT}^{-1}(1/\Phi_u(e^{j\omega})).$$
3) When $\Phi_u(e^{j\omega})$ turns out to be a rational filter, it is obvious that if $\Phi_u(e^{j\omega}) \neq 0$, then $1/\Phi_u(e^{j\omega})$ is stable (possibly noncausal).
4) Evidently, we cannot make both $\Phi_u(e^{j\omega})$ and $1/\Phi_u(e^{j\omega})$ trigonometric polynomials unless $\Phi_u(e^{j\omega}) = e^{-jk\omega}$. Therefore, if $\phi(t)$ is compactly supported, i.e., if $\Phi_u(e^{j\omega})$ is a trigonometric polynomial, $S(t)$ cannot have compact support. This restriction will be lifted in the schemes we propose.

C. Periodically Nonuniform Sampling

Now, we consider the case of periodically nonuniform sampling. For this, let us choose $u_m \in [0,L)$ for $m = 0,1,\cdots,L-1$, and set $t_{kL+m} = kL+u_m$. From the abstract point of view that we developed in the previous subsection, $\mathcal{T}$ can be viewed as the analysis filter bank of an L-channel maximally decimated filter bank (see [18]). To see this, notice that $f(t_{kL+m})$ using Lemma 2.1 is

$$f(t_{kL+m}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jkL\omega}C(e^{j\omega})\Phi_m(e^{j\omega})\,d\omega \tag{2.9}$$

where $\Phi_m(e^{j\omega}) = \Phi_{u_m}(e^{j\omega})$ for $m = 0.1,\cdots,L-1$. In terms of multirate filter banks, $\{f(t_{kL+m})\}$ is the mth subband signal in Fig. 2.

There is no fundamental difference between this scheme and that of Janssen. Fig. 1 is a trivial one-channel filter bank. The aim here is the same. We want to get back $\{c_n\}$ from $\{f(t_n)\}$ in a stable way (notice that since $\{f(t_{kL+m})\} \in l^2$ for $m = 0.1,\cdots,L-1$, the sequence $\{f(t_n)\} \in l^2$ as well). In Fig. 1, we inverted a single filter $\Phi_u(u)$. Here, we have to find a stable synthesis filter bank (which is the inverse of the analysis filter bank)[5]. For this, we will use standard techniques from the multirate filter bank theory. Let

$$\Phi_m(z) = \sum_{k=0}^{L-1} z^{-k}E_{mk}(z^L) \quad \text{and}$$
$$G_m(z) = \sum_{k=0}^{L-1} z^{k}R_{km}(z^L)$$

be the polyphase decompositions of analysis and synthesis filters (for more discussion on polyphase decompositions, see [18]). Then, we define the polyphase matrices as $[\boldsymbol{E}(z)]_{kl} = E_{kl}(z)$ and $[\boldsymbol{R}(z)]_{kl} = R_{kl}(z)$. Therefore, when this filter bank has perfect reconstruction (PR) property, the operator $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ can be represented by $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$, respectively (see Fig. 3). Now that we have this filter bank interpretation of $\mathcal{T}$, we can return to the problem of uniqueness and stability.

Lemma 2.2: In the case of periodically nonuniform sampling, the sequence of sampling points $\{t_n\}$ is a sequence of uniqueness if the matrix $\boldsymbol{E}(e^{j\omega})$, as defined above, is nonsingular a.e.

Proof: Let

$$\boldsymbol{c_k} = [c_{kL} \quad c_{kL-1} \quad \cdots \quad c_{kL-L+1}]^T \quad \text{and}$$
$$\boldsymbol{f_k} = [f(t_{kL}) \quad f(t_{kL+1}) \quad \cdots f(t_{kL+L+1})]^T$$

be blocked versions of $\{c_k\}$ and $\{f(t_k)\}$. Then, using Noble identities (see [18]), the system from Fig. 2 can be transformed into that in Fig. 3.

Notice that $\|c_k\|_2^2 = \|\boldsymbol{c_k}\|_2^2 = \Sigma_k\ \boldsymbol{c}_k^\dagger\boldsymbol{c}_k$ and that $\boldsymbol{f}(e^{j\omega}) = \boldsymbol{E}(e^{j\omega})\boldsymbol{c}(e^{j\omega})$, where $\boldsymbol{c}(e^{j\omega}) = \Sigma_k\ \boldsymbol{c}_k e^{-jk\omega}$, and

[4] In this paper, z is just a formal argument, and it stands for $e^{j\omega}$. $\Phi(z)$ should not be interpreted as Z transform in the conventional sense [16]. Most of our signals are assumed to be in l^2 so that their Fourier transform exists in l^2 sense only.

[5] Stable in our context does not necessarily mean BIBO stable. We want a bounded transformation of l^2 into l^2. However, it turns out that if the synthesis filter bank is BIBO stable, i.e., if the filter coefficients are in l^1, then it also represents a bounded transformation from l^2 into l^2; see Appendix A.

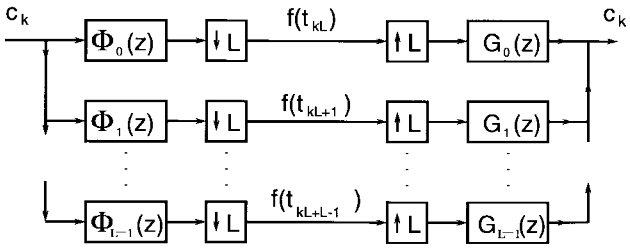

Fig. 2. Filter-bank interpretation of periodically nonuniform sampling.

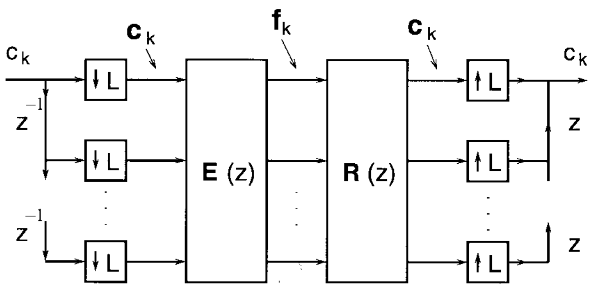

Fig. 3. Polyphase representation.

$\boldsymbol{f}(e^{j\omega}) = \Sigma_k\ \boldsymbol{f}_k e^{-jk\omega}$ [$\boldsymbol{f}(e^{j\omega})$ and $\boldsymbol{c}(e^{j\omega})$ are elements of $(L^2[-\pi,\pi])^L$]. Again, using Parseval's equality, we have

$$\|f_k\|_2^2 = \|\boldsymbol{f}_k\|_2^2 = \sum_k \boldsymbol{f}_k^\dagger \boldsymbol{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \boldsymbol{f}^\dagger(e^{j\omega})\boldsymbol{f}(e^{j\omega})\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \boldsymbol{c}^\dagger(e^{j\omega})\boldsymbol{E}^\dagger(e^{j\omega})\boldsymbol{E}(e^{j\omega})\boldsymbol{c}(e^{j\omega})\, d\omega. \quad (2.10)$$

Suppose $\boldsymbol{E}^\dagger(e^{j\omega})\boldsymbol{E}(e^{j\omega})$ (a positive semidefinite matrix) is nonsingular a.e.. Then, $\|f_k\|_2$ can be zero if and only if $\boldsymbol{c}^\dagger(e^{j\omega})\boldsymbol{c}(e^{j\omega}) = 0$ a.e., which implies that $\{c_k\}$ is a zero sequence itself.

The question of stable reconstruction can be answered using Wiener's theorem (see Appendix A and [19]). What we want is BIBO stability of $\boldsymbol{E}^{-1}(e^{j\omega}) = \boldsymbol{R}(e^{j\omega})$.

Lemma 2.3: A stable recovery from periodically nonuniform samples $\{f(t_n)\}$ of any $f \in V_0$ is possible if there exist BIBO stable synthesizing filters $G_k(z)$. This will be the case if and only if $[\det \boldsymbol{E}(e^{j\omega})] \neq 0$ for all $\omega \in [-\pi,\pi]$.

Proof: Since entries of $\boldsymbol{E}(e^{j\omega})$ are Fourier transforms of l^1 sequences, so is the determinant $[\det \boldsymbol{E}(e^{j\omega})]$ (operation of convolution is closed in l^1, as explained in Appendix A). Now, by Wiener's theorem, the convolutional inverse of $[\det \boldsymbol{E}(e^{j\omega})]$ is in l^1 if and only if $[\det \boldsymbol{E}(e^{j\omega})] \neq 0$. Then, the entries of $\boldsymbol{E}^{-1}(e^{j\omega})$ are Fourier transforms of l^1 sequences. This means that $\boldsymbol{R}(z)$ is a multi-input multioutput (MIMO), BIBO stable system, and therefore, $\boldsymbol{c}(e^{j\omega})$ can be recovered from $\boldsymbol{f}(e^{j\omega})$ in a stable way. ◇

Remark: The existence of FIR PR filter banks allows for the possibility of having both $\phi(t)$ and $S_n(t)$'s compactly supported, unlike the schemes in [7] and [11]. In particular, if $\Phi_m(z) = \Sigma_{k=0}^{L-1}\ \phi(u_m+n)z^{-n}$ for $m = 0,1,\cdots,L-1$, then the polyphase matrix will be just a constant matrix. In this case, the inverse filter bank is guaranteed to be FIR. This constraint on the number of channels will not be necessary in the schemes we propose next.

D. Expressions for Synthesis Functions

Let us now construct synthesis functions and show that they have some shift property as well. $G_k(e^{j\omega}) = \Sigma_n\ g_n^k e^{-jn\omega}$ are the synthesis filters in our PR filter bank (see Fig. 2). The PR property implies that (see [18])

$$c_m = \sum_{k=0}^{L-1}\sum_n f(t_{nL+k})g_{m-nL}^k. \quad (2.11)$$

Then, the reconstructed function is

$$f(t) = \sum_{k=0}^{L-1}\sum_n f(t_{nL+k})\sum_m g_{m-nL}^k \phi(t-m)$$
$$= \sum_{k=0}^{L-1}\sum_n f(t_{nL+k})S_{kn}(t). \quad (2.12)$$

If we define $S_k(t) = \Sigma_m\ g_m^k\phi(t-m)$, then

$$S_{kn}(t) = \sum_m g_{m-nL}^k \phi(t-m+nL-nL) = S_k(t-nL),$$
$$0 \le k \le L-1. \quad (2.13)$$

Therefore, all the synthesizing functions are obtained as shifts of the L basic functions $S_k(t)$. The above lemmas are summarized in the following theorem.

Theorem 2.2: Let $\Phi_m(z)$ and $\boldsymbol{E}(z)$ be analysis filters and their polyphase matrix as shown in Figs. 2 and 3. A stable reconstruction of $\{c_n\}$ from the samples $\{f(t_n)\}$ exists if the determinant $[\det \boldsymbol{E}(e^{j\omega})] \neq 0$ for all $\omega \in [-\pi,\pi]$. Furthermore, all synthesizing functions are shifts of L fixed functions, as given by (2.13).

Next, we are going to illustrate the above theory with some practical examples, Before this, let us first give a short summary of the algorithm.

E. Summary of the Algorithm for Recovery of $\{c_n\}$ from $\{f(t_n)\}$

1) Choose $u_m \in [0, L-1)$ for $m = 0,1,\cdots,L-1$.
2) Obtain filters $\Phi_m(e^{j\omega}) = \Sigma_n\ \phi(n+u_m)e^{-jn\omega}$, and form $\boldsymbol{E}(z)$.
3) Find $\boldsymbol{E}^{-1}(z)$ (provided Theorem 2.2 is satisfied), and calculate synthesis filters $G_m(e^{j\omega})$'s.
4) Construct synthesis functions $S_{kn}(t)$'s as given by (2.13).

Example 2.1: Consider the MRA generated by linear splines (see [15]). The scaling function is

$$\phi(t) = \begin{cases} t, & \text{for } 0 \le t < 1, \\ 2-t, & \text{for } 1 \le t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in the case where there exist compactly supported synthesizing functions. For this, let us choose $L = 2, u_0 = 0$ and $u_1 = 0.5$. Fig. 4 shows $\phi(t)$ and samples $\phi(n+u_m)$ that determine filter coefficients. Namely, from (2.4)

$$\Phi_m(z) = \sum_n \phi(n+u_m)z^{-n} = \sum_n \phi_n^m z^{-n}$$

so that $\Phi_0(z) = z^{-1}$, and $\Phi_1(z) = \frac{1}{2}(1+z^{-1})$.

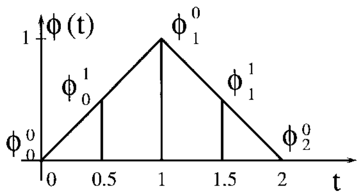

Fig. 4. Linear spline and its samples at $n + u_m$.

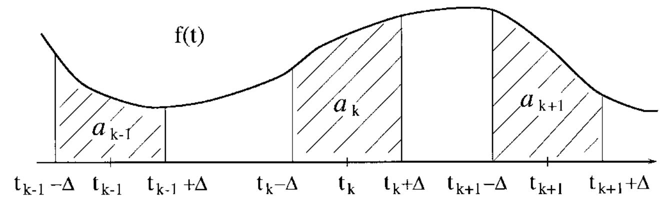

Fig. 5. Local averaging scheme.

The polyphase matrix is

$$\boldsymbol{E}(z) = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2. \end{pmatrix}$$

The inverse of $\boldsymbol{E}(z)$ is

$$\boldsymbol{R}(z) = \boldsymbol{E}^{-1}(z) = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Then, the synthesis filters are $G_0(z) = z - 1$, and $G_1(z) = 2$. The synthesis functions are $S_0(t-2n)$ and $S_1(t-2n)$, where $S_0(t) = \phi(t+1) - \phi(t)$, and $S_1(t) = 2\phi(t)$.

Example 2.2: Let us now consider the case of quadratic splines [15]. The scaling function is

$$\phi(t) = \begin{cases} t^2/2, & \text{for } 0 \le t < 1, \\ -(t-3/2)^2 + 3/4, & \text{for } 1 \le t < 2, \\ \frac{1}{2}(t-3)^2, & \text{for } 2 \le t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

As usual, we want to have compactly supported synthesizing functions. Since $\phi(t)$ is supported on [0,3], we choose $L = 3$ and $u_0 = 0, u_1 = 1/3$, and $u_2 = 2/3$. Then

$$\begin{aligned} \Phi_0(z) &= \tfrac{1}{2}(z^{-1} + z^{-2}), \\ \Phi_1(z) &= \tfrac{1}{18} + \tfrac{13}{18}z^{-1} + \tfrac{2}{9}z^{-2}, \quad \text{and} \\ \Phi_2(z) &= \tfrac{2}{9} + \tfrac{13}{18}z^{-1} + \tfrac{1}{18}z^{-2}. \end{aligned}$$

The polyphase matrix and its inverse are

$$\boldsymbol{E}(z) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/18 & 13/18 & 2/9 \\ 2/9 & 13/18 & 1/18 \end{pmatrix} \quad \text{and}$$

$$\boldsymbol{E}^{-1}(z) = \boldsymbol{R}(z) = \begin{pmatrix} 13/4 & -9 & 27/4 \\ -5/4 & 3 & -3/4 \\ 13/4 & -3 & 3/4 \end{pmatrix}.$$

From $\boldsymbol{R}(z)$, we get $G_k(z)$'s and synthesizing functions $S_0(t-3n), S_1(t-3n)$ and $S_2(t-3n)$, where

$$\begin{aligned} S_0(t) &= \tfrac{13}{4}\phi(t) - \tfrac{5}{4}\phi(t+1) + \tfrac{13}{4}\phi(t+2), \\ S_1(t) &= -9\phi(t) + 3\phi(t+1) - 3\phi(t+2), \quad \text{and} \\ S_2(t) &= \tfrac{27}{4}\phi(t) - \tfrac{3}{4}\phi(t+1) + \tfrac{3}{4}\phi(t+2). \end{aligned}$$

Therefore, we see that the actual implementation of the algorithm can be simple. The above examples show how we can achieve compactly supported synthesizing functions, but the problem is that if the support of $\phi(t)$ contains interval $[N_1, N_2]$, where N_1, N_2 are integers, then we need $L \ge N_2 - N_1$ channels. In other words, the number of channels L necessary for $\boldsymbol{E}(z)$ to be a constant matrix grows linearly with the length of the support of $\phi(t)$.

F. Reconstruction from Local Averages

In this subsection, we consider another discrete representation of functions in V_0. The motivation for this subsection comes from [20], where it was shown that a bandlimited function can be recovered from its local averages. We extend this representation to wavelet subspaces. This scheme offers three advantages over the previous ones.

1) *Compactly supported synthesizing functions*: If $\phi(t)$ is compactly supported, then we can guarantee existence of compactly supported synthesizing functions $S_n(t)$'s. Unlike in the case of nonuniform sampling, we can guarantee this even when we have only two channels, regardless of the length of the support of $\phi(t)$.
2) *Shaping of the frequency response*: We are able to shape frequency responses of analysis/synthesis filters. Namely, we can force filters $\Phi_m(z)$ to be anything we want (provided a length constraint is satisfied).
3) *Robustness*: This scheme has reduced sensitivity toward the input noise, compared with pure sampling.

In the first part of the subsection, we make introductory derivations that are similar to those in the case of uniform and periodically nonuniform sampling. Then, at the expense of a small increase in complexity, we modify the scheme in order to gain control over the filter coefficients of $\Phi_m(z)$'s. This discrete representation has the same average rate of sampling as in previous sections.

The main idea is to find ways of reconstructing $f(t)$ not from samples $f(t_k)$ but from local averages $a_k = \int_{t_k-\Delta}^{t_k+\Delta} f(t)\,dt$ around t_k (as shown in Fig. 5). In the uniform case, we can choose $\Delta = 1/2$ and $t_k = k + 1/2 + u$. Then,[6]

$$a_k = \int_{k+u}^{k+1+u} f(t)\,dt = \int_{k+u}^{k+1+u} \sum_n c_n \phi(t-n)\,dt. \quad (2.14)$$

If $\phi_k^u = \int_{k+u}^{k+1+u} \phi(t)\,dt$, then (2.1) implies that $\{\phi_k^u\} \in l^1$. The signal $\{a_k\}$ can be viewed as a convolution of $\{c_k\} \in l^2$ and $\{\phi_k^u\}$ so that $\{a_k\} \in l^2$. Therefore, its Fourier transform $A(e^{j\omega}) = \Sigma_n\, a_n e^{jn\omega}$ is well defined. Then, just as in the case of uniform sampling (Section II-B), we have the following relation:

$$A(e^{j\omega}) = C(e^{j\omega})\Phi_u(e^{j\omega})$$

where, now, $\Phi_u(e^{j\omega}) = \Sigma_n\, \phi_n^u e^{-jn\omega}$ is the Fourier transform of $\{\phi_n^u\}$. The only difference from the case of uniform sampling is that the operator $\mathcal{T}$ now maps the sequence $\{c_k\}$ into the sequence of local averages $\{a_k\}$. So this system is

[6]This integral exists because $f(t) \in L^2([u, u+1]) \subset L^1([u, u+1])$ for all $u \in \mathcal{R}$.

the same as that in Fig. 1, except for the definition of $\Phi_u(z)$ and the meaning of its output $\{a_k\}$. Therefore, we have the following theorem.

Theorem 2.3: If $\Phi_u(e^{j\omega})$ as defined above is nonzero on $[-\pi,\pi]$, then the representation of a function $f \in V_0$ by its local averages a_k is unique. Moreover, there exists a stable reconstruction algorithm.

Remark: This scheme has the same problems as that of uniform sampling. Namely, $S_n(t)$'s and $\phi(t)$ cannot be simultaneously compactly supported.

Let us now consider periodically nonuniform averaging. The idea is to partition the interval $[0, L]$ into L subintervals $I_m, m = 0, 1, \cdots, L-1$. The sequence of local averages is defined as

$$a_{kL+m} = \int_{kL+I_m} f(t)\, dt. \tag{2.15}$$

Let

$$\phi_n^m = \int_{n+I_m} \phi(t)\, dt \quad \text{and} \quad \Phi_m(e^{j\omega}) = \sum_n \phi_n^m e^{-jn\omega}.$$

Using this notation, we have

$$a_{nL+m} = \int_{nL+I_m} \sum_k c_k \phi(t-k)\, dt = \sum_k c_k \phi_{nL-k}^m. \tag{2.16}$$

This equation can be viewed as a convolution of $\{c_k\}$ with $\{\phi_k^m\}$ followed by L-fold decimation. Accordingly, in the frequency domain, this means $A_m(e^{j\omega}) = (C(e^{j\omega})\phi_m(e^{j\omega})) \downarrow_L$, where $A(z) = \Sigma_{k=0}^{L-1} z^{-k} A_k(z^L)$ is the polyphase decomposition of $A(z)$, and $\downarrow_L$ denotes L-fold decimation. The situation is completely identical to that in the case of periodically nonuniform sampling (which is shown in Fig. 2), and Theorem 2.2 holds true for this case (except that the filters $\Phi_m(e^{j\omega})$'s are obtained in a different way). Again, if $\Phi_m(e^{j\omega})$'s are FIR and such that the synthesis filter bank is FIR as well, the synthesizing functions will be compactly supported.

However, if everything is the same as in the case of nonuniform sampling, what is the point of doing all this? The answer is given in the rest of the subsection: *At the expense of a little more complexity, we will be able to force the filters $\Phi_m(e^{j\omega})$ to be anything we want.*

Nonuniform sampling gave us very little control over the filters $\Phi_m(e^{j\omega})$. Consequently, it is unlikely that filters $\Phi_m(e^{j\omega})$, in that scheme, will have an FIR inverse filter bank (unless we choose L big enough to make $\boldsymbol{E}(z)$ a constant matrix). In order to make this happen for any $L \geq 2$, we have to work a little harder. For this, notice that what we have done so far, in this section, is equivalent to finding inner products of $f(t)$ and windows $w_k(t - nL)$, where $w_k(t) = \chi_{I_k}(t)$ is the characteristic function of the interval I_k. If we use some other window, can we use this freedom to make sure that the filter bank has an FIR inverse? We will show that the answer is in the affirmative.

Let $\phi(t)$ be compactly supported on an interval $[0, N]$. We divide [0, 1] into N subintervals $I_k, k = 0, 1, \cdots, N-1$. Let the windows be piecewise constant functions

$$w_m(t) = \sum_{l=0}^{N-1} \alpha_{ml} \chi_{I_l}(t). \tag{2.17}$$

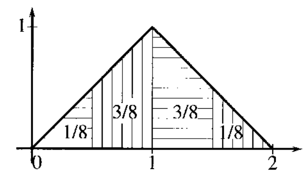

Fig. 6. Areas under $\phi(t)$ over intervals I_l are the entries of Γ.

We are going to show how the coefficients α_{ml} can be used to gain more control over the filters $\Phi_m(e^{j\omega})$'s.

Let $\{H_m(e^{j\omega})\}$ and $\{G_m(e^{j\omega})\}$ be the analysis and synthesis filters of an FIR PR, L-channel filter bank. We assume that filters $H_m(e^{j\omega})$ have lengths N. Note that except for the length constraint, $\{H_m(z)\}, \{G_m(e^{j\omega})\}$ is an arbitrary FIR PR filter bank.

The idea is to choose α_{ml}'s so that $\Phi_m(e^{j\omega}) = H_m(e^{j\omega})$. Let us see how to achieve this. The filter coefficients of $\Phi_m(e^{j\omega})$ are

$$\phi_n^m = \int w_m(t)\phi(t+n)\, dt.$$

Using (2.17), this can be written as

$$\phi_n^m = \sum_{l=0}^{N-1} \alpha_{ml} \int_{I_l} \phi(n+t)\, dt = \sum_{l=0}^{N-1} \alpha_{ml}\gamma_{nl} \tag{2.18}$$

where

$$\gamma_{nl} = \int_{I_l} \phi(n+t)\, dt.$$

Now, there are L systems of linear equations (for $m = 0, 1, \cdots, L-1$), each in N unknowns $\alpha_{ml}, l = 0, 1, \cdots, N-1$. All these systems have the same system matrix $[\Gamma]_{nl} = \gamma_{nl}$. If Γ is nonsingular, there is a unique solution for the α_{ml}'s. If Γ turns out to be singular (which is very unlikely), we can change subintervals I_l and get a nonsingular Γ (almost surely). This way, we can shape the filters $\Phi_m(e^{j\omega})$ into anything we want. The synthesizing functions can be obtained from $\{G(z)\}$ as in the previous subsection (see (2.11)–(2.13)). Even though the equations look messy, the application of the algorithm is pretty straightforward. Let us summarize the steps.

G. Summary of the Algorithm

1) Partition the interval [0, 1] into N arbitrary subintervals I_k.
2) Form the matrix Γ with entries $[\Gamma]_{n,l} = \int_{I_l} \phi(n+t)\, dt$. If Γ is singular, return to step 1; otherwise, determine α_{nl}'s from the filter coefficients of desired filters $\{H(z)\}$.
3) Determine synthesizing functions $S_{kn}(t)$'s from $\{G_k(z)\}$ as given by (2.13).

We demonstrate the algorithm on simple examples of spline-generated MRA's.

Example 2.3: Let the scaling function be the linear spline, as in Example 2.1, and let $L = 2$ (two channels), $I_0 = [0, 0.5]$, and $I_1 = [0.5, 1]$. The entries of Γ are easy to calculate. They are the areas shown in Fig. 6. More precisely

$$\gamma_{00} = \gamma_{11} = \int_0^{0.5} \phi(t)\, dt = 1/8, \quad \gamma_{01} = \gamma_{10} = \int_{0.5}^{1} = 3/8.$$

Then

$$\Gamma = \tfrac{1}{8}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

is nonsingular, and its inverse is

$$\Gamma^{-1} = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}.$$

Suppose we wish to choose windows $w_k(t)$ such that

$$\Phi_0(z) = H_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1}) \quad \text{and} \quad \Phi_1(z) = H_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1}).$$

($\Phi_0(z)$ and $\Phi_1(z)$ are analysis filters of the simplest two-channel paraunitary filter bank.) Then, the coefficients α_{nl} are obtained as

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \end{pmatrix} = \Gamma^{-1}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha_{10} \\ \alpha_{11} \end{pmatrix} = \Gamma^{-1}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}.$$

Therefore, the windows are

$$w_0(t) = \sqrt{2}\chi_{[0,1]}(t) \quad \text{and} \quad w_1(t) = 2\sqrt{2}(-\chi_{0,1/2]}(t) + \chi_{[1/2,1]}).$$

Since $\{\Phi_k(z)\}$ form a paraunitary filter bank, the synthesis filters are time-reversed versions of $\Phi_k(z)$'s, i.e.

$$G_0(z) = \frac{1}{\sqrt{2}}(z+1) \quad \text{and} \quad G_1(z) = \frac{1}{\sqrt{2}}(-z+1).$$

Therefore, the compactly supported synthesizing functions are $S_0(t - 2n)$ and $S_1(t - 2n)$, where

$$S_0(t) = \frac{1}{\sqrt{2}}(\phi(t) + \phi(t+1)) \quad \text{and} \quad S_1(t) = \frac{1}{\sqrt{2}}(\phi(t) - \phi(t+1)).$$

This example shows how easy it is to shape filters $\Phi(z)$'s. In a similar way, if we consider some scaling function with a bigger support, we can have longer filters with better frequency responses.

Example 2.4: In this example, we want to show that the number of channels may be smaller than the length of filters $\Phi_m(z)$'s. For demonstration, we choose quadratic splines as given in Example 2.2. Choose $L = 3$, and $I_0 = [0, 1/3], I_1 = [1/3, 2/3]$, and $I_2 = [2/3, 1]$; it is straightforward to check that

$$\Gamma = \frac{1}{162}\begin{pmatrix} 1 & 7 & 19 \\ 34 & 40 & 34 \\ 19 & 7 & 1 \end{pmatrix}$$

is nonsingular. Therefore, we can force filters $\Phi_m(z)$ to be any desired filters of length 3. Notice that there are no assumptions about the number of channels. In particular, compactly supported synthesizing functions can be guaranteed with a two channel filter bank (unlike in the case of periodically nonuniform sampling). All this generalizes for $\phi(t)$'s with bigger supports.

Remarks:

1) *Complexity*: Notice that the windows $w_k(t)$ are step functions. It is therefore not necessary to perform true inner products. A simple "integrate and dump" circuit with weighted output will do.
2) *An additional advantage*: We wanted complete control over the filters $\Phi_k(z)$ in order to be able to guarantee compact supports for the synthesizing functions $S_k(t)$'s. However, this freedom can be used to achieve even more. Namely, we can design $\Phi_m(z)$'s with good frequency characteristics and then use standard subband coding techniques for signal compression.
3) *Limitations*: This extended local averaging technique works for compactly supported scaling functions only. As the length of the support of $\phi(t)$ increases, we have to subdivide the interval [0,1] into increasingly more subintervals. This makes the scheme increasingly more sensitive to errors in limits of integration.

At the beginning of the subsection, we announced three main advantages of local averages over the previous schemes. So far, we justified the first two. Intuitively, it is clear that if the input signal $f(t)$ is contaminated with a zero mean noise $n(t)$, local integration will tend to eliminate the effect of the noise. This can be more rigorously justified, and the details are provided in Appendix B.

III. Further Extensions of Sampling in Wavelet Subspaces

In this section, ideas of sampling in wavelet subspaces are extended to three more cases. Namely, we consider oversampling, derivative sampling, and multiband sampling. As in Section II, we mainly work in V_0, except for the multiband case. All the assumptions from Section II-A are kept, and whenever we make some additional assumption, it will be explicitly stated.

A. Oversampling

We already saw two schemes in which both $\phi(t)$ and $S_n(t)$'s can be compactly supported. Here, we introduce another such scheme: oversampling. So far, we considered L-channel maximally decimated filter banks only (Fig. 1 is a

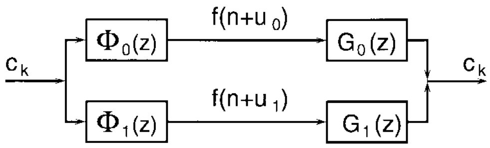

Fig. 7. Filter-bank interpretation of oversampling.

special case with $L = 1$). Let us see what happens in the case of nonmaximally decimated filter banks. Instead of doing derivations for the most general case, we will demonstrate the idea on the example of a two-channel nondecimated filter bank. For this, we choose $u_0, u_1 \in [0, 1)$ and $u_0 \neq u_1$. If we sample the signal $f(t) \in V_0$ at the two sets of points $\{n + u_0\}$ and $\{n+u_1\}$, we obtain the two sequences $\{f_o(n)\} = \{f(n+u_0)\}$ and $\{f_1(n)\} = \{f(n + u_1)\}$. We know from Section II-B that $f(t)$ can be recovered from either of these two sequences (provided conditions of Theorem 2.1 are satisfied). The idea is to use the redundancy to achieve reconstruction of $f(t)$ with compactly supported functions. Let

$$\Phi_i(e^{j\omega}) = \sum_k \phi(k + u_i)e^{-jk\omega} \quad \text{for } i = 0, 1.$$

Using our abstract approach, this situation can be represented as in Fig. 7.

In Fig. 7, if $\Phi_i(z)$ are polynomials, Euclid's algorithm guarantees existence of polynomial $G_i(z)$'s if and only if $\Phi_0(z)$ and $\Phi_1(z)$ are coprime. In the general case, when $\Phi_i(z)$'s are not necessarily FIR, the following theorem gives us solution to the reconstruction problem.

Theorem 3.1: For the system in Fig. 7, operator $\mathcal{T}$ maps $L^2[-\pi,\pi]$ into $L^2[-\pi,\pi] \times L^2[-\pi,\pi]$. If $|\Phi_0(e^{j\omega})| + |\Phi_1(e^{j\omega})| \neq 0$ for all $\omega \in [-\pi,\pi], \mathcal{T}$ has a bounded inverse. Furthermore, if $\phi(t)$ is compactly supported, there *always* exists an FIR inverse so that $S_n(t)$'s can be compactly supported.

Proof: In this case, operator $\mathcal{T}$ maps $L^2[-\pi,\pi]$ into $L^2[-\pi,\pi] \times L^2[-\pi,\pi]$. What we need is a bounded inverse operator $\mathcal{T}^{-1}$. If we can guarantee that there always exists a 1×2 matrix ($(G_0(z)\ G_1(z))$ in our notation in Fig. 7), whose entries are Fourier transforms of sequences in l^1, then we are done (see Appendix A).

For the case of polynomial $\Phi_i(z)$'s, Euclid's algorithm guarantees existence of polynomials $G_0(z)$ and $G_1(z)$ such that $\Phi_0(z)G_0(z) + \Phi_1(z)G_1(z) = 1$, provided $\Phi_0(z)$ and $\Phi_1(z)$ are coprime. The same is true for rational $\Phi_i(z)$'s.

In the general case, an extension of the Wiener's theorem (the basic Wiener theorem is stated in Appendix A) is as follows. If $|\Phi_0(e^{j\omega}| + \Phi_1(e^{j\omega})| \neq 0$ for all $\omega \in [-\pi,\pi]$ and $\Phi_0(e^{j\omega}), \Phi_1(e^{j\omega})$ have absolutely summable Fourier series, then there exist functions $G_i(e^{j\omega})$ $(i = 0, 1)$ with absolutely summable Fourier series such that $\Phi_0(e^{j\omega})G_0(e^{j\omega}) + \Phi_1(e^{j\omega})G_1(e^{j\omega}) = 1$ (see [21] for the proof). ◇

Remarks:

1) *Local averages*: It is clear that we can conceive the idea of oversampled local averages. Namely, let us divide the interval [0,1) by some $u (0 < u < 1)$ and consider the following sequences:

$$a_n^0 = \int_n^{n+u} f(t)\,dt \quad \text{and} \quad a_n^1 = \int_{n+u}^{n+1} f(t)\,dt.$$

Then, we can have a similar structure as in Fig. 7, and a theorem analogous to Theorem 3.1. holds.

2) *Oversampling at a lower rate*: Compact supports of $\phi(t)$ and $S_n(t)$'s can be guaranteed even if we oversample at a lower rate. In the discussions above, we considered a nondecimated two-channel filter bank. However, both Euclid's and Wiener's theorem can be generalized for the matrix case. For example, if we choose L points $u_0, u_1, \cdots, u_{L-1} \in [0, M]$ and sample at t_{kM+m} for $m = 0, 1, \cdots, L-1$, the scheme can be viewed as an L-channel filter bank, with decimation ratio $M < L$. The operator $\mathcal{T}$ is an $L \times M$ matrix $\boldsymbol{E}(e^{j\omega})$ whose entries are Fourier transforms of l_1 sequences. In the case of polynomial matrices, an extended Euclid's theorem guarantees existence of a polynomial inverse $M \times L$ matrix if rank $[\boldsymbol{E}(e^{j\omega})] = M$ for all $\omega \in [-\pi,\pi]$. More generally, an extended version of Wiener's theorem guarantees the existence of a bounded inverse operator $\mathcal{T}^{-1}$ if rank $[\boldsymbol{E}(e^{j\omega})] = M$ for all $\omega \in [-\pi,\pi]$. The sampling rate in this case is $L/M > 1$. Therefore, we see that compact supports for $\phi(t)$ and $S_n(t)$'s can be guaranteed if we sample at a rate $1 + \epsilon$ for any $\epsilon > 0$.

B. Reconstruction from Samples of Functions and Their Derivatives

It is well known that a bandlimited function $f(t)$ can be recovered from samples of $f(t)$ and its derivative at *half* the Nyquist rate (see [10] for further references). In this subsection, we want to show how this idea can be extended for the case of wavelet subspaces. This problem can be treated the same way as that of periodically nonuniform sampling. We will derive expressions for the analysis bank filters since that is the only difference from Section II-C. Let us demonstrate the idea on the example of reconstruction of $f(t) \in V_0$ from the samples of $f(t)$ and its derivative at rate 1/2.

We assume that the scaling function $\phi(t)$ is compactly supported and that it has a derivative $\phi'(t)$. It is also assumed that $\phi'(t)$ itself satisfies Janssen's conditions stated in Section II-A. Consider uniform sampling, i.e., $t_n = n + u$. The above assumptions enable us to differentiate (2.2) term by term and get

$$f'(t_n) = \sum_k c_k \phi'(t_n - k). \tag{3.1}$$

Using Lemma 2.1, we have

$$f(t_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jn\omega}C(e^{j\omega})\Phi_0(e^{j\omega})\,d\omega \quad \text{and}$$
$$f'(t_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jn\omega}C(e^{j\omega})\Phi_1(e^{j\omega})\,d\omega \tag{3.2}$$

where

$$\Phi_0(e^{j\omega}) = \sum_n \phi(u+n)e^{-jn\omega} \quad \text{and}$$

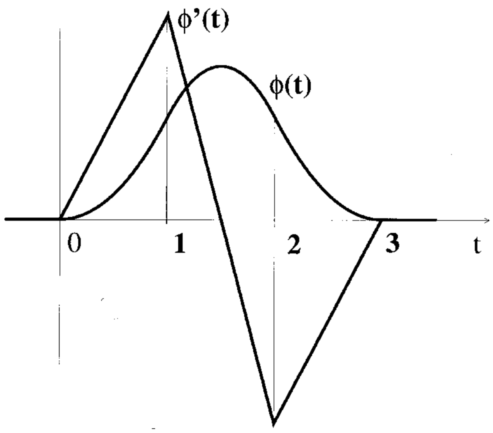

Fig. 8. Functions $\phi(t), \phi'(t)$ and their samples at integer points.

$$\Phi_1(e^{j\omega}) = \sum_n \phi'(u+n)e^{-jn\omega}. \tag{3.3}$$

From [11], we know that $f(t)$ can be reconstructed from $\{f(t_n)\}$. This means that the sequence of derivatives $\{f'(t_n)\}$ is redundant. The idea is to use this redundancy to reconstruct $f(t)$ from subsampled sequences $\{f(t_{2n})\}$ and $\{f'(t_{2n})\}$. Notice that this scheme corresponds to a two-channel maximally decimated filter bank with $\Phi_0(e^{j\omega})$ and $\Phi_1(e^{j\omega})$ as analysis filters. Therefore, everything is the same as in Fig. 2, and Theorem 2.2 provides us with conditions for the existence of a stable reconstruction scheme. Namely, stable reconstruction is possible if the polyphase matrix of $\Phi_0(e^{j\omega})$ and $\Phi_1(e^{j\omega})$ is nonsinglar for all $\omega \in [-\pi, \pi]$.

This can be easily generalized to the case of higher derivatives. Assume that the scaling function $\phi(t)$ and its $M-1$ derivatives satisfy Janssen's conditions from Section II-A. Then, $f(t)$ can be reconstructed from the samples of $f(t)$ and its $M-1$ derivatives at $1/M$th Nyquist rate, provided conditions of Theorem 2.2 are satisfied. Synthesizing functions can be constructed as in Section II-C. Let us illustrate the above derivations for the case of quadratic splines.

Example 3.1: Consider the MRA generated by the quadratic spline as in Example 2.2. Then, $\phi(t)$, its derivative, and integer samples are shown in Fig. 8. From the figure, it is easy to see that

$$\Phi_0(z) = \sum_n \phi(n)z^{-n} = 1/2(z^{-1} + z^{-2}) \quad \text{and}$$
$$\Phi_1(z) = \sum_n \phi'(n)z^{-n} = z^{-1} - z^{-2}.$$

The polyphase matrix is

$$\boldsymbol{E}(z) = \begin{pmatrix} z^{-1}/2 & 1/2 \\ -z^{-1} & 1 \end{pmatrix}.$$

Its inverse is

$$\boldsymbol{R}(z) = \begin{pmatrix} z & -z/2 \\ 1 & 1/2 \end{pmatrix}$$

so that the FIR synthesis filters are

$$G_0(z) = z + z^2 \quad \text{and} \quad G_1(z) = z/2 = z^2/2.$$

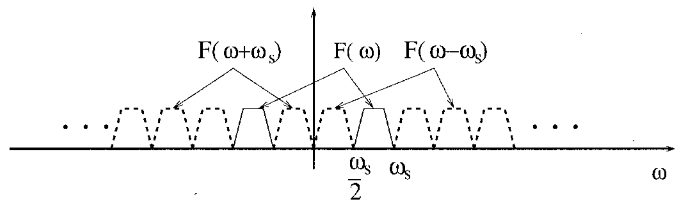

Fig. 9. Ideal bandpass signal and its aliasing copies.

Finally, the synthesis functions are $S_0(t-2n)$ and $S_1(t-2n)$, where

$$S_0(t) = \phi(t+1) + \phi(t+2) \quad \text{and}$$
$$S_1(t) = \tfrac{1}{2}\phi(t+1) - \tfrac{1}{2}\phi(t+2).$$

Remark: If one uses longer filters, $\boldsymbol{E}^{-1}(z)$ may turn out to be IIR. Then, the synthesis functions are not compactly supported. In that case, one can use techniques of nonuniform sampling or reconstruction from local averages together with sampling of derivatives to ensure compactly supported synthesizing functions.

C. Multiband or Multiscale Sampling

Consider a signal $\mathcal{F}(\omega)$ as in Fig. 9. If $\mathcal{F}(\omega)$ is regarded as a lowpass signal, the minimum necessary sampling rate is $2\omega_s$. If it is regarded as a bandpass signal, it can be verified that aliasing copies $\mathcal{F}(\omega + k\omega_s)$ caused by sampling at the rate ω_s do not overlap. Therefore, the minimum sampling rate in this case is ω_s. The aim of this subsection is to find the equivalent of this situation in wavelet subspaces.

Spaces V_k, roughly speaking, contain lowpass signals, whereas the W_k spaces contain bandpass signals. So far, we have examined lowpass signals only (ones that belong to V_0). The necessary sampling rate for exact reconstruction was unity. We will show that if a signal does not occupy the whole frequency range that V_0 covers, it can be sampled at a lower rate. For example, if $f(t) \in W_{-1}$, we can sample it at the rate 1/2. More generally, assume that $f(t) \in W_{-1} + W_{-2} + \cdots + W_{-J} \subset V_0$. This means that there are sequences $\{c_{-1,n}\}, \{c_{-2,n}\}, \cdots, \{c_{J,n}\} \in l^2$ such that

$$f(t) = \sum_{k=-J}^{-1} \sum_n c_{k,n} 2^{k/2} \psi(2^k t - n). \tag{3.4}$$

From Walter's work, we know that since $f(t) \in V_0$, it can be recovered from its integer samples. Here, the aim is to exploit the fact that $f(t)$ belongs to a subspace of V_0 and sample it at a lower rate. As in previous subsections, the idea is to find an invertible map from the sequences $\{c_{k,n}\}$ to a sequence of samples of $f(t)$. For this, let us sample $f(t)$ at $t_{k,n} = n2^{-k} + u_k, k = -J, -J+1, \cdots, -1$, where $u_k \in [0, 2^{-k})$. Intuitively, this rate should be enough since we can project $f(t)$ onto each of W_k's and then sample those projections at rates 2^k. As in the previous sections, we would like to find a nice interpretation of the operator $\mathcal{T}$ in terms of digital filter banks. The samples $\{c_{k,n}\}$ are spaced apart by 2^k, and this spacing depends on k. The same is true for the sequence $\{f(t_{k,n})\}$. In order to simplify the analysis that

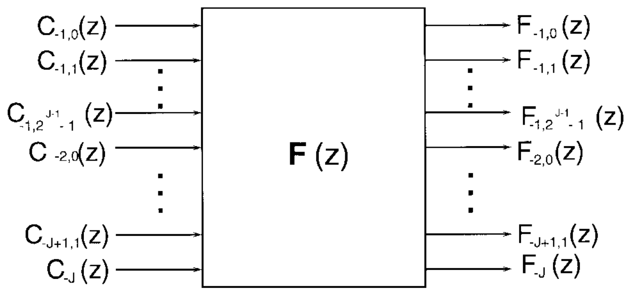

Fig. 10. Polyphase representation of a MIMO nonuniform filter bank.

follows, let us find some equivalent system where all the inputs/outputs operate at the same rate. For this, let

$$C_k(e^{j\omega}) = \sum_n c_{k,n} e^{-j\omega n} \quad \text{and}$$
$$F_m(e^{j\omega}) = \sum_n f(t_{m,n}) e^{-j\omega n} \tag{3.5}$$

be the Fourier transforms of $\{c_{k,n}\}$ and $\{f(t_{m,n})\}$. In order to bring all these signals to the same rate, we expand $C_k(e^{j\omega})$'s and $F_k(e^{j\omega})$'s into their 2^{J+k}-fold polyphase components

$$C_k(e^{j\omega}) = \sum_{l=0}^{2^{J+k}-1} e^{-j\omega l} C_{k,l}(e^{j\omega 2^{J+k}}) \quad \text{and}$$
$$F_k(e^{j\omega}) = \sum_{l=0}^{2^{J+k}-1} e^{-j\omega l} F_{k,l}(e^{j\omega 2^{J+k}}), \qquad -J \le k \le -1. \tag{3.6}$$

Now that all the inputs and outputs are brought to the same rate, our system can be represented as that in Fig. 10.

Let us find the entries of $\boldsymbol{F}(z)$. Using (3.4), the qth element of $F_m(z)$ is

$$f(2^{-m}q + u_m) = \sum_{k=-J}^{-1} \sum_l c_{k,l} 2^{k/2} \psi(2^k(2^{-m}q - 2^{-k}l + u_m)). \tag{3.7}$$

Substituting $q = 2^{J+m}n + p$ in the above expression and using

$$c_{k,l} = \frac{2^{-k}}{2\pi} \int_{-\pi}^{\pi} C_k(e^{j\omega e^{-k}}) e^{j\omega l 2^{-k}}\, d\omega$$

we get

$$\begin{aligned} f(2^J n + 2^{-m}p + u_m) &= \sum_{k=-J}^{-1} \frac{2^{-k/2}}{2\pi} \int_{-\pi}^{\pi} C_k(e^{j\omega 2^{-k}}) \\ &\quad \cdot \left(\sum_l e^{j\omega(2^{-k}l - 2^J n)} \psi(2^k(2^J n + 2^{-m}p - 2^{-k}l + u_m) \right) \\ &\quad \cdot e^{j\omega 2^J n}\, d\omega. \end{aligned} \tag{3.8}$$

Finally, after expanding $C_k(e^{k\omega 2^{-k}})$'s into their 2^{J+k}-fold polyphase components, we get

$$\begin{aligned} f(2^J n + 2^{-m}p + u_m) &= \sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+k}-1} \frac{1}{2\pi} \\ &\quad \cdot \int_{-\pi}^{\pi} C_{k,l}(e^{j\omega 2^J}) H_{k,l}^{m,p}(e^{j\omega 2^{-k}}) e^{j\omega 2^J n}\, d\omega \end{aligned} \tag{3.9}$$

where

$$H_{k,l}^{m,p}(e^{j\omega 2^{-k}}) = 2^{-k/2} e^{-j\omega 2^{-k} l} \sum_n e^{-j\omega 2^{-k} n} \cdot \psi(2^k(2^{-m}p + 2^{-k}n + u_m)).$$

Therefore, the output polyphase components are

$$\begin{aligned} F_{m,p}(e^{j\omega}) &= \left(\sum_{k=-J}^{-1} \sum^{2^{J+k}-1} C_{k,l}(e^{j\omega 2^J}) H_{k,l}^{m,p}(e^{j\omega 2^{-k}}) \right) \Big\downarrow_{2^J} \\ &= \sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+k}-1} C_{k,l}(e^{j\omega}) \left(H_{k,l}^{m,p}(e^{j\omega}) \right) \Big\downarrow_{2^{J+k}}. \end{aligned} \tag{3.10}$$

Hence, the entries of the matrix $\boldsymbol{F}(z)$ in Fig. 10 are

$$[\boldsymbol{F}(z)]_{i_1,i_2} = (H_{k,l}^{m,n}(z)) \downarrow_{2^{J+k}}$$

where

$$i_1 = 2^J - 2^{J+m+1} + n \quad \text{and}$$
$$i_2 = 2^J - 2^{J+k+1} + l.$$

As before, the MIMO version of the Wiener's theorem [21] gives us sufficient conditions for the existence of a stable inversion scheme.

Theorem 3.2: If a function

$$f(t) \in W_{-i_1} + W_{-i_2} + \cdots + W_{-i_J} \subset V_0 (1 \le i_1 < i_2 < \cdots < i_J)$$

is sampled at the rate $2^{-i_1} + 2^{-i_2} + \cdots + 2^{-i_J} < 1$, then there exists a stable reconstruction scheme if $\boldsymbol{F}(e^{j\omega})$, as defined above, is nonsingular for all $\omega \in [-\pi, \pi]$.

Remark: Notice that if the projection of $f(t)$ onto some of W_k's is zero, we can drop the corresponding term $C_k(e^{j\omega})$ and sample at an even lower rate.

Synthesizing functions can be determined from $\boldsymbol{F}^{-1}(z)$ as follows. Let

$$\boldsymbol{G}(z) = \boldsymbol{F}^{-1}(z) \quad \text{and} \quad G_{m,n}^{k,l}(z) = [\boldsymbol{G}(z)]_{i_1,i_2}$$

where

$$i_1 = 2^J - 2^{J+k+1} + l \quad \text{and} \quad i_2 = 2^J - 2^{J+m+1} + n.$$

Then

$$c_{k,2^{J+k}p+l} = \sum_{m=-J}^{-1} \sum_{n=0}^{2^{J+m}-1} \sum_q g_{m,n}^{k,l}(p-q) f(2^J q + 2^{-m}n + u_m) \tag{3.11}$$

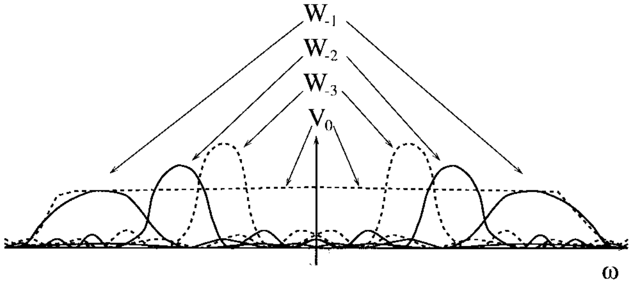

Fig. 11. $W_{-1} \cup W_{-2} \subset V_0$ in the frequency domain.

where

$$G_{m,n}^{k,l}(z) = \sum_p g_{m,n}^{k,l}(p) z^{-p}.$$

In addition

$$f(t) = \sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+K}-1} \sum_p c_{k,2^{J+K}p+l} 2^{k/2} \psi(2^k t - 2^{J+K} p - l). \tag{3.12}$$

Substituting (3.11) into the last equation, we get

$$f(t) = \sum_{m=-J}^{-1} \sum_{n=0}^{2^{J+m}-1} \sum_q f(2^J q + 2^{-m} n + u_m) S_{m,n}^q(t) \tag{3.13}$$

where

$$S_{m,n}^q(t) = \sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+k}-1} \sum_p g_{m,n}^{k,l}(p-q) 2^{k/2} \psi \cdot (2^k t - 2^{J+k} p - l). \tag{3.14}$$

These synthesizing functions have a shift property as well. Notice that $S_{m,n}^{q+1}(t) = S_{m,n}^q(t - 2^J)$, i.e., all the functions are obtained as shifts (for 2^J) of 2^J basic functions $s_{m.n}^0(t) = S_{m,n}(t)$ for $n = 0, 1, \cdots, 2^{J+m}$ and $m = -J, -J+1, \cdots, -1$. Using $S_{m.n}(t)$, the synthesis algorithm (3.14) can be written as

$$f(t) = \sum_{m=-J}^{-1} \sum_{n=0}^{2^{J+m}-1} \sum_q f(2^J q + 2^m n + u_m) S_{m,n}(t - 2^J q). \tag{3.15}$$

This provides the formula for $f(t)$ in terms of the samples $f(t_{n,k})$.

Example 3.2: Let $i_1 = 1$ and $i_2 = 2$ in Theorem 3.2. Then, our function is $f(t) \in W_{-1} \cup W_{-2} \subset V_0$.The situation is schematically sketched in Fig. 11. In this case, matrix $\boldsymbol{F}(e^{j\omega})$ is of the form

$$\boldsymbol{F}(e^{j\omega}) = \begin{pmatrix} (H_{-1,0}^{-1,0}(e^{j\omega})) \downarrow_2 & (H_{-1,1}^{-1,0}(e^{j\omega})) \downarrow_2 & H_{-2,0}^{-1,0}(e^{j\omega}) \\ (H_{-1,0}^{-1,1}(e^{j\omega})) \downarrow_2 & (H_{-1,1}^{-1,1}(e^{j\omega})) \downarrow_2 & H_{-2,0}^{-1,1}(e^{j\omega}) \\ (H_{-1,0}^{-2,0}(e^{j\omega})) \downarrow_2 & (H_{-1,1}^{-2,0}(e^{j\omega})) \downarrow_2 & H_{-2,0}^{-2,0}(e^{j\omega}) \end{pmatrix}.$$

The entries $H_{k,l}^{m,p}(e^{j\omega})$ are functions of the chosen $\phi(t)$ and $\psi(t)$, i.e., the underlying MRA. If $[\det \boldsymbol{F}(e^{j\omega})]$ turns out to be nonzero for all $\omega \in [-\pi, \pi]$, then $\boldsymbol{F}^{-1}(e^{j\omega})$ is stable, and we can obtain synthesizing functions as given by (3.11)–(3.15).

D. Efficient Computation of Inner Products in MRA

As one of the inventors of MRA, Mallat was the first to face the problem of computing inner products $c_{k,n} = (f(t), \phi_{k,n}(t))$, where $\phi_{k,n}(t) = 2^{k/2}\phi(2^k t - n)$. There exists a computationally very efficient method for getting $c_{k,n}$ from $c_{0,n}$ for any $k < 0$: the so-called "fast wavelet transform" (FWT) (tree structured filter banks in other words). Therefore, the problem is to compute $c_{0,n}$. Mallat showed that under mild conditions on regularity of $f(t)$, the samples $f(n/2^J)$ approach $c_{J,n}$ when $J \to \infty$. However, obtaining coefficients $c_{0,n}$ was an intermediate step in reconstructing $f(t)$ from its samples (or local averages) in the methods we developed earlier. This means that the schemes we proposed can be used for computation of inner products $(f(t), \phi(t-n))$. In the rest of the subsection, we will compare our methods with the existing ones.

1) Existing Schemes: Instead of computing inner products, Mallat samples $f(t)$ at the rate 2^J (notice that this rate is usually much higher than necessary, if $f(t) \in V_0$) and then uses FWT to get $c_{0,n}$ and other lower resolution coefficients. Therefore, in this case, we have very high sampling rate, moderate complexity, and relatively good approximation of true inner products. Daubechies ([9, p. 166]) outlined a method of getting $c_{0,n}$ from the integer samples $f(n)$. Walter [7] provided a detailed derivation for this method in the context of sampling in wavelet subspaces. In terms of Section II-A, Theorem 2.1 says that the coefficients $c_{0,n}$ can be exactly determined from samples of $f(n)$ (unit rate) by filtering them with $1/\Phi(e^{j\omega})$ (provided $\Phi(e^{j\omega})$ has no zeros on the unit circle). However, the problem is that $1/\Phi(e^{j\omega})$ is an IIR filter, and one has to make a truncation error. Shensa [13] proposed a method that is somewhere in between the above two. His idea is to approximate the input $f(t)$ by $\tilde{f}(t) = \Sigma_n f(n)\chi(t-n)$. It is easy to see that the inner products $(\tilde{f}(t), \phi(t-n))$ are filtered integer samples $f(n)$. The main problem with this method is in finding a good approximation $\tilde{f}(t)$ of $f(t)$, which is the only source of errors. Let us now see how our methods perform in terms of complexity, sampling rate, and approximation error.

2) New Schemes: Mallat's algorithm has a rather low complexity. Therefore, from the point of view of complexity, direct computation of inner products and Mallat's algorithm are two extreme cases. The gap between those two extremes is bridged by our local averaging scheme. It has higher complexity than Mallat's algorithm, but it gives zero error and minimal possible rate, and it has some other nice features, as explained in Section II-D. Mallat's and the Daubechies/Walter algorithm are the two extreme cases in terms of the sampling rate. While the Daubechies/Walter algorithm has a higher complexity, it has minimum possible rate and very small error. The gap between those two methods in terms of the sampling rate is bridged by our oversampling scheme. In its simplest form (for sampling rate 2), it is just Euclid's algorithm, as was described in detail in Section III-A. It achieves zero error with negligible complexity (FIR filtering)[7] at the expense of a slightly higher sampling rate (which is still small when compared with Mallat's $2^J \gg 2$). Finally, nonuniform sampling, when $\boldsymbol{E}(z)$

[7] When we say FIR filtering, it is assumed that $\phi(t)$ is compactly supported.

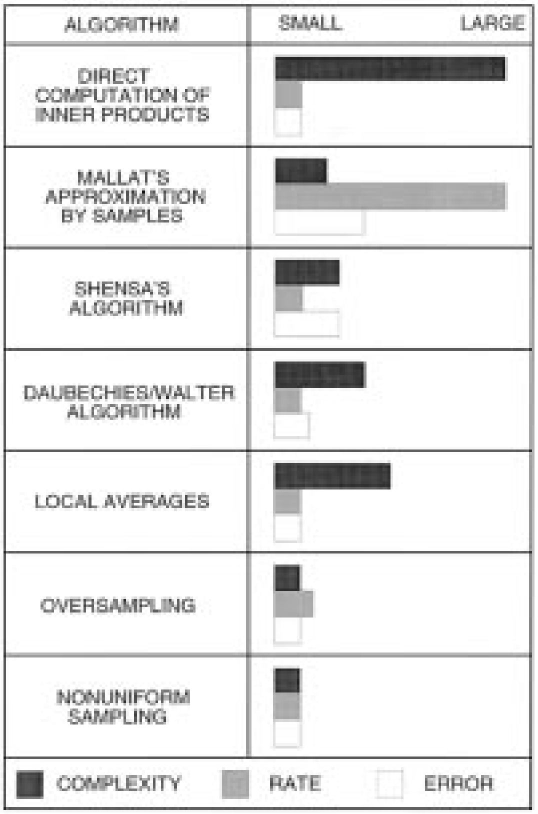

Fig. 12. Qualitative comparison of different methods for computing $c_{0,n}$'s.

is forced to be a constant matrix, achieves minimal sampling rate, small complexity (FIR filtering), and zero error.

Depending on the application, one or another scheme that we propose gives results better than any of the previous schemes. In particular, all our schemes achieve zero error and use FIR filters. The above discussion is summarized in Fig. 12. It shows relative merits of the schemes in terms of computational complexity, sampling rate, and approximation error.

It should be mentioned that all the above discussion holds true only for the case when we know that $f(t) \in V_0$; otherwise, we make an "aliasing" error [7].

E. Errors in Sampling Times

Here, we want to see what happens if we reconstruct $f(t)$ thinking that the sampling times are t_n but, in actuality, are $t'_n \neq t_n$. It will be shown that the error is bounded and tends to zero when $t'_n \to t_n$. In this subsection, we assume that $\phi(t)$ is compactly supported on an interval of length D and that it satisfies $|\phi(t_h) - \phi(t)| < C|h|^\alpha$ for every t and some $0 < \alpha < 1$ (i.e., $\phi(t)$ is Lipschitz(α) continuous). Let the actual sampling times be

$$t'_n = t_n + \delta_n \tag{3.16}$$

where $|\delta_n| < \delta$. Then, we have

$$|f(t_n) - f(t'_n)| = \left|\sum_k c_k(\phi(t_n - k) - \phi(t'_n - k))\right|. \tag{3.17}$$

From the last equation and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \|f(t_n) - f(t'_n)\|_2 &= \sum_n |f(t_n) - f(t'_n)|^2 \\ &= \sum_n \left|\sum_k c_k(\phi(t_n - k) - \phi(t'_n - k))\right|^2 \\ &\leq D^2 \sup |\phi(t_n)|^2 \sum |c_k|^2 \\ &\leq K^2 \|c_n\|_2^2 \sup_n |t_n - t'_n|^{2\alpha} \leq K\|c_n\|_2 \delta^\alpha \end{aligned} \tag{3.18}$$

where K is some constant. Because of the stability of the reconstruction algorithm, we also have

$$\|f(t) - \tilde{f}(t)\|_2 \leq K_1 \delta^\alpha \tag{3.19}$$

where $\tilde{f}(t) = \Sigma_n f(t'_n) S_n(t)$. The last inequality means that the L^2 norm of the error can be controlled by δ. As for the supremum norm of the error, we use (2.7). Then, it is immediate that

$$\sup_t |f(t) - \tilde{f}(t)| \leq K_2 \delta^\alpha. \tag{3.20}$$

We see that both L^2 and supremum norms of the error can be controlled by δ.

IV. Sampling of WSS Random Processes

The problem of sampling of random processes was thoroughly investigated by the end of 1960's. Uniform and nonuniform sampling of WSS and nonstationary bandlimited random processes was considered (for an overview, see [10]). In this section, we want to look at the problem of uniform and nonuniform sampling of WSS random processes related to wavelet subspaces.

Let us first specify more precisely the class of random processes to which the derivations will apply. Let $\phi_a(t)$ be the deterministic autocorrelation function of $\phi(t)$, i.e.

$$\phi_a(\tau) = \int \phi(t + \tau)\phi^*(t)\, dt.$$

Since $\phi(t) \in L^1(\mathcal{R}) \cap L^2(\mathcal{R})$, then $\phi_a(\tau) \in L^1(\mathcal{R}) \cap L^2(\mathcal{R})$ as well. In addition, the Cauchy–Schwarz inequality implies $|\phi_a(\tau)| \leq \|\phi(t)\|_2^2 < \infty$ for all $\tau \in \mathcal{R}$. We keep assumptions from Section II, namely, that $\{\phi(t - n)\}$ is a Riesz bases for $V_0 = \overline{\text{span}\{\phi(t - n)\}}$ and that $\phi(t)$ satisfies Janssen's conditions from Section II-A. We will consider random processes whose autocorrelation functions have the following form:

$$R_{ff}(t) = \sum_n c_n \phi_a(t - n) \tag{4.1}$$

where $\{c_n\} \in l^1$. The above series converges absolutely and uniformly on $\mathcal{R}$ so that samples of $R_{ff}(\tau)$ are well defined. Notice, in particular, that $\{c_n\} \in l^1$ implies that $R_{ff}(\tau) \in L^1(\mathcal{R})$ and $R_{ff}(n) \in l^1$. Therefore, the Fourier transform of $R_{ff}(\tau)$ exists and is equal to $\mathcal{S}_{ff}(\omega)$ (we also assume that

$R_{ff}(\tau)$ is the inverse Fourier transform of $\mathcal{S}_{ff}(\omega)$). The PSD function has the following form:

$$\mathcal{S}_{ff}(\omega) = C(e^{j\omega})\Phi_a(\omega) \tag{4.2}$$

where $C(e^{j\omega}) \geq 0$ and $\Phi_a(\omega) = |\Phi(\omega)|^2$.

In the first part, we are going to consider uniform sampling of random processes from the specified class. We show that the PSD function can be recovered but not the random process itself, unless it is bandlimited. In the second part, we consider nonuniform sampling. We have to introduce randomness into the sampling times in order to preserve wide sense stationarity of the sampled random process. As in the case of uniform sampling, we show how to reconstruct the PSD function.

A. Uniform Sampling

Consider a random processes $\{f(t), -\infty < t < \infty\}$ with autocorrelation functions $R_{ff}(\tau)$ of the form (4.1). The discrete parameter autocorrelation function is

$$r_{ff}(m,n) = E[f(n+m)f^*(m)] = R_{ff}(n). \tag{4.3}$$

Since $r_{ff}(m,n)$ is not a function of $m, \{f(n)\}$ is a discrete parameter WSS random process.

Let

$$s_{ff}(e^{j\omega}) = \sum_n \mathcal{S}_{ff}(\omega + 2\pi n).$$

Then, the Fourier coefficients of $s_{ff}(e^{j\omega})$ are integer samples of the autocorrelation function $R_{ff}(\tau)$, i.e.,

$$r_{ff}(n) = R_{ff}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_{ff}(e^{j\omega})e^{jn\omega}\, d\omega.$$

Since $r_{ff}(n) \in l^1$, we also have

$$s_{ff}(e^{j\omega}) = \sum_n r_{ff}(n)e^{-jn\omega}.$$

We will use this relationship later on.

First, we show that the random process $\{f(t)\}$ cannot be reconstructed from the samples $\{f(n)\}$ if the synthesizing functions are restricted to be shifts of one function (unless, of course, the random process is bandlimited). In order to show this, assume the contrary. Let there be a function $g(t) \in L^2(\mathcal{R})$ such that $\{f(t)\}$ is equal to $\Sigma_n f(n)g(t-n)$ in the MS sense. The error random process

$$e(t) = f(t) - \sum_n f(n)g(t-n) \tag{4.4}$$

has autocorrelation function

$$\begin{aligned} R_{ee}(t,\tau) =& E\left[\left(f(t+\tau) - \sum_n f(n)g(t+\tau-n)\right)\right.\\ &\left.\cdot\left(f^*(t) - \sum_n f^*(n)g^*(t-n)\right)\right]\\ =& R_{ff}(\tau) - \sum_n R_{ff}(t+\tau-n)g^*(t-n)\\ &- \sum_n R_{ff}(n-t)g(t+\tau-n)\\ &+ \sum_{n,m} R_{ff}(n-m)g(t+\tau-n)g^*(t-m). \end{aligned} \tag{4.5}$$

It is easy to check that $R_{ee}(t+1,\tau) = R_{ee}(t,\tau)$; threrfore, the error is a cyclo-WSS random process with period $T = 1$. We can average $R_{ee}(t,\tau)$ with respect to t to get the autocorrelation function $R_{ee}(\tau)$. Its variance is the value of $R_{ee}(\tau)$ at $\tau = 0$, i.e., $\sigma^2 = R_{ee}(0)$. Now, using the relation

$$R_{ff}(\tau) = \frac{1}{2\pi}\int e^{j\tau\omega}\mathcal{S}_{ff}(\omega)\, d\omega$$

we have (the order of integration and summation in the last term can be switched because $\{R_{ff}(n)\} \in l^1$)

$$\begin{aligned} \sigma^2 =& R_{ff}(0) - \int R_{ff}(t-n)g^*(t-n)\, dt\\ &- \int R_{ff}(n-t)g(t-n)\, dt\\ &+ \sum_l R_{ff}(l)\int g(t-l)g^*(t)\, dt. \end{aligned} \tag{4.6}$$

Using Parseval's identity, the above expressions can be simplified to

$$\begin{aligned} \sigma^2 =& \frac{1}{2\pi}\int \mathcal{S}_{ff}(\omega)\\ &\cdot\left(1 - \mathcal{G}^*(\omega) - \mathcal{G}(\omega) + \sum_k |\mathcal{G}(\omega+2\pi k)|^2\right) d\omega. \end{aligned} \tag{4.7}$$

This cannot be zeroed for any choice of $\mathcal{G}(\omega)$ unless $\phi(t)$ is bandlimited.

Even though we cannot recover the random process in the MS sense, we can still recover the PSD function $\mathcal{S}_{ff}(\omega)$. We know that $r_{ff}(n) = R_{ff}(n)$ and that

$$s_{ff}(e^{j\omega}) = \sum_k \mathcal{S}_{ff}(\omega + 2\pi k).$$

Substituting the special form of $\mathcal{S}_{ff}(\omega)$ into the last formula, we get

$$\begin{aligned} s_{ff}(e^{j\omega}) &= \sum_k C(e^{j\omega})|\Phi(\omega+2\pi k)|^2\\ &= C(e^{j\omega})\sum_k |\Phi(\omega+2\pi k)|^2. \end{aligned} \tag{4.8}$$

Since

$$s_{ff}(e^{j\omega}) = \sum_n r_{ff}(n)e^{-jn\omega}$$

we can recover $C(e^{j\omega})$ as follows:

$$C(e^{j\omega}) = \frac{\sum_n r_{ff}(n)e^{-jn\omega}}{\sum_k |\Phi(\omega+2\pi k)|^2}. \tag{4.9}$$

Notice that the division is legal because of the following: $\{\phi(t-n)\}$ is assumed to form a Riesz basis for its span. Therefore, there are constants $0 < A \leq B < \infty$ such that $A \leq \Sigma_k|\Phi(\omega+2\pi k)|^2 \leq B$ a.e., and the result of the division is in $L^2[-\pi,\pi]$. Finally, the reconstructed spectrum is

$$\mathcal{S}_{ff}(\omega)=\frac{\sum_n r_{ff}(n)e^{-jn\omega}}{\sum_k |\Phi(\omega+2\pi k)|^2}|\Phi(\omega)|^2. \tag{4.10}$$

The above derivation can be summarized in the following theorem.

Theorem 4.1: Let autocorrelation function $R_{ff}(\tau)$ of random process $\{f(t)\}$ be of the form (4.1). The PSD function $\mathcal{S}_{ff}(\omega)=C(e^{j\omega})\Phi(\omega)$ can be recovered from integer samples $\{r_{ff}(n)\}$ of $R_{ff}(\tau)$, as given by (4.10).

Remark: If $\{\phi(t-n)\}$ forms an orthonormal basis, then $\Sigma_k|\Phi(\omega+2\pi k)|^2=1$ a.e., and the above equation simplifies to

$$\mathcal{S}_{ff}(\omega)=|\Phi(\omega)|^2\sum_n r_{ff}(n)e^{-jn\omega}. \tag{4.11}$$

B. Nonuniform Sampling

It can be easily seen that a deterministic nonuniform sampling of a random process produces a nonstationary discrete parameter random process. In order to preserve stationarity, we introduce randomness (i.e., jitter) into the sampling times. These so-called stationary point random processes were investigated in [22]. One special case is when the sampling times are $t_n=n+u_n$, where u_n are independent random variables with some distribution function $p(u)$. Let $\gamma(\omega)=E_u[e^{-ju\omega}]$ be its characteristic function. The autocorrelation sequence of the discrete parameter random process $\{f(t_n)\}$ is

$$\begin{aligned} r_{ff}(m,n) &= E_u[E[f(t_{n+m})f^*(t_m)]] \\ &= E_u[R_{ff}(t_{m+n}-t_m)] \\ &= \frac{1}{2\pi}\int \mathcal{S}_{ff}(\omega)E_u[e^{j(u_{n+m}-u_m)\omega}]e^{jn\omega}\,d\omega. \end{aligned} \tag{4.12}$$

Since u_n's are mutually independent, we have

$$E_u[e^{ju_{n+m}\omega}e^{-ju_m\omega}]=\begin{cases}|\gamma(\omega)|^2, & \text{if } n\neq 0\\ 1, & \text{if } n=0\end{cases}$$

so that we finally get

$$r_{ff}(m,n)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int e^{jn\omega}|\gamma(\omega)|^2\mathcal{S}_{ff}(\omega)\,d\omega, & \text{if } n\neq 0;\\ \dfrac{1}{2\pi}\displaystyle\int \mathcal{S}_{ff}(\omega)\,d\omega & \text{if } n=0.\end{cases} \tag{4.13}$$

Since $r_{ff}(m,n)$ is independent of $m, \{f(t_n)\}$ is a WSS random process, and we will just leave out index m in (4.13). Our PSD function has a special form $\mathcal{S}_{ff}(\omega) = C(e^{j\omega})|\Phi(\omega)|^2$, and putting this in (4.13), we get

$$\begin{aligned} r_{ff}(n) &= \frac{1}{2\pi}\int e^{jn\omega}|\gamma(\omega)|^2C(e^{j\omega})|\Phi(\omega)|^2\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jn\omega}C(e^{j\omega}) \\ &\quad\cdot\sum_k|\gamma(\omega+2\pi k)|^2|\Phi(\omega+2\pi k)|^2\,d\omega \end{aligned} \tag{4.14}$$

for $n\neq 0$. It is clear now how to recover the PSD function. First, we recover $C(e^{j\omega})$

$$C(e^{j\omega})=\frac{r(0)+\sum_{n\neq 0}^{n} r_{ff}(n)e^{-jn\omega}}{\sum_k|\gamma(\omega+2\pi k)|^2|\Phi(\omega+2\pi k)|^2} \tag{4.15}$$

where $r(0)$ is given at the bottom of the page. Then, the original PSD function is

$$\mathcal{S}_{ff}(\omega)=\frac{r(0)+\sum_{n\neq 0}^{n} r_{ff}(n)e^{-jn\omega}}{\sum_k|\gamma(\omega+2\pi k)|^2|\Phi(\omega+2\pi k)|^2}|\Phi(\omega)|^2. \tag{4.16}$$

We summarize these derivations in the following theorem.

Theorem 4.2: In addition to assumptions of Theorem 3.1, assume that there is a random jitter and that its statistics are known. Then, the PSD function can be recovered from nonuniform samples $\{f(t_n)\}$ of random process $\{f(t)\}$, as given by (4.16), where $r_{ff}(n)=E[f(t_{m+n}f^*(t_m)]$.

APPENDIX A
DEFINITIONS AND THEOREMS FROM MATHEMATICAL ANALYSIS

This appendix contains some definitions and theorems from mathematical analysis that we use in the paper. It is intended to provide readers with a quick reference. For a more detailed discussion, see the corresponding references.

1) Reproducting Kernel Hilbert Spaces

Some Hilbert spaces have an additional structure built in. Functional Hilbert spaces are one such example. Here is the definition from [8].

$$r(0)=\frac{r_{ff}(0)-\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\frac{\sum_k|\Phi(\omega+2\pi k)|^2}{\sum_k|\gamma(\omega+2\pi k)\Phi(\omega+2\pi k)|^2}\sum_{n\neq 0}^{n} r_{ff}(n)e^{-jn\omega}\,d\omega}{\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\frac{\sum_k|\Phi(\omega+2\pi k)|^2}{\sum_k|\gamma(\omega+2\pi k)\Phi(\omega+2\pi k)|^2}\,d\omega}.$$

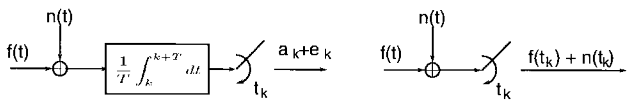

Fig. 13. Block diagrams of local averaging and sampling schemes.

Definition A.1: Let $\mathcal{H}$ be a Hilbert space of functions on X. $\mathcal{H}$ is called a *functional Hilbert* space if "point-evaluation" functionals $\Phi_x(f) = f(x)$ are bounded on $\mathcal{H}, \forall x \in X$.

It can be shown that in this case there exists a function $K(x,y)$ on $X \times X$ called the reproducting kernel such that $f(x) = (f(y), K(x,y))$ for all $f \in \mathcal{H}$. Here, $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}$. In this case, $\mathcal{H}$ is called a reproducing kernel Hilbert space (RKHS).

For a more detailed discussion of the role of the RKHS in the sampling theory, see [10] and references therein.

2) Facts From the Analysis

1) *Fourier transform*: The Fourier transform, as usually defined, exists for functions in $L^1(\mathcal{R})$ only. However, it can also be defined on $L^2(\mathcal{R})$, and in this case, it is an isometrical isomorphism from $L^2(\mathcal{R})$ onto itself. In this case, any equality is understood in the L^2 sense. Many equalities in this paper are in the L^2 sense, and it is usually clear from the context. The same is true for the Fourier transform of sequences in l^2 as well [15].

2) *Convolutions*: As we mentioned in Section II, stability in this paper does not mean BIBO stability. All we really need is for $\mathcal{T}^{-1}$ to be a bounded linear transformation from l^2 into itself. However, if $F(z)$ is a representation of $\mathcal{T}^{-1}$ and if $F(z)$ is BIBO stable, it implies that $\mathcal{T}^{-1}$ is a bounded transformation of l^2 into itself. This follows from the following theorem.

Theorem A.1 [19]: Let $f(t) \in L^p$ and $g(t) \in L^1$ $(1 \leq p \leq \infty)$. Then, the convolution $f(t) * g(t) \in L^p$. As a special case, the convolution of a sequence from l^1 and a sequence from l^p is a sequence in l^p for all $1 \leq p \leq \infty$.

3) *Wiener's Theorem [19]:* From the previous theorem, it is clear that the operation of convolution is closed in l^1. Wiener's theorem gives us a necessary and sufficient condition for the existence of a convolutional inverse of some $\{x_n\} \in l^1$.

Thorem A.2: A sequence $\{x_n\} \in l^1$ has a convolutional inverse $\{y_n\} \in l^1$ and only if $\sum_n x_n e^{-jn\omega} \neq 0$ for all $\omega \in [-\pi, \pi]$. In this case

$$\frac{1}{\sum_n x_n e^{-jn\omega}} = \sum_n y_n e^{-jn\omega}.$$

Appendix B
Sensitivity to the Input Noise

In this Appendix, we show that the "local averages" scheme is indeed less sensitive to the input noise than plain sampling. In the analysis that follows, we use the additive noise model. The setup is shown in Fig. 13. For a fair comparison, we scale the output of the integrator by $1/T$. We assume that noise $n(t)$ is a zero mean, WSS random process with finite variance and autocorrelation function $R_{nn}(\tau)$.

In the case of sampling, the output is not $f(t_k)$ as we expect but $f(t_k) + n(t_k)$. Therefore, the error term is simply a sample of the noise $n(t_k)$. The variance of this error is $\sigma_s^2 = E[|n(t_k)|^2] = R_{nn}(0)$. In the case of local averaging, the noise passes through the integrator. The output is $1/T \int_k^{k+T} (f(t) + n(t))\, dt$. Then, the error term $e_k = 1/T \int_k^{k+T} n(t)\, dt$ has variance

$$\begin{aligned} \sigma_a^2 &= E\left[\frac{1}{T}\int_0^T n(t+k)\, dt \frac{1}{T}\int_0^T n^*(s+k)\, ds\right] \\ &= \frac{1}{T^2}\int_0^T\int_0^T R_{nn}(t-s)\, dt\, ds \\ &= \frac{1}{T^2}\int_{-T}^T (T - |\tau|) R_{nn}(\tau)\, d\tau. \end{aligned} \quad \text{(B.1)}$$

We also know that $|R_{nn}(\tau)| \leq R_{nn}(0)$ for all $\tau \in \mathcal{R}$. Therefore, it follows that

$$\frac{\sigma_a^2}{\sigma_s^2} = \frac{1}{T}\int_{-T}^T \left(1 - \frac{|\tau|}{T}\right)\frac{R_{nn}(\tau)}{R_{nn}(0)} d\tau \leq 1 \quad \text{(B.2)}$$

i.e., the error due to noise in the "local averages" scheme has variance smaller or equal to that in the case of sampling. It is equal if and only if $R_{nn}(\tau) = R_{nn}(0)$ for all $\tau \in [-T, T]$.

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